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Research article

A study on the estimator for the extreme value index of heavy-tailed distribution generated from moment statistic

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Abstract: In extreme value statistics, the extreme value index of heavy-tailed distribution is closely related to the probability of occurrence of extreme events, and its estimator has become a major research topic. Based on the moment statistic, we constructed a class of estimators with four parameters for the extreme value index of heavy-tailed distribution. The consistency and asymptotic normality of the proposed estimator were proved under the first-order regular variation condition and the second-order regular variation condition. Specific expressions for ten estimators were given by the specific values of the parameters, which contain both existing estimators in the literature and newly derived ones. The asymptotical unbiasedness of specific new estimators was discussed, and some of the asymptotical unbiased estimators were compared with existing ones in terms of asymptotic variance. The results show that the new estimators perform better. In addition, in the finite sample case, using Monte-Carlo simulation, it can be seen from the simulated mean value and mean square error that the obtained results are in line with the theoretical analysis among the asymptotical unbiased estimators compared. Furthermore, it can be concluded that some of the new estimators perform better at the optimal level.

Keywords: heavy-tailed distribution; extreme value index; asymptotic unbiased estimator; asymptotic normality; Monte-Carlo simulation

1. Introduction

The heavy-tailed distribution is an important model in extreme value statistics with applications in finance, insurance, meteorology, and hydrology. Its primary parameter is the positive extreme value index (also known as extreme value index of heavy-tailed distribution, abbreviated as heavy-tailed index), which characterizes the probability of extreme events such as the catastrophic flood on the 100-year return period, enormous earthquakes, large insurance claims, and so on (see [1, 2]). Therefore, estimators for the extreme value index has become one of the main research problems in extreme value statistics.

The estimation of the extreme value index started earlier. However, in the 1970s, researchers started working on semi-parametric estimators of extreme value index. The seminal and most famous estimator is the Hill estimator [3], which is favored by many scholars for its simplicity in form; however, it is sensitive to the threshold k. In general, the Hill estimator has a large variance for small values of k, while large values of k usually induce a high bias. This means that an inadequate selection of k can result in large expected errors. Therefore, the threshold selection is also one of the most fundamental problems in extreme value statistics. The existing methods include graphical diagnostics and heuristic procedures based on sample paths' stability as a function of k, as well as minimization of the estimator of the mean square error, also as functions of k. More comprehensive reviews about threshold selection can be found in [4, 5].

Afterward, Dekkers et al. [6] proposed a moment estimator by constructing a moment statistic, which improved the Hill estimator. By use of the moment statistic, many estimators for the extreme value index have been proposed, such as the moment ratio estimator [7], the estimator proposed by Gomes and Martins [8], the estimator proposed by Caeiro and Gomes [9], and Lehmer's mean-of-order-$p(L_p)$ estimator [10]; more estimators can be found in [11, 12]. As a further study, based on the moment statistic, this paper constructs a class of heavy-tailed index estimators with four parameters. Many estimators can be obtained from specific parameter values, which include not only the existing estimators in the literature but also newly derived ones. The consistency and asymptotic normality of the proposed estimators are investigated under the first-order regular variation and second-order regular variation conditions. The asymptotical unbiasedness of the specified estimators is discussed, and a comparison of the main components of the asymptotic variance is performed in the asymptotical unbiased estimators. In addition, their finite sample performance is analyzed through a Monte-Carlo simulation in terms of the simulated mean value and mean square error. The results show that some of the new estimators perform better.

Let X be a non-negative random variable with a distribution function $F(x)$, for sufficiently large $x > 0$, satisfying

$$\overline{F}(x) := 1 - F(x) = x^{-1/\gamma}L(x), \gamma > 0, \tag{1.1}$$

we call $F(x)$ a heavy-tailed distribution, and γ is the heavy-tailed index, one of the primary parameters of extreme events, where $L(x) > 0$ is a slow varying function at infinity, that is, for all $t > 0$,

$$\lim_{x\to+\infty} \frac{L(tx)}{L(x)} = 1.$$

If a positive measure function $g(x)$ has infinite right endpoints and satisfies that for all $t > 0$, there are

$$\lim_{x\to+\infty} \frac{g(tx)}{g(x)} = t^{\alpha},$$

then $g(x)$ is said to be a regularly varying function with index α, denoted $g \in \mathcal{R}_{\alpha}$.

Let us denote $U(t) := F^{\leftarrow}(1 - 1/t)$, $t > 1$ as the tail quantile function, and $F^{\leftarrow}(t) := \inf\{x : F(x) \geq t\}, 0 < t < 1$ as a generalized inverse function of $F(x)$; from [13], the heavy-tailed distribution $F(x)$ has the following equivalence relation

$$\overline{F} \in \mathcal{R}_{-1/\gamma} \Longleftrightarrow U \in \mathcal{R}_{\gamma}. \tag{1.2}$$

In general, the equivalence relation (1.2) is called the first-order regular variation condition.

The classical heavy-tailed index estimators are all composed of the number k of the top order statistics, whose consistency are obtained under the first-order regular variation condition (1.2) and the following condition on the sequences k,

$$k = k(n) \to +\infty, \frac{k}{n} \to 0, n \to +\infty. \tag{1.3}$$

In general, we call this k intermediate if it satisfies (1.3).

In addition, in order to obtain the asymptotic normality of the heavy-tailed index estimator, the second-order regular variation condition is often needed. The $F(x)$ is said to satisfy the second-order regular variation condition if there exists an eventually positive regular varying function A with index $\rho \leq 0$, that is $|A(t)| \in \mathcal{R}_\rho$, such that

$$\lim_{t\to+\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \begin{cases} \dfrac{x^\rho - 1}{\rho}, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{1.4}$$

for every $x > 0$, where ρ is the second-order parameter. In this paper, we only consider the case $\rho < 0$.

To construct new estimators, we shall consider the following moment statistic and give some heavy-tailed index estimators.

Let $X_1, X_2, \cdots, X_n$ be a sample of n independent observations from a common unknown distribution function $F(x)$, $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ denote the ascending order statistics associated with the sample $X_1, X_2, \cdots, X_n$.

Let us consider the moment statistic

$$M_n^{(\alpha)}(k) := \frac{1}{k}\sum_{i=1}^{k}(\ln X_{n-i+1,n} - \ln X_{n-k,n})^\alpha, \alpha \geq 0. \tag{1.5}$$

Now we introduce several heavy-tailed index estimators constructed from the moment statistic.

- The Hill estimator is expressed as follows:

$$\hat{\gamma}^H(k) \equiv H(k) := M_n^{(1)}(k). \tag{1.6}$$

- The monent estimator is expressed as follows:

$$\hat{\gamma}^M(k) \equiv M(k) := M_n^{(1)}(k) + 1 - \frac{1}{2}\left(1 - \frac{\left(M_n^{(1)}(k)\right)^2}{M_n^{(2)}(k)}\right)^{-1}. \tag{1.7}$$

The moment estimator is an asymptotical unbiased estimator when γ, ρ satisfy $\gamma - \gamma\rho + \rho = 0$.

- The moment ratio estimator is expressed as follows:

$$\hat{\gamma}^{MR}(k) \equiv MR(k) := \frac{M_n^{(2)}(k)}{2H(k)}. \tag{1.8}$$

• The estimators in [8] are expressed as follows:

$$\hat{\gamma}^{\alpha}_{GM_1}(k) = GM_1(\alpha) := \frac{M_n^{(\alpha)}(k)}{\Gamma(\alpha+1)[M_n^{(1)}(k)]^{\alpha-1}}, \alpha > 0, \tag{1.9}$$

and

$$\hat{\gamma}^{\alpha}_{GM_2}(k) = GM_2(\alpha) := \left(\frac{M_n^{(\alpha)}(k)}{\Gamma(\alpha+1)}\right)^{1/\alpha}, \alpha > 0, \tag{1.10}$$

where for any $\rho < 0$, when $\alpha > 2$, there exists α_0 satisfying $(1-\rho)^{\alpha_0-1}[1+\rho(\alpha_0-2)] = 1$, such that $GM_1(\alpha_0)$ is an asymptotical unbiased estimator. In addition, note that the estimator $GM_1(\alpha)$ becomes the moment ratio estimator when $\alpha = 2$.

