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Research article

Sincere wide τ-tilting modules

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Abstract: Enomoto and Sakai introduced wide τ-tilting modules, which are τ-tilting modules over functorially finite wide subcategories. They also proved that wide τ-tilting modules bijection with doubly functorially finite image-cokernel-extension-closed (ICE-closed) subcategories, which extended Adachi-Iyama-Reiten's result. In this paper, we show that this bijection can be restricted to the support sets. As a consequence, we establish bijections between sincere wide τ-tilting modules, sincere ICE-closed subcategories, and sincere epibricks, and then we show that its number is related to the little Schr*ö*der number for Nakayama algebras.

Keywords: wide τ-tilting modules; ICE-closed subcategories; epibricks; Nakayama algebras

1. Introduction

In 2014, Adachi, Iyama and Reiten [1] introduced the concept of support τ-tilting modules, and showed that there exists a bijection between support τ-tilting modules and functorially finite torsion classes. Recently, Enomoto [2] studied image-cokernel-extension-closed (ICE-closed) subcategories that contain torsion classes and wide subcategories. Moreover, it is shown that a subcategory is ICE-closed if and only if it is a sincere torsion class for some wide subcategory (see, [2, Corollary 2.5, Proposition 4.2]). Note that every functorially finite wide subcategory is equivalent to a module category. Enomoto and Sakai introduced the concept of wide τ-tilting modules that are τ-tilting modules over some functorially finite wide subcategory. Moreover, a bijection was established between wide τ-tilting modules and doubly functorially finite ICE-closed subcategories, which extended Adachi et al.'s bijection. Note that τ-tilting modules are exactly sincere support τ-tilting modules. Adachi et al. also showed that there is a bijection between τ-tilting modules and sincere functorially finite torsion classes.

$$\begin{tikzcd}
\{wide\ \tau\text{-}tilting\ modules\} \arrow[r, leftrightarrow, "ES's\ bijection"] & \{doubly\ functorially\ finite\ ICE\text{-}closed\ subcategories\} \\
\{support\ \tau\text{-}tilting\ modules\} \arrow[u, hook] \arrow[r, leftrightarrow, "AIR's\ bijection"] & \{functorially\ finite\ torsion\ classes\} \arrow[u, hook] \\
\{\tau\text{-}tilting\ modules\} \arrow[u, hook] \arrow[r, leftrightarrow] & \{sincere\ functorially\ finite\ torsion\ classes\} \arrow[u, hook]
\end{tikzcd}$$

Bricks and semibricks are considered, and they have long been studied in [3, 4]. Asai proved that a bijection exists between support τ-tilting modules and left finite semibricks (see [5, Theorem 2.3]). Gao [6] also showed that Asai's bijiection can be restricted to a sincere case. Moreover, considering the dual definition of Enomoto about monobricks, Gao [7] introduced the concept of epibricks and then established a bijection between wide τ-tilting modules and doubly finite epibricks.

$$\begin{tikzcd}
\{wide\ \tau\text{-}tilting\ modules\} \arrow[r, leftrightarrow, "Gao's\ bijection"] & \{doubly\ finite\ epibricks\} \\
\{support\ \tau\text{-}tilting\ modules\} \arrow[u, hook] \arrow[r, leftrightarrow, "Asai's\ bijection"] & \{left\ finite\ semibricks\} \arrow[u, hook] \\
\{\tau\text{-}tilting\ modules\} \arrow[u, hook] \arrow[r, leftrightarrow] & \{sincere\ left\ finite\ semibricks\} \arrow[u, hook]
\end{tikzcd}$$

The aim of this paper is to show that Enomoto and Sakai's bijection and Gao's bijection can be restricted to support sets. In particular, it will be proved that there are bijections between sincere wide τ-tilting modules, sincere ICE-closed subcategories, and sincere epibricks. Then we show that its number is related to the little Schröder number for Nakayama algebras.

