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https://www.aimspress.com/journal/era

ERA, 33(3): 1610–1632.
DOI: 10.3934/era.2025076
Received: 10 December 2024
Revised: 22 February 2025
Accepted: 25 February 2025
Published: 20 March 2025

Research article

Global dynamics of a predator-prey model with prey-taxis and hunting cooperation

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Abstract: The mathematical analysis of spatiotemporal distributions in many species exhibiting different predation mechanisms has attracted considerable attention in biology and ecology. In this article, we investigated a prey-taxis model involving hunting cooperation, which has more strong coupling structures. Utilizing energy estimates and semigroup theory, the global boundness of its classical solution was established when the hunting cooperation is weak in two dimensions. By means of Lyapunov functionals, the global asymptotically stability of the non-negative constant steady-state solution for the discussed model was established under certain assumptions on parameters. These results enrich the related researches on the prey-taxis model with Lotka-Volterra functional response, which has been studied by Jin and Wang.

Keywords: prey-taxis; hunting cooperation; boundedness; long time behavior

1. Introduction

In natural habitats, there exist diverse interspecific interactions, such as competition, cooperation, predation, and so on. Predation serves as a fundamental interspecific relationship, which plays a pivotal role in maintaining ecological balance by regulating the populations of predators and prey. To capture the pattern of the quality of the species and the dynamics of predation, Lotka and Volterra presented a classical ordinary differential equation (ODE) model, called the predator-prey model, in the 1920s. Subsequently, many scholars began to study various predator-prey models with different predatory mechanisms, such as direct predation [1], selective predation [2, 3], cooperative predation [4, 5], and so on.

The random motion of species in space is a natural movement behavior, and scholars introduce random diffusion of predators and prey into the classical predator-prey model [6–8] to depict the spatiotemporal distribution of species. In addition, predators seek to improve their survival prospects by making directional movements toward areas with higher density of prey. To describe this phenomenon, the following partial differential equation model, called the prey-taxis model:

$$\begin{cases} u_t = d_1\Delta u - vF(u,v) + f(u), \\ v_t = d_2\Delta v - \chi\nabla\cdot(v\nabla u) + bvF(u,v) - vg(v), \end{cases} \tag{1.1}$$

was proposed by Kareiva and Odell [9]. The functions $u(x,t)$ and $v(x,t)$ stand for the density of prey and predators, respectively. The parameters $d_i (i = 1,2) > 0$ mean the random diffusion coefficients. The term $-\chi\nabla\cdot(v\nabla u)$ signifies the directional movements of the predator toward the prey with prey-taxis sensitivity coefficient $\chi > 0$. $f(u)$ measures the growth rate of prey, while $g(v)$ is the mortality rate of the predator. $F(u,v)$ denotes the interspecific interaction, also called the functional response function, and $b > 0$ indicates the conversion efficiency. In recent years, for model (1.1) with various functional response functions (for example, Holling type [10], ratio-dependent [11] and Beddington-DeAngelis type [12, 13]), many scholars have achieved numerous results, involving topics such as the global existence, boundedness, stability, traveling waves, global bifurcation, and so on. Specifically, Jin and Wang [14] provided that the solution of (1.1) with some certain assumptions on the functions $F(u,v)$, $f(u)$, and $g(v)$ is globally bounded in two dimensions. In addition, they also investigated that the prey-only steady state of (1.1) with weak predation is globally asymptotically stable, while the coexistence steady state of (1.1) with strong predation and weak prey-taxis is globally asymptotically stable.

In nature, cooperative behavior of populations is a widespread and important phenomenon in ecosystems. To ensure their survival, reproduction, and development, predators often engage in cooperative hunting, such as wolves, African wild dogs, Harris' hawks, etc. In order to investigate the impact of hunting cooperation on the density of the predator and the dynamics of the ecological community, Alves and Hilker [4] discussed an ODE model, which reads

$$\begin{cases} \frac{du}{dt} = \sigma u\left(1 - \frac{u}{\kappa}\right) - F(u,v)v, \\ \frac{dv}{dt} = bF(u,v)v - \beta v, \end{cases} \tag{1.2}$$

with various functional response functions $F(u,v)$, such as Lotka-Volterra type, Holling type II, etc. Subsequently, many scholars have conducted extensive research on the properties for the solutions of (1.2) with various functional response functions, including local existence, global boundedness, stability, global bifurcation, pattern formation. Moreover, scholars also have studied a wide range of mechanisms on (1.2), such as the Allee effect [15, 16], spatial diffusion [17], time delay [18], and so on. Specifically, inspired by (1.1) and (1.2), Zhang et al. [19] studied a prey-taxis model with Holling type II hunting cooperative functional response function $F(u,v) = \frac{(1+av)v}{1+h(1+av)u}$, where $a > 0$ represents the intensity of cooperative hunting among predators and $h > 0$ is the average handing time of the predator for the prey, which is formulated as follows:

$$\begin{cases} u_t = d_1\Delta u + \sigma u\left(1 - \frac{u}{\kappa}\right) - \frac{(1+av)uv}{1+h(1+av)u}, \\ v_t = d_2\Delta v - \nabla\cdot(\chi v\nabla u) + \frac{(1+av)uv}{1+h(1+av)u} - v. \end{cases} \tag{1.3}$$

They demonstrated the uniform boundedness and global existence of time-varying solutions for (1.3). Concurrently, they also analyzed the stability and prey-taxis-driven instability of positive equilibrium through linearization analysis. When $\chi = 0$, Zhang [20] incorporated predator-taxis into (1.3) and established the global existence of the classical solution for (1.3) in any spatial dimension. Additionally, he analyzed the instability induced by predator-taxis.

Hunting cooperation not only affects the population size of predators, but also has a significant impact on their spatiotemporal distribution. Therefore, considering (1.1) with the Lotka-Volterra-type hunting cooperative functional response function $F(u, v) = (1 + av)u$ (e.g., see [4]), we also introduce the random movement of species, the directional movement of predators toward prey, and the intra-specific competition within predator populations (e.g., see [14, 21]) into (1.2). This derived

$$\begin{cases} u_t = d_1\Delta u + \sigma u\left(1 - \frac{u}{\kappa}\right) - (1 + av)\,uv, & x \in \Omega, t > 0, \\ v_t = d_2\Delta v - \chi\nabla\cdot(v\nabla u) + (1 + av)\,uv - \beta v - \gamma v^2, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u\,(x, 0) = u_0\,(x)\,, v\,(x, 0) = v_0\,(x)\,, & x \in \Omega, \end{cases} \tag{1.4}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^2$. The constant $\sigma > 0$ is the intrinsic growth rate on prey, $\kappa > 0$ stands for the carrying capacity of prey, $\beta > 0$ means the mortality rate of predators, and $\gamma > 0$ denotes the mortality rate caused by intra-specific competition. The initial date satisfies

$$u_0 \in W^{1,\infty}(\Omega) \text{ and } v_0 \in C^0\left(\overline{\Omega}\right) \text{ with } u_0, v_0 \geq 0. \tag{1.5}$$

In what follows, without confusion, we shall abbreviate $\int_\Omega f dx$ as $\int_\Omega f$ for simplicity.

Main ideas and results: It is straightforward to achieve that $\|u\|_{L^\infty(\Omega)}$ is bounded by utilizing the comparison principle on parabolic equations. We remark that from a mathematical perspective, the analysis of the global boundedness of solutions for the Lotka-Volterra-type functional response function is more difficult than that for the Holling type II functional response function derived from (1.3). After all, the holling type II functional response function allows for the estimates $\left|\frac{(1+av)u}{1+h(1+av)u}\right| \leq \frac{1}{h}$ due to a priori estimates for u and v. However, the Lotka-Volterra-type functional response function does not possess such a directly useful property for establishing the global boundedness of solutions. Therefore, to derive the $L^\infty(\Omega)$ estimate for ∇u in two-dimensional space, we need an a priori $\|v\|_{L^6(\Omega)}$ estimate. First, we construct an energy function $\int_\Omega v^2 + \int_\Omega |\nabla u|^2$, which can be utilized to prove the boundness of $\|\nabla u\|_{L^2(\Omega)}$ and $\|v\|_{L^2(\Omega)}$ when a is suitably small and also establish the boundedness of $\int_t^{t+\tau}\int_\Omega |\Delta u|^2$ and $\int_t^{t+\tau}\int_\Omega v^3$ for some appropriately small $\tau \in (0, 1]$. Based on the estimate of $\|v\|_{L^3(\Omega)}$, we construct the energy function $\int_\Omega |\nabla u|^4 + \int_\Omega v^4$ and demonstrate that v and ∇u are bounded in $L^4(\Omega)$, which can help to get the a priori estimate of $\|v\|_{L^6(\Omega)}$. So far, the estimates of $\|\nabla u\|_{L^\infty(\Omega)}$ and $\|v\|_{L^\infty(\Omega)}$ are proved by Neumann heat semigroups. Zhang et al. primarily discussed the stability of the coexistence equilibrium for (1.3) in [19]. In this paper, by constructing suitable Lyapunov functionals, we demonstrate the long-time behavior of the prey-only and coexistence steady state of (1.4) when a falls within a specific range. These results mean that the weak hunting cooperation of predators can avoid population overcrowding, enrich the diversity of biological populations, and enhance ecological balance.