• The estimator in [9] is expressed as follows:

$$\hat{\gamma}^{\alpha}_{CG}(k) = CG(\alpha) := \frac{\Gamma(\alpha)}{M_n^{(\alpha-1)}(k)}\left(\frac{M_n^{(2\alpha)}(k)}{\Gamma(2\alpha+1)}\right)^{1/2}, \alpha \geq 1, \tag{1.11}$$

where for any $\rho < 0$, $CG(\alpha)$ is an asymptotical unbiased estimator when $\alpha = \alpha_0 = -\frac{\ln\left[1-\rho-\sqrt{(1-\rho)^2-1}\right]}{\ln(1-\rho)}$.

• The L_p estimator in [10] is expressed as follows:

$$\hat{\gamma}_{L_p}(k) = L_p(k) := \frac{M_n^{(p)}(k)}{pM_n^{(p-1)}(k)}, p \geq 1, \tag{1.12}$$

where the L_p estimator becomes the moment ratio estimator when $p = 2$.

2. New estimators

Based on the moment statistic (1.5), we construct a class of heavy-tailed index estimators with four parameters, whose expression is as follows:

$$\hat{\gamma}^{MG}(k,a,b,\alpha,\beta) \equiv MG(a,b,\alpha,\beta) := H(k)\left(\frac{M_n^{(a)}(k)}{\Gamma(a+1)H^a(k)}\right)^{\alpha}\left(\frac{\Gamma(b+1)H^b(k)}{M_n^{(b)}(k)}\right)^{\beta}, \tag{2.1}$$

where $a, b \geq 0, \alpha, \beta \in R$, $M_n^{(a)}(k)$ and $H(k)$, see (1.5) and (1.6), respectively.

Before discussing the properties of the proposed estimator, the following two lemmas are introduced.

Lemma 2.1. *[8] Let $X_1, X_2, \cdots, X_n$ be a sample of n independent observations from a common unknown distribution function $F(x)$, $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ denote the ascending order statistics associated with the sample $X_1, X_2, \cdots, X_n$. If $F(x)$ satisfies the first-order regular variation condition* (1.2) *and k is intermediate, then the moment statistic*

$$M_n^{(\alpha)}(k) \xrightarrow{P} \Gamma(\alpha+1)\gamma^{\alpha}. \tag{2.2}$$

Lemma 2.2. *[8] Let $X_1, X_2, \cdots, X_n$ be a sample of n independent observations from a common unknown distribution function $F(x)$, $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ denote the ascending order statistics associated with the sample $X_1, X_2, \cdots, X_n$. If $F(x)$ satisfies the second-order regular variation condition* (1.4) *and k is intermediate, then the moment statistic has the following asymptotic distributional representation*

$$M_n^{(\alpha)}(k) \stackrel{d}{=} \Gamma(\alpha+1)\gamma^\alpha + \gamma^\alpha \frac{\sigma_M(\alpha)}{\sqrt{k}} Z_k^{(\alpha)} + \gamma^{\alpha-1} b_M(\alpha) A(n/k) + o_p(A(n/k)),$$

where $b_M(\alpha) = \frac{\Gamma(\alpha+1)}{\rho}\left(\frac{1}{(1-\rho)^\alpha} - 1\right)$, $\sigma_M(\alpha) = \sqrt{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}$, $Z_k^{(\alpha)} := \frac{\frac{1}{k}\sum_{i=1}^{k}(E_i^\alpha - \Gamma(\alpha+1))}{\sigma_M(\alpha)/\sqrt{k}}$ *is an asymptotical standard normal random variable,* $\{E_i\}$, $i = 1, 2, \cdots, n$ *are independent identically distributed standard exponential random variables, and the covariance of* $Z_k^{(\alpha)}$ *and* $Z_k^{(\beta)}$ *is denoted by*

$$\sigma_M(\alpha, \beta) := cov(Z_k^{(\alpha)}, Z_k^{(\beta)}) = \frac{\Gamma(\alpha+\beta+1) - \Gamma(\alpha+1)\Gamma(\beta+1)}{\sigma_M(\alpha)\sigma_M(\beta)}.$$

The consistency and asymptotic normality of the new estimators are discussed below.

Theorem 2.1. *Let $X_1, X_2, \cdots, X_n$ be a sample of n independent observations from a common unknown distribution function $F(x)$, $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ denote the ascending order statistics associated with the sample $X_1, X_2, \cdots, X_n$. If $F(x)$ satisfies the first-order regular variation condition* (1.2) *and k is intermediate, then*

$$\hat{\gamma}^{MG}(k, a, b, \alpha, \beta) \stackrel{P}{\longrightarrow} \gamma. \tag{2.3}$$

Proof. According to Lemma 2.1 and the continuous mapping theorem, we have $H(k) \stackrel{P}{\longrightarrow} \gamma$, $\left(\frac{M_n^{(a)}(k)}{\Gamma(a+1)H^a(k)}\right)^\alpha \stackrel{P}{\longrightarrow} 1$, and $\left(\frac{\Gamma(b+1)H^b(k)}{M_n^{(b)}(k)}\right)^\beta \stackrel{P}{\longrightarrow} 1$. Using the continuous mapping theorem again, we obtain $\hat{\gamma}^{MG}(k, a, b, \alpha, \beta) \stackrel{P}{\longrightarrow} \gamma$. □

Theorem 2.2. *Let $X_1, X_2, \cdots, X_n$ be a sample of n independent observations from a common unknown distribution function $F(x)$, $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ denote the ascending order statistics associated with the sample $X_1, X_2, \cdots, X_n$. If $F(x)$ satisfies the second-order regular variation condition* (1.4) *and k is intermediate, then*

$$\hat{\gamma}^{MG}(k, a, b, \alpha, \beta) \stackrel{d}{=} \gamma + \frac{\sigma_{MG}(a, b, \alpha, \beta)}{\sqrt{k}} P_k^{MG} + b_{MG}(a, b, \alpha, \beta) A(n/k)(1 + o_p(1)), \tag{2.4}$$

where

$$b_{MG}(a, b, \alpha, \beta) = \frac{\alpha}{\rho}\frac{1-(1-\rho)^a}{(1-\rho)^a} - \frac{\beta}{\rho}\frac{1-(1-\rho)^b}{(1-\rho)^b} + (\beta b - \alpha a + 1)\frac{1}{1-\rho},$$

$$\sigma_{MG}^2(a, b, \alpha, \beta) = \frac{\alpha^2\gamma^2(\Gamma(2a+1) - \Gamma^2(a+1))}{\Gamma^2(a+1)} + \frac{\beta^2\gamma^2(\Gamma(2b+1) - \Gamma^2(b+1))}{\Gamma^2(b+1)} +$$

$$- \frac{2\alpha\beta\gamma^2(\Gamma(a+b+1)-\Gamma(a+1)\Gamma(b+1))}{\Gamma(a+1)\Gamma(b+1)} + \left[1-(\beta b-\alpha a)^2\right]\gamma^2,$$

$$P_k^{MG} := \frac{\frac{\alpha\gamma\sigma_M(a)}{\Gamma(a+1)}Z_k^{(a)} - \frac{\beta\gamma\sigma_M(b)}{\Gamma(b+1)}Z_k^{(b)} + (\beta b-\alpha a+1)\gamma\sigma_M(1)Z_k^{(1)}}{\sigma_{MG}(a,b,\alpha,\beta)}$$

is an asymptotical standard normal random variable.