All algebras will be basic, finite dimensional K-algebras over an algebraically closed field K. Let $A := KQ/I$ be an algebra, $\mathrm{mod}\,A$ be the category of finitely generated right A-modules and τ be the Auslander-Reiten translation of A. We also denote the number of pairwise nonisomorphic indecomposable summands of M by $|M|$, and the Loewy length of M by $l(M)$. We use S_i, P_i, and I_i to denote the indecomposable simple, projective, and injective modules of an algebra corresponding to the vertex i, respectively. For any $i, j \in \{1, 2, \cdots, n\}$, we denote by $[i, j] = \{i, i+1, \cdots, j\}$ if $i \leqslant j$; otherwise, $[i, j] = \emptyset$. Let e_i be the primitive idempotent element of an algebra corresponding to the vertex i. We write $e_{[i,j]} := e_i + e_{i+1} + \cdots + e_j$.

2. Some bijections

We recall some notions and results about support τ-tilting modules, wide τ-tilting modules, and some subcategories.

Let A be an algebra. An A-module M is called τ-tilting if $\mathrm{Hom}_A(M, \tau M) = 0$ and $|M| = |A|$. M is support τ-tilting if it is a τ-tilting A/AeA-module for some idempotent e of A. The set of all support τ-tilting A-modules (respectively, τ-tilting A-modules) will be denoted by $\mathrm{s}\tau\text{-tilt}\,A$ (respectively, $\mathrm{s}\tau\text{-tilt}\,A$). Enomoto showed that every functorially finite wide subcategory $\mathcal{W}$ is equivalent to a module category (i.e., $\mathcal{W}$ is equivalent to $\mathrm{mod}\,\Gamma$ for some algebra Γ), and then he introduced the concept of wide τ-tilting modules as follows.

Definition 2.1. ([2, Definition 4.11])

(1) *Let $\mathcal{W}$ be a functorially finite wide subcategory of* $\operatorname{mod} A$ *and $M \in \mathcal{W}$. Fix an equivalent F : $\mathcal{W} \simeq$* $\operatorname{mod}\Gamma$. *M is called $\tau_{\mathcal{W}}$-tilting if $F(M)$ is a τ-tilting Γ-module.*

(2) *An A-module M is wide τ-tilting if M is $\tau_{\mathcal{W}}$-tilting for some functorially finite wide subcategory $\mathcal{W}$ of* $\operatorname{mod} A$. *The set of all wide τ-tilting A-modules will be denoted by* $\mathrm{w}\tau\text{-tilt}\, A$.

An A-module E is called a brick if $\operatorname{Hom}_A(E, E)$ is a K-division algebra. A set $\mathcal{E}$ of isomorphism classes of bricks in $\operatorname{mod} A$ is said to be a epibrick if every morphism between elements of $\mathcal{E}$ is zero or a surjection in $\operatorname{mod} A$. We denote by $\operatorname{ebrick} A$ the set of epibricks in $\operatorname{mod} A$. A subcategory C of $\operatorname{mod} A$ is right Schur if it is closed under extensions and, for every simple object M in C, every morphism $X \to M$ with $X \in C$ is zero or a surjection in $\operatorname{mod} A$. Here, M is a simple object in C if there no exists any short exact sequence with M as the middle term. The set of right Schur subcategories of $\operatorname{mod} A$ will denoted by $\operatorname{schur}_{\mathrm{R}} A$. The following proposition follows from the dual of [8, Theorem 2.11].

Proposition 2.2. *Let A be an algebra. Then, there is a bijection:*

$$\operatorname{schur}_{\mathrm{R}} A \underset{\operatorname{Filt}(-)}{\overset{\operatorname{Sim}(-)}{\rightleftarrows}} \operatorname{ebrick} A$$

where $\operatorname{Filt}(\mathcal{E})$ *stands for the minimal extension-closed subcategory that contains $\mathcal{E}$ for $\mathcal{E} \in$* $\operatorname{ebrick} A$ *and* $\operatorname{Sim}(C)$ *stands for the set of all simple objects of the right Schur subcategory C.*