The first main result is as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then (u_0, v_0) satisfies (1.5), if*

$$a < \frac{\gamma}{K_1}\ (\text{ weak hunting cooperation })\,, \tag{1.6}$$

where $K_1 := \max\{\kappa, \|u_0\|_{L^\infty(\Omega)}\}$, and then (1.4) possesses a positive global classical solution

$$(u,v)\in\left[C^{0}\left(\bar{\Omega}\times[0,\infty)\right)\cap C^{2,1}\left(\Omega\times(0,\infty)\right)\right]^{2}.$$

Furthermore, the solution satisfies

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|v(\cdot,t)\|_{L^{\infty}(\Omega)}\le K,$$

where constant $K>0$ *does not depend on t. In particular, we also have* $0<u\le K_1$.

Our next result aims to present the impact of the cooperative hunting on the predator-prey model for its dynamic behavior. By a simple calculation, the constant equilibrium point (u_s,v_s) of (1.4) satisfies

$$\begin{cases}u_s\left(\sigma-\frac{\sigma u_s}{\kappa}-v_s-av_s^2\right)=0,\\ v_s\left(u_s+au_sv_s-\beta-\gamma v_s\right)=0,\end{cases}\tag{1.7}$$

which admits three possible homogeneous equilibria:

- Extinction steady state $(0,0)$.
- Prey-only steady state $(\kappa,0)$.
- Coexistence steady state (u^*,v^*).

Set

$$h(x):=a^2\kappa x^3+2a\kappa x^2+(\sigma\gamma+\kappa-a\sigma\kappa)\,x+\sigma(\beta-\kappa),$$

with $x\in(0,+\infty)$, and its derivative functions are

$$h'(x)=3a^2\kappa x^2+4a\kappa x+\sigma\gamma+\kappa-a\sigma\kappa,\quad h''(x)=6a^2\kappa x+4a\kappa.$$

It is easy to verify that $h''(x)>0$, which implies that $h'(x)$ is strictly monotonically increasing on $(0,\infty)$.

On the other hand, from the equations presented in (1.7), we can calculate that the equilibrium point u^* satisfies

$$u^*=\frac{\beta+\gamma v^*}{1+av^*},$$

where v^* is the positive root of the equation $h(x)=0$. In order to compute the value of v^*, we undertake an analysis encompassing the following three cases:

Case 1: $\kappa>\beta$. It follows from $h(0)<0$ and the Descartes' rule of signs [22] that $h(x)=0$ has a unique root $v_*\in(0,\sigma)$ and then $u_*=\frac{\beta+\gamma v_*}{1+av_*}\in(\beta,\beta+\gamma\sigma)$.

Case 2: $\kappa=\beta$. It holds that $h(0)=0$. When $a\le\frac{\sigma\gamma+\kappa}{\sigma\kappa}$, $h'(x)>0$. Thus $h(x)=0$ does not have a positive root. If $a>\frac{\sigma\gamma+\kappa}{\sigma\kappa}$, this deduces that $h'(x)<0$ on $x\in(0,v_1)$ and $h'(x)>0$ on $x\in(v_1,+\infty)$, where

$$v_1=\frac{\sqrt{\kappa(\kappa+3a\sigma\kappa-3\sigma\gamma)}-2\kappa}{3a\kappa}.$$

Therefore, $h(x)=0$ has a unique root $v_2=\frac{\sqrt{\sigma\kappa(a\kappa-\gamma)}-\kappa}{a\kappa}>0$.

Case 3: $\kappa<\beta$. Note that $h(0)>0$. If $a\le\frac{\sigma\gamma+\kappa}{\sigma\kappa}$, $h(x)=0$ has no positive root. While $a>\frac{\sigma\gamma+\kappa}{\sigma\kappa}$, $h(x)=0$ admits three statuses: no positive root as $h(v_1)>0$, one positive root v_1 as $h(v_1)=0$, and two positive roots v_3,v_4 as $h(v_1)<0$, where $v_3\in(0,v_1)$ and $v_4\in(v_1,\sigma)$.

All in all, the coexistence steady state of (1.4) satisfies

$$(u^*, v^*) = \begin{cases} (u_*, v_*), & \text{if } \kappa > \beta, \\ \left(\frac{\beta+\gamma v_2}{1+a v_2}, v_2\right), & \text{if } \kappa = \beta \text{ and } a > \frac{\sigma\gamma+\kappa}{\sigma\kappa}, \\ \left(\frac{\beta+\gamma v_1}{1+a v_1}, v_1\right), & \text{if } \kappa < \beta, a > \frac{\sigma\gamma+\kappa}{\sigma\kappa}, \text{ and } h(v_1) = 0, \\ \left(\frac{\beta+\gamma v_3}{1+a v_3}, v_3\right) \text{ and } \left(\frac{\beta+\gamma v_4}{1+a v_4}, v_4\right), & \text{if } \kappa < \beta, a > \frac{\sigma\gamma+\kappa}{\sigma\kappa}, \text{ and } h(v_1) < 0. \end{cases}$$

Theorem 1.2. *Suppose that the assumptions of Theorem 1.1 hold. Then:*
1) Let $\kappa \le \beta$. If the model parameters satisfy

$$a < \frac{\gamma}{K_1} \quad (\textit{weak hunting cooperation}),$$

then for all $t > 0$, the classical solution (u, v) in (1.4) converges to $(\kappa, 0)$ in an exponential manner, as described below:

$$\|u-\kappa\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le Ce^{-\lambda t},$$

where constants $C > 0$ and $\lambda > 0$ are independent of t.
2) Let $\kappa > \beta$. If the model parameters satisfy

$$a < \min\left\{\frac{\gamma}{K_1}, \frac{2\sqrt{(\beta+\sigma\gamma)^2+\sigma\kappa\gamma} - 2(\beta+\sigma\gamma)}{\sigma\kappa}\right\} (\textit{weaker hunting cooperation})$$

and

$$\chi^2 < \frac{4d_1 d_2 u_*}{K_1^2 v_*},$$

where u_* and v_* are independent of χ, then for all $t > 0$, the classical solution (u, v) converges to (u_*, v_*) in an exponential manner, as described below:

$$\|u-u_*\|_{L^\infty(\Omega)} + \|v-v_*\|_{L^\infty(\Omega)} \le Ce^{-\lambda t},$$

where constants $C > 0$ and $\lambda > 0$ are independent of t.

Remark 1.1. *Without the hunting cooperation (i.e., $a = 0$), our results of Theorem 1.2 are consistent with the results of Proposition 1.6 in [14]. In fact, Theorem 1.2 shows that the hunting cooperation mechanism does not change the stability of (1.4) solutions when a is appropriately small.*

Remark 1.2. *When $\kappa \le \beta$, we are unable to verify the long-time behavior of the coexistence steady state (u_*, v_*) on account of the range of the intensity of cooperative hunting among predators conflicts with the condition of Theorem 1.1. However, for one-dimensional cases, the long-time behavior of the coexistence steady state is an open problem when $\kappa \le \beta$ and a is appropriately large.*

2. Preliminaries

First, we provide the local existence of solutions for (1.4).

Lemma 2.1. *Provided that $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and (u_0, v_0) satisfies (1.5), if the condition (1.6) holds, there exists $T_{max} \in (0, \infty]$ ensuring that (1.4) possesses a classical solution*

$$(u, v) \in \left[C^0\left(\bar{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\Omega \times (0, T_{max})\right)\right]^2$$

satisfying $u, v > 0$. Furthermore, if $T_{max} < \infty$, then

$$\limsup_{t \nearrow T_{max}} \left\{\|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\right\} = \infty.$$

The conclusions of Lemma 2.1 are established from Amann's theorem [23, 24].

