Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(3): 1541–1565.
DOI: 10.3934/era.2025073
Received: 14 June 2024
Revised: 28 February 2025
Accepted: 03 March 2025
Published: 19 March 2025

Research article

Existence results for a discrete fractional boundary value problem

David Barilla[1], **Martin Bohner**[2,*], **Giuseppe Caristi**[1], **Shapour Heidarkhani**[3] **and Shahin Moradi**[3]

[1] Department of Economics, University of Messina, via dei Verdi, Messina, Italy
[2] Department of Mathematics and Statistics, Missouri S&T, Rolla, MO 65409, USA
[3] Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran

* **Correspondence:** Email: bohner@mst.edu.

Abstract: In this study, we investigate the existence of at least one solution and the existence of an infinite number of solutions for a discrete fractional boundary value problem. Requiring an algebraic condition on the nonlinear term for small values of the parameter, and requiring an additional asymptotical behavior of the potential at zero, we investigate the existence of at least one nontrivial solution for the problem. Moreover, under suitable assumptions on the oscillatory behavior of the nonlinearity at infinity, for exact collections of the parameter, we discuss the existence of a sequence of solutions for the problem. We also present some examples that illustrate the applicability of the main results.

Keywords: discrete fractional; existence of one solution; existence of an infinite number of solutions; variational methods

1. Introduction

In this paper, we discuss certain discrete fractional boundary value problems in the form

$$\begin{cases} {}_{L+1}\nabla_\ell^\alpha({}_\ell\nabla_0^\alpha(v(\ell))) + {}_\ell\nabla_0^\alpha({}_{L+1}\nabla_\ell^\alpha(v(\ell))) + \varphi_p(v(\ell)) = \lambda f(\ell, v(\ell)), & \ell \in [1, L]_{\mathbb{N}_0}, \\ v(0) = v(L+1) = 0, \end{cases} \qquad (P_\lambda^f)$$

where $1 < p < \infty$, $\alpha \in (0,1)$, $\lambda > 0$, ${}_{L+1}\nabla_\ell^\alpha$ and ${}_\ell\nabla_0^\alpha$ are the right and left discrete nabla fractional difference operators, $f : [1, L]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is continuous, and φ_p stands for the operator that is defined in the usual way by $\varphi_p(s) = |s|^{p-2}s$.

In this paper, we want to utilize a version of Ricceri's variational principle as given in [1]. We build the energy functional and we take some assumptions on the nonlinear term to get a functional that satisfies the conditions in the key theorem. In fact, by first requiring a simple algebraic inequality condition on the nonlinear term for small values of the parameter and requiring an additional asymptotical behavior of the potential at zero if $f(\ell, 0) = 0$, the existence of one nontrivial solution is achieved. Moreover, we deduce the existence of solutions for small positive values of the parameter such that the corresponding solutions have smaller and smaller energies as the parameter goes to zero. Then, under an appropriate oscillating behavior of the nonlinear term, we discuss the existence of an unbounded sequence of solutions. We give exact collections for the parameter for each results, which cannot be found in other works in the literature related to these types of discrete problems.

As is well known, fractional differential equations (FDEs) are valuable tools when modeling many phenomena in various areas of science and engineering. We refer to [2–8] and the references therein for a range of applications of FDEs in control, electrochemistry, viscoelasticity, electromagneticity, porous media, and other fields. Classical tools that have been employed in the study of nonlinear FDEs (see [9–11] and their references) include fixed point theory, monotone iterative methods, coincidence theory, and the upper and lower solutions method.

Thanks to their wide range of applications in many fields such as science, economics, ecology, neural networks, cybernetics, etc., nonlinear difference equations have been studied extensively for the last 50 years. In addition, boundary value problems involving difference equations have received a lot of attention, see, e.g., [12–15] and references therein. Difference equations subjected to many different kinds of boundary conditions have also been extensively studied by using various techniques. A popular technique has been to use variational methods [16–18]. In the last few years, many researchers have investigated nonlinear problems of this type through various approaches. Moreover, such boundary value problems for ordinary differential equations, difference equations, and dynamic equations on time scales have been studied extensively, but there are only a few papers dealing with fractional boundary value problems, besides [19–21], especially for discrete fractional boundary value problems involving Caputo fractional difference operators. For example, Lv [22], by using the fixed point theorem of Schaefer, under certain nonlinear growth assumptions, obtained the existence of solutions to a discrete fractional boundary value problem. Furthermore, in [23, 24], some existence and multiplicity results for (p, q)-Laplacian problems were considered. For example, in [23], by using suitable variational arguments and Ljusternik–Schnirelmann category theory, the multiplicity and concentration of positive solutions for (p, q)-Laplacian problems were obtained.

In view of the facts presented above, in the current study, we discuss the existence of at least one solution to the boundary value problem (P^f_λ), as well as the existence of an infinite number of solutions of (P^f_λ). Our primary tool is [1, Theorem 2.1], which is a more precise variant of the famous variational principle of Ricceri [25]. In Theorem 9 below, we show that, subject to certain assumptions, the boundary value problem (P^f_λ) possesses at least one nontrivial solution. We also offer Example 17, where all hypotheses of our Theorem 9 are satisfied. We present a series of remarks concerning our results. Moreover, under suitable assumptions of the oscillatory behavior at infinity of the nonlinearity, we investigate the existence of an infinite number of solutions for the boundary value problem (P^f_λ). We prove the existence of a definite interval about λ, in which the boundary value problem (P^f_λ) admits a sequence of solutions, which is unbounded in the space $\mathcal{V}$, to be introduced later (Theorem 18). We present an example that illustrates Theorem 20 (see Example 21). Moreover, some corollaries of Theorem 18 are offered. Under different assumptions, we ensure the existence of a sequence of pairwise different solutions that strongly converges to zero (see Theorem 26).