A subcategory C of $\operatorname{mod} A$ is called ICE-closed if it is closed under images, cokernels, and extensions. Both torsion classes and wide subcategories are ICE-closed, and ICE-closed subcategories are right Schur. Moreover, it is shown that C is ICE-closed if and only if there is a wide subcategory $\mathcal{W}$ such that C is a torsion class over $\mathcal{W}$. Enomoto called C is doubly functorially finite if C is a functorially finite torsion class over some functorially finite wide subcategory $\mathcal{W}$. The set of all doubly functorially finite ICE-closed subcategories of $\operatorname{mod} A$ will be denoted by $\operatorname{df-ice} A$.Let $\mathcal{E}$ be an epibrick in $\operatorname{mod} A$. We say $\mathcal{E}$ is doubly finite if $\operatorname{Filt}(\mathcal{E})$ is a doubly functorially finite ICE-closed subcategory. The set of all doubly finite epibricks in $\operatorname{mod} A$ will be denoted by $\operatorname{df-ebrick} A$. The following result extends Adachi et al.'s bijection and Asai's bijection.

Theorem 2.3. ([2, Theorem 4.13] and [7, Theorem 2.6]) *Let A be an algebra. Then, there are bijections:*

$$\mathrm{w}\tau\text{-tilt}\, A \underset{P(-)}{\overset{\operatorname{cok}(-)}{\rightleftarrows}} \operatorname{df-ice} A \underset{\operatorname{Filt}(-)}{\overset{\operatorname{Sim}(-)}{\rightleftarrows}} \operatorname{df-ebrick} A$$

where $\operatorname{cok}(M)$ denotes the subcategory of $\operatorname{mod}\Lambda$ *consisting of cokernels of morphisms in* $\operatorname{add} M$, *and $P(C)$ denotes the maximal Ext-projective object of C.*

3. Main results

For $M \in \operatorname{mod} A$, let $\operatorname{Supp} M := \{i \mid \operatorname{Hom}_A(P_i, M) \neq 0\}$, which is called the support of M. In addition, a set $\mathcal{E}$ consists of some modules is called sincere if $\operatorname{Supp} \mathcal{E} = \operatorname{Supp} A = \{1, 2, 3, \cdots, n\}$ where $\operatorname{Supp} \mathcal{E} := \bigcup\limits_{M \in \mathcal{E}} \operatorname{Supp} M$ is called the support of $\mathcal{E}$. It is clear that $\operatorname{Supp} \mathcal{E} = \emptyset$ if and only if $\mathcal{E} = \{0\}$. The set of sincere wide τ-tilting modules (respectively, sincere right Schur subcategories, sincere doubly functorially finite ICE-closed subcategories, sincere ICE-closed subcategories, sincere doubly finite epibricks, and sincere epibricks) in $\operatorname{mod} A$ will be denoted by $\mathrm{sw}\tau\text{-tilt}\, A$ (respectively, $\operatorname{sschur}_{\mathrm{R}} A$, $\operatorname{sdf-ice} A$, $\operatorname{sice} A$, $\operatorname{sdf-ebrick} A$, and $\operatorname{sebrick} A$).

Theorem 3.1. *Let A be a finite dimensional algebra. Then, those bijections in Proposition 2.2 and Theorem 2.3 can be restricted to the support sets. In particular, there are bijections:*

$$\operatorname{sschur}_{\mathrm{R}} A \underset{\mathrm{Filt}(-)}{\overset{\mathrm{Sim}(-)}{\rightleftarrows}} \operatorname{sebrick} A\ ,$$

and

$$\mathrm{sw}\tau\text{-tilt}\,A \underset{P(-)}{\overset{\mathrm{cok}(-)}{\rightleftarrows}} \operatorname{sdf-ice} A \underset{\mathrm{Filt}(-)}{\overset{\mathrm{Sim}(-)}{\rightleftarrows}} \operatorname{sdf-ebrick} A\ .$$