Lemma 2.2. *Provided that the assumptions of Lemma 2.1 hold, then for all $t \in (0, T_{max})$, we have*

$$\|u\|_{L^\infty(\Omega)} \leq K_1, \tag{2.1}$$

where $K_1 := \max\{\kappa, \|u_0\|_{L^\infty(\Omega)}\}$.

Proof. This lemma has been proven by using an approach similar to [25]. □

The following lemma gives a fundamental inequality.

Lemma 2.3. *[26] Assume that Ω is a smooth bounded domain, and let $g \in C^2\left(\bar{\Omega}\right)$ satisfy $\frac{\partial g}{\partial \nu} = 0$ on $\partial\Omega$. Then there exists an upper bound $l = l(\Omega)$ of the curvatures of $\partial\Omega$ guaranteeing that*

$$\frac{\partial |\nabla g|^2}{\partial \nu} \leq l |\nabla g|^2.$$

In order to prove Lemma 3.3, we need a lemma.

Lemma 2.4. *[27] Suppose that $T > 0, \tau \in (0, T), m_1 > 0$, and $m_2 > 0$. Provided that $\varphi : [0, T) \to [0, \infty)$ is absolutely continuous, and satisfies*

$$\varphi'(t) + \varphi^{1+\theta}(t) \leq \phi(t)\varphi(t) + \psi(t), \quad t \in \mathbb{R},$$

where the constant $\theta > 0$, the functions $\phi(t), \psi(t) \in L^1_{loc}([0, T))$ are nonnegative and

$$\int_{t-\tau}^{t} \phi(s)\, ds \leq m_1, \int_{t-\tau}^{t} \psi(s)\, ds \leq m_2, \quad t \in [\tau, T).$$

Then we can obtain

$$\varphi(t) \leq \varphi(t_0)\, e^{\int_{t_0}^{t} \phi(s) ds} + \int_{t_0}^{t} \psi(\tau)\, e^{\int_{\tau}^{t} \phi(s) ds} d\tau$$

and

$$\sup_t \varphi(t) \leq \theta\left(\frac{2A}{1+\theta}\right)^{\frac{1+\theta}{\theta}} + 2B \quad t > t_0,$$

where

$$A = \tau^{-\frac{1}{1+\theta}} (1 + m_1)^{\frac{1}{1+\theta}} e^{2m_1}, B = \tau^{-\frac{1}{1+\theta}} m_2^{\frac{1}{1+\theta}} e^{2m_1} + 2m_2 e^{2m_1} + \varphi(0)\, e^{m_1}.$$

3. Global boundedness

This section is devoted to establishing Theorem 1.1.

Lemma 3.1. *Provided that the assumptions of Lemma 2.1 are fulfilled, if the condition (1.6) holds, then for all $t \in (0, T_{max})$, the classical solution (u, v) of (1.4) satisfies*

$$\|u\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)} \le K_2, \tag{3.1}$$

where constant $K_2 > 0$ does not depend on t.

Proof. (1.4) implies

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u + \int_\Omega v\right) + \beta\left(\int_\Omega u + \int_\Omega v\right) &= (\sigma + \beta)\int_\Omega u - \frac{\sigma}{\kappa}\int_\Omega u^2 - \gamma\int_\Omega v^2 \\ &\le (\sigma + \beta)\int_\Omega u - \frac{\sigma}{\kappa}\int_\Omega u^2.\end{aligned} \tag{3.2}$$

Then, using Young's inequality yields

$$(\sigma + \beta)\int_\Omega u \le \frac{\sigma}{2\kappa}\int_\Omega u^2 + \frac{\kappa(\sigma + \beta)^2|\Omega|}{2\sigma}. \tag{3.3}$$

Combining (3.3) and (3.2), we deduce

$$\frac{d}{dt}\left(\int_\Omega u + \int_\Omega v\right) + \beta\left(\int_\Omega u + \int_\Omega v\right) \le \frac{\kappa(\sigma + \beta)^2|\Omega|}{2\sigma},$$

which implies (3.1) by ODE comparison. □

Then we prove that $\|\nabla u\|_{L^2(\Omega)}$ and $\|v\|_{L^2(\Omega)}$ are bounded.

Lemma 3.2. *Provided that the assumptions of Lemma 2.1 are fulfilled, if the condition (1.6) holds, then for all $t \in (0, T_{max})$, the classical solution (u, v) of (1.4) satisfies*

$$\|\nabla u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \le K_3, \tag{3.4}$$

and for all $t \in (0, T_{max} - \tau)$, it holds that

$$\int_t^{t+\tau}\int_\Omega |\Delta u|^2 + \int_t^{t+\tau}\int_\Omega v^3 \le K_4, \ \textit{with}\ 0 < \tau < \min\{1, \frac{1}{2}T_{max}\}, \tag{3.5}$$

where constants $K_3 > 0$ and $K_4 > 0$ do not depend on t.

Proof. Intergating the sum of $-\frac{\Delta u}{u}$ times the first equation of (1.4) by parts, we have

$$\begin{aligned}-\int_\Omega \frac{u_t}{u}\Delta u + d_1\int_\Omega \frac{|\Delta u|^2}{u} &= \sigma\int_\Omega\left(\frac{u}{\kappa} - 1\right)\Delta u + \int_\Omega (1 + av)\, v\Delta u \\ &= -\frac{\sigma}{\kappa}\int_\Omega |\nabla u|^2 + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u \cdot \nabla v.\end{aligned} \tag{3.6}$$

Note that

$$-\int_\Omega \frac{u_t}{u}\Delta u = \int_\Omega \nabla u\cdot\left(\frac{\nabla u}{u}\right)_t = \frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega \frac{\left(|\nabla u|^2\right)_t}{u} = \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}u_t,$$

which substituted into (3.6) gives us

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1\int_\Omega \frac{|\Delta u|^2}{u} + \frac{\sigma}{\kappa}\int_\Omega |\nabla u|^2 \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}u_t + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u\cdot\nabla v \\
&= \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{\sigma}{2}\int_\Omega \frac{|\nabla u|^2\left(1-\frac{u}{\kappa}\right)}{u} - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2 v\,(1+av)}{u} + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u\cdot\nabla v \\
&\le \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{\sigma}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u\cdot\nabla v.
\end{aligned} \tag{3.7}$$

Using $\nabla u\cdot\nabla\Delta u = \frac{1}{2}\Delta|\nabla u|^2 - |D^2u|^2$, we conclude

$$\begin{aligned}
&d_1\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u \\
&= d_1\int_\Omega \frac{|\Delta u|^2}{u} - d_1\int_\Omega \frac{|D^2u|^2}{u} + \frac{d_1}{2}\int_\Omega \frac{\Delta|\nabla u|^2}{u} \\
&= d_1\int_\Omega \frac{|\Delta u|^2}{u} - d_1\int_\Omega \frac{|D^2u|^2}{u} + \frac{d_1}{2}\int_\Omega \frac{\nabla|\nabla u|^2\cdot\nabla u}{u^2} + \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds \\
&= d_1\int_\Omega \frac{|\Delta u|^2}{u} - d_1\int_\Omega \frac{|D^2u|^2}{u} + d_1\int_\Omega \frac{|\nabla u|^4}{u^3} - \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds.
\end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}
\int_\Omega u|D^2\ln u|^2 &= \int_\Omega \frac{|D^2u|^2}{u} - 2\int_\Omega \frac{\left(D^2u\cdot\nabla u\right)\cdot\nabla u}{u^2} + \int_\Omega \frac{|\nabla u|^4}{u^3} \\
&= \int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} + \left(\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u - 2\int_\Omega \frac{|\nabla u|^4}{u^3}\right) \\
&= \int_\Omega \frac{|D^2u|^2}{u} - \int_\Omega \frac{|\nabla u|^4}{u^3} + \int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u.
\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) yields

$$\frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u = \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds + d_1\int_\Omega \frac{|\nabla u|^2}{u} - d_1\int_\Omega u|D^2\ln u|^2. \tag{3.10}$$

Bringing (3.10) into (3.7), it holds that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1\int_\Omega u|D^2\ln u|^2 + \frac{\sigma}{\kappa}\int_\Omega |\nabla u|^2 \\
&\le \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds + \frac{\sigma}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v\Delta u - 2a\int_\Omega v\nabla u\cdot\nabla v.
\end{aligned} \tag{3.11}$$

