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Research article

Global well-posedness of the 3D nonlinearly damped Boussinesq magneto-micropolar system without heat diffusion

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Abstract: This paper establishes the global well-posedness of strong solutions to the three dimensional damped Boussinesq magneto-micropolar system with zero heat diffusion for large initial data. We prove that the nonlinear damping term $|u|^{\beta-1}u$, for $\beta \geq 4$, ensures sufficient regularity to establish the global well-posedness of the system.

Keywords: global well-posedness; Boussinesq magneto-micropolar system; strong solutions

1. Introduction

This paper establishes the global well-posedness of the three dimensional (3D) nonlinearly damped Boussinesq magneto-micropolar (BMM) system without heat diffusion:

$$\begin{cases}\partial_t u + (u\cdot\nabla)u - (\mu+\chi)\Delta u + \alpha|u|^{\beta-1}u + \nabla p = \chi\nabla\times w + b\cdot\nabla b + e_3\theta,\\ \partial_t w + (u\cdot\nabla)w - \gamma\Delta w + 2\chi w - \kappa\nabla\nabla\cdot w = \chi\nabla\times u,\\ \partial_t b + u\cdot\nabla b - \nu\Delta b - b\cdot\nabla u = 0,\\ \partial_t\theta + u\cdot\nabla\theta = 0,\\ \nabla\cdot u = \nabla\cdot b = 0,\ \ (u,w,B,\theta)|_{t=0} = (u_0,w_0,b_0,\theta_0),\ \ x\in\Omega=\mathbb{R}^3, t\in\mathbb{R}^+,\end{cases} \tag{1.1}$$

where $(u,w,b,\theta)(t,x)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}$ is the unknown which represents the fluid velocity, micro-rotational velocity, magnetic field, and temperature, and p is the scalar pressure of the flow. The parameter $\mu > 0$ is the kinematic viscosity, χ is the micro-rotation viscosity, γ and κ are the angular viscosities, $\nu > 0$ is the magnetic diffusion coefficient, $e_3 = (0,0,1)$ is the unit vector, and $e_3\theta$ represents the role of the buoyancy force in fluid motion. The nonlinear damping term $\alpha|u|^{\beta-1}u$ arises from the

Darcy-Forchheimer law, which models the resistance to fluid motion in porous media [1–3]. This term plays a crucial role in high velocity flows, as the linear drag (Stokes flow) fails to adequately account for the nonlinear effects that arise from inertia and turbulence. Moreover, this system is a combination of the incompressible Boussinesq equation with the magneto-micropolar system modeling the dynamics of electrically conductive micropolar fluids under the influence of a magnetic field, where the heat diffusion is set to zero.

When the temperature is not considered (i.e., $\theta \equiv 0$), the system reduces to the magneto-micropolar system to study interesting phenomena in fluid motion, such as in liquid crystals and dilute aqueous polymers [4]. Liu et al. [2] proved the existence and uniqueness of a strong solution for 3D magneto-micropolar equations with damping. When the micro-rotational velocity is zero, the system becomes the Boussinesq equation for magnetohydrodynamic convection (Boussinesq-MHD) system, which has been also extensively studied in recent years. The global well-posedness and decay, the reader can refer to [5, 6]. When both θ and w vanish, the system becomes the famous magnetohydrodynamic (MHD) system of velocity and magnetic field. In particular, Titi and Trabelsi [3] obtained the global well-posedness of the MHD model in porous media.

For system (1.1), to the best of our knowledge, there are no results on the existence of the global weak solutions in the lower regularity spaces. In the following theorem, we first state the existence of global weak solutions for system (1.1).