Proof. By Lemma 2.2, we obtain

$$H(k) = M_n^{(1)}(k) \stackrel{d}{=} \gamma + \frac{\gamma\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} + b_M(1)A(n/k)(1+o_p(1)), \tag{2.5}$$

and

$$\frac{M_n^{(a)}(k)}{\Gamma(a+1)} \stackrel{d}{=} \gamma^a\left[1 + \frac{\sigma_M(a)}{\Gamma(a+1)}\frac{Z_k^{(a)}}{\sqrt{k}} + \frac{b_M(a)}{\gamma\Gamma(a+1)}A(n/k)(1+o_p(1))\right]. \tag{2.6}$$

Using $(1+x)^a = 1 + ax + o(x), x \to 0$, one gets

$$H^a(k) \stackrel{d}{=} \gamma^a\left[1 + \frac{a\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} + \frac{ab_M(1)}{\gamma}A(n/k)(1+o_p(1))\right].$$

From $\frac{1}{1+x} = 1 - x + o(x), x \to 0$, it follows that

$$\frac{1}{H^a(k)} \stackrel{d}{=} \gamma^{-a}\left[1 - \frac{a\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} - \frac{ab_M(1)}{\gamma}A(n/k)(1+o_p(1))\right]. \tag{2.7}$$

Combining with (2.6) and (2.7), we obtain

$$\begin{aligned}\frac{M_n^{(a)}(k)}{\Gamma(a+1)H^a(k)} &\stackrel{d}{=} \gamma^a\left[1 + \frac{\sigma_M(a)}{\Gamma(a+1)}\frac{Z_k^{(a)}}{\sqrt{k}} + \frac{b_M(a)}{\gamma\Gamma(a+1)}A(n/k)(1+o_p(1))\right]\times \\ &\quad \gamma^{-a}\left[1 - \frac{a\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} - \frac{ab_M(1)}{\gamma}A(n/k)(1+o_p(1))\right] \\ &= 1 + \frac{\sigma_M(a)}{\Gamma(a+1)}\frac{Z_k^{(a)}}{\sqrt{k}} - \frac{a\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} + \\ &\quad \left[\frac{b_M(a)}{\gamma\Gamma(a+1)} - \frac{ab_M(1)}{\gamma}\right]A(n/k)(1+o_p(1)).\end{aligned}$$

Applying $(1+x)^\alpha = 1 + \alpha x + o(x), x \to 0$ again, we obtain

$$\left(\frac{M_n^{(a)}(k)}{\Gamma(a+1)H^a(k)}\right)^\alpha \stackrel{d}{=} 1 + \frac{\alpha\sigma_M(a)}{\Gamma(a+1)}\frac{Z_k^{(a)}}{\sqrt{k}} - \frac{\alpha a\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} + \frac{\alpha}{\gamma}\left[\frac{b_M(a)}{\Gamma(a+1)} - ab_M(1)\right]A(n/k)(1+o_p(1)). \tag{2.8}$$

Similar to the proof of (2.8), we have

$$\left(\frac{\Gamma(b+1)H^b(k)}{M_n^{(b)}(k)}\right)^\beta \stackrel{d}{=} 1 - \frac{\beta\sigma_M(b)}{\Gamma(b+1)}\frac{Z_k^{(b)}}{\sqrt{k}} + \frac{\beta b\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} - \frac{\beta}{\gamma}\left[\frac{b_M(b)}{\Gamma(b+1)} - bb_M(1)\right]A(n/k)(1+o_p(1)). \tag{2.9}$$

Thus, combining with (2.5), (2.8), and (2.9), it follows that

$$
\begin{aligned}
\hat{\gamma}^{MG}(k,a,b,\alpha,\beta) &= H(k)\left(\frac{M_n^{(a)}(k)}{\Gamma(a+1)H^a(k)}\right)^{\alpha}\left(\frac{\Gamma(b+1)H^b(k)}{M_n^{(b)}(k)}\right)^{\beta} \\
&\stackrel{d}{=} \left[\gamma + \frac{\gamma\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} + b_M(1)A(n/k)(1+o_p(1))\right]\times \\
&\left\{1 + \frac{\alpha\sigma_M(a)}{\Gamma(a+1)}\frac{Z_k^{(a)}}{\sqrt{k}} - \frac{\alpha a\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} + \frac{\alpha}{\gamma}\left[\frac{b_M(a)}{\Gamma(a+1)} - ab_M(1)\right]A(n/k)(1+o_p(1))\right\}\times \\
&\left\{1 - \frac{\beta\sigma_M(b)}{\Gamma(b+1)}\frac{Z_k^{(b)}}{\sqrt{k}} + \frac{\beta b\sigma_M(1)}{\sqrt{k}}Z_k^{(1)} - \frac{\beta}{\gamma}\left[\frac{b_M(b)}{\Gamma(b+1)} - bb_M(1)\right]A(n/k)(1+o_p(1))\right\} \\
&= \gamma + \frac{\alpha\gamma\sigma_M(a)}{\Gamma(a+1)}\frac{Z_k^{(a)}}{\sqrt{k}} - \frac{\beta\gamma\sigma_M(b)}{\Gamma(b+1)}\frac{Z_k^{(b)}}{\sqrt{k}} + (\beta b - \alpha a + 1)\gamma\sigma_M(1)\frac{Z_k^{(1)}}{\sqrt{k}} + \\
&\left[\frac{\alpha b_M(a)}{\Gamma(a+1)} - \frac{\beta b_M(b)}{\Gamma(b+1)} + (\beta b - \alpha a + 1)\sigma_M(1)\right]A(n/k)(1+o_p(1)) \\
&=: \gamma + \frac{\sigma_{MG}(a,b,\alpha,\beta)}{\sqrt{k}}P_k^{MG} + b_{MG}(a,b,\alpha,\beta)A(n/k)(1+o_p(1)).
\end{aligned}
$$

From the above equation, we can see that

$$
\begin{aligned}
b_{MG}(a,b,\alpha,\beta) &= \frac{\alpha b_M(a)}{\Gamma(a+1)} - \frac{\beta b_M(b)}{\Gamma(b+1)} + (\beta b - \alpha a + 1)\sigma_M(1) \\
&= \frac{\alpha}{\rho}\frac{1-(1-\rho)^a}{(1-\rho)^a} - \frac{\beta}{\rho}\frac{1-(1-\rho)^b}{(1-\rho)^b} + (\beta b - \alpha a + 1)\frac{1}{1-\rho},
\end{aligned}
$$

$$
P_k^{MG} := \frac{\dfrac{\alpha\gamma\sigma_M(a)}{\Gamma(a+1)}Z_k^{(a)} - \dfrac{\beta\gamma\sigma_M(b)}{\Gamma(b+1)}Z_k^{(b)} + (\beta b - \alpha a + 1)\gamma\sigma_M(1)Z_k^{(1)}}{\sigma_{MG}(a,b,\alpha,\beta)}.
$$

Let $A = \frac{\alpha\gamma}{\Gamma(a+1)}$, $B = \frac{\beta\gamma}{\Gamma(b+1)}$ and $C = (\beta b - \alpha a + 1)\gamma$, we have

$$
\begin{aligned}
\sigma_{MG}(a,b,\alpha,\beta)P_k^{MG} &= A\sigma_M(a)Z_k^{(a)} + B\sigma_M(b)Z_k^{(b)} + C\sigma_M(1)Z_k^{(1)} \\
&= \frac{A}{\sqrt{k}}\sum_{i=1}^{k}(E_i^a - \Gamma(a+1)) + \frac{B}{\sqrt{k}}\sum_{i=1}^{k}(E_i^b - \Gamma(b+1)) + \frac{C}{\sqrt{k}}\sum_{i=1}^{k}(E_i - 1) \\
&= \frac{1}{\sqrt{k}}\sum_{i=1}^{k}[AE_i^a + BE_i^b + CE_i - (A\Gamma(a+1) + B\Gamma(b+1) + C)] \\
&=: \frac{1}{\sqrt{k}}\sum_{i=1}^{k}[Y_i - (A\Gamma(a+1) + B\Gamma(b+1) + C)]
\end{aligned}
$$

From Lemma 2.2, one can deduce that $\{Y_i\}$, $i = 1, 2, \cdots, k$ are independent, identically distributed random variables. It is easy to obtain that $E(Y_i) = A\Gamma(a+1) + B\Gamma(b+1) + C$ and $Var(Y_i) = \sigma_{MG}^2(a,b,\alpha,\beta)$.