Multiplying both sides of the second equation of (1.4) by $\frac{2av}{\chi}$, we derive

$$\begin{aligned}
&\frac{a}{\chi}\frac{d}{dt}\int_\Omega v^2+\frac{2ad_2}{\chi}\int_\Omega|\nabla v|^2\\
&=\frac{2a}{\chi}\int_\Omega(1+av)uv^2-\frac{2a\beta}{\chi}\int_\Omega v^2-\frac{2a\gamma}{\chi}\int_\Omega v^3+2a\int_\Omega v\nabla u\cdot\nabla v\\
&\le\frac{2aK_1}{\chi}\int_\Omega v^2-\frac{2a(\gamma-aK_1)}{\chi}\int_\Omega v^3+2a\int_\Omega v\nabla u\cdot\nabla v.
\end{aligned}\tag{3.12}$$

Utilizing the condition $a<\frac{\gamma}{K_1}$, (3.11), and (3.12), one has

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{a}{\chi}\int_\Omega v^2\right)+\frac{2ad_2}{\chi}\int_\Omega|\nabla v|^2+d_1\int_\Omega u|D^2\ln u|^2\\
&\quad+\frac{\sigma}{\kappa}\int_\Omega|\nabla u|^2+\frac{2a(\gamma-aK_1)}{\chi}\int_\Omega v^3\\
&\le\frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds+\frac{\sigma}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\int_\Omega v\Delta u+\frac{2aK_1}{\chi}\int_\Omega v^2.
\end{aligned}\tag{3.13}$$

In the light of the inequality $\int_\Omega u|D^2\ln u|^2\ge C_1\left(\int_\Omega\frac{|D^2u|^2}{u}+\int_\Omega\frac{|\nabla u|^4}{u^3}\right)$ with the constant $C_1>0$, plugged into (3.13) gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{a}{\chi}\int_\Omega v^2\right)+\frac{2ad_2}{\chi}\int_\Omega|\nabla v|^2+\frac{\sigma}{\kappa}\int_\Omega|\nabla u|^2\\
&\quad+d_1C_1\left(\int_\Omega\frac{|D^2u|^2}{u}+\int_\Omega\frac{|\nabla u|^4}{u^3}\right)+\frac{2a(\gamma-aK_1)}{\chi}\int_\Omega v^3\\
&\le\frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds+\frac{\sigma}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\int_\Omega v\Delta u+\frac{2aK_1}{\chi}\int_\Omega v^2.
\end{aligned}\tag{3.14}$$

Combining the trace inequality [28], the Cauchy-Schwarz inequality, and Lemma 2.3, we have

$$\begin{aligned}
\frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}ds&\le ld_1\int_{\partial\Omega}\frac{|\nabla u|^2}{u}ds\\
&\le\frac{d_1C_1}{2}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u}\right)+C_2\int_\Omega\frac{|\nabla u|^2}{u},
\end{aligned}\tag{3.15}$$

where constant $C_2>0$. According to (2.1) and Hölder's inequality, then

$$\begin{aligned}
\left(\frac{1}{2}+\frac{\sigma+2C_2}{2}\right)\int_\Omega\frac{|\nabla u|^2}{u}&\le\left(\frac{\sigma+1+2C_2}{2}\right)\left(\int_\Omega\frac{|\nabla u|^4}{u^3}\right)^{\frac{1}{2}}\left(\int_\Omega u\right)^{\frac{1}{2}}\\
&\le\frac{d_1C_1}{4}\int_\Omega\frac{|\nabla u|^4}{u^3}+C_3,
\end{aligned}\tag{3.16}$$

where constant $C_3>0$. Ustilizing $|\Delta u|\le\sqrt{2}|D^2u|$, (2.1), and Young's inequality, we achieve

$$\int_\Omega v\Delta u\le\frac{d_1C_1}{8}\int_\Omega\frac{|\Delta u|^2}{u}+\frac{2}{d_1C_1}\int_\Omega uv^2\le\frac{d_1C_1}{4}\int_\Omega\frac{|D^2u|^2}{u}+\frac{2K_1}{d_1C_1}\int_\Omega v^2.\tag{3.17}$$

We infer from the Gagliardo-Nirenberg inequality and (3.1) that

$$\begin{aligned}\left(\frac{a}{\chi}+\frac{2K_1}{d_1C_1}+\frac{2aK_1}{\chi}\right)\int_\Omega v^2 &= C_4\|v\|^2_{L^2(\Omega)}\\ &\le C_4C_5\left(\|\nabla v\|_{L^2(\Omega)}\|v\|_{L^1(\Omega)}+\|v\|^2_{L^1(\Omega)}\right)\\ &\le C_4C_5K_2\|\nabla v\|_{L^2(\Omega)}+C_4C_5K_2^2\\ &\le \frac{ad_2}{\chi}\int_\Omega|\nabla v|^2+C_6,\end{aligned}\tag{3.18}$$

where constants C_i $(i = 4, 5, 6) > 0$. Substituting (3.14)–(3.18) and $|\Delta u| \le \sqrt{2}|D^2u|$ yields

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{a}{\chi}\int_\Omega v^2\right)+\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{a}{\chi}\int_\Omega v^2\right)+\frac{ad_2}{\chi}\int_\Omega|\nabla v|^2\\ &\quad+\frac{d_1C_1}{8}\left(\int_\Omega\frac{|\Delta u|^2}{u}+\int_\Omega\frac{|\nabla u|^4}{u^3}\right)+\frac{\sigma}{\kappa}\int_\Omega|\nabla u|^2+\frac{2a(\beta-aK_1)}{\chi}\int_\Omega v^3\le C_3+C_6,\end{aligned}\tag{3.19}$$

which gives (3.4) by the fact $0 < u \le K_1$ and ODE comparison. Then integrating (3.19) over $(t, t + \tau)$ gives (3.5). □

Next, we get the a priori estimate of $\|v\|_{L^3(\Omega)}$.

Lemma 3.3. *Provided that the assumptions of Lemma 2.1 are fulfilled, if the condition (1.6) holds, then, the classical solution* (u, v) *of (1.4) satisfies*

$$\|v\|_{L^3(\Omega)} \le K_5,\tag{3.20}$$

where constant $K_5 > 0$ *does not depend on* t.

Proof. By direct computations, we have

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega v^3+\beta\int_\Omega v^3 &\le -d_2\int_\Omega v|\nabla v|^2+\frac{\chi^2}{d_2}\int_\Omega v^3|\nabla u|^2+K_1\int_\Omega v^3-(\gamma-aK_1)\int_\Omega v^4\\ &\le -d_2\int_\Omega v|\nabla v|^2+\frac{\chi^2}{d_2}\left(\int_\Omega v^6\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla u|^4\right)^{\frac{1}{2}}+K_1\int_\Omega v^3.\end{aligned}\tag{3.21}$$

Applying the Gagliardo-Nirenberg inequality as $n = 2$ yields

$$\|v^{\frac{3}{2}}\|^2_{L^4(\Omega)} \le C_1\left(\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)}+\|v^{\frac{3}{2}}\|^2_{L^2(\Omega)}\right)\tag{3.22}$$

and

$$\|\nabla u\|^2_{L^4(\Omega)} \le C_2\left(\|\Delta u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(\Omega)}\right)\le C_2\left(K_3\|\Delta u\|_{L^2(\Omega)}+K_3^2\right)\tag{3.23}$$

where constants $C_1 > 0$ and $C_2 > 0$. Thanks to (3.22), (3.23), and Young's inequality, one gets

$$\begin{aligned}
&\frac{\chi^2}{d_2}\left(\int_\Omega v^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} \\
&\leq \frac{\chi^2 C_1 C_2}{d_2}\left(\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)} + \|v^{\frac{3}{2}}\|^2_{L^2(\Omega)}\right)\left(K_3\|\Delta u\|_{L^2(\Omega)} + K_3^2\right) \\
&\leq \frac{\chi^2 C_1 C_2 K_3}{d_2}\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} + \frac{\chi^2 C_1 C_2 K_3^2}{d_2}\|v^{\frac{3}{2}}\|^2_{L^2(\Omega)} \\
&\quad + \frac{\chi^2 C_1 C_2 K_3^2}{d_2}\|\nabla v^{\frac{3}{2}}\|_{L^2(\Omega)}\|v^{\frac{3}{2}}\|_{L^2(\Omega)} + \frac{\chi^2 C_1 C_2 K_3}{d_2}\|v^{\frac{3}{2}}\|^2_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} \\
&\leq d_2\int_\Omega v|\nabla v|^2 + C_3\|\Delta u\|^2_{L^2(\Omega)}\|v\|^3_{L^3(\Omega)} + C_4\|v\|^3_{L^3(\Omega)},
\end{aligned} \tag{3.24}$$

where constants $C_3 > 0$ and $C_4 > 0$. Substituting (3.24) into (3.21) results in

$$\frac{d}{dt}\|v\|^3_{L^3(\Omega)} \leq C_3\|\Delta u\|^2_{L^2(\Omega)}\|v\|^3_{L^3(\Omega)} + (C_4 + K_1)\|v\|^3_{L^3(\Omega)},$$

which gives (3.20) by (3.5) and Lemma 2.4. □

Subsequently, we achieve the bounds $\|v\|_{L^4(\Omega)}$ and $\|\nabla u\|_{L^4(\Omega)}$.