Theorem 1.1. *Let $\beta \geq 0$ and $\alpha \geq 0$. Assume that $(u_0, w_0, b_0, \theta_0) \in L^2 \times L^2 \times L^2 \times L^2$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, then there exists a global weak solution $(u(t), w(t), b(t), \theta(t))$ of the BMM system* (1.1) *satisfying*

$$\begin{aligned} u \in & L^\infty(0,\infty; L^2) \cap L^2_{loc}(0,\infty; H^1) \cap L^{\beta+1}_{loc}(0,\infty; L^{\beta+1}), \\ (w,b) \in & L^\infty((0,\infty); L^2) \cap L^2_{loc}((0,\infty); H^1), \quad \textit{and} \quad \theta \in L^\infty((0,\infty); L^2). \end{aligned} \tag{1.2}$$

In [7], Shou and Zhong obtained a unique strong solution using the energy method under the $H^1 \times H^1 \times H^1 \times L^2$ framework. We shall extend the result to the global strong solutions in the higher regularity spaces. To obtain the closed energy estimates, we need to carefully analyze the dissipative effect of the nonlinear damping term $\alpha |u|^{\beta-1} u$ to control the nonlinear term. The main result is stated in the following theorem.

Theorem 1.2. *Let $(u_0, w_0, b_0, \theta_0) \in H^s \times H^s \times H^s \times H^s$ for $s \geq 3$, such that $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ and $\beta \geq 4$, then there exists a unique strong solution (u, w, b, θ) for the system* (1.1) *satisfying*

$$(u, w, b) \in L^\infty_{loc}([0,\infty); H^s) \cap L^2_{loc}([0,\infty); H^{s+1}), \quad \textit{and} \quad \theta \in L^\infty_{loc}([0,\infty); H^s). \tag{1.3}$$

In the above results, the nonlinear damping term $\alpha |u|^{\beta-1} u$ acts as a dissipative mechanism that controls the growth of the fluid velocity u. This term is sufficiently strong to ensure that the velocity u remains bounded in $L^{\beta+1}$ and H^s spaces for weak and strong solutions, respectively. It may prevent the formation of singularities, where the lack of sufficient dissipation can lead to blow-up in finite time. Moreover, our research may provide a rigorous theoretical foundation for certain astrophysical plasmas or industrial processes that involve rapid cooling. The nonlinear damping mechanism ($\beta \geq 4$) could inspire new approaches to stabilize turbulent flows in MHD systems. Future work may explore the critical case where $\beta = 3$. Additionally, we will extend this analysis to domains with boundaries.

2. Global weak solutions

First of all, by taking the inner product of $(1.1)_4$ with $|\theta|^{p-2}\theta$, it is easy to show that if $\theta_0 \in L^1 \cap L^\infty$, then

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad 1 < p \leq \infty, \ \forall t \geq 0. \tag{2.1}$$

Indeed, for any $1 < p < \infty$, we have

$$\frac{d}{dt}\int |\theta(\tau)|^p dx = 0,$$

which implies the estimate (2.1). Given that the estimate does not depend on p, we can obtain (2.1) when taking $p \to \infty$.

Lemma 2.1. *Let $(u_0, w_0, b_0, \theta_0) \in L^2$, then there exists some constant $C = C(t, u_0, w_0, b_0, \theta_0) > 0$, such that*

$$\|(u, w, b)(t)\|^2 \leq e^t \|(u_0, w_0, b_0)\|^2 + te^t \|\theta_0\|^2, \tag{2.2}$$

and

$$\begin{aligned} &\int_0^t \|(\nabla u, \nabla w, \nabla b, \nabla\cdot w)(\tau)\|^2 d\tau + \int_0^t \|u(\tau)\|_{L^{\beta+1}}^{\beta+1} d\tau + \int_0^t \|w\|^2 d\tau \\ &\leq Ce^{Ct}\|(u_0, w_0, b_0, \theta_0)(\tau)\|^2 + Cte^{Ct} \leq C(t, u_0, w_0, b_0, \theta_0). \end{aligned} \tag{2.3}$$

Proof. Applying the L^2-inner product to the equations with (u, w, b) and using integration by parts yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(u, w, b)\|^2 + (\mu + \chi)\|\nabla u\|^2 + \gamma\|\nabla w\|^2 + \nu\|\nabla b\|^2 + \alpha\|u\|_{L^{\beta+1}}^{\beta+1} + \kappa\|\nabla\cdot w\|^2 + 2\chi\|w\|^2 \\ &= \chi\int(\nabla\times w)\cdot u dx + \chi\int(\nabla\times u)\cdot w dx + \int \theta u_3 dx. \end{aligned} \tag{2.4}$$

