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Research article

The Ricci curvature and the normalized Ricci flow on the Stiefel manifolds $\mathrm{SO}(n)/\mathrm{SO}(n-2)$

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Abstract: We proved that on every Stiefel manifold $V_2\mathbb{R}^n \cong \mathrm{SO}(n)/\mathrm{SO}(n-2)$ with $n \geq 3$ the normalized Ricci flow preserves the positivity of the Ricci curvature of invariant Riemannian metrics with positive Ricci curvature. Moreover, the normalized Ricci flow evolves all metrics with mixed Ricci curvature into metrics with positive Ricci curvature in finite time. From the point of view of the theory of dynamical systems, we proved that for every invariant set Σ of the normalized Ricci flow on $V_2\mathbb{R}^n$ defined as $x_1^{n-2}x_2^{n-2}x_3 = c$, $c > 0$, there exists a smaller invariant set $\Sigma \cap \mathscr{R}_+$ for every $n \geq 3$, where $\mathscr{R}_+$ is the domain in $\mathbb{R}^3_+$ responsible for parameters $x_1, x_2, x_3 > 0$ of invariant Riemannian metrics on $V_2\mathbb{R}^n$ admitting positive Ricci curvature.

Keywords: Stiefel manifold; Riemannian metric; normalized Ricci flow; Ricci curvature; dynamical system; invariant set; singular point

1. Introduction

One of powerful tools to study the evolution of Riemannian metrics on a given Riemannian manifold $\mathscr{M}^d$ is the normalized Ricci flow equation

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2\,\mathrm{Ric}_{\mathbf{g}} + 2\mathbf{g}(t)\frac{S_{\mathbf{g}}}{d}, \tag{1.1}$$

introduced by Hamilton [1] for a family of a Riemannian metrics $\mathbf{g}(t)$ on $\mathscr{M}$, where $\mathrm{Ric}_{\mathbf{g}}$ and $S_{\mathbf{g}}$ are the Ricci tensor and the scalar curvature of a metric $\mathbf{g}$, respectively. Many papers are devoted to a class of Riemannian manifolds $\mathscr{M}$ called homogeneous, on which the isometry group $\mathrm{Isom}(\mathscr{M})$ acts transitively, see [2–5]. Any Riemannian homogeneous manifold $\mathscr{M}$ can be identified (is diffeomorphic) to some homogeneous space G/H with $G = \mathrm{Isom}(\mathscr{M})$ being a Lie group according to the Myers and Steenrod theorem and $H = G_m$ the isotropy subgroup at a given point $m \in \mathscr{M}$. A large class consists of reductive homogeneous spaces G/H, where G is a compact and semisimple Lie group and H is a connected closed subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras of G and H, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be a reductive decomposition of $\mathfrak{g}$ orthogonal with respect to the Killing form B defined on $\mathfrak{g}$ so that $\mathrm{Ad}(H)\mathfrak{p} \subset \mathfrak{p}$. Since B is negative definite, $-B$ defines an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle = -B(\cdot, \cdot)$ on $\mathfrak{g}$. Assume that $\mathfrak{p}$ admits a decomposition $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$ into pairwise inequivalent irreducible $\mathrm{Ad}(H)$-modules of dimensions $d_i = \dim \mathfrak{p}_i$ such that $d_1 + \cdots + d_k = d = \dim(G/H)$. Then, any G-invariant symmetric covariant 2-tensor on G/H can be written as $c_1 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_1} + \cdots + c_k \langle \cdot, \cdot \rangle|_{\mathfrak{p}_k}$, where $c_1, \ldots, c_k$ are some real numbers. In particular, so is any G-invariant metric $\mathbf{g}(\cdot, \cdot) = x_1 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_1} + \cdots + x_k \langle \cdot, \cdot \rangle|_{\mathfrak{p}_k}$ on G/H, where $x_i \in \mathbb{R}_+$, $i = 1, \ldots, k$. The Ricci tensor of this metric has the form $\mathrm{Ric}_{\mathbf{g}}(\cdot, \cdot) = x_1 \mathbf{r}_1 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_1} + \cdots + x_k \mathbf{r}_k \langle \cdot, \cdot \rangle|_{\mathfrak{p}_k}$, where $\mathbf{r}_i = \mathbf{r}_i(x_1, \ldots, x_k)$ are components of the Ricci tensor [6]. The decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$ allows (1.1) to split into a system of k nonlinear autonomous differential equations

$$\dot{x}_i = -2x_i \left(\mathbf{r}_i - \frac{\sum_{i=1}^k d_i \mathbf{r}_i}{\sum_{i=1}^k d_i} \right), \qquad i = 1, \ldots, k. \tag{1.2}$$

In particular, an interesting question is finding if the normalized Ricci flow preserves the positivity of the curvature of Riemannian metrics on a given manifold. In [7, 8] some results were obtained concerning this question on the Wallach spaces $\mathrm{SU}(3)/T^2$, $\mathrm{Sp}(3)/\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, $F_4/\mathrm{Spin}(8)$. The case of generalized Wallach spaces with coinciding parameters $a_1 = a_2 = a_3 \in (0, 1/2)$ was considered in [9]. Some extensions of the results of [9] can be found in [10, 11]. Recently, in [12] the results of [9] were generalized to the general case $a_1, a_2, a_3 \in (0, 1/2)$ according to the classification of [13]:

Theorem 1 (Theorem 5 in [12]). *The normalized Ricci flow* (1.1) *evolves certain invariant Riemannian metrics with positive Ricci curvature to metrics with mixed Ricci curvature on every generalized Wallach space with* $a_1 + a_2 + a_3 \le 1/2$.

Theorem 2 (Theorem 6 in [12]). *The following assertions hold for generalized Wallach spaces with* $a_1 + a_2 + a_3 > 1/2$*:*

1) The normalized Ricci flow (1.1) *evolves all invariant Riemannian metrics with positive Ricci curvature to metrics with positive Ricci curvature if* $\theta \ge \max\{\theta_1, \theta_2, \theta_3\}$*;*

2) The normalized Ricci flow (1.1) *evolves certain metrics with positive Ricci curvature to metrics with positive Ricci curvature if* $\theta \ge \max\{\theta_1, \theta_2, \theta_3\}$ *fails,*

where $\theta := a_1 + a_2 + a_3 - 1/2$ *and* $\theta_i := a_i - \dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{1 - 2a_i}{1 + 2a_i}}$*,* $i = 1, 2, 3$.

Theorem 3 (Theorem 7 in [12]). *The following assertions hold for* $\mathrm{SO}(k+l+m)/\mathrm{SO}(k) \times \mathrm{SO}(l) \times \mathrm{SO}(m)$*,* $k \ge l \ge m > 1$ *(denoted by* GWS **1** *according to [13]). Under the normalized Ricci flow* (1.1)

1) All invariant Riemannian metrics with positive Ricci curvature can be evolved into metrics with positive Ricci curvature if either $k \le 11$ *or one of the conditions* $2 < l + m \le X(k)$ *or* $l + m \ge Y(k)$ *is satisfied at each fixed* $k \in \{12, 13, 14, 15, 16\}$*;*

2) At least some invariant Riemannian metrics with positive Ricci curvature can be evolved into metrics with positive Ricci curvature if $k \ge 17$ *or if* $X(k) < l + m < Y(k)$ *for* $k \in \{12, 13, 14, 15, 16\}$*;*

3) The number of GWS **1** *spaces on which any original metric with* $\mathrm{Ric} > 0$ *maintains* $\mathrm{Ric} > 0$ *is finite, whereas there are infinitely (countably) many* GWS **1** *on which* $\mathrm{Ric} > 0$ *can be preserved at least for some original metrics with* $\mathrm{Ric} > 0$*,*

where $X(k) := \dfrac{2k(k-2)}{k + 2 + \sqrt{k^2 - 12k + 4}} - k + 2$ *and* $Y(k) := \dfrac{2k(k-2)}{k + 2 - \sqrt{k^2 - 12k + 4}} - k + 2.$

Since $(l, m) \neq (1, 1)$, the case of the space $\mathrm{SO}(n)/\mathrm{SO}(n-2) \times \mathrm{SO}(1) \times \mathrm{SO}(1)$ cannot be covered by Theorem 3. In the present paper we study the above question on the preservation of positivity of the Ricci curvature for the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ diffeomorphic to the Stiefel manifold $V_2\mathbb{R}^n$. In general $V_k\mathbb{R}^n$, $k \leq n$, can be defined as the set of $n \times k$ matrices A with real entries such that $A^t A = I_k$, where A^t means the transpose of A and I_k is the $k \times k$ identity matrix. $V_k\mathbb{R}^n$ becomes a compact smooth manifold in the subspace topology inherited from the topology of $\mathbb{R}^{n\times k}$. It is known that for $k < n$, the group $\mathrm{SO}(n)$ acts on $V_k\mathbb{R}^n$ transitively and $V_k\mathbb{R}^n$ is diffeomorphic to the reductive homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-k)$. In the case of these spaces there are equivalent submodules which may cause a complicated decomposition of the Ricci tensor. Although the decomposition of $\mathfrak{p}$ admits two equivalent submodules $\mathfrak{p}_1$ and $\mathfrak{p}_2$ causing the dependence of an $\mathrm{SO}(n)$-invariant metric of $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ on four parameters, it was proved in [14] that $\mathrm{SO}(n)$-invariant metrics on $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ can be reduced to a diagonal form and described by an $\mathrm{Ad}(\mathrm{SO}(n-2))$-invariant inner product

$$(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}, \quad x_i \in \mathbb{R}_+, \tag{1.3}$$

on $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ (see also [15]). The corresponding Ricci tensor is $\mathrm{Ric}_{\mathbf{g}}(\cdot,\cdot) = x_1\mathbf{r}_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\mathbf{r}_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\mathbf{r}_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}$. Clearly, $d_1 = \dim\mathfrak{p}_1 = n-2 = \dim\mathfrak{p}_2 = d_2$ and $d_3 = \dim\mathfrak{p}_3 = 1$ with $d = 2n-3$.