Applying Lindeberg-Lévy central limit theorem, we get that P_k^{MG} is an asymptotical standard normal random variable, and

$$\begin{aligned}\sigma_{MG}^2(a,b,\alpha,\beta) &= Var\left(\frac{\alpha\gamma\sigma_M(a)}{\Gamma(a+1)}Z_k^{(a)} - \frac{\beta\gamma\sigma_M(b)}{\Gamma(b+1)}Z_k^{(b)} + (\beta b-\alpha a+1)\gamma\sigma_M(1)Z_k^{(1)}\right)\\ &= \frac{\alpha^2\gamma^2\sigma_M^2(a)}{\Gamma^2(a+1)} + \frac{\beta^2\gamma^2\sigma_M^2(b)}{\Gamma^2(b+1)} + (\beta b-\alpha a+1)^2\gamma^2\sigma_M^2(1) - \\ &\quad \frac{2\alpha\beta\gamma^2\sigma_M(a)\sigma_M(b)}{\Gamma(a+1)\Gamma(b+1)}\sigma_M(a,b) + \frac{2\alpha(\beta b-\alpha a+1)\gamma^2\sigma_M(a)\sigma_M(1)}{\Gamma(a+1)}\sigma_M(a,1) - \\ &\quad \frac{2\beta(\beta b-\alpha a+1)\gamma^2\sigma_M(b)\sigma_M(1)}{\Gamma(b+1)}\sigma_M(b,1)\\ &= \frac{\alpha^2\gamma^2(\Gamma(2a+1)-\Gamma^2(a+1))}{\Gamma^2(a+1)} + \frac{\beta^2\gamma^2(\Gamma(2b+1)-\Gamma^2(b+1))}{\Gamma^2(b+1)} + \\ &\quad (\beta b-\alpha a+1)^2\gamma^2 - \frac{2\alpha\beta\gamma^2(\Gamma(a+b+1)-\Gamma(a+1)\Gamma(b+1))}{\Gamma(a+1)\Gamma(b+1)} + \\ &\quad 2(\alpha a-\beta b)(\beta b-\alpha a+1)\gamma^2\\ &= \frac{\alpha^2\gamma^2(\Gamma(2a+1)-\Gamma^2(a+1))}{\Gamma^2(a+1)} + \frac{\beta^2\gamma^2(\Gamma(2b+1)-\Gamma^2(b+1))}{\Gamma^2(b+1)} + \\ &\quad - \frac{2\alpha\beta\gamma^2(\Gamma(a+b+1)-\Gamma(a+1)\Gamma(b+1))}{\Gamma(a+1)\Gamma(b+1)} + \left[1-(\beta b-\alpha a)^2\right]\gamma^2.\end{aligned}$$

□

Corollary 2.1. *Under the conditions of Theorem 2.2, suppose that* $\sqrt{k}A(n/k) \to \lambda < +\infty$*, as* $n \to +\infty$*, then*

$$\sqrt{k}\left(\hat{\gamma}^{MG}(k,a,b,\alpha,\beta)-\gamma\right) \xrightarrow{d} N(\lambda b_{MG}(a,b,\alpha,\beta), \sigma_{MG}^2(a,b,\alpha,\beta)).$$

Remark 1. *For every* $\rho < 0$*, if there exist* a, b, α, β*, such that* $b_{MG}(a,b,\alpha,\beta) = 0$*, then the corresponding* $MG(a,b,\alpha,\beta)$ *is an asymptotical unbiased estimator of* γ*, even when* $\lambda \neq 0$*.*

3. Specific expression of new estimators

The estimator $MG(a,b,\alpha,\beta)$ has four parameters, and for the sake of dealing with practical problems, one can consider specifying the parameters to obtain different specific estimator expressions. We give ten specific estimators below.

(E1) $MG(a,b,0,0) = H(k) := M_n^{(1)}(k)$;

(E2) $MG(a,b,1,0) = GM_1(a) := \dfrac{M_n^{(a)}(k)}{\Gamma(a+1)[M_n^{(1)}(k)]^{a-1}}, a > 0$;

(E3) $MG(a,b,1/a,0) = GM_2(a) := \left(\dfrac{M_n^{(a)}(k)}{\Gamma(a+1)}\right)^{1/a}, a > 0$;

(E4) $MG(2a,a-1,1/2,1) = CG(a) := \dfrac{\Gamma(a)}{M_n^{(a-1)}(k)}\left(\dfrac{M_n^{(2a)}(k)}{\Gamma(2a+1)}\right)^{1/2}, a \geq 1$;

(E5) $MG(a, a-1, 1, 1) = L_a(k) := \dfrac{M_n^{(a)}(k)}{aM_n^{(a-1)}(k)}, a \geq 1;$

(E6) $MG(2a, a, 1/2, 1) = H(k)\left(\dfrac{M_n^{(2a)}(k)}{\Gamma(2a+1)}\right)^{1/2} \dfrac{\Gamma(a+1)}{M_n^{(a)}(k)};$

(E7) $MG(2a, a, 1/2, 0) = H(k)\left(\dfrac{M_n^{(2a)}(k)}{\Gamma(2a+1)H^{2a}(k)}\right)^{1/2};$

(E8) $MG(2, b, a, 0) = H(k)\left(\dfrac{M_n^{(2)}(k)}{2H^2(k)}\right)^{a};$

(E9) $MG(a, 2, 1, a/2) = \dfrac{2^{a/2}H(k)M_n^{(a)}(k)}{\Gamma(a+1)\left(M_n^{(2)}(k)\right)^{a/2}};$

(E10) $MG(2, a, 1, -1) = \dfrac{M_n^{(2)}(k)M_n^{(a)}(k)}{2\Gamma(a+1)H^{a+1}(k)}.$

Among the above-mentioned ten estimators , (E1)–(E5) are existing estimators and (E6)–(E10) are new estimators. As described in the literature [3, 8–10], (E1), (E3), and (E5) are asymptotical biased estimators, while (E2) and (E4) are asymptotical unbiased estimators when appropriate parameters are chosen. The asymptotic unbiasedness of the new estimators (E6)–(E10) is discussed below.

Let $f(x) = \frac{1-(1-\rho)^x}{\rho(1-\rho)^x}, x > 0, \rho < 0$, then $f'(x) = -\frac{\ln(1-\rho)}{\rho}\frac{1}{(1-\rho)^x} > 0$. Therefore, $f(x)$ is an increasing function for a given ρ.

Thus, the main component of the asymptotic bias of the new estimator $b_{MG}(a, b, \alpha, \beta)$ in Theorem 2.2 can be rewritten as

$$b_{MG}(a, b, \alpha, \beta) = \alpha f(a) - \beta f(b) + (\beta b - \alpha a + 1) f(1).$$

Since the main component of the asymptotic bias of the estimator (E6)–(E10) involves only one parameter, it can be abbreviated as $b_{MGi}(a), i = 6, 7, 8, 9, 10$ for convenience. The asymptotic unbiasedness of the estimators (E6)–(E10) is discussed below.