By Hölder inequality, we directly have

$$\left|\int(\nabla\times w)\cdot u dx\right| + \left|\int(\nabla\times u)\cdot w dx\right| \leq \|\nabla u\|^2 + \|w\|^2,$$

and

$$\left|\int \theta u_3 dx\right| \leq \frac{1}{2}(\|\theta\|^2 + \|u\|^2) \leq \frac{1}{2}(\|\theta_0\|^2 + \|u\|^2).$$

Therefore, we have

$$\frac{d}{dt}\|(u, w, b)\|^2 + 2\mu\|\nabla u\|^2 + 2\gamma\|\nabla w\|^2 + 2\nu\|\nabla b\|^2 + 2\alpha\|u\|_{L^{\beta+1}}^{\beta+1} + 2\kappa\|\nabla\cdot w\|^2 + 2\chi\|w\|^2 \leq \|\theta_0\|^2 + \|u\|^2.$$

By integrating in time, we finish the proof of Lemma 3.2. □

Based on the above estimates (2.1) and (2.3), the global existence of weak solutions to the BMM system with zero heat diffusion can be established through the classical Faedo-Galerkin approximation method and the Aubin-Lions Lemma. This procedure is standard, so we omit the details here. For a detailed explanation of similar techniques, we refer the reader to [8] for the application of the Faedo-Galerkin method and the Aubin-Lions Lemma in the context of Navier-Stokes equations.

3. Global smooth solutions-Proof of Theorem 1.2

This section is devoted to establishing higher order *a priori* energy estimates for the solutions of the BMM system (1.1); henceforth, we assume that the solution is smooth on $[0, T]$.

Lemma 3.1. *Let $\beta \geq 4$ and $(u_0, w_0, b_0, \theta_0) \in L^2$ such that u_0 and b_0 are divergence free. Additionally, if $b_0 \in L^{\frac{3(\beta+1)}{\beta-1}}$, then there exists some constant $C_2 > 0$ such that*

$$\sup_{t\in[0,T]} \|b(t)\|_{L^{\frac{3(\beta+1)}{\beta-1}}} + \int_0^t \|\nabla |b(\tau)|^{\frac{3(\beta+1)}{2(\beta-1)}}\|^2 d\tau \leq C_2, \quad \forall t \in [0, T]. \tag{3.1}$$

The proof is similar to that in [5]. We need to take the inner product of the magnetic equation $(1.1)_3$ with $|b|^{\frac{\beta+5}{\beta-1}} b$, by integrating by parts and using Hölder inequality, Gagliardo-Nirenberg interpolation inequality [9] and Gronwall inequality [10].

3.1. First order estimates

In the following, we give the H^1 estimates for the solutions.

Lemma 3.2. *Let $\beta \geq 4$, $(u_0, w_0, b_0) \in H^1$, $\theta_0 \in L^2$ and (u, w, b, θ) be a smooth solution, then there exists a constant $C > 0$ such that the following estimate holds*

$$\sup_{0\leq\tau\leq t} \|(\nabla u, \nabla w, \nabla b)\|^2 + \int_0^t \|(\Delta u, \Delta w, \Delta b)\|^2 + \||u|^{\frac{\beta-1}{2}} \nabla u\|_{L^2}^2 d\tau \leq C(t, \|(u_0, w_0, b_0)\|_{H^1}, \|\theta_0\|_{L^2}). \tag{3.2}$$

Proof. Multiplying the first equation in (1.1) by Δu yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + (\mu + \chi)\|\Delta u\|^2 + \alpha\beta\||u|^{\frac{\beta-1}{2}} \nabla u\|_{L^2}^2 \\ &= \int_\Omega (u \cdot \nabla u) \cdot \Delta u dx - \int_\Omega (b \cdot \nabla b) \cdot \Delta u dx - \chi \int_\Omega (\nabla \times w) \cdot \Delta u dx - \int_\Omega \theta \Delta u_3 dx, \end{aligned} \tag{3.3}$$

where we used

$$-\alpha \int |u|^{\beta-1} u \cdot \Delta u dx = \alpha\beta\||u|^{\frac{\beta-1}{2}} \nabla u\|_{L^2}^2. \tag{3.4}$$