The main result of this paper is contained in the following theorem.

Theorem 4. *On every Stiefel manifold* $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ *with* $n \geq 3$*, the normalized Ricci flow* (1.1) *evolves any metric* (1.3) *to a metric with positive Ricci curvature.*

2. Results

The following expressions were found in [16] for the components of the Ricci tensor of the metric (1.3):

$$\begin{aligned}
\mathbf{r}_1 &= \frac{1}{2x_1} - \frac{1}{4(n-2)}\left(\frac{x_3}{x_1x_2} + \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\right),\\
\mathbf{r}_2 &= \frac{1}{2x_2} - \frac{1}{4(n-2)}\left(\frac{x_3}{x_1x_2} + \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3}\right),\\
\mathbf{r}_3 &= \frac{1}{2x_3} - \frac{1}{4}\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right).
\end{aligned} \tag{2.1}$$

Then, the scalar curvature $S_{\mathbf{g}} = d_1\mathbf{r}_1 + d_2\mathbf{r}_2 + d_3\mathbf{r}_3$ takes the form

$$S_{\mathbf{g}} = \frac{1}{2x_3} - \frac{1}{4}\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right) + \frac{n-2}{2}\left(\frac{1}{x_1} + \frac{1}{x_2}\right). \tag{2.2}$$

Substituting (2.1) and (2.2) into (1.2) the following system of ordinary differential equations can be obtained on $\mathrm{SO}(n)/\mathrm{SO}(n-2)$, $n \geq 3$:

$$\dot{x}_1 = f_1(x_1,x_2,x_3), \qquad \dot{x}_2 = f_2(x_1,x_2,x_3), \qquad \dot{x}_3 = f_3(x_1,x_2,x_3), \tag{2.3}$$

where

$$f_1 = \frac{(n-1)(x_3^2+x_2^2)-(3n-5)x_1^2-2(n-2)\Big[x_1x_2+(n-2)x_3x_1-(n-1)x_3x_2\Big]}{2(n-2)(2n-3)x_2x_3},$$
$$f_2 = \frac{(n-1)(x_3^2+x_1^2)-(3n-5)x_2^2-2(n-2)\Big[x_1x_2+(n-2)x_3x_2-(n-1)x_3x_1\Big]}{2(n-2)(2n-3)x_1x_3},$$
$$f_3 = \frac{(n-2)(x_1-x_2)^2+(n-2)(x_1+x_2)x_3-(n-1)x_3^2}{(2n-3)x_1x_2}.$$

2.1. The set $\mathscr{R}_+$ and its structural properties

Denote by $\mathscr{R}_+$ the set of metrics (1.3) which admit positive Ricci curvature

$$\mathscr{R}_+ = \{(x_1,x_2,x_3)\in\mathbb{R}^3_+ \;\big|\; \mathbf{r}_1>0,\ \mathbf{r}_2>0,\ \mathbf{r}_3>0\}.$$

Observe that $\mathbf{r}_3 = 0$ is equivalent to $(x_1 - x_2 + x_3)(x_2 - x_1 + x_3) = 0$ for every $n \geq 3$. Therefore, $\mathscr{R}_+$ is bounded by two planes $-x_1 + x_2 + x_3 = 0$ and $x_1 - x_2 + x_3 = 0$ in any case, where $x_1, x_2, x_3 > 0$. Denote them by Π_1 and Π_2, respectively. Let Γ_1 and Γ_2 be those components of the cones $\mathbf{r}_1 = 0$ and $\mathbf{r}_2 = 0$ defined by the equations (see Figure 1)

$$x_3 = (n-2)x_2 + \sqrt{x_1^2+(n-1)(n-3)x_2^2} \quad\text{and}\quad x_3 = (n-2)x_1 + \sqrt{x_2^2+(n-1)(n-3)x_1^2}, \tag{2.4}$$

respectively. We denote the other components

$$x_3 = (n-2)x_2 - \sqrt{x_1^2+(n-1)(n-3)x_2^2} \quad\text{and}\quad x_3 = (n-2)x_1 - \sqrt{x_2^2+(n-1)(n-3)x_1^2}$$

by Γ_1^- and Γ_2^-, respectively (the lower surfaces in Figure 1 in magenta and aquamarine). Let Π_3 be the plane $x_1 + x_2 - x_3 = 0$ and L be the ray $(p, p, (2n-4)p)$, $p > 0$ (depicted in blue color in Figure 1).

Lemma 1. *For all $n \geq 3$, the set $\mathscr{R}_+$ has the boundary*

$$\partial(\mathscr{R}_+) = \begin{cases} \Pi_1\cup\Pi_2\cup\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2, & \text{if } n\geq 4,\\ \Pi_1\cup\Pi_2\cup\Pi_3, & \text{if } n=3,\end{cases}$$

with $\widetilde{\Gamma}_1 = \Big\{(\mu t,\nu t,t)\;\Big|\; t>0,\ 0<\nu\leq\widetilde{\nu}\Big\}\subset\Gamma_1$ and $\widetilde{\Gamma}_2 = \Big\{(\nu t,\mu t,t)\;\Big|\; t>0,\ 0<\nu\leq\widetilde{\nu}\Big\}\subset\Gamma_2$, where $\mu = \sqrt{\nu^2-2(n-2)\nu+1}$ and $\widetilde{\nu} = (2n-4)^{-1}$. Moreover, $\Pi_1\cap\Pi_2 = \Pi_1\cap\widetilde{\Gamma}_1 = \Pi_2\cap\widetilde{\Gamma}_2 = \emptyset$ and $\widetilde{\Gamma}_1\cap\widetilde{\Gamma}_2 = L$ for all $n \geq 3$.

Proof. Let us consider $\Gamma_1 \cap \Gamma_2$. The system of equations

$$\begin{cases} x_1^2-x_2^2-x_3^2+2(n-2)x_2x_3=0,\\ x_1^2-x_2^2+x_3^2-2(n-2)x_1x_3=0\end{cases}$$

is equivalent to the system of $\mathbf{r}_1 = 0$ and $\mathbf{r}_2 = 0$. It is easy to see that $x_3 = (n-2)(x_1 + x_2)$ is a consequence of the system corresponding to $\Gamma_1\cap\Gamma_2$. Substituting this expression for x_3 into the second equation in the system, we obtain

$$(n-1)(n-3)(x_1+x_2)(x_1-x_2) = 0.$$

If $n \geq 4$, then $x_1 = x_2$. Hence, the straight line $x_1 = x_2 = p > 0$, $x_3 = 2(n-2)p$ denoted by L is a common part of the cones Γ_1 and Γ_2. If $n = 3$, then (2.4) implies that $\Gamma_1 = \Gamma_2 = \Pi_3$.

The system above also admits the extra solution $x_1 = x_2$, $x_3 = 0$. Obviously, it corresponds to $\Gamma_1^- \cap \Gamma_2^-$.

The component Γ_1 of the cone $\mathbf{r}_1 = 0$ can be parameterized as

$$x_1 = \mu t, \quad x_2 = \nu t, \quad x_3 = t, \qquad t > 0, \qquad 0 < \nu < l,$$

where $\mu = \sqrt{\nu^2 - 2(n-2)\nu + 1}$ and

$$l = n - 2 - \sqrt{(n-1)(n-3)} \tag{2.5}$$

is the smallest root of the quadratic equation $\nu^2 - 2(n-2)\nu + 1 = 0$ (by symmetry, for the component Γ_2 of $\mathbf{r}_2 = 0$, we have $x_2 = \mu t$, $x_1 = \nu t$, $x_3 = t$).