Lemma 3.4. *Provided that the assumptions of Lemma 2.1 are fulfilled, if the condition (1.6) holds, then the classical solution* (u, v) *of (1.4) satisfies*

$$\|v\|_{L^4(\Omega)} + \|\nabla u\|_{L^4(\Omega)} \leq K_6, \tag{3.25}$$

where constant $K_6 > 0$ *does not depend on t.*

Proof. Multiplying both side of the second equation of (1.4) by v^3, we deduce

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega v^4 + \int_\Omega v^4 &\leq -3d_2\int_\Omega v^2|\nabla v|^2 + 3\chi\int_\Omega v^3\nabla u\cdot\nabla v + (K_1+1)\int_\Omega v^4 - (\gamma - aK_1)\int_\Omega v^5 \\
&\leq -2d_2\int_\Omega v^2|\nabla v|^2 + \frac{9\chi^2}{4d_2}\int_\Omega v^4|\nabla u|^2 + (K_1+1)\int_\Omega v^4.
\end{aligned} \tag{3.26}$$

Integrating the first equation of (1.4), while taking into account the fact that $0 < u \le K_1$, it follows that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 + \frac{d_1}{2}\int_\Omega |\nabla|\nabla u|^2|^2 + d_1\int_\Omega |\nabla u|^2|D^2u|^2 \\
&= \frac{d_1}{2}\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds + \sigma\int_\Omega |\nabla u|^4\left(1-\frac{2}{\kappa}u\right) \\
&\quad - \int_\Omega |\nabla u|^2\nabla u\cdot\nabla(uv) - a\int_\Omega |\nabla u|^2\nabla u\cdot\nabla\left(uv^2\right) \\
&= \frac{d_1}{2}\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds + \sigma\int_\Omega |\nabla u|^4\left(1-\frac{2}{\kappa}u\right) + \int_\Omega uv|\nabla u|^2\Delta u \\
&\quad + a\int_\Omega uv\nabla\left(|\nabla u|^2\right)\nabla u + \int_\Omega uv^2|\nabla u|^2\Delta u + a\int_\Omega uv^2\nabla\left(|\nabla u|^2\right)\nabla u \\
&\le \frac{d_1}{2}\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds + K_1\int_\Omega v\left(|\nabla u|^2|\Delta u| + |\nabla|\nabla u|^2||\nabla u|\right) \\
&\quad + aK_1\int_\Omega v^2\left(|\nabla u|^2|\Delta u| + |\nabla|\nabla u|^2||\nabla u|\right) + \sigma\int_\Omega |\nabla u|^4.
\end{aligned} \tag{3.27}$$

Thanks to the trace inequality and Lemma 2.3, we know

$$\frac{d_1}{2}\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}ds \le \frac{ld_1}{2}\||\nabla u|^2\|^2_{L^2(\partial\Omega)} \le \frac{d_1}{8}\int_\Omega |\nabla|\nabla u|^2|^2 + C_1\int_\Omega |\nabla u|^4. \tag{3.28}$$

Utilizing $|\Delta u| \le \sqrt{2}|D^2u|$ and $\nabla|\nabla u|^2 = 2D^2u\cdot\nabla u$ gives rise to

$$\begin{aligned}
&K_1\int_\Omega v\left(|\nabla u|^2|\Delta u| + |\nabla|\nabla u|^2||\nabla u|\right) \\
&\le \sqrt{2}K_1\int_\Omega v|\nabla u|^2|D^2u| + 2K_1\int_\Omega v|\nabla u|^2|D^2u| \\
&= \left(\sqrt{2}+2\right)K_1\int_\Omega v|\nabla u|^2|D^2u| \\
&\le \frac{d_1}{4}\int_\Omega |\nabla u|^2|D^2u|^2 + K_1\int_\Omega v^4 + \frac{\left(\sqrt{2}+2\right)^4K_1^3}{4d_1^2}\int_\Omega |\nabla u|^4
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&aK_1\int_\Omega v^2\left(|\nabla u|^2|\Delta u| + |\nabla|\nabla u|^2||\nabla u|\right) \\
&\le \sqrt{2}aK_1\int_\Omega v^2|\nabla u|^2|D^2u| + 2aK_1\int_\Omega v|\nabla u|^2|D^2u| \\
&= \left(\sqrt{2}+2\right)aK_1\int_\Omega v^2|\nabla u|^2|D^2u| \\
&\le \frac{d_1}{4}\int_\Omega |\nabla u|^2|D^2u|^2 + \frac{\left(\sqrt{2}+2\right)^2a^2K_1^2}{d_1}\int_\Omega v^4|\nabla u|^2.
\end{aligned} \tag{3.30}$$

The Gagliardo-Nirenberg inequality implies

$$\begin{aligned}
&\left(\sigma+1+C_1+\frac{\left(\sqrt{2}+2\right)^4K_1^3}{4d_1^2}\right)\int_\Omega|\nabla u|^4\\
&=\left(\sigma+1+C_1+\frac{\left(\sqrt{2}+2\right)^4K_1^3}{4d_1^2}\right)\||\nabla u|^2\|_{L^2(\Omega)}^2\\
&\le C_2\|\nabla|\nabla u|^2\|_{L^2(\Omega)}\||\nabla u|^2\|_{L^1(\Omega)}+C_2\||\nabla u|^2\|_{L^1(\Omega)}^2\\
&\le C_2K_3^2\|\nabla|\nabla u|^2\|_{L^2(\Omega)}+C_2K_3^4\\
&\le\frac{d_1}{8}\int_\Omega|\nabla|\nabla u|^2|^2+C_3,
\end{aligned}\tag{3.31}$$

where constants $C_2>0$ and $C_3>0$. Adding (3.27)–(3.31) and (3.26), we conclude

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\int_\Omega v^4+\int_\Omega|\nabla u|^4\right)+\int_\Omega v^4+\int_\Omega|\nabla u|^4+2d_2\int_\Omega v^2|\nabla v|^2\\
&\quad+\frac{d_1}{4}\int_\Omega|\nabla|\nabla u|^2|^2+\frac{d_1}{2}\int_\Omega|\nabla u|^2|D^2u|^2\\
&\le\left(\frac{9\chi^2}{4d_2}+\frac{\left(\sqrt{2}+2\right)^2a^2K_1^2}{d_1}\right)\int_\Omega v^4|\nabla u|^2+(2K_1+1)\int_\Omega v^4+C_3\\
&\le C_4\left(\int_\Omega v^6\right)^{\frac{2}{3}}\left(\int_\Omega|\nabla u|^6\right)^{\frac{1}{3}}+(2K_1+1)\int_\Omega v^4+C_3,
\end{aligned}\tag{3.32}$$

where constant $C_4>0$. Applying (3.4) and the Gagliardo-Nirenberg inequality yields

$$\begin{aligned}
(2K_1+1)\int_\Omega v^4&=(2K_1+1)\,\|v^2\|_{L^2(\Omega)}\\
&\le(2K_1+1)\,C_5\left(\|\nabla v^2\|_{L^2(\Omega)}\|v^2\|_{L^1(\Omega)}+\|v^2\|_{L^1(\Omega)}^2\right)\\
&\le(2K_1+1)\,K_3^2C_5\|\nabla v^2\|_{L^2(\Omega)}+(2K_1+1)\,K_3^4C_5\\
&\le d_2\int_\Omega v^2|\nabla v|^2+C_6,
\end{aligned}\tag{3.33}$$