Using Hölder and Young's inequalities, we have

$$\begin{aligned} \left|\int_\Omega (u \cdot \nabla u) \cdot \Delta u dx\right| \leq& \||u||\nabla u|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}} \||\nabla u|^{\frac{\beta-3}{\beta-1}}\|_{L^{\frac{2(\beta-1)}{\beta-3}}} \|\Delta u\|_{L^2} \\ \leq& \frac{\mu}{8}\|\Delta u\|_{L^2}^2 + \frac{\alpha\beta}{2}\||u|^{\frac{\beta-1}{2}} \nabla u\|_{L^2}^2 + C\mu^{\frac{\beta-1}{\beta-3}}\|\nabla u\|_{L^2}^2 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \left|\int_\Omega (b \cdot \nabla b) \cdot \Delta u dx\right| \leq& \frac{\mu}{8}\|\Delta u\|_{L^2}^2 + \frac{2}{\mu}\|b\|_{L^{\frac{3(\beta+1)}{\beta-1}}}^2 \|\nabla b\|_{L^{\frac{6(\beta+1)}{\beta-5}}}^2 \\ \leq& \frac{\mu}{8}\|\Delta u\|_{L^2}^2 + \frac{2}{\mu}\|b\|_{L^{\frac{3(\beta+1)}{\beta-1}}}^2 \|\nabla b\|_{L^2}^{\frac{4}{\beta+1}} \|\Delta b\|_{L^2}^{\frac{2(\beta-1)}{\beta+1}} \\ \leq& \frac{\mu}{8}\|\Delta u\|_{L^2}^2 + \frac{\nu}{6}\|\Delta b\|^2 + C_3\|\nabla b\|_{L^2}^2, \end{aligned} \tag{3.6}$$

thanks to the estimate (3.1). Similarly, for the last two terms in (3.3), we have

$$\left|\chi\int_\Omega(\nabla\times w)\cdot\Delta u dx+\int_\Omega\theta\Delta u_3 dx\right|\leq\frac{\mu}{8}\|\Delta u\|_{L^2}^2+\frac{4\chi^2}{\mu}\|\nabla w\|^2+\frac{4}{\mu}\|\theta_0\|_{L^2}^2, \tag{3.7}$$

thanks to (2.1). Applying the L^2-inner product to $(1.1)_3$ with Δb and using integration by parts yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla b\|^2+\gamma\|\Delta b\|^2&=\int_\Omega(u\cdot\nabla b)\cdot\Delta b dx-\int_\Omega(b\cdot\nabla u)\cdot\Delta b dx\\&=\int_\Omega(\partial_i u\cdot\nabla\partial_i b)\cdot b dx-\int_\Omega(b\cdot\nabla u)\cdot\Delta b dx\\&\leq\frac{\mu}{8}\|\Delta u\|^2+\frac{\nu}{3}\|\Delta b\|^2+C\|\nabla u\|^2,\end{aligned} \tag{3.8}$$

owing to the divergence-free condition. Putting the estimates (3.3) and (3.8) together, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b)\|^2+(\frac{\mu}{2}+\chi)\|\Delta u\|^2+\frac{\alpha\beta}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{\gamma}{2}\|\Delta b\|^2\leq C\|\theta_0\|^2+C\|(\nabla u,\nabla w,\nabla b)\|_{L^2}^2,$$

from which we have by Gronwall inequality [10] that

$$\sup_{0\leq\tau\leq t}\|(\nabla u,\nabla b)\|^2+\int_0^t(\frac{\mu}{2}+\chi)\|\Delta u\|^2+\frac{\alpha\beta}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{\gamma}{2}\|\Delta b\|^2 d\tau\leq C(t,u_0,w_0,b_0,\theta_0), \tag{3.9}$$