For all $n \geq 4$ the conic components Γ_1 and Γ_2 have extra pieces not relating to the set $\mathscr{R}_+$. Since $\Gamma_1 \cap \Gamma_2 = L$ then necessarily $\mu = \nu$, implying the critical value $\widetilde{\nu} = (2n-4)^{-1}$. It is easy to see that the useful part $\widetilde{\Gamma}_1$ of Γ_1 can be parameterized by values $t > 0$ and $0 < \nu \leq \widetilde{\nu}$. The extra part of Γ_1 corresponds to the values $t > 0$ and $\tilde{\nu} < \nu < l$. The description of $\widetilde{\Gamma}_2$ is obvious from symmetry. The equality $\widetilde{\Gamma}_1 \cap \widetilde{\Gamma}_2 = L$ is obvious too. The system of the equations $-x_1 + x_2 + x_3 = 0$ and $x_1 - x_2 + x_3 = 0$ implies that $x_3 = 0$. Analogously, it follows from $\Pi_1 \cap \Gamma_1$ and $\Pi_2 \cap \Gamma_2$ that $x_2 = 0$ and $x_1 = 0$. Therefore, $\Pi_1 \cap \Pi_2 = \Pi_1 \cap \Gamma_1 = \Pi_2 \cap \Gamma_2 = \emptyset$ for $x_i > 0$ implying $\Pi_1 \cap \Pi_2 = \Pi_1 \cap \widetilde{\Gamma}_1 = \Pi_2 \cap \widetilde{\Gamma}_2 = \emptyset$. □

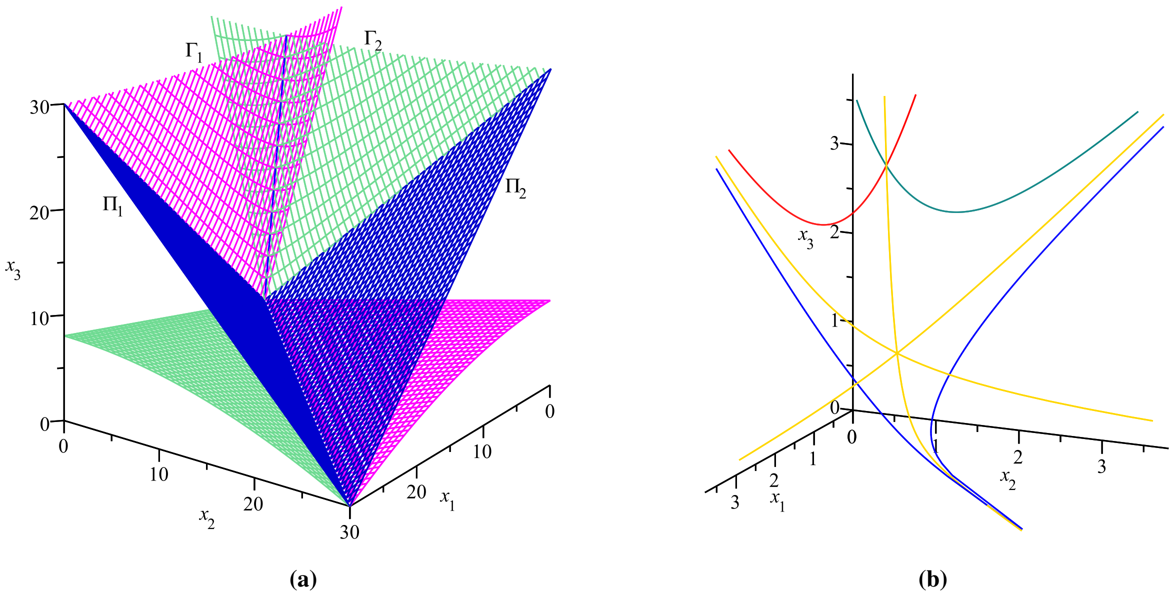

Figure 1. The case $n = 4$: the surfaces Γ_1, Γ_2 and Π_1, Π_2 (left panel); the curves $\Sigma \cap \Gamma_1$, $\Sigma \cap \Gamma_2$ and $\Sigma \cap \Pi_1$, $\Sigma \cap \Pi_2$ (right panel).

2.2. *The vector field* $\mathbf{V}$ *on the boundary* $\partial(\mathscr{R}_+)$

Introduce the vector field

$$\mathbf{V}\colon \mathbf{x} \mapsto (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})), \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3_+,$$

associated with the differential system (2.3).

Lemma 2. *For every $n \geq 3$, the vector field* $\mathbf{V}$ *associated to the system* (2.3)*, when restricted to its boundary $\partial(\mathscr{R}_+)$ takes values in the domain $\mathscr{R}_+$.*

Proof. By Lemma 1 the planes Π_1 and Π_2 bound the domain $\mathscr{R}_+$ for every $n \geq 3$. Denote by $\mathbf{n}_1$ and $\mathbf{n}_2$ their normals $(-1, 1, 1)$ and $(1, -1, 1)$. We claim that the inner product $(\mathbf{V}, \mathbf{n}_1)$ is positive at every point of Π_1 for all $n \geq 3$. Indeed, Π_1 can be parameterized as $x_2 = u$, $x_3 = v$, $x_1 = u + v$ with $u, v > 0$. Then,

$$(\mathbf{V}, \mathbf{n}_1) = -f_1\big|_{\Pi_1} + f_2\big|_{\Pi_1} + f_3\big|_{\Pi_1} = \frac{2}{n-2} > 0$$

on Π_1 for $n \geq 3$, where

$$\begin{aligned} f_1\big|_{\Pi_1} &= \frac{(n-2)(n-3)v - (3n-5)u}{(2n-3)(n-2)u}, \\ f_2\big|_{\Pi_1} &= \frac{(n-1)(u-(n-3)v)}{(2n-3)(n-2)(u+v)}, \\ f_3\big|_{\Pi_1} &= \left(\frac{v}{2n-3}\right)\frac{2(n-2)u + (n-3)v}{(u+v)u}. \end{aligned}$$

The same inequality $(\mathbf{V}, \mathbf{n}_2) = \frac{2}{n-2} > 0$ holds on the plane Π_2 for all $n \geq 3$. Indeed, Π_2 can be parameterized as $x_1 = u$, $x_3 = v$, $x_2 = u + v$ and $f_1\big|_{\Pi_2} = f_2\big|_{\Pi_1}$, $f_2\big|_{\Pi_2} = f_1\big|_{\Pi_1}$, $f_3\big|_{\Pi_2} = f_3\big|_{\Pi_1}$.

The case $n = 3$. According to Lemma 1, we have one more plane Π_3 that bounds the domain $\mathscr{R}_+$. Note that f_1, f_2, and f_3 take the following forms on Π_3:

$$f_1\big|_{\Pi_3} = \frac{u}{u+v}, \qquad f_2\big|_{\Pi_3} = \frac{v}{u+v}, \qquad f_3\big|_{\Pi_3} = -\frac{4}{3}.$$

Then, we obtain $(\mathbf{V}, \mathbf{n}_3) = f_1\big|_{\Pi_3} + f_2\big|_{\Pi_3} - f_3\big|_{\Pi_3} = 2 > 0$ on Π_3, where $\mathbf{n}_3 = (1, 1, -1)$ is the normal of Π_3.

Since the normals $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are directed inside the domain $\mathscr{R}_+$ the inequalities $(\mathbf{V}, \mathbf{n}_i) > 0$ imply that the vector field $\mathbf{V}$ is directed towards $\mathscr{R}_+$ on every point of the boundary $\partial(\mathscr{R}_+) = \Pi_1 \cup \Pi_2 \cup \Pi_3$.

The case $n \geq 4$. Then, $\partial(\mathscr{R}_+) = \Pi_1 \cup \Pi_2 \cup \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2$ by Lemma 1. It suffices to consider the surface $\widetilde{\Gamma}_1 = \{(\mu t, \nu t, t) \mid t > 0,\ \nu \in (0, \widetilde{\nu}]\}$, where $\widetilde{\nu} = (2n-4)^{-1}$.

The components of the normal $\mathbf{m} = \nabla \mathbf{r}_1 = \left(\frac{\partial \mathbf{r}_1}{\partial x_1}, \frac{\partial \mathbf{r}_1}{\partial x_2}, \frac{\partial \mathbf{r}_1}{\partial x_3}\right)$ to the conic component Γ_1 have the forms

$$\begin{aligned} \frac{\partial \mathbf{r}_1}{\partial x_1} &= 2x_1 > 0, \\ \frac{\partial \mathbf{r}_1}{\partial x_2} &= 2(n-2)x_3 - 2x_2, \\ \frac{\partial \mathbf{r}_1}{\partial x_3} &= 2(n-2)x_2 - 2x_3. \end{aligned}$$

It is easy to check that $\frac{1}{2(n-2)} < n - 2 - \sqrt{(n-1)(n-3)} \leq \frac{1}{n-2}$ for $n \geq 3$. Then, the inequalities $0 < \nu \leq \widetilde{\nu} < l < \frac{1}{n-2} < n - 2$ yield that

$$\frac{\partial \mathbf{r}_1}{\partial x_2} = 2t\,(n-2-\nu) > 0, \qquad \frac{\partial \mathbf{r}_1}{\partial x_3} = 2t\,((n-2)\nu - 1) < 0$$

on Γ_1 and, in particular, on $\widetilde{\Gamma}_1$ implying that $\mathbf{m}$ is directed inside the domain $\mathscr{R}_+$ for all $n \geq 4$.