The paper is set up as follows. In Section 2, we remind the reader of some basic definitions and our main tool. In Sections 3 and 4, we state and prove our main results.

2. Preliminaries

The key argument in our results is the next version of the variational principle by Ricceri [25, Theorem 2.1], as presented in [1].

Theorem 1. *Suppose that X is a reflexive real Banach space. Assume that $\Phi, \Psi : X \to \mathbb{R}$ are Gâteaux-differentiable functionals such that Ψ is sequentially weakly upper semicontinuous, and Φ is strongly continuous, sequentially weakly lower semicontinuous, and coercive. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{v \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) - \Psi(v)}{r - \Phi(v)}$$

and

$$\theta := \liminf_{r \to \infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

One then has the following.

(*a*) *For every $r > \inf_X \Phi$ and each $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, the restriction of the functional $I_\lambda = \Phi - \lambda \Psi$ to $\Phi^{-1}(] - \infty, r[)$ possesses a global minimum, and this global minimum is a critical point, i.e., a local minimum, of I_λ in X.*

(*b*) *If $0 < \theta < \infty$, then, for all $\lambda \in \left(0, \frac{1}{\theta}\right)$, either*

(*b_1*) *I_λ possesses a global minimum, or*

(*b_2*) *for all $n \in \mathbb{N}$, I_λ has a critical point u_n, and*

$$\lim_{n \to \infty} \Phi(v_n) = \infty.$$

(*c*) *If $0 < \delta < \infty$, then, for all $\lambda \in \left(0, \frac{1}{\delta}\right)$, either*

(*c_1*) *Φ possesses a global minimum, which is a local minimum of I_λ, or*

(*c_2*) *a sequence of pairwise different critical points of I_λ exists, and this sequence converges weakly to a global minimum of Φ.*

We refer to [9, 26–28], in which Theorem 1 was applied successfully in order to prove the existence of at least one nontrivial solution for certain boundary value problems, and [29–33], in which Theorem 1 has been successfully employed in order to prove the existence of an infinite number of solutions for certain boundary value problems.

In this section, we present several foundational definitions, notations, and results that are used in the remainder of this paper.

Definition 2 (see [34]). (i) For $m \in \mathbb{N}$, the m rising factorial of t is defined as

$$t^{\overline{m}} = \prod_{k=0}^{m-1} (t + k), \quad t^{\overline{0}} = 1.$$

(ii) For $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $\alpha \in \mathbb{R}$, the α rising function is increasing on $\mathbb{N}_0$ and

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

Definition 3 (see [34])**.** For f defined on $\mathbb{N}_{a-1} \cap {}_{b+1}\mathbb{N}$, $a < b$, $\alpha \in (0,1)$, the left Caputo discrete fractional nabla difference operator is given by

$$\left({}^{\mathrm{C}}_{\ell}\nabla^{\alpha}_{a-1} f\right)(\ell) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=a}^{\ell} \nabla_s f(s)(\ell - \rho(s))^{\overline{-\alpha}} \quad (\ell \in \mathbb{N}_a), \tag{1}$$

the right Caputo discrete fractional nabla difference operator is given by

$$\left({}^{\mathrm{C}}_{b+1}\nabla^{\alpha}_{\ell} f\right)(\ell) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=\ell}^{b} (-\Delta_s f)(s)(s - \rho(\ell))^{\overline{-\alpha}} \quad (\ell \in {}_b\mathbb{N}), \tag{2}$$

the left Riemann discrete fractional nabla difference operator is given by

$$\begin{aligned}\left({}^{\mathrm{R}}_{\ell}\nabla^{\alpha}_{a-1} f\right)(\ell) &= \frac{1}{\Gamma(1-\alpha)} \nabla_\ell \sum_{s=a}^{\ell} f(s)(\ell - \rho(s))^{\overline{-\alpha}} \\ &= \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{\ell} f(s)(\ell - \rho(s))^{\overline{-\alpha-1}} \quad (\ell \in \mathbb{N}_a),\end{aligned}$$

the right Riemann discrete fractional nabla difference operator is given by

$$\begin{aligned}\left({}^{\mathrm{R}}_{b+1}\nabla^{\alpha}_{\ell} f\right)(\ell) &= \frac{1}{\Gamma(1-\alpha)} (-\Delta_\ell) \sum_{s=\ell}^{b} (f(s))(s - \rho(\ell))^{\overline{-\alpha}} \\ &= \frac{1}{\Gamma(-\alpha)} \sum_{s=\ell}^{b} (f(s))(s - \rho(\ell))^{\overline{-\alpha-1}} \quad (\ell \in {}_b\mathbb{N}),\end{aligned}$$

where $\rho(\ell) = \ell - 1$ is the backward jump operator.