thanks to the estimate (2.3). Now, we treat the estimate for w. Multiply the second equation with $-\Delta w$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\gamma\|\Delta w\|^2+2\chi\|\nabla w\|^2+\kappa\|\nabla\nabla\cdot w\|^2&=\int_\Omega(u\cdot\nabla w)\cdot\Delta w dx-\chi\int_\Omega(\nabla\times u)\cdot\Delta w dx\\&\leq\frac{\gamma}{2}\|\Delta w\|^2+C\|\nabla u\|\|\Delta u\|\|\nabla w\|^2+C\|\nabla u\|^2.\end{aligned} \tag{3.10}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\frac{\gamma}{2}\|\Delta w\|^2+2\chi\|\nabla w\|^2+\kappa\|\nabla\nabla\cdot w\|^2\leq C\|\nabla u\|\|\Delta u\|\|\nabla w\|^2+C\|\nabla u\|^2, \tag{3.11}$$

which implies from Gronwall inequality and (3.9) that

$$\sup_{0\leq\tau\leq t}\|\nabla w(\tau)\|^2+\int_0^t\|\Delta w(\tau)\|^2+\|\nabla\nabla\cdot w(\tau)\|^2 d\tau\leq C(t,u_0,w_0,b_0,\theta_0). \tag{3.12}$$

This completes the proof of Lemma 3.2. □

A direct corollary of these estimates implies that

$$\int_0^t\|u,w,b\|_{L^\infty}^2 d\tau\leq C(\|(u_0,w_0,b_0)\|_{H^1},\|\theta_0\|_{L^2}).$$

3.2. Time derivative estimates

Now, we estimate the time derivatives of the solutions.

Lemma 3.3. *Let $\beta \geq 4$ and (u, w, b, θ) be a smooth solution, then there holds*

$$\int_0^t \|\partial_t u, \partial_t w, \partial_t b\|^2 d\tau \leq C(t, \|(u_0, w_0, b_0)\|_{H^1}, \|\theta_0\|_{L^2}). \tag{3.13}$$

Proof. Multiply the first equation in (1.1) with u_t and then integrate over $\mathbb{R}^3$ to obtain

$$\begin{aligned}
&\|\partial_t u\|^2 + \frac{\mu+\chi}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{2\alpha}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1} \\
&= \int_\Omega (b\cdot\nabla b)\cdot\partial_t u dx - \int_\Omega (u\cdot\nabla u)\cdot\partial_t u dx + \chi\int_\Omega \nabla\times w\cdot\partial_t u dx + \int_\Omega \theta e_3\cdot\partial_t u dx \\
&\leq \frac{1}{4}\|\partial_t u\|^2 + 4\|u\cdot\nabla u\|^2 + 4\|b\cdot\nabla b\|^2 + 4\chi^2\|\nabla w\|^2 + 4\|\theta\|^2.
\end{aligned} \tag{3.14}$$

For the equation of magnetic field b, we have

$$\begin{aligned}
\|\partial_t b\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla b\|^2 &= \int_\Omega (b\cdot\nabla u)\cdot\partial_t b dx - \int_\Omega (u\cdot\nabla b)\cdot\partial_t b dx \\
&\leq \frac{1}{4}\|\partial_t b\|^2 + 2\|b\cdot\nabla u\|^2 + 2\|u\cdot\nabla b\|^2.
\end{aligned} \tag{3.15}$$

By Gagliardo-Nirenberg interpolation inequality [9], one has

$$\|u\cdot\nabla b\| \leq \|u\|_{L^\infty}\|\nabla b\| \leq C\|u\|^{\frac{1}{4}}\|\Delta u\|^{\frac{3}{4}}\|\nabla b\| \leq C\|\Delta u\| + C\|u\|\|\nabla b\|^4.$$

A similar treatment can be applied to other nonlinear terms; then, we have

$$\begin{aligned}
&\|\partial_t u\|^2 + \|\partial_t b\|^2 + (\mu+\chi)\frac{d}{dt}\|\nabla u\|^2 + \nu\frac{d}{dt}\|\nabla b\|^2 + \frac{4\alpha}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1} \\
&\leq C\|\Delta u, \Delta b\|^2 + C\|u, b\|^2\|\nabla u, \nabla b\|_{L^2}^8 + C(\|\nabla w\|^2 + \|\theta\|^2).
\end{aligned} \tag{3.16}$$