We claim that the inner product $(\mathbf{V}, \mathbf{m})$ is positive on every point of $\widetilde{\Gamma}_1$. Indeed, by calculations in Maple, we obtain that $\operatorname{sign}(\mathbf{V}, \mathbf{m}) = \operatorname{sign}(F - G)$, where

$$\begin{aligned} F = F(\nu) &:= -(n-1)(n-2-\nu)((n-2)\nu - 1), \\ G = G(\nu) &:= (n-2)(n-3)(\nu+1)\sqrt{\nu^2 - 2(n-2)\nu + 1}. \end{aligned}$$

Since $F > 0$ and $G > 0$ for all $n \geq 4$ and all $\nu \in (0, \widetilde{\nu}]$ the sign of $F - G$ coincides with the sign of the polynomial

$$\begin{aligned} p(\nu) := F^2 - G^2 = 4(n-2)^3\nu^4 - 2(n-2)(5n^3 - 31n^2 + 67n - 49)\nu^3 \\ + (n^6 - 6n^5 - 5n^4 + 136n^3 - 453n^2 + 630n - 327)\nu^2 \\ - 2(n-2)(5n^3 - 31n^2 + 67n - 49)\nu + 4(n-2)^3. \end{aligned}$$

We claim that $p(\nu) > 0$ for all $\nu \in (0, \widetilde{\nu}]$ and all $n \geq 4$. It is more convenient to consider the interval $[0, 1] \supset (0, \widetilde{\nu}]$ instead of $(0, \widetilde{\nu}]$. Since the discriminant

$$(n-2)^4(n-3)^9(n-1)^{10}(4n^2 - 13n + 11)^2 \left[8 + 20(n-3) + 12(n-3)^2 + (n-3)^3\right]$$

of the polynomial $p(\nu)$ is positive for every $n \geq 4$ then $p(\nu)$ has no multiple roots in $[0, 1]$. Therefore, we can use Sturm's method for our goal. Construct the sequence of Sturm polynomials $p_0 = p$, $p_1 = p_0'$, $p_{i+1} = -\operatorname{rem}(p_{i-1}, p_i)$, where $\operatorname{rem}(p_{i-1}, p_i)$ is the remainder of the division of p_{i-1} by p_i, $i = 1, 2, 3$. Then, for $\nu = 0$ we obtain using Maple that

$$\begin{aligned} p_0(0) &= 4(n-2)^3 > 0, \\ p_1(0) &= -2(n-2)\alpha_1 < 0, \\ p_2(0) &= \frac{(n-1)(5n-9)(n-3)^2(5n^2 - 18n + 17)}{16(n-2)} > 0, \\ p_3(0) &= 32(n-1)(n-2)^2(n-3)(4n^2 - 13n + 11)\frac{\alpha_1\alpha_2}{\alpha_3^2} > 0, \\ p_4(t) &\equiv \frac{(n-1)^6(n-2)(n-3)^3}{16}\left(\frac{\alpha_3}{\alpha_4}\right)^2 \alpha_5 > 0, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= 5(n-3)^3 + 14(n-3)^2 + 16(n-3) + 8 > 0, \\ \alpha_2 &= (n-3)^3 + 10(n-3)^2 + 20(n-3) + 12 > 0, \\ \alpha_3 &= 8n^4 - 83n^3 + 285n^2 - 413n + 219 \neq 0 \text{ (the divisor 73 of 219 is not a root)}, \\ \alpha_4 &= n^9 - 5n^8 - 24n^7 + 256n^6 - 802n^5 + 930n^4 + 692n^3 - 3268n^2 + 3589n - 1401 \neq 0, \\ \alpha_5 &= (n-3)^3 + 12(n-3)^2 + 20(n-3) + 8 > 0 \end{aligned}$$

for $n \geq 4$, where $\alpha_4 \neq 0$ for $n \geq 4$ since the unique acceptable divisor $n = 467$ of 1401 does not satisfy $\alpha_4 = 0$.

Evaluate now p_i at $\nu = 1$:

$$\begin{aligned} p_0(1) &= (n-3)^3\alpha_5 > 0, \\ p_1(1) &= 2(n-3)^3\alpha_5 > 0, \\ p_2(1) &= -\frac{(n-1)(n-3)^2(5n^2-18n+17)\alpha_5}{16(n-2)^2} < 0, \\ p_3(1) &= -32(n-1)(n-2)(n-3)^2(4n^2-13n+11)\,\frac{\alpha_5\alpha_6}{\alpha_3^2} < 0, \end{aligned}$$

where $\alpha_6 = 24 + 60(n-3) + 63(n-3)^2 + 35(n-3)^3 + 10(n-3)^4 + (n-3)^5 > 0$.

It follows that the Sturm sequence has the same number of sign changes, namely 1, for both $\nu = 0$ and $\nu = 1$ under the condition $n \geq 4$. Therefore, $p(\nu)$ has no real roots in the interval $[0, 1]$ for all $n \geq 4$. Since $p(0) > 0$ we have $p(\nu) > 0$ for all $\nu \in [0, 1]$, in particular, this is true for all $\nu \in (0, \widetilde{\nu}]$. Therefore, $(\mathbf{V}, \mathbf{m}) > 0$ on $\widetilde{\Gamma}_1$. Since the normal $\mathbf{m}$ is directed into the domain $\mathscr{R}_+$, as shown above the vector field $\mathbf{V}$ is directed towards $\mathscr{R}_+$ on the surface $\widetilde{\Gamma}_1$. By symmetry, the analogous assertion holds for $\mathbf{V}$ on the surface $\widetilde{\Gamma}_2 \subset \Gamma_2$. □

2.3. Invariant sets and singular points of the system (2.3)

Let Σ be the surface defined by the equation $\mathrm{Vol}(x_1, x_2, x_3) = c$, $c > 0$, where

$$\mathrm{Vol}(x_1, x_2, x_3) = x_1^{n-2} x_2^{n-2} x_3$$

is the volume function for the metric (1.3). Introduce the following subsets of Σ (curves on Σ) as well

$$\begin{aligned} I_1 &= \left\{ \left(c^{\frac{1}{n-2}}\,\tau^{\frac{1-n}{n-2}}, \tau, \tau \right) \in \mathbb{R}_+^3 \;\middle|\; \tau > 0 \right\}, \\ I_2 &= \left\{ \left(\tau, c^{\frac{1}{n-2}}\,\tau^{\frac{1-n}{n-2}}, \tau \right) \in \mathbb{R}_+^3 \;\middle|\; \tau > 0 \right\}, \\ I_3 &= \left\{ \left(\tau, \tau, c\tau^{4-2n} \right) \in \mathbb{R}_+^3 \;\middle|\; \tau > 0 \right\}. \end{aligned}$$

In Figure 1 the curves I_1, I_2, I_3 are depicted in gold for $c = 1$ and $n = 4$.

Lemma 3. *The following assertions hold for the system* (2.3)*:*

1) The algebraic surface Σ is invariant for every $n \geq 3$ and every $c > 0$;

2) The curve I_3 is invariant for every $n \geq 3$ and every $c > 0$;

3) The curves I_1 and I_2 are also invariant for every $c > 0$ if $n = 3$.

4) For all $n \geq 3$ the following inclusions are true:

$$\begin{cases} I_1, I_2 \subset \mathscr{R}_+, & \text{if } \tau > \tau_1 := 2^{2-n}c, \\ I_3 \subset \mathscr{R}_+, & \text{if } \tau > \tau_2 := \left(\dfrac{2n-4}{c} \right)^{\frac{1}{3-2n}}. \end{cases}$$

Proof. 1) Actually the first assertion can easily be proved for system (1.2). Put $\mathrm{Vol} = x_1^{d_1} \cdots x_k^{d_k}$ and $U := \mathrm{Vol} - c$. Then, the invariance of Σ is equivalent to the inner product $(\nabla U, \mathbf{V})$ vanishing on it. Indeed

$$(\nabla U, \mathbf{V}) = \sum_{i=1}^{k} \frac{\partial U}{\partial x_i} f_i = (U + c) \sum_{i=1}^{k} \frac{d_i}{x_i} f_i = -2(U + c) \sum_{i=1}^{k} d_i \left(\mathbf{r}_i - \frac{S_{\mathbf{g}}}{d} \right) = 0$$

due to $S_{\mathbf{g}} = \sum_{i=1}^{k} d_i \mathbf{r}_i$ and $d = \sum_{i=1}^{k} d_i$ (see also Lemma 2 in [17]).