Proof. We only need to prove that four mappings preserve support sets. First, for an exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} A$, it is clear $\operatorname{Supp} M = \operatorname{Supp} L \cup \operatorname{Supp} N$ since $\operatorname{Hom}(P_i, -)$ is exact. Note that $\operatorname{mod} A$ has finite length, we claim $\mathrm{Filt}(-)$ preserve support sets (i.e., $\operatorname{Supp}\operatorname{Filt}\mathcal{E} = \operatorname{Supp}\mathcal{E}$). Second, $\operatorname{Supp} N \subset \operatorname{Supp} M$ for any epimorphism from M to N. This implies that $\operatorname{Supp}\operatorname{cok} M \subset \operatorname{Supp} M$. Thus $\operatorname{Supp}\operatorname{cok} M = \operatorname{Supp} M$ since $M \in \operatorname{cok} M$. Similarly, $\operatorname{Supp} P(C) = \operatorname{Supp} C$ since every functorially finite subcategory has enough Ext-projective objects. Finally, $\operatorname{Supp}\operatorname{Sim} C = \operatorname{Supp}\operatorname{Filt}\operatorname{Sim} C = \operatorname{Supp} C$. $\square$

Remark 3.2. *(1) Adachi et al.'s bijection can be restricted to the support sets since $Fac(-)$ preserve support sets.*

(2) Enomoto showed that there exists a bijection between rigid modules and ICE-closed subcategories with enough Ext-projective objects [9, Theorem 2.3] for hereditary algebras. This bijection also can be restricted to the support sets.

Example 3.3. *Let Q be the quiver: $1 \leftarrow 2 \rightarrow 3$ and $A = KQ$. The Auslander-Reiten quiver of* $\operatorname{mod} A$ *is as follows:*

$$\begin{tikzcd}
3 \arrow[dr] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & \\
& \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \arrow[ur] \arrow[dr] & & 2. \\
1 \arrow[ur] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

For a full subcategory C of $\operatorname{mod} A$*, we use $\bullet$ or $\circ$ instead of each indecomposable module in the Auslander-Reiten quiver, and $\bullet$ means that the module belongs to C and $\circ$ means not. For example,* $\operatorname{add}\{\begin{smallmatrix}&2&\\1&&3\end{smallmatrix}, 2\}$ *will be denoted by* $\begin{smallmatrix}\circ&&\circ&\\&\bullet&&\bullet\\\circ&&\circ&\end{smallmatrix}$. *Now, we list* $\mathrm{w}\tau\text{-tilt}\,A$*,* $\operatorname{df-ice} A$*, and* $\operatorname{df-ebrick} A$ *in table 1 (sincere cases) and 2 (non-sincere cases).*

$\#\,\mathrm{w}\tau\text{-tilt}\,A = 22$ *and* $\#\,\mathrm{sw}\tau\text{-tilt}\,A = 11$ *(in Table 1), but* $\#\operatorname{ebrick} A = 26$ *and* $\#\operatorname{sebrick} A = 15$. *In fact, there are four sincere epibricks* $\mathcal{E}_1 = \{2, \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}\}$, $\mathcal{E}_2 = \{2, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}\}$, $\mathcal{E}_3 = \{2, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}\}$, $\mathcal{E}_4 = \{\begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}\}$ *corresponding to four sincere right Schur subcategories* $\begin{smallmatrix}\circ&&\circ&\\&\bullet&&\bullet\\\circ&&\circ&\end{smallmatrix}$, $\begin{smallmatrix}\circ&&\bullet&\\&\bullet&&\bullet\\\circ&&\circ&\end{smallmatrix}$, $\begin{smallmatrix}\circ&&\circ&\\&\bullet&&\bullet\\\circ&&\bullet&\end{smallmatrix}$, $\begin{smallmatrix}\circ&&\bullet&\\&\bullet&&\circ\\\circ&&\bullet&\end{smallmatrix}$ *are not ICE-closed.*

An algebra is Nakayama if its quiver is

$$A_n:\quad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \qquad\qquad \text{or} \qquad\qquad \widetilde{A}_n:\quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend right] \end{tikzcd}$$

see [10, V.3.2]. [8, Theorem 6.1] stated that all left Schur subcategories are IKE-closed (i.e., closed under extensions, kernels and images) for Nakayama algebras. Further, we have all right Schur subcategories are ICE-closed. Hence, all ebricks are doubly finite for Nakayama algebras. Therefore, we have

Table 1. wτ-tilt$A$, df-ice$A$ and df-eprick$A$.