where constants $C_5>0$ and $C_6>0$. In addition,

$$\begin{aligned}
\|\nabla u\|_{L^6(\Omega)}^2=\||\nabla u|^2\|_{L^3(\Omega)}&\le C_7\|\nabla|\nabla u|^2\|_{L^2(\Omega)}^{\frac{2}{3}}\||\nabla u|^2\|_{L^1(\Omega)}^{\frac{1}{3}}+C_7\||\nabla u|^2\|_{L^1(\Omega)}\\
&\le C_7K_3^{\frac{2}{3}}\|\nabla|\nabla u|^2\|_{L^2(\Omega)}^{\frac{2}{3}}+C_7K_3^2,
\end{aligned}\tag{3.34}$$

where constant $C_7>0$. Applying Young's inequality along with (3.34), one has

$$\begin{aligned}
C_4\|v^2\|_{L^3(\Omega)}^2\|\nabla u\|_{L^6(\Omega)}^2&\le C_4C_7K_3^{\frac{2}{3}}\|v^2\|_{L^3(\Omega)}^2\|\nabla|\nabla u|^2\|_{L^2(\Omega)}^{\frac{2}{3}}+C_4C_7K_3^2\|v^2\|_{L^3(\Omega)}^2\\
&\le\frac{d_1}{4}\|\nabla|\nabla u|^2\|_{L^2(\Omega)}^2+C_8\|v^2\|_{L^3(\Omega)}^3+C_9,
\end{aligned}\tag{3.35}$$

where constants $C_8 > 0$ and $C_9 > 0$. Utilizing the Gagliardo-Nirenberg inequality and Lemma 3.3 yields

$$\begin{aligned} C_8\|v^2\|^3_{L^3(\Omega)} &\leq C_8C_{10}\left(\|\nabla v^2\|^{\frac{3}{2}}_{L^2(\Omega)}\|v^2\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\Omega)} + \|v^2\|^2_{L^{\frac{3}{2}}(\Omega)}\right) \\ &\leq C_8C_{10}K_5^3\|\nabla v^2\|^{\frac{3}{2}}_{L^2(\Omega)} + C_8C_{10}K_5^4 \\ &\leq d_2\int_\Omega v^2|\nabla v|^2 + C_{11}, \end{aligned} \tag{3.36}$$

where constants $C_{10} > 0$ and $C_{11} > 0$. Plugging (3.33), (3.35), and (3.36) into (3.32), there exists a constant $C_{12} = C_3 + C_6 + C_9 + C_{11}$ such that

$$\frac{1}{4}\frac{d}{dt}\left(\int_\Omega v^4 + \int_\Omega |\nabla u|^4\right) + \int_\Omega v^4 + \int_\Omega |\nabla u|^4 \leq C_{12},$$

which in conjunction with ODE comparison results in (3.25). □

Lemma 3.5. *Provided that the assumptions of Lemma 2.1 are fulfilled, if the condition (1.6) holds, then for all $t \in (0, T_{max})$, we can get*

$$\|v\|_{L^6(\Omega)} \leq K_7, \tag{3.37}$$

where constant $K_7 > 0$ does not depend on t.

Proof. We can directly compute

$$\begin{aligned} &\frac{1}{6}\frac{d}{dt}\int_\Omega v^6 + \beta\int_\Omega v^6 + 5d_2\int_\Omega v^4|\nabla v|^2 \\ &= 5\chi\int_\Omega v^5\nabla u\cdot\nabla v + \int_\Omega uv^6 + a\int_\Omega uv^7 - \gamma\int_\Omega v^7 \\ &\leq d_2\int_\Omega v^4|\nabla v|^2 + \frac{25\chi^2}{4d_2}\int_\Omega v^6|\nabla u|^2 + K_1\int_\Omega v^6 - (\gamma - aK_1)\int_\Omega v^7 \\ &\leq d_2\int_\Omega v^4|\nabla v|^2 + \frac{25\chi^2}{4d_2}\left(\int_\Omega v^{12}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} + K_1\int_\Omega v^6 \\ &\leq d_2\int_\Omega v^4|\nabla v|^2 + \frac{25\chi^2K_6^2}{4d_2}\left(\int_\Omega v^{12}\right)^{\frac{1}{2}} + K_1\int_\Omega v^6. \end{aligned} \tag{3.38}$$

Utilizing the Gagliardo-Nirenberg inequality, it follows that

$$\begin{aligned} \frac{25\chi^2K_6^2}{4d_2}\left(\int_\Omega v^{12}\right)^{\frac{1}{2}} &= \frac{25\chi^2K_6^2}{4d_2}\|v^3\|^2_{L^4(\Omega)} \\ &\leq \frac{25\chi^2K_6^2C_1}{4d_2}\left(\|\nabla v^3\|^{\frac{3}{2}}_{L^2(\Omega)}\|v^3\|^{\frac{1}{2}}_{L^1(\Omega)} + \|v^3\|^2_{L^1(\Omega)}\right) \\ &\leq \frac{25\chi^2C_1K_6^2K_5^{\frac{3}{2}}}{4d_2}\|\nabla v^3\|^{\frac{3}{2}}_{L^2(\Omega)} + \frac{25\chi^2C_1K_6^2K_5^6}{4d_2} \\ &\leq d_2\int_\Omega v^4|\nabla v|^2 + C_2, \end{aligned} \tag{3.39}$$

where constants $C_1 > 0$ and $C_2 > 0$. Then

$$\begin{aligned} K_1 \int_\Omega v^6 &= K_1 \|v^3\|^2_{L^2(\Omega)} \\ &\le K_1 C_3 \left(\|\nabla v^3\|_{L^2(\Omega)} \|v^3\|_{L^1(\Omega)} + \|v^3\|^2_{L^1(\Omega)} \right) \\ &\le C_3 K_1 K_5^3 \|\nabla v^3\|_{L^2(\Omega)} + C_3 K_1 K_5^6 \\ &\le d_2 \int_\Omega v^4 |\nabla v|^2 + C_4, \end{aligned} \tag{3.40}$$

where constants $C_3 > 0$ and $C_4 > 0$. Plugging (3.39) and (3.40) into (3.38), we conclude

$$\frac{1}{6} \frac{d}{dt} \int_\Omega v^6 + \beta \int_\Omega v^6 \le C_2 + C_4,$$

which results in (3.37) with ODE comparison. □

We are currently capable of deducing the bounded property of $\|\nabla u\|_{L^\infty(\Omega)}$ and $\|v\|_{L^\infty(\Omega)}$ in the case where the dimension $n = 2$.

Lemma 3.6. *Provided that the assumptions of Lemma 2.1 are fulfilled, if the condition (1.6) holds, then for all $t \in (0, T_{max})$, we have*

$$\|\nabla u\|_{L^\infty(\Omega)} \le K_8 \tag{3.41}$$

and

$$\|v\|_{L^\infty(\Omega)} \le K_9, \tag{3.42}$$

where constants $K_8 > 0$ and $K_9 > 0$ do not depend on t.

Proof. The variation-of-constants formula implies

$$u(\cdot, t) = e^{d_1 t \Delta} u_0 + \int_0^t e^{d_1 (t-s) \Delta} \left[\sigma u \left(1 - \frac{u}{\kappa} \right) - (1 + av) uv \right] ds,$$

and hence

$$\nabla u(\cdot, t) = \nabla e^{d_1 t \Delta} u_0 + \int_0^t \nabla e^{d_1 (t-s) \Delta} \left[\sigma u \left(1 - \frac{u}{\kappa} \right) - (1 + av) uv \right] ds.$$

Then by (3.1), (3.20), and (3.37), it holds that

$$\begin{aligned} \|\sigma u \left(1 - \frac{u}{\kappa} \right) - (1 + av) u)\|_{L^3(\Omega)} &\le \|\sigma u - \frac{\sigma}{\kappa} u^2\|_{L^3(\Omega)} + \|uv\|_{L^3(\Omega)} + a\|uv^2\|_{L^3(\Omega)} \\ &\le \sigma K_1 \left(1 + \frac{K_1}{\kappa} \right) |\Omega|^{\frac{1}{3}} + K_1 K_5 + a K_1 K_7^2. \end{aligned} \tag{3.43}$$