2) The invariance of I_3 is equivalent to the vector field $\mathbf{V}$ being parallel to the tangent of I_3 at its every point. Direct calculations confirm this property:

$$\frac{d}{d\tau}\left(c\tau^{4-2n}\right) = -2c(n-2)\tau^{-2n+3} = \left.\frac{f_3}{f_1}\right|_{I_3} = \left.\frac{f_3}{f_2}\right|_{I_3}.$$

3) It suffices to observe that I_1, I_2, and I_3 can be obtained by cyclic permutations in $(\tau, \tau, c\tau^{-2})$ in the case $n = 3$. Moreover, $\mathbf{x}^0 = (\sqrt[3]{c}, \sqrt[3]{c}, \sqrt[3]{c})$ by Lemma 4. However, for $n \geq 4$, the curves I_1 and I_2 cannot pass through the point $\mathbf{x}^0$ and hence cannot be invariant.

4) Choose I_1. Substituting $x_1 = c^{\frac{1}{n-2}}\tau^{\frac{1-n}{n-2}}$, $x_2 = \tau$ and $x_3 = \tau$ into Π_1 and Π_2 we obtain

$$\left.\Pi_1\right|_{I_1} = \tau\left(2 - c^{\frac{1}{n-2}}\tau^{-\frac{1}{n-2}}\right), \qquad \left.\Pi_2\right|_{I_1} = c^{\frac{1}{n-2}}\tau^{-\frac{n-1}{n-2}}. \tag{2.6}$$

Analogously $\mathbf{r}_1$ and $\mathbf{r}_2$ take the following forms on I_1:

$$\begin{aligned} \left.\mathbf{r}_1\right|_{I_1} &= \frac{1}{4(n-2)}\left(2(n-3) + c^{\frac{2}{n-2}}\tau^{-\frac{2}{n-2}}\right)\tau^{\frac{n-1}{n-2}}c^{-\frac{1}{n-2}}, \\ \left.\mathbf{r}_2\right|_{I_1} &= \frac{1}{4(n-2)\tau}\left(2(n-2) - c^{\frac{1}{n-2}}\tau^{-\frac{1}{n-2}}\right). \end{aligned} \tag{2.7}$$

Then, $\left.\Pi_2\right|_{I_1} > 0$ and $\left.\mathbf{r}_1\right|_{I_1} > 0$ for $\tau > 0$ and $n \geq 3$. But $\left.\Pi_1\right|_{I_1} > 0$ only for $\tau > \tau_1$, where $\tau_1 := 2^{2-n}c$ is the single root of the equation $2 - c^{\frac{1}{n-2}}\tau^{-\frac{1}{n-2}} = 0$.

Note that $\left.\mathbf{r}_2\right|_{I_1} > 0$ for all $n \geq 3$ if $\tau > \tau_1$:

$$2(n-2) - c^{\frac{1}{n-2}}\tau^{-\frac{1}{n-2}} = 2(n-3) + 2 - c^{\frac{1}{n-2}}\tau^{-\frac{1}{n-2}} > 0.$$

By symmetry, for I_2 we have

$$\left.\Pi_1\right|_{I_2} = \left.\Pi_2\right|_{I_1}, \qquad \left.\Pi_2\right|_{I_2} = \left.\Pi_1\right|_{I_1}, \qquad \left.\mathbf{r}_1\right|_{I_2} = \left.\mathbf{r}_2\right|_{I_1}, \qquad \left.\mathbf{r}_2\right|_{I_2} = \left.\mathbf{r}_1\right|_{I_1}. \tag{2.8}$$

Now consider I_3. Then,

$$\begin{aligned} \left.\Pi_1\right|_{I_3} = \left.\Pi_2\right|_{I_3} &= c\tau^{-2(n-2)} > 0, \\ \left.\mathbf{r}_1\right|_{I_3} = \left.\mathbf{r}_2\right|_{I_3} &= \frac{1}{4(n-2)\tau}\left(2n - 4 - c\tau^{3-2n}\right). \end{aligned}$$

Therefore, $\left.\mathbf{r}_1\right|_{I_3} = \left.\mathbf{r}_2\right|_{I_3} > 0$ for $n \geq 3$ and $\tau > \tau_2 := \left(\frac{2n-4}{c}\right)^{\frac{1}{3-2n}}$. □

Remark 1. *As follows from* (2.7)*, I_1 intersects the cone $\mathbf{r}_2 = 0$ at the point $\left(c^{\frac{1}{n-2}}\tau_0^{\frac{1-n}{n-2}}, \tau_0, \tau_0\right)$, where τ_0 is defined from the equation $c^{\frac{1}{n-2}}\tau_0^{-\frac{1}{n-2}} = 2(n-2)$ for all $n \geq 3$. However, this point does not belong* *to the set* $\mathscr{R}_+$ *since* $\Pi_1\big|_{I_1} = \tau\left(2 - c^{\frac{1}{n-2}}\tau^{-\frac{1}{n-2}}\right) \leq 0$ *at* $\tau = \tau_0$*. Indeed,* $2 - 2(n-2) = 6 - 3n \leq 0$ *for* $n \geq 3$*. Actually, here we deal with the intersection of* I_1 *with another component* Γ_2^- *of the cone* $\mathbf{r}_2 = 0$*, establishing that* Γ_2^- *is the extra component which has no relation to the set* $\mathscr{R}_+$ *(the lower surface in aquamarine in Figure 1). Thus,* I_1 *does not intersect the useful component* Γ_2 *of* $\mathbf{r}_2 = 0$ *(see* (2.4)*). A similar conclusion can be obtained for* I_2*, by interchanging the indices by symmetry.*

Lemma 4. *The following assertions hold for the system* (2.3) *at a fixed* $c > 0$*:*

1) For every $n \geq 3$ *the system* (2.3) *has the unique family of one-parameter singular (equilibrium) points given by the formula*

$$\mathbf{x}^0 = (q, q, \kappa q) \in \mathscr{R}_+, \quad \textit{where} \quad \kappa := 2(n-2)(n-1)^{-1}, \quad q \in \mathbb{R}_+.$$

The actual value $q = q_0$ *of the parameter* q *at which* $\mathbf{x}^0$ *belongs to* Σ *is*

$$q_0 = \sqrt[2n-3]{c\kappa^{-1}}.$$

2) If $n = 3$*, then* Σ *is the stable manifold of* $\mathbf{x}^0$ *with the tangent space* $E^s = \operatorname{Span}\{(-1, 1, 0), (-1, 0, 1)\}$ *and, hence,* $\mathbf{x}^0 \in \Sigma$ *is a stable node. Moreover,* I_1, I_2 *and* I_3 *are one-dimensional stable submanifolds for* $\mathbf{x}^0$*.*

3) If $n \geq 4$*, then* $\mathbf{x}^0$ *is a saddle with the tangent spaces* $E^s = \operatorname{Span}\left\{(1, 1, -4(n-2)^2(n-1)^{-1})\right\}$ *and* $E^u = \operatorname{Span}\{(-1, 1, 0)\}$ *of corresponding manifolds (separatrices) on* Σ*. Moreover, the stable separatrix is exactly* I_3 *for all* $n \geq 4$*.*

4) For all $n \geq 3$*, the singular point* $\mathbf{x}^0$ *also admits the center manifold with the tangent* $E^c = \operatorname{Span}\{(1, 1, \kappa)\}$*.*

Proof. Fix $c > 0$ and use ideas of [17].

1) Every singular (equilibrium) point of the system (1.2) is also an invariant Einstein metric of a considered space G/H. Conversely, every Einstein metric of G/H is a singular point of (1.2). Such a conclusion easily follows from the observation that the equalities $f_1 = \cdots = f_k = 0$ and $\mathbf{r}_1 = \cdots = \mathbf{r}_k = d^{-1}S_{\mathbf{g}}$ are equivalent. In particular, according to [14, 18], the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ admits the unique $\mathrm{SO}(n)$-invariant Einstein metric $(1, 1, \kappa)$ up to scale for every $n \geq 3$ which is also the unique singular point of the system (2.3). The actual value q_0 providing $\mathbf{x}^0 \in \Sigma$ can easily be found from the condition

$$\operatorname{Vol}(q, q, \kappa q) = c.$$

To establish that $\mathbf{x}^0 \in \mathscr{R}_+$ for all $n \geq 3$, it suffices to check the definition of the set $\mathscr{R}_+$:

$$\mathbf{r}_1(\mathbf{x}^0) = \mathbf{r}_2(\mathbf{x}^0) = \frac{2(n-2) - \kappa}{4q(n-2)} = \frac{n-2}{2q(n-1)} > 0, \qquad \mathbf{r}_3(\mathbf{x}^0) = \frac{\kappa}{4q} > 0.$$

Denote by $\chi(\lambda, \mathbf{x})$ the characteristic polynomial of the Jacobian matrix $\mathbf{J}(\mathbf{x})$ of the vector field $\mathbf{V}(\mathbf{x})$ associated with the differential system (2.3), $\mathbf{x} = (x_1, x_2, x_3)$. Then, the cubic equation

$$\chi(\lambda, \mathbf{x}^0) = \lambda^3 + \frac{n-1}{q(n-2)^2}\lambda^2 - \frac{n^2 - 5n + 5}{q^2(n-2)^2}\lambda = 0$$

admits three real roots

$$\lambda_1 = (n^2 - 5n + 5)(n-2)^{-2}q^{-1}, \qquad \lambda_2 = -q^{-1}, \qquad \lambda_3 = 0$$

for $\mathbf{x} = \mathbf{x}^0 = (q, q, \kappa q)$.