For the estimator (E6), the main component of the asymptotic bias is $b_{MG6}(a) = \frac{1}{2}f(2a) - f(a) + f(1)$. Since $\frac{db_{MG6}(a)}{da} = f'(2a) - f'(a) = -\frac{\ln(1-\rho)}{\rho}\left(\frac{1}{(1-\rho)^{2a}} - \frac{1}{(1-\rho)^{a}}\right) < 0$, $b_{MG6}(a)$ is a decreasing function on a for a given $\rho < 0$. Also, $b_{MG6}(0) = \frac{1}{1-\rho} > 0$, and $\lim\limits_{a\to+\infty} b_{MG6}(a) = \frac{1+\rho}{2\rho(1-\rho)}$. It is easy to see that $\frac{1+\rho}{2\rho(1-\rho)} < 0$ when $1 + \rho > 0$. There exists a_0 such that $b_{MG6}(a_0) = 0$, i.e., the estimator (E6) with the parameter a_0 is an asymptotical unbiased estimator.

For the estimator (E7), the main component of the asymptotic bias is $b_{MG7}(a) = \frac{1}{2}f(2a) + (1-a)f(1)$. Since $b_{MG7}(0) = \frac{1}{1-\rho} > 0$, and $\lim\limits_{a\to+\infty} b_{MG7}(a) = -\infty$, there always exists a_0 such that $b_{MG7}(a_0) = 0$, i.e., the estimator (E7) with the parameter a_0 is asymptotically unbiased, where a_0 satisfies $(1-\rho)^{2a_0-1}[(2a_0 - 3)\rho + 1] = 1$.

For the estimator (E8), the main component of the asymptotic bias is $b_{MG8}(a) = af(2) + (1-2a)f(1)$. Since $b_{MG8}(0) = \frac{1}{1-\rho} > 0$, and $\lim\limits_{a\to+\infty} b_{MG8}(a) = -\infty$, there always exists a_0 such that $b_{MG8}(a_0) = 0$, i.e., the estimator (E8) with the parameter a_0 is asymptotically unbiased, where $a_0 = 1 - 1/\rho$.

For the estimator (E9), the main component of the asymptotic bias is $b_{MG9}(a) = f(a) - \frac{a}{2}f(2) + f(1)$. Since $b_{MG9}(0) = \frac{1}{1-\rho} > 0$, and $\lim\limits_{a\to+\infty} b_{MG9}(a) = -\infty$, there always exists a_0 such that $b_{MG9}(a_0) = 0$, i.e., the estimator (E9) with the parameter a_0 is asymptotically unbiased, where a_0 satisfies $(1-\rho)^{a_0-2}[(4 - a_0)\rho^2 + (2a_0 - 6)\rho + 2] = 2$.

For the estimator (E10), the main component of the asymptotic bias is $b_{MG10}(a) = f(a) + f(2) - (1+a)f(1)$. Since $b_{MG10}(0) = f(2) - f(1) > 0$, and $\lim\limits_{a\to+\infty} b_{MG10}(a) = -\infty$, there always exists a_0 such that $b_{MG10}(a_0) = 0$, i.e., the estimator (E10) with the parameter a_0 is asymptotically unbiased, where a_0 satisfies $(1-\rho)^{a_0-2}[(1-a_0)\rho^2 + (a_0-3)\rho + 1] = 1$.

In summary, the estimator (E6) can be asymptotically unbiased only if it satisfies $1+\rho > 0$ and a suitable a is chosen. However, for any $\rho < 0$, one can always find a such that the estimators (E7)–(E10) are asymptotically unbiased and the value of a depends only on ρ.

Next, in the above-mentioned asymptotical unbiased estimators, we compare the main components of their asymptotic variances $\sigma^2_{MG}(a, b, \alpha, \beta)$. The asymptotical unbiased estimators (E2), (E4), (E7), (E8), (E9), and (E10) are considered here. For convenience, the main components of the asymptotic variance of the estimators involved are denoted as $\sigma_i^2, i = 2, 4, 7, 8, 9, 10$. Given different values of ρ, we compute the ratio of $\sigma_i^2, i = 2, 4, 7, 8, 9, 10$ to γ^2 in the compared estimators (the ratio depends only on ρ) and the corresponding values of a_0. The results of the calculations are shown in Table 1. From Table 1, we can draw the following conclusions.

1) All ratios decrease as ρ decreases.

2) For a given ρ, σ_2^2/γ^2, σ_8^2/γ^2 and σ_{10}^2/γ^2 are smaller, followed by σ_4^2/γ^2 and σ_7^2/γ^2, and finally σ_9^2/γ^2.

3) For σ_2^2/γ^2, σ_8^2/γ^2 and σ_{10}^2/γ^2, when $\rho > -1$, $\sigma_8^2/\gamma^2 < \sigma_{10}^2/\gamma^2 < \sigma_2^2/\gamma^2$; when $\rho < -1$, $\sigma_{10}^2/\gamma^2 < \sigma_8^2/\gamma^2 < \sigma_2^2/\gamma^2$.

Overall, among the asymptotical unbiased estimators compared, the estimators (E8) and (E10) perform better with smaller values of asymptotic variance.

Table 1. Comparison of $\sigma_i^2/\gamma^2, i = 2, 4, 7, 8, 9, 10$ in the asymptotic unbiased estimators and the corresponding values of a_0.

ρ	-0.25	-0.5	-0.75	-1	-1.25	-1.5	-1.75	-2
σ_2^2/γ^2	47.22	14.64	8.707	6.443	5.287	4.596	4.140	3.818
(a_0)	(3.91)	(3.17)	(2.86)	(2.69)	(2.58)	(2.50)	(2.44)	(2.39)
σ_4^2/γ^2	277.8	38.75	17.21	10.92	8.133	6.609	5.661	5.019
(a_0)	(3.11)	(2.37)	(2.07)	(1.90)	(1.79)	(1.71)	(1.65)	(1.60)
σ_7^2/γ^2	121.7	29.62	16.37	11.72	9.452	8.131	7.274	6.678
(a_0)	(2.78)	(2.26)	(2.05)	(1.93)	(1.86)	(1.80)	(1.76)	(1.73)
σ_8^2/γ^2	26.00	10.00	6.444	5.000	4.240	3.778	3.469	3.250
(a_0)	(5.00)	(3.00)	(2.33)	(2.00)	(1.80)	(1.67)	(1.57)	(1.50)
σ_9^2/γ^2	120.5	42.60	28.70	23.74	21.49	20.38	19.84	19.60
(a_0)	(4.74)	(4.12)	(3.91)	(3.81)	(3.76)	(3.73)	(3.72)	(3.71)
σ_{10}^2/γ^2	35.72	11.18	6.700	5.000	4.142	3.637	3.310	3.083
(a_0)	(3.52)	(2.64)	(2.24)	(2.00)	(1.84)	(1.72)	(1.63)	(1.56)

4. Comparison of finite sample properties

In order to investigate the performance of the estimators (E2), (E4), (E7)–(E10) mentioned in the previous section in the finite sample case, Monte-Carlo simulations are used to generate $N = 100$ samples of sample size $n = 1000$ from the following model.

1) Fréchet(γ) model, $F(x) = \exp(-x^{-1/\gamma}), x \geq 0, \rho = -1$;

2) Burr(γ, ρ) model, $F(x) = 1 - (1 + x^{-\rho/\gamma})^{1/\rho}, x \geq 0$.

For convenience, the estimators involved are denoted as $\hat{\gamma}_i, i = 2, 4, 7, 8, 9, 10$. And for each estimator, we computed the simulated mean value (E) and mean square error (MSE) of the estimators. The calculation formulas are as follows:

$$\mathrm{E}[\hat{\gamma}_i] := \frac{1}{N}\sum_{j=1}^{N}\hat{\gamma}_{ij}, \quad \mathrm{MSE}[\hat{\gamma}_i] := \frac{1}{N}\sum_{j=1}^{N}(\hat{\gamma}_{ij} - \gamma)^2.$$

From the discussion in the previous section, it can be seen that the parameter values of the asymptotic unbiased estimators $\hat{\gamma}_i, i = 2, 4, 7, 8, 9, 10$ only depend on the parameter ρ. Therefore, the following simulations will be divided into two cases: ρ is known and ρ is unknown. For the case where ρ is unknown, we will adopt the approach of the literature [14, 15] to give an estimator of ρ, which in turn can be used to obtain an estimator of the parameter a_0 in the estimators compared. Specifically, the following is discussed.