Example 4. Let us give an example concerning Definition 3. Consider $f : \mathbb{N}_{a-1} \cap {}_{b+1}\mathbb{N} \to \mathbb{R}$ defined by $f(\ell) \equiv 1$. For this f, from (1) and (2), we have

$${}^{\mathrm{C}}_{\ell}\nabla^{\alpha}_{a-1} 1 = {}^{\mathrm{C}}_{b+1}\nabla^{\alpha}_{\ell} 1 = 0, \quad \ell \in \mathbb{N}_a \cap {}_b\mathbb{N}. \tag{3}$$

The relations among the right and left Riemann and Caputo nabla fractional difference operators are

$$\left({}^{\mathrm{C}}_{\ell}\nabla^{\alpha}_{a-1} f\right)(\ell) = \left({}^{\mathrm{R}}_{\ell}\nabla^{\alpha}_{a-1} f\right)(\ell) - \frac{(\ell - a + 1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a-1), \tag{4}$$

$$\left({}^{\mathrm{C}}_{b+1}\nabla^{\alpha}_{\ell} f\right)(\ell) = \left({}^{\mathrm{R}}_{b+1}\nabla^{\alpha}_{\ell} f\right)(\ell) - \frac{(b + 1 - \ell)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(b+1). \tag{5}$$

Thus, by (3)–(5), we have

$${}^{\mathrm{R}}_{b+1}\nabla^{\alpha}_{\ell} 1 = \frac{(b+1-\ell)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}, \quad {}^{\mathrm{R}}_{\ell}\nabla^{\alpha}_{a-1} 1 = \frac{(\ell - a + 1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}, \quad \ell \in \mathbb{N}_a \cap {}_b\mathbb{N}.$$

With respect to the domains of the various fractional-type difference operators, we observe the following:

(i) The nabla left fractional operator ${}_{\ell}\nabla^{\alpha}_{a-1}$ maps functions defined on ${}_{a-1}\mathbb{N}$ to functions defined on ${}_{a}\mathbb{N}$.
(ii) The nabla right fractional operator ${}_{b+1}\nabla^{\alpha}_{\ell}$ maps functions defined on ${}_{b+1}\mathbb{N}$ to functions defined on ${}_{b}\mathbb{N}$.

One can show that for $\alpha \to 0$, we have ${}_{\ell}\nabla^{\alpha}_{a-1}f(\ell) \to f(\ell)$ and for $\alpha \to 1$, we have ${}_{\ell}\nabla^{\alpha}_{a-1}f(\ell) \to \nabla f(\ell)$. We note that the Caputo and Riemann nabla fractional difference operators for $0 < \alpha < 1$ coincide when f vanishes at the end points, i.e., $f(a-1) = 0 = f(b+1)$ (see (4), (5), and [2]). So, for convenience, for the remainder of this paper, we use the symbol ${}_{\ell}\nabla^{\alpha}_{a-1}$ instead of ${}^{\mathrm{C}}_{\ell}\nabla^{\alpha}_{a-1}$ or ${}^{\mathrm{R}}_{\ell}\nabla^{\alpha}_{a-1}$ and ${}_{b+1}\nabla^{\alpha}_{\ell}$ instead of ${}^{\mathrm{C}}_{b+1}\nabla^{\alpha}_{\ell}$ or ${}^{\mathrm{R}}_{b+1}\nabla^{\alpha}_{\ell}$.

Now we present the discrete fractional summation by parts formula.

Theorem 5 (see [35, Theorem 4.4])**.** *For $f, g : \mathbb{N}_a \cap {}_{b}\mathbb{N} \to \mathbb{R}$, $a < b$, and $\alpha \in (0,1)$, the formulas*

$$\sum_{\ell=a}^{b} f(\ell)\left({}_{\ell}\nabla^{\alpha}_{a-1}g\right)(\ell) = \sum_{\ell=a}^{b} g(\ell)\left({}_{b+1}\nabla^{\alpha}_{\ell}f\right)(\ell)$$

and

$$\sum_{\ell=a}^{b} f(\ell)\left({}_{b+1}\nabla^{\alpha}_{\ell}g\right)(\ell) = \sum_{\ell=a}^{b} g(\ell)\left({}_{\ell}\nabla^{\alpha}_{a-1}f(\ell)\right)$$

hold.

In order to give the variational formulation of the boundary value problem (P^{f}_{λ}), let us introduce the finite L-dimensional Banach space

$$\mathcal{V} = \{v : [0, L+1]_{\mathbb{N}_0} \to \mathbb{R} :\ v(0) = v(L+1) = 0\},$$

which is equipped with the norm

$$\|v\| = \left(\sum_{\ell=1}^{L} |v(\ell)|^2\right)^{\frac{1}{2}}.$$

According to the definition of the norm, the following lemma is obvious.

Lemma 6. *For all $\alpha \in (0,1)$ and for all $v \in \mathcal{V}$, we have*

$$\|v\|_\infty = \max_{\ell \in [1,L]_{\mathbb{N}_0}} |v(\ell)| \le \|v\|. \tag{6}$$