Applying the Neumann heat semigroup [29] and (3.43), there exist constants $\gamma_1 > 0$ and $\lambda_1 > 0$

ensuring that

$$\begin{aligned}\|\nabla u\|_{L^\infty(\Omega)} &\leq \|\nabla e^{d_1 t\Delta}u_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{d_1(t-s)\Delta}\left[\sigma u\left(1-\frac{u}{\kappa}\right)-(1+av)\,uv\right]\|_{L^\infty(\Omega)}ds \\ &\leq 2\gamma_1 e^{-d_1\lambda_1 t}\|u_0\|_{L^\infty(\Omega)} \\ &\quad + \gamma_1\int_0^t\left(1+(t-s)^{-\frac{1}{2}-\frac{1}{3}}\right)e^{-d_1\lambda_1 t}\|\sigma u\left(1-\frac{u}{\kappa}\right)-(1+av)\,uv\|_{L^3(\Omega)}ds \\ &\leq 2\gamma_1\|u_0\|_{L^\infty(\Omega)} \\ &\quad + \gamma_1\left[\sigma K_1\left(1+\frac{K_1}{\kappa}\right)|\Omega|^{\frac{1}{3}} + K_1K_5 + aK_1K_7^2\right]\int_0^\infty\left(1+(t-s)^{-\frac{5}{6}}\right)e^{-d_1\lambda_1 t}ds \\ &\leq 2\gamma_1\|u_0\|_{L^\infty(\Omega)} \\ &\quad + \frac{\gamma_1}{d_1\lambda_1}\left[\sigma K_1\left(1+\frac{K_1}{\kappa}\right)|\Omega|^{\frac{1}{3}} + K_1K_5 + aK_1K_7^2\right]\left(1+(d_1\lambda_1)^{\frac{5}{6}}\,\Gamma\left(\frac{1}{6}\right)\right),\end{aligned}$$

which gives (3.41) and gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. We use (3.20) and (3.41) to obtain

$$\|v\nabla u\|_{L^3(\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)}\|v\|_{L^3(\Omega)} \leq K_5K_8 \tag{3.44}$$

and combining with (3.1), (3.20), and (3.37), we conclude

$$\|(1+av)\,uv\|_{L^3(\Omega)} \leq \|u\|_{L^\infty(\Omega)}\left(\|v\|_{L^3(\Omega)} + a\|v^2\|_{L^3(\Omega)}\right) \leq K_1(K_5 + aK_7^2). \tag{3.45}$$

Applying the Neumann heat semigroup [29], (3.44), and(3.45), there exist constants $\gamma_2 > 0$, $\gamma_3 > 0$, and $\lambda_1 > 0$ ensuring that

$$\begin{aligned}\|v\|_{L^\infty(\Omega)} &\leq \|e^{t(d_2\Delta-1)}v_0\|_{L^\infty(\Omega)} + \chi\int_0^t \|e^{(t-s)(d_2\Delta-1)}\nabla\cdot(v\nabla u)\|_{L^\infty(\Omega)}ds \\ &\quad + \int_0^t \|e^{(t-s)(d_2\Delta-1)}\,(1+av)\,uv\|_{L^\infty(\Omega)}ds \\ &\leq \|v_0\|_{L^\infty(\Omega)} + \gamma_2\chi\int_0^t\left(1+(t-s)^{-\frac{5}{6}}\right)e^{-(\lambda_1 d_2+1)(t-s)}\|v\nabla u\|_{L^3(\Omega)}ds \\ &\quad + \gamma_3\int_0^t\left(1+(t-s)^{-\frac{1}{3}}\,e^{-(t-s)}\right)\|(1+av)\,uv\|_{L^3(\Omega)}ds \\ &\leq \|v_0\|_{L^\infty(\Omega)} + \gamma_2\chi K_5K_8\int_0^\infty\left(1+(t-s)^{-\frac{5}{6}}\right)e^{-(t-s)}ds \\ &\quad + \gamma_3K_1(K_5+aK_7^2)\int_0^\infty\left(1+(t-s)^{-\frac{1}{3}}\right)e^{-(t-s)}ds \\ &\leq \|v_0\|_{L^\infty(\Omega)} + \gamma_2\chi K_5K_8\left(1+\Gamma\left(\frac{1}{6}\right)\right) + \gamma_3K_1(K_5+aK_7^2)\left(1+\Gamma\left(\frac{2}{3}\right)\right),\end{aligned}$$

which gives (3.42). □

Proof of Theorem 1.1. Combining Lemma 2.2 and Lemma 3.6, then, there exists a positive constant K which guarantees that

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq K,$$

which proves Theorem 1.1 with the extension criterion outlined in Lemma 2.1.

4. Long time behavior

Based on Theorem 1.1, this part aims to establish the convergence property of the solution. Before this, we introduce a lemma as follows.

Lemma 4.1. *For all given $\theta \in (0,1)$, we have*

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} \le C, \quad t \ge 1$$

with a constant $C > 0$.

Proof. The derived conclusion stems directly from the regularity properties of parabolic equations as outlined in [30]. □

4.1. *The prey-only steady state:* $(\kappa, 0)$.

In this part, we will demonstrate that the solution (u,v) converges to $(\kappa,0)$ under certain conditions. To achieve this aim, we introduce the Lyapunov functional, denoted as

$$\mathcal{F}_1(t) := \int_\Omega \left(u - \kappa - \kappa \ln \frac{u}{\kappa}\right) + \int_\Omega v, \quad t > 0.$$

Lemma 4.2. *Under the assumed conditions of Theorem 1.1, if the parameter satisfies $\kappa \le \beta$ and $a < \frac{\gamma}{K_1}$, this ensures that*

$$\|u-\kappa\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le K_{10} e^{-\delta_1 t}, \quad t > 0,$$

where constants $K_{10} > 0$ and $\delta_1 > 0$.

Proof. From (1.4), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(u - \kappa - \kappa \ln \frac{u}{\kappa}\right)\\
&= \int_\Omega \left(1 - \frac{\kappa}{u}\right)\left(d_1 \Delta u + \sigma u \left(1 - \frac{u}{\kappa}\right) - (1+av)uv\right)\\
&= -d_1\kappa \int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{\sigma}{\kappa}\int_\Omega (u-\kappa)^2 + \kappa \int_\Omega v + a\kappa \int_\Omega v^2 - \int_\Omega uv - a\int_\Omega uv^2
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v &= \int_\Omega \left(d_2\Delta v - \chi \nabla\cdot(v\nabla u) + (1+av)uv - \beta v - \gamma v^2\right)\\
&= \int_\Omega uv + a\int_\Omega uv^2 - \beta \int_\Omega v - \gamma \int_\Omega v^2.
\end{aligned} \tag{4.2}$$

Adding (4.1) and (4.2) results in

$$\frac{d}{dt}\mathcal{F}_1(t) \le -d_1\kappa \int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{\sigma}{\kappa}\int_\Omega (u-\kappa)^2 - (\beta-\kappa)\int_\Omega v - (\gamma - a\kappa)\int_\Omega v^2. \tag{4.3}$$

Then, due to $\kappa \le \beta$ and $a < \frac{\gamma}{K_1}$, one can choose a constant $c_1 > 0$ such that

$$\frac{d}{dt}\mathcal{F}_1(t) \le -c_1\left(\int_\Omega (u-\kappa)^2 + \int_\Omega v + \int_\Omega v^2\right), \quad t > 0,$$

which provides

$$\int_1^{+\infty}\int_\Omega (u-\kappa)^2+\int_1^{+\infty}\int_\Omega v^2 \le c_2, \tag{4.4}$$

where constant $c_2>0$. Using (4.4) and the uniform continuity of u and v due to Lemma 4.1 yields

$$\int_\Omega (u-\kappa)^2+\int_\Omega v^2 \to 0,\ \text{as } t\to+\infty.$$

Utilizing the Gagliardo-Nirenberg inequality, for all $t>1$, we derive

$$\|u-\kappa\|_{L^\infty(\Omega)} \le c_3\|u-\kappa\|_{W^{1,\infty}(\Omega)}^{\frac{2}{N+2}}\|u-\kappa\|_{L^2(\Omega)}^{\frac{2}{N+2}} \tag{4.5}$$

and

$$\|v\|_{L^\infty(\Omega)} \le c_4\|v\|_{W^{1,\infty}(\Omega)}^{\frac{2}{N+2}}\|v\|_{L^2(\Omega)}^{\frac{2}{N+2}}, \tag{4.6}$$

where constants $c_3>0$ and $c_4>0$, which proves the claim with Lemma 4.1 and (4.4). Furthermore, applying L'Hoptial's rule, it holds that

$$\lim_{s\to s_0}\frac{s-s_0-s_0\ln\frac{s}{s_0}}{(s-s_0)^2}=\lim_{s\to s_0}\frac{1-\frac{s_0}{s}}{2(s-s_0)}=\lim_{s\to s_0}\frac{1}{2s}=\frac{1}{2s_0},\quad s_0>0.$$