We estimate the parameter ρ by the following estimator proposed in [14].

$$\hat{\rho} = \hat{\rho}_n^{(\tau)}(k) := -\left|\frac{3(R_n^{(\tau)}(k) - 1)}{R_n^{(\tau)}(k) - 3}\right| \tag{4.1}$$

where

$$R_n^{(\tau)}(k) := \begin{cases} \dfrac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}}, & \text{if } \tau \neq 0, \\ \dfrac{\ln\left(M_n^{(1)}(k)\right) - (1/2)\ln\left(M_n^{(2)}(k)/2\right)}{(1/2)\ln\left(M_n^{(2)}(k)/2\right) - (1/3)\ln\left(M_n^{(3)}(k)/6\right)}, & \text{if } \tau = 0. \end{cases}$$

To decide which value (0 or 1) of the parameter τ to take in the aforementioned estimator, we use the algorithm provided in [15]. And for the estimator $\hat{\rho}$, following the recommendation from [15], we use $k = \min\{n - 1, [2n^{0.995}/\ln(\ln(n))]\}$.

The simulation results are shown in Figures 1–4. Figure 1 shows the simulated mean values and MSEs of the estimators involved under study for sample of size $n = 1000$ from the Fréchet(1) model when ρ is unknown. The simulated mean values show that all estimators are almost asymptotically unbiased. Regarding the simulated MSEs, for almost all values of k, we can see that

$$\mathrm{MSE}(\hat{\gamma}_{10}) < \mathrm{MSE}(\hat{\gamma}_8) < \mathrm{MSE}(\hat{\gamma}_2) < \mathrm{MSE}(\hat{\gamma}_4) < \mathrm{MSE}(\hat{\gamma}_7) < \mathrm{MSE}(\hat{\gamma}_9)$$

and $\mathrm{MSE}(\hat{\gamma}_{10})$ is almost equal to $\mathrm{MSE}(\hat{\gamma}_8)$. This is generally consistent with the theoretical analysis.

Figure 2 is the equivalent of Figure 1 when ρ is known and similar to Figure 7 in [9]. The simulated mean values show that all estimators are asymptotically unbiased. Regarding the simulated MSEs, for every k, we can see that

$$\mathrm{MSE}(\hat{\gamma}_{10}) < \mathrm{MSE}(\hat{\gamma}_8) < \mathrm{MSE}(\hat{\gamma}_2) < \mathrm{MSE}(\hat{\gamma}_4) < \mathrm{MSE}(\hat{\gamma}_7) < \mathrm{MSE}(\hat{\gamma}_9).$$

In addition, $\mathrm{MSE}(\hat{\gamma}_{10})$ is almost equal to $\mathrm{MSE}(\hat{\gamma}_8)$ and $\mathrm{MSE}(\hat{\gamma}_4)$ is closer to $\mathrm{MSE}(\hat{\gamma}_7)$. This is consistent with the theoretical analysis.

Figures 3 and 4 show the simulated mean values and MSEs of the estimators involved under study for sample of size $n = 1000$ from the Burr(1, -1) model when ρ is unknown and known, respectively. They perform essentially the same in terms of the simulated mean value and MSEs. The simulated mean values show that the estimators seem to be asymptotically unbiased for some of the values of k.

At this time, for most of the values of k, it is clear from the simulated MSEs that

$$\mathrm{MSE}(\hat{\gamma}_{10}) < \mathrm{MSE}(\hat{\gamma}_8) < \mathrm{MSE}(\hat{\gamma}_2) < \mathrm{MSE}(\hat{\gamma}_4) < \mathrm{MSE}(\hat{\gamma}_7) < \mathrm{MSE}(\hat{\gamma}_9).$$

For other different values of Burr's model, the conclusions obtained are similar to the results of the above analysis.

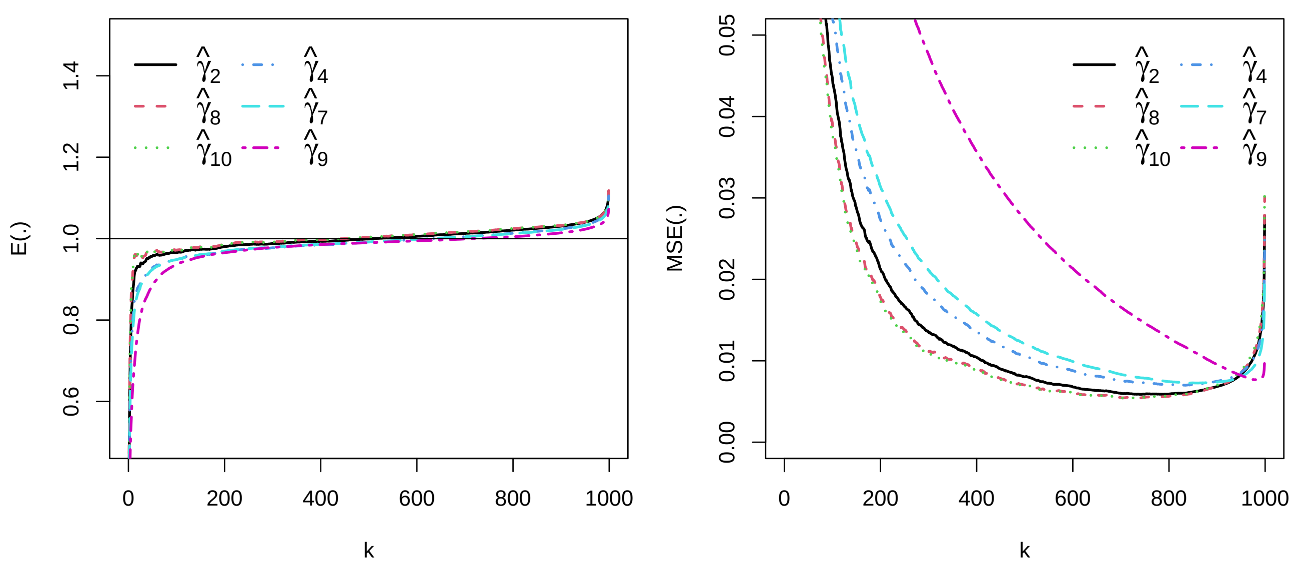

Figure 1. Simulated mean values (left) and MSEs (right) of the estimators involved under study for sample of size $n = 1000$ from Fréchet(1) model (ρ is unknown).