For every $v \in \mathcal{V}$, we define the functionals Φ and Ψ as

$$\Phi(v) = \frac{1}{2}\sum_{\ell=1}^{L} \left|\left({}_{\ell}\nabla^{\alpha}_{0}v\right)(\ell)\right|^2 + \left|\left({}_{L+1}\nabla^{\alpha}_{\ell}v\right)(\ell)\right|^2 + \frac{1}{p}\sum_{\ell=1}^{L} |v(\ell)|^p \tag{7}$$

and

$$\Psi(v) = \sum_{\ell=1}^{L} F(\ell, v(\ell)), \tag{8}$$

and we put

$$I_\lambda(v) = \Phi(v) - \lambda\Psi(v).$$

Definition 7. A (weak) solution of (P_λ^f) is defined to be any function $v \in \mathcal{V}$ such that

$$\sum_{\ell=1}^{L} \{({}_\ell\nabla_0^\alpha v)(\ell)({}_\ell\nabla_0^\alpha \tilde{v})(\ell) + ({}_{L+1}\nabla_\ell^\alpha v)(\ell)({}_{L+1}\nabla_\ell^\alpha \tilde{v})(\ell)\}$$
$$+ \sum_{\ell=1}^{L} |v(\ell)|^{p-2} v(\ell)\tilde{v}(\ell) - \lambda \sum_{\ell=1}^{L} f(\ell, v(\ell))\tilde{v}(\ell) = 0 \quad (9)$$

for every $\tilde{v} \in \mathcal{V}$.

Note that since $\mathcal{V}$ is a finite-dimensional space, every weak solution is a usual solution of the boundary value problem (P_λ^f).

Lemma 8. *Let $v \in \mathcal{V}$. Then v is a critical point of I_λ in $\mathcal{V}$ if and only if v solves (P_λ^f).*

Proof. First assume that $v \in \mathcal{V}$ is a critical point of I_λ. Then, for any $\tilde{v} \in \mathcal{V}$, (9) holds. Bearing in mind that $\tilde{v} \in \mathcal{V}$ is arbitrary, we get

$${}_{L+1}\nabla_\ell^\alpha \left({}_\ell\nabla_0^\alpha (v(\ell))\right) + {}_\ell\nabla_0^\alpha \left({}_{L+1}\nabla_\ell^\alpha (v(\ell))\right) + |v(\ell)|^{p-1} v(\ell) - \lambda f(\ell, v(\ell)) = 0$$

for all $\ell \in [1, L]_{\mathbb{N}}$. Therefore, v solves (P_λ^f). Since v was chosen arbitrarily, we deduce that all critical points of the functional I_λ in $\mathcal{V}$ solve (P_λ^f). Conversely, if v solves (P_λ^f), then, by reversing the above steps, the proof is completed.

Put

$$F(\ell, \xi) = \int_0^\xi f(\ell, x)\mathrm{d}x \quad \text{for all} \quad (\ell, \xi) \in [1, L]_{\mathbb{N}_0} \times \mathbb{R}.$$

3. Existence of one solution

The following is our main result concerning the existence of a solution of (P_λ^f).

Theorem 9. *Assume that $f(\ell, 0) = 0$ and*

$$\sup_{\theta>0} \frac{\theta^p}{\sum_{\ell=1}^{L} \max_{|x|\le\theta} F(\ell, x)} > \frac{p}{(L+1)^{\frac{p(p-2)}{4}}} \quad (10)$$

and there are discrete intervals $D = [1, L_1]_{\mathbb{N}_0} \subseteq [1, L]_{\mathbb{N}_0}$ and $B = [1, L_2]_{\mathbb{N}_0} \subset [1, L_1]_{\mathbb{N}_0}$ with $L_1, L_2 \ge 2$, such that

$$\limsup_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{\ell\in B} F(\ell, \xi)}{|\xi|^p} = \infty$$

and

$$\liminf_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{\ell\in D} F(\ell, \xi)}{|\xi|^p} > -\infty.$$

Then, for every

$$\lambda \in \Lambda := \left(0, \frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \sup_{\theta>0} \frac{\theta^p}{\sum_{\ell=1}^{L} \max_{|x|\le\theta} F(\ell,x)}\right),$$

the boundary value problem (P_λ^f) *admits at least one nontrivial solution* $v_\lambda \in \mathcal{V}$.

Proof. Our goal is to apply Theorem 1 to (P_λ^f). We utilize the functionals Φ and Ψ as introduced in (7) and (8), respectively. Let us demonstrate that Φ and Ψ meet the required assumptions of Theorem 1. As $\mathcal{V}$ is embedded compactly in $(C^0([1,L]_{\mathbb{N}_0}),\mathbb{R})$, we know that Ψ is Gâteaux-differentiable, and its Gâteaux derivative $\Psi'(v) \in \mathcal{V}^*$ at $v \in \mathcal{V}$ is given by

$$\Psi'(v)(v) = \sum_{\ell=1}^{L} f(\ell, v(\ell))\tilde{v}(\ell)$$

for each $\tilde{v} \in \mathcal{V}$, and Ψ is sequentially weakly upper semicontinuous. Furthermore, Φ is also Gâteaux-differentiable, and its Gâteaux derivative at $v \in \mathcal{V}$ is the functional $\Phi'(v) \in \mathcal{V}^*$ given by

$$\begin{aligned}\Phi'(v)(\tilde{v}) = &\sum_{\ell=1}^{L} \left\{\left({}_{\ell}\nabla_0^{\alpha} v(\ell)\right)\left({}_{\ell}\nabla_0^{\alpha}\tilde{v}(\ell)\right) + \left({}_{L+1}\nabla_{\ell}^{\alpha} v(\ell)\right)\left({}_{L+1}\nabla_{\ell}^{\alpha}\tilde{v}(\ell)\right)\right\} \\ &+ \sum_{\ell=1}^{L} |v(\ell)|^{p-2} v(\ell)\tilde{v}(\ell)\end{aligned}$$

for every $\tilde{v} \in \mathcal{V}$. Furthermore, by the definition of Φ, we observe that it is sequentially weakly lower semicontinuous and strongly continuous. Now, in light of (7), for each $v \in \mathcal{V}$, we obtain

$$\frac{1}{p}(L+1)^{\frac{p(p-2)}{4}} \|v\|^p \le \Phi(v) \le 2L(L+1)\|v\|^2 + \frac{1}{p}(L+1)^{\frac{2-p}{2}} \|v\|^p. \tag{11}$$