Then, Gronwall inequality [10] implies the following estimate

$$\sup_{0\leq\tau\leq t}(\|u(\tau)\|_{L^{\beta+1}}^{\beta+1} + \|(\nabla u, \nabla b)\|_{L^2}^2) + \int_0^t \|\partial_t u\|^2 + \|\partial_t b\|^2 d\tau \leq C(u_0, w_0, b_0, \theta_0), \tag{3.17}$$

where the constant C depends on $\|(u_0, w_0, b_0)\|_{H^1}$, $\|\theta_0\|_{L^2}$ and $\|u_0\|_{L^{\beta+1}}$, thanks to Lemma 3.2. Then, we multiply $(1.1)_2$ for w in (1.1) and integrate by parts to obtain

$$\begin{aligned}
&\|\partial_t w\|^2 + \frac{\gamma}{2}\frac{d}{dt}\|\nabla w\|^2 + \frac{\kappa}{2}\frac{d}{dt}\|\nabla\cdot w\|^2 + \chi\frac{d}{dt}\|w\|^2 \\
&= \chi\int_\Omega (\nabla\times u)\cdot\partial_t w dx - \int_\Omega (u\cdot\nabla w)\cdot\partial_t w dx \\
&\leq \frac{1}{2}\|\partial_t w\|^2 + C\|\Delta u\|^2 + C\|\nabla u\|^2 + C\|u\|^2\|\nabla w\|^8.
\end{aligned} \tag{3.18}$$

Integration in time, together with (3.17) implies the results, thanks to Lemma 3.2. □

Lemma 3.4. *Let (u, w, b, θ) be a smooth solution, then there holds*

$$\sup_{0\le\tau\le t} \|\partial_t u, \partial_t w, \partial_t b\|^2 + \int_0^t \|\nabla u_t, \nabla b_t, \nabla w_t\|^2 d\tau \le C. \tag{3.19}$$

Proof. Differentiating $(1.1)_1$ with respect to t and then applying L^2-inner product with $\partial_t u$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t u, \partial_t b\|^2 + (\mu+\chi)\|\nabla u_t\|^2 + \nu\|\nabla b_t\|^2 + \alpha\int_\Omega (|u|^{\beta-1}u)_t \cdot u_t dx \\
&= \underbrace{\int_\Omega (b\cdot\nabla b)_t \cdot u_t dx - \int_\Omega (u\cdot\nabla u)_t \cdot u_t dx + \int_\Omega (b\cdot\nabla u)_t \cdot b_t dx - \int_\Omega (u\cdot\nabla b)_t \cdot b_t dx}_{\star} \\
&\quad + \chi\int_\Omega \nabla\times w_t \cdot u_t dx + \int_\Omega \theta_t u_{3,t} dx.
\end{aligned} \tag{3.20}$$

By integration by parts, the first four integrals on the right hand side (RHS) can be bounded by

$$\begin{aligned}
|\star| =& \left|\int_\Omega (b_t\cdot\nabla b)\cdot u_t dx - \int_\Omega (u_t\cdot\nabla u)\cdot u_t dx + \int_\Omega (b_t\cdot\nabla u)\cdot b_t dx - \int_\Omega (u_t\cdot\nabla b)\cdot b_t dx\right| \\
\le& \|(\nabla u, \nabla b)\|_{L^2}\|(u_t, b_t)\|_{L^4}^2 \le C\|(u_t, b_t)\|_{L^4}^2 \le C\|(u_t, b_t)\|_{L^2}^{1/2}\|(\nabla u_t, \nabla b_t)\|_{L^2}^{3/2} \\
\le& \frac{\mu}{4}\|\nabla u_t\|^2 + \frac{\nu}{2}\|\nabla b_t\|^2 + C\|(u_t, b_t)\|^2,
\end{aligned} \tag{3.21}$$

and the last two integrals can be bounded by

$$\left|\chi\int_\Omega \nabla\times w_t \cdot u_t dx\right| \le \frac{\chi}{2}\|\nabla u_t\|_{L^2}^2 + C\|w_t\|_{L^2}^2$$

and

$$\left|\int_\Omega \theta_t u_{3,t} dx\right| = \left|\int_\Omega (u\cdot\nabla\theta) u_{3,t} dx\right| = \left|\int_\Omega \theta u\cdot\nabla u_{3,t} dx\right| \le \frac{\mu}{4}\|\nabla u_{3,t}\|^2 + C\|\theta\|_{L^\infty}^2\|u\|_{L^2}^2.$$