wτ-tilt A	df-ice A	df-ebrick A
$1 \oplus 3 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	● ● ● ● ● ●	$\{1, 2, 3\}$
$1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	○ ● ● ● ● ●	$\{2, \begin{smallmatrix}2\\3\end{smallmatrix}, 1\}$
$\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	○ ● ● ● ○ ●	$\{2, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}\}$
$\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$	○ ● ○ ○ ○ ●	$\{\begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}\}$
$\begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus 3$	● ● ● ○ ○ ○	$\{\begin{smallmatrix}2\\1\end{smallmatrix}, 3\}$
$1 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	○ ○ ● ○ ● ●	$\{\begin{smallmatrix}2\\3\end{smallmatrix}, 1\}$
$2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	○ ● ○ ● ○ ●	$\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2, \begin{smallmatrix}2\\1\end{smallmatrix}\}$
$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	● ● ● ● ○ ●	$\{\begin{smallmatrix}2\\1\end{smallmatrix}, 2, 3\}$
$\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	○ ● ● ○ ○ ○	$\{\begin{smallmatrix}&2&\\1&&3\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}\}$
$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	○ ○ ● ○ ○ ●	$\{\begin{smallmatrix}&2&\\1&&3\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}\}$
$\begin{smallmatrix}&2&\\1&&3\end{smallmatrix}$	○ ○ ● ○ ○ ○	$\{\begin{smallmatrix}&2&\\1&&3\end{smallmatrix}\}$

Table 2. wτ-tilt$A$, df-ice$A$ and df-eprick$A$.

wτ-tilt A	df-ice A	df-ebrick A
$2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$	○ ● ○ ● ○ ○	$\{2, \begin{smallmatrix}2\\1\end{smallmatrix}\}$
$2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$	○ ○ ○ ● ○ ●	$\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2\}$
$\begin{smallmatrix}2\\1\end{smallmatrix} \oplus 1$	○ ● ○ ● ● ○	$\{1, 2\}$
$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$	● ○ ○ ● ○ ●	$\{3, 2\}$
$1 \oplus 3$	● ○ ○ ○ ● ○	$\{1, 3\}$
$\begin{smallmatrix}2\\1\end{smallmatrix}$	○ ● ○ ○ ○ ○	$\{\begin{smallmatrix}2\\1\end{smallmatrix}\}$
$\begin{smallmatrix}2\\3\end{smallmatrix}$	○ ○ ○ ○ ○ ●	$\{\begin{smallmatrix}2\\3\end{smallmatrix}\}$
2	○ ○ ○ ● ○ ○	$\{2\}$
3	● ○ ○ ○ ○ ○	$\{3\}$
1	○ ○ ○ ○ ● ○	$\{1\}$
0	○ ○ ○ ○ ○ ○	$\{0\}$

Theorem 3.4. *There are bijections for a Nakayama algebra A :*

$$\mathrm{sw}\tau\text{-tilt}\, A \underset{P(-)}{\overset{\mathrm{cok}(-)}{\rightleftarrows}} \mathrm{sice}\, A \underset{\mathrm{Filt}(-)}{\overset{\mathrm{Sim}(-)}{\rightleftarrows}} \mathrm{sebrick}\, A\ .$$

Ler $\Lambda_n^r := KA_n/\operatorname{rad}^r$ and $\tilde{\Lambda}_n^r := K\widetilde{A}_n/\operatorname{rad}^r$. Enomoto showed that

$$\#\,\mathrm{w}\tau\text{-tilt}\, KA_n = LS_n = \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n+i}{i}$$

which is the n-th large Schröder number. Next, the number of sincere wide τ-tilting modules will be obtained by calculating the number of sincere epibricks over Λ_n^r and $\tilde{\Lambda}_n^r$. Further, we show that

$$\#\,\mathrm{sw}\tau\text{-tilt}\, KA_n = ls_n = \frac{1}{2}LS_n$$

which is the n-th little Schröder number, and

$$\#\,\mathrm{sw}\tau\text{-tilt}\, \tilde{\Lambda}_n^n = \sum_{i=0}^{n-1}\binom{n-1}{i}\binom{n+i}{i+1}.$$