By employing the left inequality of (11), we get

$$\lim_{\|v\|\to\infty} \Phi(v) = \infty.$$

In other words, Φ is coercive. Using (10), there is $\bar{\theta} > 0$ with

$$\frac{\bar{\theta}^p}{\sum_{\ell=1}^{L} \max_{|x|\le\bar{\theta}} F(\ell,x)} > \frac{p}{(L+1)^{\frac{p(p-2)}{4}}}.$$

Put

$$r := \frac{(L+1)^{\frac{p(p-2)}{4}}}{p}\bar{\theta}^p.$$

From the way Φ is defined and considering (6), (7), and (11), since $r > 0$, we have

$$\begin{aligned}\Phi^{-1}(-\infty, r) &= \{v \in \mathcal{V} : \ \Phi(v) < r\} \\ &\subseteq \left\{ v \in \mathcal{V} : \ \|v\|^p \le \frac{pr}{(L+1)^{\frac{p(p-2)}{4}}} \right\} \subseteq \left\{ v \in \mathcal{V} : \ \|v\|_\infty^p \le \frac{pr}{(L+1)^{\frac{p(p-2)}{4}}} \right\} \\ &= \left\{ v \in \mathcal{V} : \ \|v\|_\infty^p \le \bar{\theta}^p \right\},\end{aligned}$$

which implies

$$\sup_{\Phi(v)<r} \Psi(v) = \sup_{\Phi(v)<r} \sum_{\ell=1}^{L} F(\ell, v(\ell)) \le \sum_{\ell=1}^{L} \max_{|x|\le\bar{\theta}} F(\ell, x).$$

By considering these computations, as $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, we get

$$\begin{aligned}\varphi(r) = \inf_{\tilde{v}\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(\tilde{v})}{r - \Phi(\tilde{v})} &\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)}{r} \\ &\le \frac{p}{(L+1)^{\frac{p(p-2)}{4}}} \frac{\sum_{\ell=1}^{L} \max_{|x|\le\bar{\theta}} F(\ell, x)}{\bar{\theta}^p}.\end{aligned}$$

Hence, we put

$$\lambda^* = \frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \sup_{\theta>0} \frac{\theta^p}{\sum_{\ell=1}^{L} \max_{|x|\le\theta} F(\ell, x)}.$$

At this point, thanks to Theorem 1, for each $\lambda \in (0, \lambda^*) \subseteq \left(0, \dfrac{1}{\varphi(r)}\right)$, Theorem 1 shows that I_λ possesses at least one critical point (local minimum) $v_\lambda \in \Phi^{-1}(-\infty, r)$. We show that v_λ cannot be the trivial function. Let us prove

$$\limsup_{\|v\|\to 0^+} \frac{\Psi(v)}{\Phi(v)} = \infty. \tag{12}$$

Due to our required conditions at zero, we can find a sequence $\{\xi_k\} \subset \mathbb{R}^+$ that converges to zero, and $\zeta > 0$ and κ with

$$\lim_{k\to\infty} \frac{\operatorname{ess\,inf}_{\ell\in B} F(\ell, \xi_k)}{|\xi_k|^p} = \infty$$

and

$$\operatorname*{ess\,inf}_{\ell\in D} F(\ell, \xi) \ge \kappa|\xi|^p, \quad \xi \in [0, \zeta].$$

Let $[1, L_3]_{\mathbb{N}_0} \subset [1, L_2]_{\mathbb{N}_0}$, where $L_3 \ge 2$, and let $\tilde{v} \in \mathcal{V}$ be such that

(i) $\tilde{v}(\ell) \in [0, 1]$ for all $\ell \in [1, L]_{\mathbb{N}_0}$,
(ii) $\tilde{v}(\ell) = 1 \in \mathbb{R}$ for all $\ell \in [1, L_3]_{\mathbb{N}_0}$,
(iii) $\tilde{v}(\ell) = 0$ for all $\ell \in [L_1 + 1, L]_{\mathbb{N}_0}$.

Therefore, we fix an arbitrary $Y > 0$ and $\eta > 0$ with

$$Y < \frac{\eta L_3 + \kappa \sum_{\ell=L_3+1}^{L} |\tilde{v}(\ell)|^p}{\frac{1}{p}(L+1)^{\frac{2-p}{2}} \|\tilde{v}\|^p}.$$

Thus, there is $n_0 \in \mathbb{N}$ with $\varepsilon^n < \zeta$ and

$$\operatorname*{ess\,inf}_{\ell \in B} F(\ell, \xi_n) \geq \eta |\xi_n|^p$$

for all $n > n_0$. Next, for all $n > n_0$, using $0 \leq \xi_n \tilde{v}(\ell) < \zeta$ for all large enough n, by (11), we get

$$\begin{aligned}
\frac{\Psi(\xi_n \tilde{v})}{\Phi(\xi_n \tilde{v})} =& \frac{\sum_{\ell=1}^{L_3} F(\ell, \xi_n) + \sum_{\ell=L_3+1}^{L} F(\ell, \xi_n \tilde{v}(\ell))}{\Phi(\xi_n \tilde{v})} \\
>& \frac{\eta L_3 + \kappa \sum_{\ell=L_3+1}^{L} |\tilde{v}(\ell)|^p}{\frac{1}{p}(L+1)^{\frac{2-p}{2}} \|\tilde{v}\|^p} > Y.
\end{aligned}$$