Therefore, we have

$$\frac{d}{dt}\|(\partial_t u, \partial_t b)\|^2 + (\mu+\chi)\|\nabla u_t\|^2 + \nu\|\nabla b_t\|^2 \le C\|u_t, b_t, w_t\|^2 + C, \tag{3.22}$$

thanks to (2.1), (2.2) and the non-negativity of the α-term. Indeed,

$$\alpha\int_\Omega (|u|^{\beta-1}u)_t \cdot u_t dx = \alpha\int_\Omega |u|^{\beta-1}|u_t|^2 dx + \alpha(\beta-1)\int_\Omega (|u|^{\beta-3}|u\cdot u_t|^2 dx \ge 0. \tag{3.23}$$

For the w-equation in (1.1), we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_t w\|^2 + \gamma\|\nabla w_t\|^2 + 2\chi\|w_t\|^2 + \kappa\|\nabla\cdot w_t\|^2 = \chi\int_\Omega \nabla\times u_t \cdot w_t dx - \int_\Omega (u\cdot\nabla w)_t \cdot w_t dx. \tag{3.24}$$

For the RHS, we have

$$\begin{aligned}
\left|\int_\Omega (u\cdot\nabla w)_t \cdot w_t dx\right| =& \left|\int_\Omega (u_t\cdot\nabla w)\cdot w_t dx\right| \le \|\nabla w\|_{L^2}\|u_t\|_{L^4}\|w_t\|_{L^4} \\
\le& C\|u_t\|^{1/4}\|\nabla u_t\|^{3/4}\|w_t\|^{1/4}\|\nabla w_t\|^{3/4} \\
\le& C(\|u_t\|^2 + \|u_t\|^2) + \frac{\mu}{8}\|\nabla u_t\|^2 + \frac{\gamma}{8}\|\nabla w_t\|^2,
\end{aligned} \tag{3.25}$$

thanks to Lemma 3.2 and

$$\left|\chi\int_{\Omega}\nabla\times u_t\cdot w_t dx\right|\le\frac{\chi}{2}(\|\nabla u_t\|^2+\|w_t\|^2). \tag{3.26}$$

Therefore, we have

$$\frac{d}{dt}\|\partial_t u,\partial_t b,\partial_t w\|^2+\|\nabla u_t,\nabla b_t,\nabla w_t\|^2+\|\nabla\cdot w_t\|^2\le C\|u_t,b_t,w_t\|^2+C. \tag{3.27}$$

Then, Grönwall inequality implies the result. □

3.3. Higher order estimates

From Lemma 3.4, we immediately have

$$\sup_t(\|\Delta u,\Delta w,\Delta b\|_{L^2}^2)+\int_0^t\|u_t,b_t,w_t\|_{L^p}^2 d\tau\le C,\quad 1\le p\le 6. \tag{3.28}$$

Indeed, we only need to note that

$$\|u\cdot\nabla b\|_{L^2}\lesssim\|u\|_{L^4}\|\nabla b\|_{L^4}\lesssim\|u\|^{1/4}\|\nabla u\|^{3/4}\|\nabla u\|^{1/4}\|\Delta u\|^{3/4}\lesssim\delta\|\Delta u\|+C_\delta\|u\|\|\nabla u\|^4, \tag{3.29}$$

where $\delta>0$ can be chosen to be sufficiently small. The pressure can be estimated in a standard way; there exists a constant $C>0$ satisfying

$$\sup_t\|\nabla p(t)\|_{L^6}\le C. \tag{3.30}$$

Indeed, taking $\nabla\Delta^{-1}\nabla\cdot$ to the equation $(1.1)_1$ and using the divergence free condition, we have

$$\begin{aligned}\nabla p=&-\nabla\Delta^{-1}\nabla\cdot(u\cdot\nabla u)+\nabla\Delta^{-1}\nabla\cdot(b\cdot\nabla b)-\alpha\nabla\Delta^{-1}\nabla\cdot(|u|^{\beta-1}u)\\&+\chi\nabla\Delta^{-1}\nabla\cdot(\nabla\times w)+\nabla\Delta^{-1}\nabla\cdot(e_3\theta),\end{aligned} \tag{3.31}$$