Proposition 3.5. *Let A be a Nakayama algebra whose quiver is A_n and $LS_0 = 1$. Then,*

$$\#\,\mathrm{sw}\tau\text{-tilt}\, A = \sum_{i=1}^{l(I_n)} LS_{i-1}\cdot \#\,\mathrm{sw}\tau\text{-tilt}(A/\langle e_{[n-i+1,n]}\rangle).$$

Proof. Let $E_{i,j}$ be the brick of A with $\operatorname{top} E_{i,j} = S_i$ and $\operatorname{soc} E_{i,j} = S_j$. Let $\mathcal{E} \in \mathrm{sebrick}\, A$. Because $\mathcal{E}$ is a sincere epibrick, it satisfies exactly one of the following conditions (i): $E_{n-i+1,n}$ belongs to $\mathcal{E}$ ($i \in \{1, 2, ...l(I_n)\}$). Define W_i as the subset of $\mathrm{sebrick}\, A$ consisting of the epibricks satisfying the condition (i). Then, $\mathrm{sebrick}\, A = \bigcup_{i=1}^{l(I_n)} W_i$. Hence $\#\,\mathrm{sebrick}\, A = \sum_{i=1}^{l(I_n)} \#W_i$.

For $i = 1, 2, 3, , \cdots , l(I_n)$, there exists a bijection

$$W_i \mapsto \mathrm{ebrick}(A/\langle 1 - e_{[n-i+1,n-1]}\rangle) \times \mathrm{sebrick}(A/\langle e_{[n-i+1,n]}\rangle)$$

given by $\mathcal{E} \mapsto (\{E \in \mathcal{E} \mid \operatorname{Supp} E \subset [n-i+1, n-1]\}, \{E \in \mathcal{E} \mid \operatorname{Supp} E \cap [n-i+1, n] = \emptyset\})$. The inverse is given by $(\mathcal{E}_1, \mathcal{E}_2) \mapsto \mathcal{E}_1 \cup \mathcal{E}_2 \cup E_{n-i+1,n}$. Theorem 3.4 and $A/\langle 1 - e_{[n-i+1,n-1]}\rangle \cong KA_{i-1}$ imply

$$\begin{aligned}\#W_i =&\#\,\mathrm{ebrick}(A/\langle 1 - e_{[n-i+1,n-1]}\rangle)\cdot \#\,\mathrm{sebrick}(A/\langle e_{[n-i+1,n]}\rangle)\\ =&\#\,\mathrm{ebrick}(KA_{i-1})\cdot \#\,\mathrm{sebrick}(A/\langle e_{[n-i+1,n]}\rangle)\\ =&LS_{i-1}\cdot \#\,\mathrm{sebrick}(A/\langle e_{[n-i+1,n]}\rangle).\end{aligned}$$

□

Let $sa_{n,r} := \#\,\mathrm{sw}\tau\text{-tilt}\, \Lambda_n^r$, and we have the following recurrence relation.

Corollary 3.6.

$$sa_{n,r} = \begin{cases} \sum_{i=1}^{r} LS_{i-1}\cdot sa_{n-i,r} & n \geq r \\ sa_{n,n} & n < r \end{cases}.$$

The number $a_n := \#\,\mathrm{w}\tau\text{-tilt}\,KA_n = LS_n$. Consider the generating function $F(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = 1$. Thus it is known that the following holds for the large Schrӧder number

$$F(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}.$$

Let $sa_n := \#\,\mathrm{sw}\tau\text{-tilt}\,KA_n$ and its generating function be $f(x) = \sum_{n=0}^{\infty} sa_n x^n$ with $sa_0 = 1$. Then, $sa_{n,n} = sa_n$. Proposition 3.5 implies the recurrence relation

$$sa_n = \sum_{i=1}^{n} a_{i-1} \cdot sa_{n-i} = \sum_{i=0}^{n-1} a_i \cdot sa_{n-i-1}.$$

Thus,

$$\begin{aligned}
f(x) =& \sum_{n=0}^{\infty} sa_n x^n \\
=& sa_0 + sa_1 x + (a_0 sa_1 + a_1 sa_0)x^2 + (a_0 sa_2 + a_1 sa_1 + a_2 sa_0)x^3 + \cdots \\
=& 1 + x + ((a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(sa_0 + sa_1 x + sa_2 x^2 + sa_3 x^3 + \cdots) - 1)x \\
=& 1 + x + (f(x)F(x) - 1)x
\end{aligned}$$

which implies $f(x) = \frac{1}{1-xF(X)} = \frac{1+x-\sqrt{x^2-6x+1}}{4x} = \frac{1}{2}F(x) + \frac{1}{2}$. This coincides with the generating function of little Schrӧder number and implies $sa_n = \frac{1}{2}a_n,\ n \geq 1$, $(sa_0 = \frac{1}{2}a_0 + \frac{1}{2} = 1)$.