Since Y can be as large as we desire, we conclude that

$$\lim_{n \to \infty} \frac{\Psi(\xi_n \tilde{v})}{\Phi(\xi_n \tilde{v})} = \infty,$$

which implies (12). Thus, we have a sequence $\{\upsilon_n\} \subset \mathcal{V}$ that converges strongly to zero, $\upsilon_n \in \Phi^{-1}(-\infty, r)$, and

$$I_\lambda(\upsilon_n) = \Phi(\upsilon_n) - \lambda \Psi(\upsilon_n) < 0.$$

Since v_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we get

$$I_\lambda(v_\lambda) < 0, \tag{13}$$

and thus v_λ is not trivial. The proof is complete.

Some comments are given next.

Remark 10. In Theorem 9, we sought for the critical points of I_λ, a functional that is intrinsically connected to the boundary value problem (P_λ^f). We observe that, in a general case, I_λ may not be bounded in $\mathcal{V}$. To see this, consider, for instance, the situation when $f(\xi) = 1 + \xi^{\gamma-1}$ for each $\xi \in \mathbb{R}$, where $\gamma > p$. Let $v \in \mathcal{V} \setminus \{0\}$ and $\mu \in \mathbb{R}$. We then find

$$\begin{aligned}
I_\lambda(\mu v) =& \Phi(\mu v) - \lambda \sum_{\ell=1}^{L} F(\mu v(\ell)) \\
\leq& 2\mu^2 L(L+1)\|v\|^2 + \frac{\mu^p}{p}(L+1)^{\frac{2-p}{2}} \|v\|^p + \lambda \mu L \|v\| - \lambda \frac{\mu^\gamma}{\gamma} L \|v\|^\gamma \to -\infty
\end{aligned}$$

as $\mu \to \infty$. Thus, [36, (I_2) in Theorem 2.2] is not satisfied. Therefore, we are unable to use direct minimization to find the critical points of I_λ.

Remark 11. Note that I_λ is not coercive. To see this, let $s \in (p, \infty)$ and define F by $F(\xi) = |\xi|^s$ for each $\xi \in \mathbb{R}$. Let $v \in \mathcal{V} \setminus \{0\}$ and $\mu \in \mathbb{R}$. We then find

$$\begin{aligned} I_\lambda(\mu v) =& \Phi(\mu v) - \lambda \sum_{\ell=1}^{L} F(\mu v(\ell)) \\ \leq& 2\mu^2 L(L+1)\|v\|^2 + \frac{\mu^p}{p}(L+1)^{\frac{2-p}{2}}\|v\|^p - \lambda \mu^s L \|v\|^s \to -\infty \end{aligned}$$

as $\mu \to -\infty$.

Remark 12. If, in Theorem 9, f satisfies $f(\ell, x) \geq 0$ for almost every $(\ell, x) \in [1, L]_{\mathbb{N}_0} \times \mathbb{R}$, then (10) assumes the simpler form

$$\sup_{\theta > 0} \frac{\theta^p}{\sum_{\ell=1}^{L} F(\ell, \theta)} > \frac{p}{(L+1)^{\frac{p(p-2)}{4}}}. \tag{14}$$

Moreover, if the assumption

$$\limsup_{\theta \to \infty} \frac{\theta^p}{\sum_{\ell=1}^{L} F(\ell, \theta)} > \frac{p}{(L+1)^{\frac{p(p-2)}{4}}}$$

is satisfied, then (14) automatically holds.

Remark 13. From (13), we can directly see that

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(v_\lambda) \tag{15}$$

is indeed a negative map. Furthermore, we have

$$\lim_{\lambda \to 0^+} \|v_\lambda\| = 0.$$

Indeed, by considering that Φ is coercive and that for $\lambda \in (0, \lambda^*)$, the solution $v_\lambda \in \Phi^{-1}(-\infty, r)$, we have the existence of $\mathcal{L} > 0$ with $\|v_\lambda\| \leq \mathcal{L}$ for every $\lambda \in (0, \lambda^*)$. Next, it is also easy to see that $\mathcal{M} > 0$ exists with

$$\left| \sum_{\ell=1}^{L} f(\ell, v_\lambda(\ell)) v_\lambda(\ell) \right| \leq \mathcal{M}\|v_\lambda\| \leq \mathcal{M}\mathcal{L} \tag{16}$$

for each $\lambda \in (0, \lambda^*)$. Since v_λ is a critical point of I_λ, $I'_\lambda(v_\lambda)(\tilde{v}) = 0$ for any $\tilde{v} \in \mathcal{V}$ and each $\lambda \in (0, \lambda^*)$. In particular, $I'_\lambda(v_\lambda)(v_\lambda) = 0$, i.e.,

$$\Phi'(v_\lambda)(v_\lambda) = \lambda \sum_{\ell=1}^{L} f(\ell, v_\lambda(\ell)) v_\lambda(\ell) \tag{17}$$