2) If $n = 3$, then $\lambda_1 = \lambda_2 = -q^{-1} < 0$. Therefore $\mathbf{x}^0$ is a stable (attracting) node on every Σ with the stable manifold $W^s = \Sigma$ admitting the tangent space spanned by the eigenvectors $(-1, 1, 0)$ and $(-1, 0, 1)$ corresponding to the eigenvalue λ_1 of multiplicity 2.

In addition, I_1, I_2 and I_3 are invariant for $n = 3$ by Lemma 3. Moreover, I_1, I_2 and I_3 are also subsets of the stable manifold Σ and hence each of them is stable.

3) For $n \geq 4$, we have $\lambda_1\lambda_2 < 0$. Therefore, $\mathbf{x}^0$ is a saddle on Σ with one dimensional stable and unstable manifolds (separatrices) both contained in Σ. The tangent space E^s of the stable separatrix is spanned by the eigenvector $(1, 1, -4(n-2)^2(n-1)^{-1})$ corresponding to $\lambda_2 = -q^{-1} < 0$. The eigenvector $(-1, 1, 0)$ corresponds to $\lambda_1 > 0$.

In addition, I_3 maintains the property of invariance for all $n \geq 4$ by Lemma 3 and in fact it is the stable manifold of the saddle $\mathbf{x}^0$. Indeed, since $\dot{x}_1(\tau) \equiv 1$, $\dot{x}_2(\tau) \equiv 1$, $\dot{x}_3(\tau) = (4 - 2n)c\tau^{3-2n}$ we have

$$\begin{aligned} \dot{x}_1(q_0) &= \dot{x}_2(q_0) = 1, \\ \dot{x}_3(q_0) &= (4 - 2n)cq_0^{3-2n} = (4 - 2n)\kappa = -4(n-2)^2(n-1)^{-1} \end{aligned}$$

for $\tau = q_0$ corresponding to the saddle point $\mathbf{x}^0 = (q_0, q_0, \kappa q_0) \in I_3 \subset \Sigma$. Therefore, the vector $(1, 1, -4(n-2)^2(n-1)^{-1})$ is tangent to I_3 at $\mathbf{x}^0$. On the other hand, this vector spans the stable eigenspace E^s of $\mathbf{x}^0$ as shown above. Therefore, the stable manifold (separatrix) W^s of $\mathbf{x}^0$ coincides with I_3 for every $n \geq 4$.

4) Finally, $\lambda_3 = 0$ is responsible for the center (slow) manifold with the tangent space spanned by the eigenvector $(1, 1, \kappa)$ for all $n \geq 3$. □

Remark 2. *Actually, we are interested in the special case* $c = 1$ *which corresponds to the metrics* (1.3) *of unit volume and, therefore, has a purely geometric meaning. Studies of* (2.3) *on* Σ *with an arbitrary* $c > 0$ *can easily be reduced to the "geometric" case,* $c = 1$*, by the change of variables* $x_i = X_i\sqrt[3]{c}$ *and* $t = \tau\sqrt[3]{c}$ *in* (2.3) *based on homogeneity of the functions* f_i *and autonomy of the system* (2.3) *(see also [17]). Thus, in the sequel we assume that* $c = 1$ *without loss of generality.*

2.4. Structural properties of the set $\Sigma \cap \mathscr{R}_+$

We need the sets $\gamma_1 = \Sigma \cap \widetilde{\Gamma}_1$, $\gamma_2 = \Sigma \cap \widetilde{\Gamma}_2$ (the curves in red and teal respectively in Figure 1) and $\pi_1 = \Sigma \cap \Pi_1$, $\pi_2 = \Sigma \cap \Pi_2$ (the curves in blue color there).

Lemma 5. *For all* $n \geq 3$*, the set* $\Sigma \cap \mathscr{R}_+$ *is bounded by the smooth and connected curves* π_1, π_2, γ_1 *and* γ_2 *such that:*

1) $\pi_1 \cap \pi_2 = \pi_1 \cap \gamma_1 = \pi_2 \cap \gamma_2 = \emptyset$ *and the components in the pairs* (π_1, π_2), (π_1, γ_1) *and* (π_2, γ_2) *approach each other arbitrarily closely at infinity (which we denote by* $\pi_1 \to \pi_2$*,* $\pi_1 \to \gamma_1$ *and* $\pi_2 \to \gamma_2$*);*

2) The curves γ_1 *and* γ_2 *have a single common point* $P_{12}(\overline{p}, \overline{p}, (2n-4)\overline{p})$*, where* $\overline{p} = \left(\dfrac{c}{2n-4}\right)^{\frac{1}{2n-3}}$.

Proof. Intersections and long time behaviors of the curves π_i, γ_i*,* $i = 1, 2$. Since $\gamma_1 \cap \gamma_2 = (\widetilde{\Gamma}_1 \cap \widetilde{\Gamma}_2) \cap \Sigma$ and $\widetilde{\Gamma}_1 \cap \widetilde{\Gamma}_2 = L$ by Lemma 1, the set $\gamma_1 \cap \gamma_2$ consists of the single point P_{12}, where $\overline{p}$ is defined as the unique root of the equation $p^{n-2}p^{n-2}(2n-4)p = c$, for a given $n \geq 3$.

By analogy, $\pi_1 \cap \pi_2 = \pi_1 \cap \gamma_1 = \pi_2 \cap \gamma_2 = \emptyset$ easily follows from $\Pi_1 \cap \Pi_2 = \Pi_1 \cap \widetilde{\Gamma}_1 = \Pi_2 \cap \widetilde{\Gamma}_2 = \emptyset$, also known from Lemma 1, for all $n \geq 3$. To prove that π_1 and γ_1 approximate each other at infinity we use the approach developed in [12]: we will show $\pi_1 \to I_2$ and $\gamma_1 \to I_2$ instead of directly showing that $\pi_1 \to \gamma_1$ (note that it is quite difficult to derive it directly from the corresponding systems which define π_1 and γ_1). According to Lemma 3, we know that $I_1, I_2, I_3 \subset \mathscr{R}_+$ for sufficiently large τ (more precisely, for all $\tau > \max\{\tau_1, \tau_2\}$).

Moreover, Lemma 3 also implies the following limits for every fixed $n \geq 3$ (see formulas (2.6)–(2.8)):

$$\lim_{\tau\to+\infty} \mathbf{r}_1\Big|_{I_2} = \lim_{\tau\to+\infty} \frac{1}{4(n-2)\tau}\left(2(n-2) - \tau^{-\frac{1}{n-2}}\right) = 0,$$
$$\lim_{\tau\to+\infty} \Pi_1\Big|_{I_2} = \lim_{\tau\to+\infty} \tau^{-\frac{n-1}{n-2}} = 0.$$

Therefore, $\pi_1 \to \gamma_1$ at infinity as close as we want. Then, $\pi_2 \to \gamma_2$ is clear from symmetry. Finally, $\Pi_1\Big|_{I_3} = \Pi_2\Big|_{I_3} = \tau^{-2(n-2)} \to 0$ as $\tau \to +\infty$, implying that $\pi_1 \to \pi_2$.