In the following, the above-mentioned estimators are considered to be compared in terms of the simulated mean value and mean square error at the optimal level k_0,

$$k_0 := \arg\min_k \mathrm{MSE}(\bullet),$$

where $\bullet$ denote $\hat{\gamma}_i, i = 2, 4, 7, 8, 9, 10$. The estimator of the parameter α is given by the estimator of ρ in (4.1). And the estimators $\hat{\gamma}_i, i = 2, 4, 7, 8, 9, 10$ are denoted as $\hat{\gamma}_i(\alpha_{i0}, k_{i0}), i = 2, 4, 7, 8, 9, 10$ at the

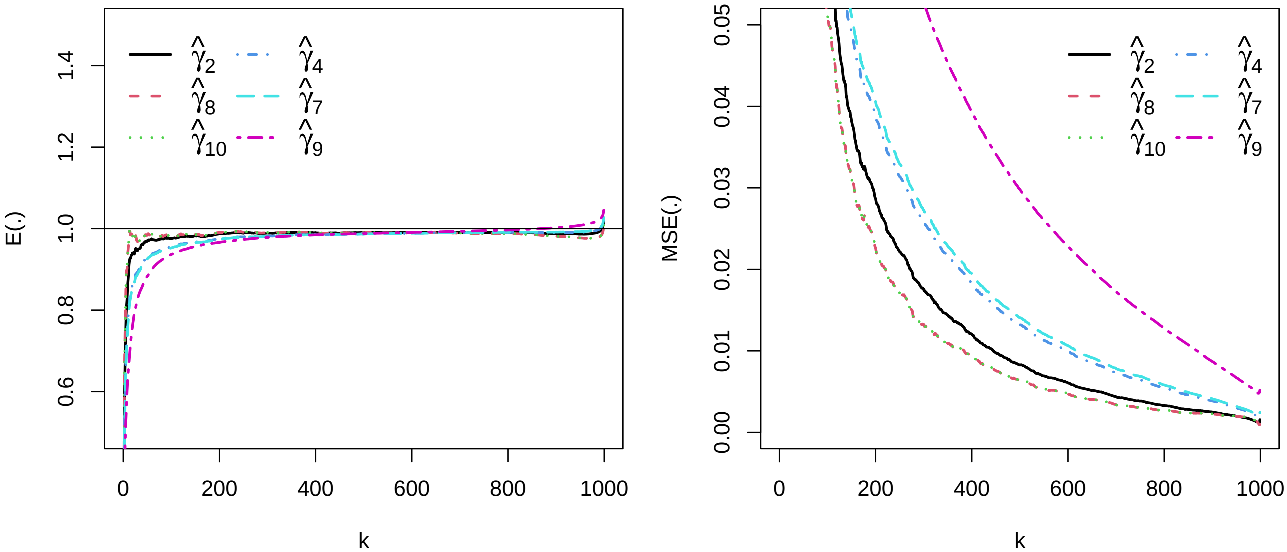

Figure 2. Simulated mean values (left) and MSEs (right) of the estimators involved under study for sample of size n = 1000 from Fréchet(1) model (ρ is known).

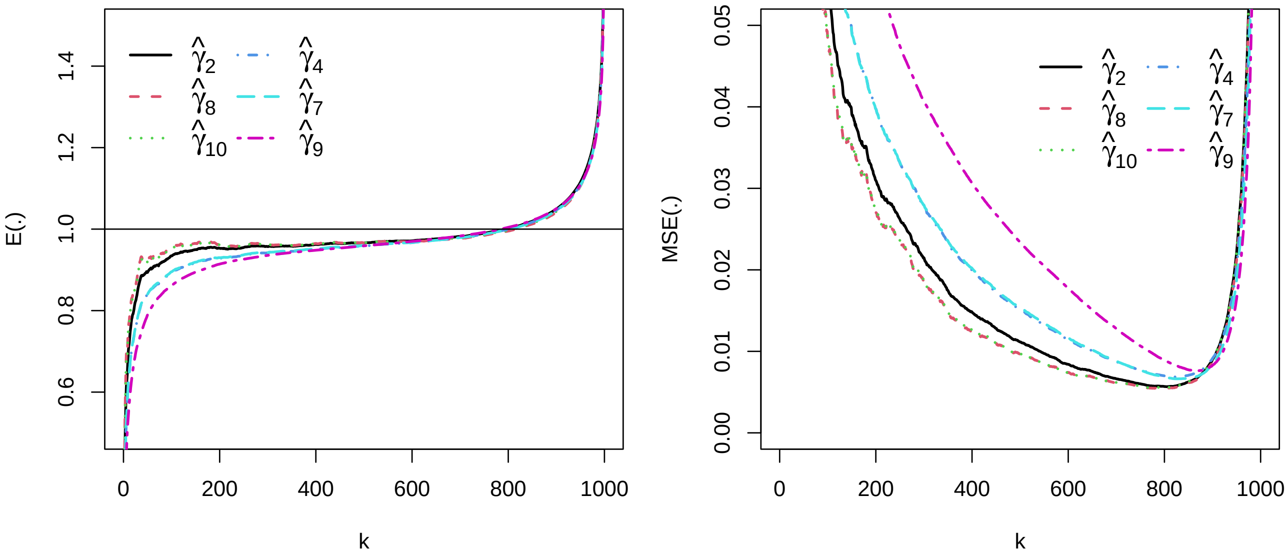

Figure 3. Simulated mean values (left) and MSEs (right) of the estimators involved under study for sample of size n = 1000 from Burr(1,-1) model (ρ is unknown).

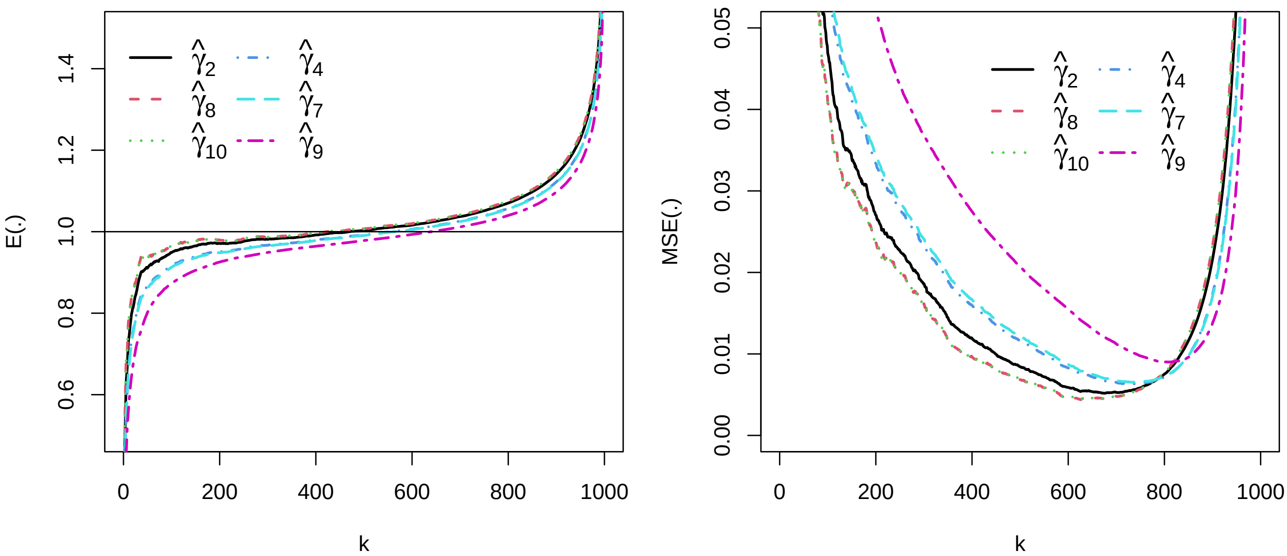

Figure 4. Simulated mean values (left) and MSEs (right) of the estimators involved under study for sample of size $n = 1000$ from Burr(1,-1) model (ρ is known).

optimal level. We have implemented Monte-Carlo simulation experiments of size 100 for sample sizes $n = 100, 200, 500, 1000$, and 2000, from the Fréchet and Burr models. For each model, mean values with the smallest squared bias and the smallest root mean square error (RMSE) are presented in bold, where RMSE := $\sqrt{\text{MSE}}$. The simulated results are shown in Tables 2–5.

From Tables 2–5, we now provide a few comments.

1) For all simulated models, the simulated values of $\hat{\gamma}_8(\alpha_{80}, k_{80})$ and $\hat{\gamma}_{10}(\alpha_{100}, k_{100})$ are almost equal especially in terms of RMSE.

2) For models with $\rho \geq -1$, $\hat{\gamma}_4(\alpha_{40}, k_{40})$ has a smaller squared bias than other estimators in terms of the simulated mean value.