Now, let $sb_{n,r} := \#\,\mathrm{sw}\tau\text{-tilt}\,\widetilde{\Lambda^r_n}$. Note that $M \in \mathrm{mod}\,\widetilde{\Lambda^r_n}$ is a brick if and only if $l(M) \leqslant n$, we get $sb_{n,r} = sb_{n,n}$ when $r \geqslant n$. Hence, we only need to calculate $sb_{n,r}$ for $r \leqslant n$.

Lemma 3.7.

$$sb_{n,r} = \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot sa_{n-i,r}.$$

Proof. Let $\mathcal{E} \in \mathrm{sebrick}\,\widetilde{\Lambda^r_n}$. Because $\mathcal{E}$ is a sincere epibrick, it satisfies exactly one of the following conditions (i): $\max\{l(S)|S \in \mathcal{E}, n \in \mathrm{Supp}\,S\} = i$, $(i \in \{1, 2, ...r\})$.

Define V_i as the subset of sebrick $\widetilde{\Lambda^r_n}$ satisfying the condition (i). Then, sebrick $\widetilde{\Lambda^r_n} = \bigcup_{i=1}^{r} V_i$. Hence $sb_{n,r} = \sum_{i=1}^{r} \#V_i$.

First, there exists a bijection $V_1 \mapsto \mathrm{sebrick}\,\widetilde{\Lambda^r_n}/\langle e_n\rangle$ given by $V_1 \ni \mathcal{E} \mapsto \mathcal{E}\backslash\{E_{n,n}\}$. Because $\widetilde{\Lambda^r_n}/\langle e_n\rangle \cong \Lambda^r_{n-1}$, we have $\#V_1 = sa_{n-1,r}$.

Second, let $i \in \{2, 3, 4, \cdots, r\}$. $E_{u,v}$ is a brick if the pair (u, v) satisfying $n \in [[u, v]]$ and $\#[[u, v]] = i$, where

$$[[u, v]] = \begin{cases} \{u, u+1, u+2, \cdots, v\} & u \leq v \\ \{u, u+1, u+2, \cdots, n, 1, 2, 3, \cdots, v\} & u > v \end{cases}.$$

Now, let $V_{u,v} \subseteq V_i$ consists of sincere epibricks $\mathcal{E}$ with $E_{u,v} \in \mathcal{E}$.

For $\mathcal{E} \in V_{u,v}$, any brick $E \in \mathcal{E}$ different from $E_{u,v}$ satisfies $\operatorname{Supp} E \in [[u, v]]\backslash\{v\}$ or $\operatorname{Supp} E \cap [[u, v]] = \emptyset$. Moreover, $E_{u,v}$ is the unique brick $E \in \mathcal{E}$ satisfying $n \in \operatorname{Supp} E$ and $\# \operatorname{Supp} E = i$.

Hence, we have a decomposition which is a disjoint union:

$$V_i = \coprod_{n\in[[u,v]],\#[[u,v]]=i} V_{u,v}.$$

For any pair (u, v) satisfying $n \in [[u, v]]$ and $\#[[u, v]] = i$, there exists a bijection

$$V_{u,v} \mapsto \operatorname{ebrick}(\widetilde{\Lambda_n^r}/\langle 1 - e_{[[u,v]]\backslash\{v\}}\rangle) \times \operatorname{sebrick}(\widetilde{\Lambda_n^r}/\langle e_{[[u,v]]}\rangle)$$

given by

$$\mathcal{E} \mapsto (\{E \in \mathcal{E} \mid \operatorname{Supp} E \subset [[u, v]]\backslash\{v\}\}, \{E \in \mathcal{E} \mid \operatorname{Supp} E \cap [[u, v]] = \emptyset\})$$

with inverse given by $(\mathcal{E}_1, \mathcal{E}_2) \mapsto \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{E_{u,v}\}$, where

$$e_{[[u,v]]} := \sum_{i\in[[u,v]]} e_i \quad \text{and} \quad e_{[[u,v]]\backslash\{v\}} := \sum_{i\in[[u,v]]\backslash\{v\}} e_i.$$