for every $\lambda \in (0, \lambda^*)$. Then, since

$$0 \leq (L+1)^{\frac{p(p-2)}{4}} \|v\|^p \leq \Phi'(v_\lambda)(v_\lambda),$$

by using (17), it is concluded that

$$0 \le (L+1)^{\frac{p(p-2)}{4}} \|v\|^p \le \Phi'(v_\lambda)(v_\lambda)$$
$$\le \lambda \sum_{\ell=1}^{L} f(\ell, v_\lambda(\ell)) v_\lambda(\ell)$$

for all $\lambda \in (0, \lambda^*)$. If we now let $\lambda \to 0^+$, by (16), then we get $\lim_{\lambda\to 0^+} \|v_\lambda\| = 0$. One has

$$\lim_{\lambda\to 0^+} \|v_\lambda\|_\infty = 0.$$

Finally, we prove that

$$\lambda \mapsto I_\lambda(v_\lambda) \tag{18}$$

decreases strictly in $(0, \lambda^*)$. We observe that for all $v \in \mathcal{V}$, we have

$$I_\lambda(v) = \lambda\left(\frac{\Phi(v)}{\lambda} - \Psi(v)\right). \tag{19}$$

Now, let $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let v_{λ_1}, v_{λ_2} be the global minima of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Moreover, let

$$m_{\lambda_i} = \left(\frac{\Phi(v_{\lambda_i})}{\lambda_i} - \Psi(v_{\lambda_i})\right) = \inf_{\tilde{v}\in\Phi^{-1}(-\infty,r)} \left(\frac{\Phi(\tilde{v})}{\lambda_i} - \Psi(\tilde{v})\right)$$

for $i = 1, 2$. Obviously, (15), in connection with (19) and $\lambda > 0$, yields

$$m_{\lambda_i} < 0 \quad \text{for} \quad i = 1, 2.$$

In addition,

$$m_{\lambda_2} \le m_{\lambda_1}, \tag{20}$$

because $0 < \lambda_1 < \lambda_2$. Then, by observing (19)–(20) and as $0 < \lambda_1 < \lambda_2$, we get

$$I_{\lambda_2}(\bar{v}_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(\bar{v}_{\lambda_1}),$$

so that (18) decreases strictly for $\lambda \in (0, \lambda^*)$. Since $\lambda < \lambda^*$ is arbitrary, we see that (18) indeed decreases strictly in $(0, \lambda^*)$.

Remark 14. We note that Theorem 9 represents a bifurcation result in the sense that $(0, 0)$ belongs to the closure of

$$\left\{(v_\lambda, \lambda) \in \mathcal{V} \times (0, \infty) : \ v_\lambda \text{ is a nontrivial solution of } (P_\lambda^f)\right\}$$

in $\mathcal{V} \times \mathbb{R}$. To observe this, by Remark 13, we get

$$\|v_\lambda\| \to 0 \quad \text{as} \quad \lambda \to 0.$$

Therefore, there are sequences $\{v_j\}$ in $\mathcal{V}$ and $\{\lambda_i\}$ in $\mathbb{R}^+$ (here, $v_j = v_\lambda$) with

$$\lambda_i \to 0^+ \quad \text{and} \quad \|v_j\| \to 0$$

as $j \to \infty$. In addition, we want to emphasize that because

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(v_\lambda)$$

is a strictly decreasing map, for all $\lambda_1, \lambda_2 \in (0, \lambda^*)$ such that $\lambda_1 \ne \lambda_2$, the solutions $\bar{v}_{\lambda_1}$ and $\bar{v}_{\lambda_2}$ ensured by Remark 13 are distinct.

Remark 15. Under the assumption that $f \geq 0$, the solution that is ensured by Theorem 9 is nonnegative. To observe this, suppose that v_0 is a nontrivial solution of (P_λ^f). Assume that

$$\mathcal{A} = \{\ell \in [1, L]_{\mathbb{N}_0} : v_0(\ell) < 0\} \neq \emptyset$$

has positive measure. Put $\bar{\bar{v}}(\ell) = \min\{0, v_0(\ell)\}$ for each $\ell \in [1, L]_{\mathbb{N}_0}$. We obtain $\bar{\bar{v}} \in \mathcal{V}$ and

$$\sum_{\ell=1}^{L} \{({}_\ell\nabla_0^\alpha v_0(\ell))\,({}_\ell\nabla_0^\alpha \bar{\bar{v}}(\ell)) + ({}_{L+1}\nabla_\ell^\alpha v_0(\ell))\,({}_\ell\nabla_0^\alpha \bar{\bar{v}}(\ell))\}$$
$$+ \sum_{\ell=1}^{L} |v_0(\ell)|^{p-2} v_0(\ell)\bar{\bar{v}}(\ell) - \lambda \sum_{\ell=1}^{L} f(\ell, v_0(\ell))\bar{\bar{v}}(\ell) = 0.$$

Thus, from our imposed data sign assumptions, we have

$$\begin{aligned} 0 \leq &(L+1)^{\frac{p(p-2)}{4}} \|v\|_{\mathcal{A}}^p \leq \sum_{\mathcal{A}} \left\{({}_\ell\nabla_0^\alpha v_0(\ell))^2 + ({}_{L+1}\nabla_\ell^\alpha v_0(\ell))^2\right\} + \sum_{\mathcal{A}} |v_0(\ell)|^p \\ =&\lambda \sum_{\mathcal{A}} f(\ell, v_0(\ell))v_0(\ell) \leq 0. \end{aligned}$$

Hence, $v_0 = 0$ in $\mathcal{A}$, which is impossible.