Smoothness and connectedness of the curves π_i, γ_i, $i = 1, 2$. Substituting $x_2 = tx_1$, where $t > 0$, into $\mathbf{r}_3 = 0$, we easily get that $(t-1)^2 x_1^2 - x_3^2 = 0$. Taking into account the condition $x_1^{n-2} x_2^{n-2} x_3 = c$ (the equation of the surface Σ), we obtain the parametric equations

$$x_1 = \phi_1(t) = c^{\frac{1}{2n-3}}\, t^{\frac{2-n}{2n-3}}\, |t-1|^{\frac{1}{3-2n}}, \qquad x_2 = \phi_2(t) = t\,\phi_1(t), \qquad x_3 = \phi_3(t) = |t-1|\,\phi_1(t)$$

for π_1 and π_2 with $t \in (0, +\infty) \setminus \{1\}$ so that the interval $(0, 1)$ corresponds to π_1 and $(1, +\infty)$ corresponds to π_2. By analogy, putting $x_2 = tx_3$ in $\mathbf{r}_1 = 0$, the equations

$$x_3 = \psi_3(t) = c^{\frac{1}{2n-3}}\, t^{-\frac{n-2}{2n-3}}\, \Psi(t)^{-\frac{n-2}{2n-3}}, \quad x_2 = \psi_2(t) = t\,\psi_3(t), \quad x_1 = \psi_1(t) = \psi_3(t)\Psi(t), \quad t \in (0, l),$$

can be obtained for the curve $\Sigma \cap \Gamma_1$ with $\Psi(t) = \sqrt{t^2 - 2(n-2)t + 1}$ and l is given in (2.5). The representation $x_1 = \psi_2(t)$, $x_2 = \psi_1(t)$, $x_3 = \psi_3(t)$ is clear for the curve $\Sigma \cap \Gamma_2$ due to symmetry, $t \in (0, l)$. It is clear that $\gamma_1 \subset \Sigma \cap \Gamma_1$ and $\gamma_2 \subset \Sigma \cap \Gamma_2$, and hence γ_1 and γ_2 can be parameterized by the same functions ψ_1, ψ_2, ψ_3, but with $t \in \Delta$, where Δ is some interval such that $\Delta \subset (0, l)$. To clarify Δ, recall Lemma 1, according to which the curves $\Sigma \cap \Gamma_1$ and $\Sigma \cap \Gamma_2$ leave tails (extra pieces) not related to the set $\Sigma \cap \mathscr{R}_+$ after their intersection in the case $n \geq 4$, and coincide if $n = 3$. So to eliminate those extra pieces, take into account $x_1 = x_2$ at the unique point $P_{12} \in \gamma_1 \cap \gamma_2$, which implies the equation $\Psi(t) = t$ with the unique root $\widetilde{t} = (2n-4)^{-1}$. Thus, in what follows we assume that $\Delta = (0, \widetilde{t}]$.

Now it is easy to see that the functions ϕ_1, ϕ_2, and ϕ_3 are differentiable on the sets $(0, 1)$ and $(1, +\infty)$. Therefore, the curves π_1 and π_2 are smooth. The curve π_1 must be connected, as it is the image of the connected set $(0, 1)$ under the continuous function $t \mapsto (\phi_1(t), \phi_2(t), \phi_3(t))$, $t \in (0, 1)$. So is π_2, being the continuous image of the connected set $(1, +\infty)$. Smoothness and connectedness of γ_1 and γ_2 follow analogously from the differentiability of the functions $t \mapsto (\psi_1(t), \psi_2(t), \psi_3(t))$ and $t \mapsto (\psi_2(t), \psi_1(t), \psi_3(t))$, where $t \in \Delta$. □

Proof of Theorem 4. Step 1. Any trajectory of (2.3) originating in $\mathscr{R}_+$ *remains* there forever. Indeed, by Lemma 2, the vector field $\mathbf{V}$ associated with the system (2.3) is directed into $\mathscr{R}_+$ on every point of $\partial(\mathscr{R}_+)$, for all $n \geq 3$. Therefore, no trajectory of (2.3) can leave $\mathscr{R}_+$: if $\mathbf{x}(0) \in \mathscr{R}_+$ then $\mathbf{x}(t) \in \mathscr{R}_+$ for all $t > 0$. According to the definition of the set $\mathscr{R}_+$, this means that (1.1) preserves the positivity of the Ricci curvature of metrics (1.3) on $\mathrm{SO}(n)/\mathrm{SO}(n-2)$: metrics with positive Ricci curvature can be evolved only into metrics with positive Ricci curvature.

Step 2. Any trajectory originating in the exterior of $\mathscr{R}_+$ *enters* $\mathscr{R}_+$ in finite time. To prove this assertion we use Lemma 4. It suffices to study (2.3) on $\Sigma \cap \mathscr{R}_+$, where Σ are the invariant surfaces of (2.3) responsible for dominant motions of its trajectories corresponding to nonzero eigenvalues λ_1 and λ_2, according to Lemmas 3 and 4. Any movement caused by $\lambda_3 = 0$, for all $n \geq 3$, can be neglected because it can only occur within the domain $\mathscr{R}_+$ itself, along the eigenvector $(1, 1, \kappa)$ (as slow transitions between different invariant surfaces):

$$\mathbf{r}_1(1, 1, \kappa) = \mathbf{r}_2(1, 1, \kappa) = \frac{n-2}{2(n-1)} > 0, \qquad \mathbf{r}_3(1, 1, \kappa) = \frac{\kappa}{4} > 0.$$

By Lemma 4, for every $n \geq 3$, there exists $q = q_0$ such that $\mathbf{x}^0 \in \Sigma \cap \mathscr{R}_+$. Therefore, every trajectory of (2.3) originated in $\Sigma \cap \operatorname{ext} \mathscr{R}_+$ must intersect the boundary $\Sigma \cap \partial(\mathscr{R}_+)$ in finite time and enter the set $\Sigma \cap \mathscr{R}_+$ governed by the stable manifold $W^s = \Sigma$ (and by the stable submanifolds I_1, I_2, and I_3 as well) of the unique stable node $\mathbf{x}^0$ in the case $n = 3$, and by the separatrices $W^s = I_3 \subset \Sigma$ and $W^u \subset \Sigma$ of $\mathbf{x}^0$ being the unique saddle for every $n \geq 4$ (see Figure 2). The domain $\Sigma \cap \mathscr{R}_+$ is able to receive all trajectories due to the fact that it is unbounded, with the boundary $\Sigma \cap \partial(\mathscr{R}_+)$ consisting of the curves π_1, π_2, γ_1, and γ_2 in $\mathbb{R}^3$ such that $\pi_1 \cap \pi_2 = \pi_1 \cap \gamma_1 = \pi_2 \cap \gamma_2 = \emptyset$, according to Lemma 5. This is also quite consistent with the facts established above. Indeed, if $\Sigma \cap \partial(\mathscr{R}_+)$ was a bounded set, then in order to satisfy Lemma 2, the dynamics of the system (2.3) would be different from those described in Lemma 4 contradicting it. Another significant circumstance that causes all trajectories to enter $\mathscr{R}_+$ and remain there forever for all $n \geq 4$ is that the separatrices W^s and W^u are subsets of $\mathscr{R}_+$ at infinity since none of them can reach (intersect or touch) the boundary $\Sigma \cap \partial(\mathscr{R}_+)$, according to Lemma 2.

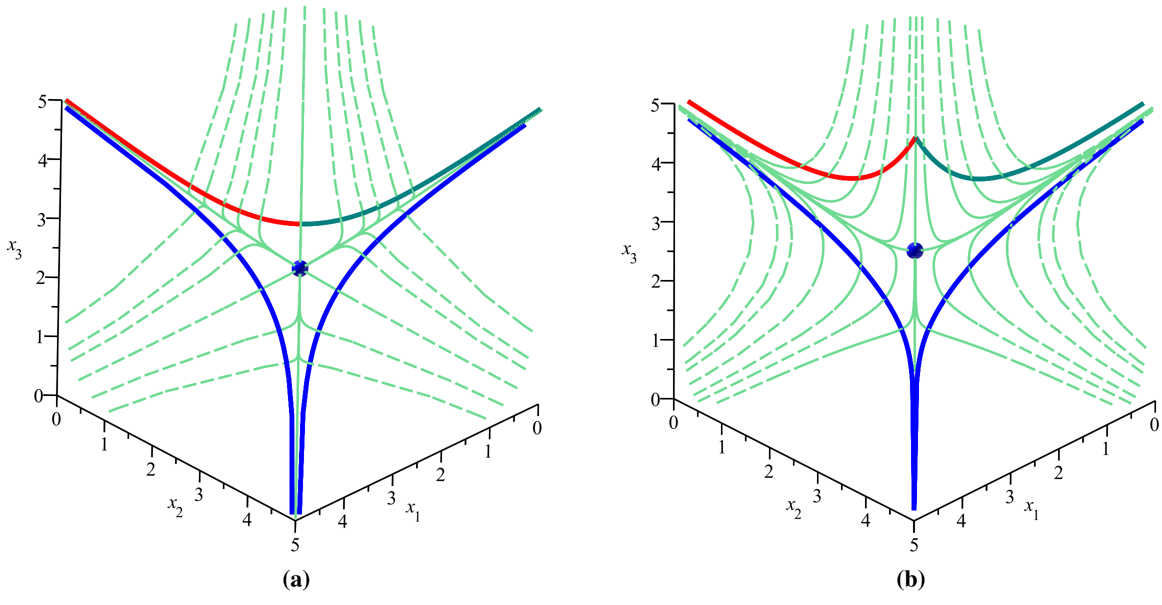

Figure 2. The dynamics of the system (2.3) on the invariant set Σ for $n = 3$ (left panel) and $n = 4$ (right panel).