Since there are isomorphisms of algebras,

$$\widetilde{\Lambda_n^r}/\langle 1 - e_{[[u,v]]\backslash\{v\}}\rangle \cong KA_{i-1} \quad \text{and} \quad \widetilde{\Lambda_n^r}/\langle e_{[[u,v]]}\rangle \cong \Lambda_{n-i}^r,$$

we have $\#V_{u,v} = LS_{i-1} \cdot sa_{n-i,r}$. There exist exactly i pair (u, v) such that $n \in [[u, v]]$ and $\#[[u, v]] = i$, so we get $\#V_i = i \cdot LS_{i-1} \cdot sa_{n-i,r}$. Finally, Theorem 3.4 implies

$$sb_{n,r} = \# \operatorname{sebrick} \widetilde{\Lambda_n^r} = \sum_{i=1}^{r} \#V_i = \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot sa_{n-i,r}.$$

□

In particular, $sb_{n,n} = \sum_{i=1}^{n} i \cdot LS_{i-1} \cdot sa_{n-i,n} = \sum_{i=1}^{n} i \cdot a_{i-1} \cdot sa_{n-i}$. Considering the generating function $g(x) = \sum_{n=1}^{\infty} sb_{n,n}x^n$, we obtain the following equality:

$$\begin{aligned}
g(x) = &\sum_{n=1}^{\infty} sb_{n,n}x^n \\
=&(1a_0 sa_0)x + (1a_0 sa_1 + 2a_1 sa_0)x^2 + (1a_0 sa_2 + 2a_1 sa_1 + 3a_2 sa_0)x^3 + \cdots \\
=&x(a_0 + 2a_1 x + 3a_2 x^2 + 4a_3 x^3 + \cdots)(sa_0 + sa_1 x + sa_2 x^2 + sa_3 x^3 + \cdots) \\
=&x\frac{d[xF(x)]}{dx} f(x) \\
=&xF(x)f(x) + x^2\frac{d[F(x)]}{dx} f(x) \\
=&\frac{1}{2}(\frac{1-x}{\sqrt{x^2 - 6x + 1}} - 1)
\end{aligned}$$

which implies $sb_{n,n} = \sum_{i=0}^{n-1} \binom{n-1}{i}\binom{n+i}{i+1}$ [11, A002002].

Note that $sb_{n,r}$ is a linear combination of some $sa_{n-i,r} (i = 1, 2, \cdots, r)$, we get the some recurrence relation on $sb_{n,r}$ with $sa_{n,r}$.

Corollary 3.8.

$$sb_{n,r} = \begin{cases} \sum_{i=1}^{r} LS_{i-1} \cdot sb_{n-i,r}, & n > r \\ sb_{n,n} & n \leq r \end{cases}.$$

Proof. It follows from the equations:

$$\begin{aligned} sb_{n,r} - \sum_{i=1}^{r} LS_{i-1} \cdot sb_{n-i,r} &\overset{Lemma\ 3.7}{=} \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r} - \sum_{i=1}^{r} LS_{i-1} \cdot (\sum_{j=1}^{r} j \cdot LS_{j-1} \cdot a_{n-i-j,r}) \\ &= \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r} - \sum_{j=1}^{r} LS_{j-1} \cdot (\sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-j-i,r}) \\ &= \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r} - \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot (\sum_{j=1}^{r} LS_{j-1} \cdot a_{n-i-j,r}) \\ &= \sum_{i=1}^{r} i \cdot LS_{i-1}(a_{n-i,r} - \sum_{j=1}^{r} LS_{j-1} \cdot a_{n-i-j,r}) \\ &\overset{Corollary\ 3.6}{=} 0. \end{aligned}$$

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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