3) For models with $\rho < -1$, $\hat{\gamma}_7(\alpha_{70}, k_{70})$ has a smaller squared bias than other estimators in terms of the simulated mean value. However, $\hat{\gamma}_9(\alpha_{90}, k_{90})$ has a smaller RMSE than other estimators when $k \geq 500$.

Overall, the new estimators perform well within a certain range.

5. Conclusions

In extreme value statistics, the estimator for the heavy-tailed index is one of the most important current research topics. By means of the moment statistic, this paper constructs a class of heavy-tailed index estimators with four parameters. The consistency and asymptotic normality of the proposed estimators are proved under first-order and second-order regular variation conditions. For the proposed estimators, ten estimators are given by the specific values of the parameters, which include both the existing estimators in the literature and new estimators. The asymptotic unbiasedness is discussed for

Table 2. Simulated mean values/RMSE of $\hat{\gamma}_i(\alpha_{i0}, k_{i0}), i = 2, 4, 7, 8, 9, 10$ at their simulated optimal level from Fréchet(1) model.

n	100	200	500	1000	2000
$\hat{\gamma}_2(\alpha_{20}, k_{20})$	1.0186 / 0.1983	1.0606 / 0.1689	1.0128 / 0.1067	1.0135 / 0.0713	1.0139 / 0.0593
$\hat{\gamma}_4(\alpha_{40}, k_{40})$	1.0059 / 0.2051	**1.0525** / 0.1772	**1.0092** / 0.1139	**1.0079** / 0.0770	1.0123 / 0.0648
$\hat{\gamma}_7(\alpha_{70}, k_{70})$	**1.0037** / 0.2062	1.0587 / 0.1817	1.0185 / 0.1158	1.0113 / 0.0778	1.0134 / 0.0658
$\hat{\gamma}_8(\alpha_{80}, k_{80})$	1.0239 / **0.1965**	1.0639 / 0.1656	1.0166 / **0.1042**	1.0165 / **0.0692**	1.0135 / 0.0574
$\hat{\gamma}_9(\alpha_{90}, k_{90})$	1.0137 / 0.2049	1.0866 / 0.1889	1.0475 / 0.1205	1.0278 / 0.0760	1.0268 / 0.0675
$\hat{\gamma}_{10}(\alpha_{100}, k_{100})$	1.0241 / 0.1966	1.0634 / **0.1650**	1.0123 / 0.1044	1.0167 / **0.0692**	**1.0119** / **0.0573**

Table 3. Simulated mean values/RMSE of $\hat{\gamma}_i(\alpha_{i0}, k_{i0}), i = 2, 4, 7, 8, 9, 10$ at their simulated optimal level from Burr(1, -0.5) model.

n	100	200	500	1000	2000
$\hat{\gamma}_2(\alpha_{20}, k_{20})$	1.2094 / 0.3335	1.0682 / 0.1914	1.1336 / 0.2139	1.1185 / 0.1626	1.0658 / 0.1133
$\hat{\gamma}_4(\alpha_{40}, k_{40})$	**1.1723** / **0.3194**	1.0807 / **0.1801**	**1.1297** / **0.2097**	**1.1073** / 0.1700	1.0608 / 0.1173
$\hat{\gamma}_7(\alpha_{70}, k_{70})$	1.1741 / 0.3219	1.0833 / 0.1805	1.1317 / 0.2097	1.1086 / 0.1690	**1.0527** / 0.1169
$\hat{\gamma}_8(\alpha_{80}, k_{80})$	1.2247 / 0.3408	1.0907 / 0.1986	1.1461 / 0.2127	1.1246 / **0.1606**	1.0566 / **0.1112**
$\hat{\gamma}_9(\alpha_{90}, k_{90})$	1.2031 / 0.3417	**1.0558** / 0.1821	1.1324 / 0.2228	1.1210 / 0.1856	1.0643 / 0.1228
$\hat{\gamma}_{10}(\alpha_{100}, k_{100})$	1.2227 / 0.3392	1.0875 / 0.1973	1.1444 / 0.2127	1.1234 / 0.1608	1.0547 / 0.1113

Table 4. Simulated mean values/RMSE of $\hat{\gamma}_i(\alpha_{i0}, k_{i0}), i = 2, 4, 7, 8, 9, 10$ at their simulated optimal level from Burr(1, -1) model.

n	100	200	500	1000	2000
$\hat{\gamma}_2(\alpha_{20}, k_{20})$	1.0693 / 0.2338	1.0050 / 0.1174	1.0125 / 0.0844	0.9968 / 0.0764	1.0142 / 0.0609
$\hat{\gamma}_4(\alpha_{40}, k_{40})$	**1.0625** / 0.2392	**1.0049** / 0.1213	1.0158 / 0.0945	**1.0078** / 0.0818	1.0158 / 0.0670
$\hat{\gamma}_7(\alpha_{70}, k_{70})$	1.0826 / 0.2444	1.0169 / 0.1191	1.0216 / 0.0931	1.0101 / 0.0795	1.0156 / 0.0671
$\hat{\gamma}_8(\alpha_{80}, k_{80})$	1.0724 / **0.2312**	1.0050 / 0.1170	**1.0053** / **0.0826**	0.9881 / **0.0770**	**1.0123** / **0.0578**
$\hat{\gamma}_9(\alpha_{90}, k_{90})$	1.1114 / 0.2863	1.0260 / **0.1162**	1.0450 / 0.1053	1.0310 / 0.0847	1.0275 / 0.0748
$\hat{\gamma}_{10}(\alpha_{100}, k_{100})$	1.0731 / 0.2313	1.0058 / 0.1176	1.0068 / 0.0834	0.9899 / 0.0773	1.0125 / 0.0581

Table 5. Simulated mean values/RMSE of $\hat{\gamma}_i(\alpha_{i0}, k_{i0}), i = 2, 4, 7, 8, 9, 10$ at their simulated optimal level from Burr(1, -2) model.

n	100	200	500	1000	2000
$\hat{\gamma}_2(\alpha_{20}, k_{20})$	0.9839 / 0.1887	0.9784 / 0.1530	0.9680 / 0.1306	0.9826 / 0.0944	0.9897 / 0.0646
$\hat{\gamma}_4(\alpha_{40}, k_{40})$	0.9753 / 0.1977	0.9833 / 0.1657	0.9774 / 0.1377	**0.9851** / 0.1026	0.9865 / 0.0706
$\hat{\gamma}_7(\alpha_{70}, k_{70})$	**0.9914** / 0.2070	**0.9960** / 0.1652	**0.9799** / 0.1362	0.9837 / 0.1030	**0.9915** / 0.0722
$\hat{\gamma}_8(\alpha_{80}, k_{80})$	0.9682 / 0.1879	0.9649 / **0.1480**	0.9680 / 0.1266	0.9820 / 0.0911	0.9860 / 0.0622
$\hat{\gamma}_9(\alpha_{90}, k_{90})$	1.0491 / 0.2175	1.0209 / 0.1553	1.0259 / **0.1136**	1.0140 / **0.0878**	1.0130 / **0.0601**
$\hat{\gamma}_{10}(\alpha_{100}, k_{100})$	0.9675 / **0.1870**	0.9652 / **0.1480**	0.9681 / 0.1265	0.9821 / 0.0911	0.9859 / 0.0615

specific new estimators. In the asymptotical unbiased estimators, some of the new estimators are compared with the existing ones in terms of asymptotic variance, and the new estimators perform better. In the finite sample case, the simulated mean and mean square error of the estimators compared were calculated by Monte-Carlo simulation. The results show that the size relationship of the simulated mean square error is consistent with the theoretical analysis in asymptotical unbiased estimators. In addition, we compare the simulated mean value and mean square error of the mentioned estimators at the optimal level. It is concluded that the new estimators perform better within a certain range. Although we propose a class of parameterized heavy-tailed index estimators, the selection of parameters is still an open problem and will require further study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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