Since $\nabla\Delta^{-1}\nabla\cdot$ is a bounded operator in L^p, by the above Lemmas, we have

$$\|\nabla\Delta^{-1}\nabla\cdot(\nabla\times w)\|_{L^p}\le C\|\nabla w\|_{L^p},\quad 1<p<\infty$$

and hence

$$\|\nabla\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\|_{L^6}\le C\|u\cdot\nabla u\|_{L^6}\le C\|u\|_{L^\infty}\|\nabla u\|_{L^6}\le C\|u\|_{H^2}^2\le C,$$

where the constant C only depends on p. Similar estimates can be applied to the terms $e_3\theta$ and $(b\cdot\nabla b)$. We estimate the nonlinear damping term

$$\|\nabla\Delta^{-1}\nabla\cdot(|u|^{\beta-1}u)\|_{L^6}\le C\||u|^{\beta-1}u\|_{L^6}\le C\|u\|_{L^\infty}^{\beta-1}\|u\|_{L^6}\le C.$$

Putting these estimates together, we indeed have (3.30).

Furthermore, we can show an increased regularity of the solutions.

Lemma 3.5. *Let (u,w,b,θ) be a smooth solution, then we obtain $(u,w,b)\in L^2_{loc}(0,\infty;W^{2,p})$ and*

$$\int_0^t\|\nabla u,\nabla w,\nabla b\|_{L^\infty}d\tau+\int_0^t\|\Delta u,\Delta b,\Delta w\|_{L^p}^2 d\tau\le C,\quad 1\le p\le 6. \tag{3.32}$$

Indeed, we only need to note that

$$\int_0^t \|u\cdot\nabla b\|_{L^6}^2 d\tau \lesssim \int_0^t \|u\|_{L^\infty}^2 \|\nabla b\|_{L^6}^2 d\tau \lesssim \int_0^t \|(u,b)\|_{H^2}^4 d\tau \le C, \tag{3.33}$$

thanks to (3.28); similar estimates holds when b is replaced by u and w, and

$$\int_0^t \||u|^{\beta-1}u\|_{L^6} d\tau \lesssim \int_0^t \|u\|_{L^\infty}^{\beta-1}\|u\|_{L^6} d\tau \le C. \tag{3.34}$$

Then, the result is proven by directly applying system (1.1) and the above estimates (3.19) and (3.30). Now, we can estimate $\|\nabla\theta\|_{L^\infty_{t,x}}$. Taking ∇ to $(1.1)_4$ and then applying the L^2-inner product with $|\nabla\theta|^r\nabla\theta$, by Gronwall inequality [10], we have

$$\|\nabla\theta(t)\|_{L^{r+2}}^{r+2} \le \|\nabla\theta_0\|_{L^{r+2}} e^{\int_0^t \|\nabla u\|_{L^\infty} d\tau} \le C, \quad \forall r \ge 0, \forall t \in [0,T],$$

which is independent of r. This implies $\sup_{t\in[0,T]} \|\nabla\theta(t)\|_{L^\infty} \le C$ by letting $r \to \infty$. This, in turn, implies that

$$\int_0^t \|\theta_t\|_{L^\infty}^2 d\tau \le \int_0^t \|u\cdot\nabla\theta\|_{L^\infty}^2 d\tau \le \sup_{\tau\in[0,t]} \|\nabla\theta\|_{L^\infty}^2 \int_0^t \|u\|_{L^\infty}^2 d\tau \le C.$$

Based on these estimates, we can estimate higher derivatives of the solutions such as (1.3) and prove Theorem 1.2. Uniqueness and continuous dependence follow from the regularity of the strong solutions by a standard estimate and the Gronwall inequality.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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