Thus, we proved that on the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n-2)$, $n \geq 3$, the normalized Ricci flow (1.1) evolves all metrics (1.3) (of mixed or positive Ricci curvature) into metrics with positive Ricci curvature. Theorem 4 is proved. □

3. Planar illustrations

Clearly, the volume function Vol $= x_1^{n-2}x_2^{n-2}x_3$ is a first integral of (2.3). Based on this fact, the system (2.3) could equivalently be reduced to the following planar system:

$$\frac{dx_1}{dt} = \widetilde{f}_1(x_1, x_2), \qquad \frac{dx_2}{dt} = \widetilde{f}_2(x_1, x_2), \tag{3.1}$$

where $\widetilde{f}_i(x_1, x_2) := f_i(x_1, x_2, \varphi(x_1, x_2))$ and $\varphi(x_1, x_2) := (x_1x_2)^{2-n}$. The corresponding singular point of (3.1) is $\widetilde{\mathbf{x}}^0 = (q_0, q_0)$, where $q_0 = \sqrt[2n-3]{(n-1)(2n-4)^{-1}}$.

On the other hand, it is well known that $\chi(\lambda, \mathbf{x}^0) = \lambda^3 - \rho\lambda^2 + \delta\lambda$, where $\rho = \operatorname{trace}\widetilde{\mathbf{J}}(\widetilde{\mathbf{x}}^0)$ and $\delta = \det\widetilde{\mathbf{J}}(\widetilde{\mathbf{x}}^0)$ are the trace and the determinant, respectively, of the Jacobian matrix $\widetilde{\mathbf{J}}(\widetilde{\mathbf{x}})$ of the system (3.1), evaluated at its singular point $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^0$. Consequently,

$$\delta = -\frac{n^2 - 5n + 5}{(n-2)^2q^2}, \qquad \rho = -\frac{n-1}{(n-2)^2q} < 0, \qquad \sigma = \rho^2 - 4\delta = \frac{(2n-3)^2(n-3)^2}{(n-2)^4q^2} \geq 0.$$

According to the theory of planar dynamical systems, $\widetilde{\mathbf{x}}^0$ is a hyperbolic stable node of (3.1) for $n = 3$ since $\delta > 0$, $\rho < 0$, and $\sigma = 0$. Clearly, $\widetilde{\mathbf{x}}^0$ is a hyperbolic saddle of (3.1) for all $n \geq 4$ since $\delta < 0$. These results are illustrated in Figure 3.

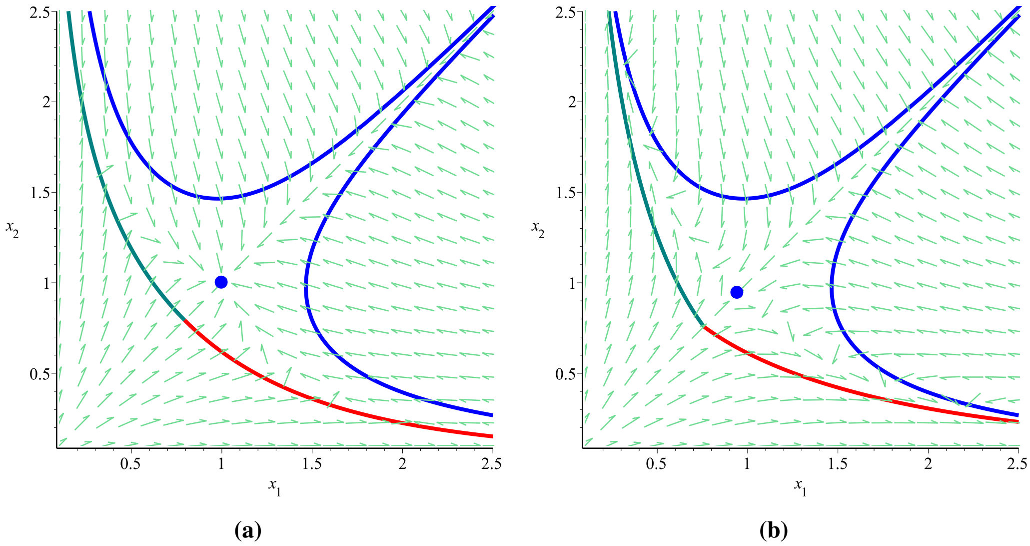

Figure 3. The phase portrait of the planar system (3.1) for $n = 3$ (left panel) and $n = 4$ (right panel).

Using the idea in [9], the dynamics of (2.3) can be illustrated on the plane $x_1 + x_2 + x_3 = 1$ (denoted by Π_0) preserving the dihedral symmetry of the problem. In Figure 4, the relevant results are depicted, where the edges of the big triangle correspond to $x_i = 0$ on Π_0; the curves $\Pi_0 \cap \widetilde{\Gamma}_1$, $\Pi_0 \cap \widetilde{\Gamma}_2$ and $\Pi_0 \cap \Pi_1$, $\Pi_0 \cap \Pi_2$ are depicted in red, teal, and blue, respectively; the yellow circle (or ellipse) represents the curve defined by the systems of the equations $x_0^2 - (2n-4)(x_1 + x_2)x_0 + (x_1 - x_2)^2 = 0$ and $x_1 + x_2 + x_3 = 1$, where the first one is equivalent to $S_{\mathbf{g}} = 0$. These illustrations confirm the well known general fact that the normalized Ricci flow preserves the positivity of the scalar curvature of invariant metrics on every compact homogeneous space (see [19, 20]).

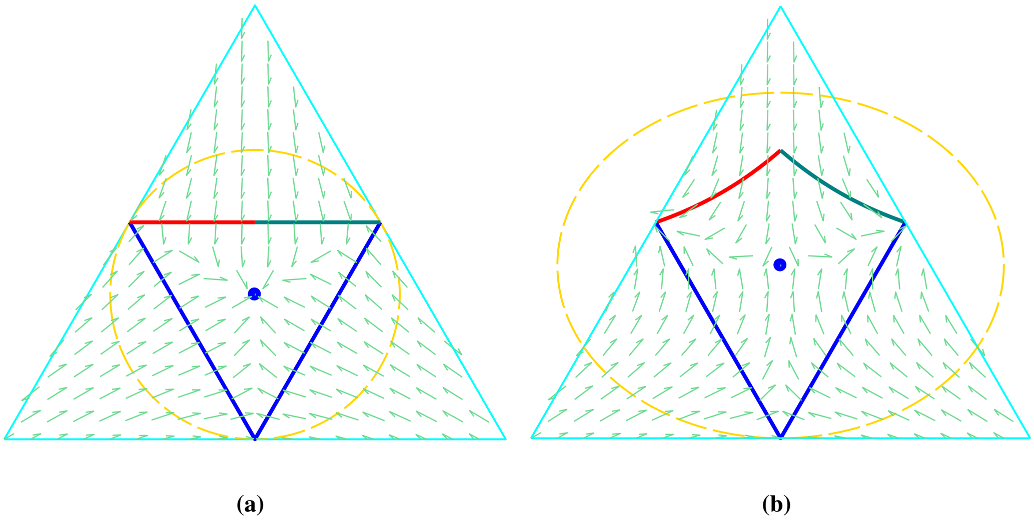

Figure 4. The phase portrait of (2.3) on the plane $x_1 + x_2 + x_3 = 1$ at $n = 3$ (left panel) and $n = 4$ (right panel).

4. Conclusions

The geometric problem considered in the article can be of some interest from the point of view of the dynamical systems. It is not easy to find the smallest invariant set (or minimal invariant sets) of the flow. In Theorem 4 we proved that, for every invariant set Σ of the system (2.3) defined as $x_1^{n-2}x_2^{n-2}x_3 = c$, there exists its invariant subset $\Sigma \cap \mathscr{R}_+$. Similar results were obtained in [9, 12, 17] for (1.1) reduced to a dynamical system on generalized Wallach spaces; see for instance the case that $a \in (1/6, 1/4) \cup (1/4, 1/2)$ in [9, Theorem 3], the case that $a = 1/6$ in [9, Theorem 4] and the case that $a \in [1/4, 1/2)$ in [17, Theorem 2] (see cases 4 and 5), which concern invariance of certain sets related to generalized Wallach spaces with $a_1 = a_2 = a_3 = a$. The case that $a_1 + a_2 + a_3 > 1/2$ was studied in [12, Theorem 6], where some additional conditions were found on the parameters a_1, a_2, a_3, which provide the invariance of a set analogous to $\mathscr{R}_+$.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflicts of interest.

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