There exists a constant $\varepsilon>0$ ensuring that

$$\frac{1}{4s_0}(s-s_0)^2 \le s-s_0-s_0\ln\frac{s}{s_0} \le \frac{1}{s_0}(s-s_0)^2 \quad \text{for all } |s-s_0|\le\varepsilon. \tag{4.7}$$

By (4.5) and (4.6), there exists $t_0>0$ ensuring that

$$\|u-\kappa\|_{L^\infty(\Omega)}+\|v\|_{L^\infty(\Omega)}\le\varepsilon,\quad t\ge t_0.$$

Therefore, by (4.7), we get

$$\frac{1}{4\kappa}\int_\Omega (u-\kappa)^2 \le \int_\Omega\left(u-\kappa-\kappa\ln\frac{u}{\kappa}\right) \le \frac{1}{\kappa}\int_\Omega (u-\kappa)^2,\quad t\ge t_0,$$

which gives

$$\mathcal{F}_1(t)\le\frac{1}{c_5}\left(\int_\Omega (u-\kappa)^2+\int_\Omega v^2\right),\quad t\ge t_0, \tag{4.8}$$

where constant $c_5>0$. Plugging (4.8) into (4.3), it follows that

$$\frac{d}{dt}\mathcal{F}_1(t)\le-\left(\int_\Omega (u-\kappa)^2+\int_\Omega v^2\right)\le -c_5\mathcal{F}_1(t),\quad t\ge t_0.$$

Therefore, we obtain

$$\mathcal{F}_1(t)\le e^{-c_5(t-t_0)}\mathcal{F}_1(t_0)+c_5(t-t_0)e^{-c_5t}\le c_6e^{-\delta_1 t},\quad t\ge t_0$$

with some constants $c_6>0$ and $\delta_1>0$. □

4.2. The coexistence steady state: (u_*, v_*).

In this part, we will demonstrate that the solution (u, v) converges to (u_*, v_*) under certain conditions. To achieve this aim, we define the Lyapunov functional

$$\mathcal{F}_2(t) := \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*}\right), \quad t > 0,$$

where $u_* \in (\beta, \beta + \gamma\sigma)$ and $v_* \in (0, \sigma)$.

Lemma 4.3. *Under the assumed conditions of Theorem 1.1, if the parameters satisfy* $\kappa > \beta$ *and*

$$a < \min\left\{\frac{\gamma}{K_1}, \frac{2\sqrt{(\beta + \sigma\gamma)^2 + \sigma\kappa\gamma} - 2(\beta + \sigma\gamma)}{\sigma\kappa}\right\}, \tag{4.9}$$

as well as

$$\chi^2 < \frac{4 d_1 d_2 u_*}{K_1^2 v_*}, \tag{4.10}$$

this ensures that

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} \le K_{11} e^{-\delta_2 t}, \quad t > 0,$$

where constants $K_{11} > 0$ *and* $\delta_2 > 0$.

Proof. Using $\sigma\left(1 - \frac{u_*}{\kappa}\right) - (1 + a v_*) v_* = 0$, it can be directly calculated that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) \\
&= \int_\Omega \left(1 - \frac{u_*}{u}\right)\left(d_1 \Delta u + \sigma u \left(1 - \frac{u}{\kappa}\right) - (1 + av) uv\right) \\
&= -d_1 u_* \int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{\sigma}{\kappa}\int_\Omega (u - u_*)^2 - \int_\Omega (u - u_*)(v - v_*) - a\int_\Omega (u - u_*)\left(v^2 - v_*^2\right).
\end{aligned} \tag{4.11}$$

Utilizing $(1 + a v_*) u_* - \beta - \gamma v_* = 0$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*}\right) \\
&= \int_\Omega \left(1 - \frac{v_*}{v}\right)\left(d_2 \Delta v - \chi \nabla \cdot (v \nabla u) + (1 + av) uv - \beta v - \gamma v^2\right) \\
&= -d_2 v_* \int_\Omega \frac{|\nabla v|^2}{v^2} + \chi v_* \int_\Omega \frac{\nabla u \cdot \nabla v}{v} - \gamma \int_\Omega (v - v_*)^2 + \int_\Omega (u - u_*)(v - v_*) \\
&\quad + a\int_\Omega (v - v_*)(uv - u_* v_*).
\end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), it follows that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) &= -d_1u_*\int_\Omega\frac{|\nabla u|^2}{u^2}-d_2v_*\int_\Omega\frac{|\nabla v|^2}{v^2}+\chi v_*\int_\Omega\frac{\nabla u\cdot\nabla v}{v}-\frac{\sigma}{\kappa}\int_\Omega(u-u_*)^2\\ &\quad-\gamma\int_\Omega(v-v_*)^2-a\int_\Omega(v-v_*)(uv_*-u_*v)\\ &= -d_1u_*\int_\Omega\frac{|\nabla u|^2}{u^2}-d_2v_*\int_\Omega\frac{|\nabla v|^2}{v^2}+\chi v_*\int_\Omega\frac{\nabla u\cdot\nabla v}{v}-\frac{\sigma}{\kappa}\int_\Omega(u-u_*)^2\\ &\quad-\int_\Omega(\gamma-au_*)(v-v_*)^2-av_*\int_\Omega(u-u_*)(v-v_*)\\ &:= -XQX^T-YMY^T,\end{aligned} \tag{4.13}$$

where $X=\left(\frac{\nabla u}{u},\frac{\nabla v}{v}\right)$, $Y=(u-u_*,v-v_*)$, and the matrices Q and M stand for

$$Q=\begin{pmatrix} d_1u_* & -\frac{\chi v_*u}{2}\\ -\frac{\chi v_*u}{2} & d_2v_*\end{pmatrix}$$

and

$$M=\begin{pmatrix} \frac{\sigma}{\kappa} & \frac{av_*}{2}\\ \frac{av_*}{2} & \gamma-au_*\end{pmatrix}.$$

If (4.9) and (4.10) hold, we check

$$|Q|=d_1d_2u_*v_*-\frac{\chi^2v_*^2u^2}{4}>d_1d_2u_*v_*-\frac{\chi^2v_*^2K_1^2}{4}>0$$

and

$$|M|=\frac{\sigma}{\kappa}(\gamma-au_*)-\frac{a^2v_*^2}{4}>0,$$

which means that the matrices Q and M are positive definite with $K_1:=\max\{\kappa,\|u_0\|_{L^\infty(\Omega)}\}$. Then for all u,v, it implies that

$$\frac{d}{dt}\mathcal{F}_2(t)\le -c_1\int_\Omega\left(\frac{|\nabla u|^2}{u^2}+\frac{|\nabla v|^2}{v^2}\right)-c_2\int_\Omega\left((u-u_*)^2+(v-v_*)^2\right),$$

where constants $c_1>0$ and $c_2>0$. Subsequently, the rest is analogous to the reasoning employed in Lemma 4.2, and we readily demonstrate that the solution (u,v) exponentially converges to (u_*,v_*) as $t\to\infty$ in $L^\infty(\Omega)$. □

Proof of Theorem 1.2. Combining Lemmas 4.2 and 4.3, we directly obtain Theorem 1.2.

5. Conclusions

In this paper, we have proposed a prey-taxis model with hunting cooperation mechanism and explored the effect of predator hunting cooperation mechanism on predator and prey populations. Through mathematical analysis, we have demonstrated that weak cooperative hunting can prevent the blow-up of the classical solutions for model (1.4) in two-dimensional space. On the one hand, we have received that the long time behavior of the prey-only steady state is established under the weak hunting cooperation. On the other hand, we have only obtained the long time behavior for the coexistence steady states under weaker hunting cooperation. However, for the strong hunting cooperation mechanism, whether the long time behavior of the coexistence steady state can be established remains an open problem that necessitates further research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Science and Technology Project of YiLi Prefecture (Grant No. YZ2022B038), the Special Project of Yili Normal University to Improve Comprehensive Strength of Disciplines (Grant No. 22XKZY14), and the Natural Science Foundation of Xinjiang Autonomous Region (Grant No: 2022D01C335).

Conflict of interest

The authors declare that there is no conflict of interest.

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