The next result concerns a particular case of the previously presented results, in which the function f depends only on the second variable, considering the nonautonomous case of the problem.

Theorem 16. *Let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative with $f(0) = 0$. Let $F(\xi) = \int_0^\xi f(x)\mathrm{d}x$ for all $x \in \mathbb{R}$. Assume that*

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} = \infty.$$

Then, for every

$$\lambda \in \left(0, \frac{(L+1)^{\frac{p(p-2)}{4}}}{Lp} \sup_{\theta > 0} \frac{\theta^p}{F(\theta)}\right),$$

the boundary value problem

$$\begin{cases} {}_{L+1}\nabla_\ell^\alpha({}_\ell\nabla_0^\alpha(v(\ell))) + {}_\ell\nabla_0^\alpha({}_{L+1}\nabla_\ell^\alpha(v(\ell))) + \varphi_p(v(\ell)) = \lambda f(v(\ell)), & \ell \in [1, L]_{\mathbb{N}_0}, \\ v(0) = v(L+1) = 0, \end{cases}$$

possesses at least one nontrivial solution $v_\lambda \in \mathcal{V}$ satisfying

$$\lim_{\lambda \to 0^+} \|v_\lambda\| = 0,$$

and the real function

$$\lambda \to \frac{1}{2} \sum_{\ell=1}^{L} \left\{ \left|({}_\ell\nabla_0^\alpha v)(\ell)\right|^2 + \left|({}_{L+1}\nabla_\ell^\alpha v)(\ell)\right|^2 \right\} + \frac{1}{p} \sum_{\ell=1}^{L} |v(\ell)|^p - \lambda \sum_{\ell=1}^{L} F(v(\ell))$$

is negative and decreases strictly in $\left(0, \frac{(L+1)^{\frac{p(p-2)}{4}}}{Lp} \sup_{\theta>0} \frac{\theta^p}{F(\theta)}\right)$.

Finally, we present the following example to illustrate Theorem 16.

Example 17. Let $p = 4$ and $L = 2$. We consider the problem

$$\begin{cases} {}_3\nabla_\ell^{\frac{1}{2}}({}_\ell\nabla_0^{\frac{1}{2}}(v(\ell))) + {}_\ell\nabla_0^{\frac{1}{2}}({}_3\nabla_\ell^{\frac{1}{2}}(v(\ell))) + \varphi_4(v(\ell)) = \lambda f(v(\ell)), & \ell \in \{1,2\}, \\ v(0) = v(3) = 0, \end{cases} \tag{21}$$

where

$$f(\xi) = 34\xi^3 + 2\tan(\xi)\sec^2(\xi) + e^\xi, \quad \xi \in \mathbb{R}.$$

By simple computations, we have

$$F(\xi) = \xi^4 + \sec^2(\xi) + e^\xi - 2, \quad \xi \in \mathbb{R}.$$

We see that all assumptions of Theorem 16 are satisfied, and this implies that the boundary value problem (21), for each $\lambda \in \left(0, \frac{9}{8}\right)$, admits at least one nontrivial solution $v_\lambda \in \mathcal{V}$ such that

$$\lim_{\lambda \to 0^+} \|v_\lambda\| = 0,$$

and the real function

$$\lambda \to \frac{1}{2}\sum_{\ell=1}^{2}\left\{\left|\left({}_\ell\nabla_0^{\frac{1}{2}}v\right)(\ell)\right|^2 + \left|\left({}_3\nabla_\ell^{\frac{1}{2}}v\right)(\ell)\right|^2\right\} + \frac{1}{4}\sum_{\ell=1}^{2}|v(\ell)|^4 - \lambda\sum_{\ell=1}^{2}F(v(\ell))$$

is negative and strictly decreasing in $\left(0, \frac{9}{8}\right)$.

4. Existence of an infinite number of solutions

Put

$$\mathcal{B}^\infty = \limsup_{\xi\to\infty} \frac{\sum_{\ell=1}^{L} F(\ell,\xi)}{\frac{\xi^2}{(\Gamma(1-\alpha))^2}\sum_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2 + \frac{L\xi^p}{p}}.$$

Our main result concerning the existence of infinitely many solutions of (P_λ^f) is as follows.

Theorem 18. *Assume that two sequences* $\{a_n\}$ *and* $\{b_n\}$ *exist with*

$$\lim_{n\to\infty} b_n = \infty$$

such that

$$\frac{a_n^2}{(\Gamma(1-\alpha))^2}\sum_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2 + \frac{La_n^p}{p} < \frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p \quad \textit{for all} \quad n \in \mathbb{N}, \tag{A$_1$}$$

$$\mathcal{A}^{\infty}=\lim_{n\to\infty}\frac{\sum_{\ell=1}^{L}\max_{|t|\le b_n}F(\ell,t)-\sum_{\ell=1}^{L}F(\ell,a_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p}b_n^p-\left(\frac{a_n^2}{(\Gamma(1-\alpha))^2}\sum_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2+\frac{La_n^p}{p}\right)}<\mathcal{B}^{\infty}. \tag{A$_2$}$$

In this case, for all $\lambda\in\left(\frac{1}{\mathcal{B}^{\infty}},\frac{1}{\mathcal{A}^{\infty}}\right)$*, the boundary value problem* (P_λ^f) *possesses an unbounded sequence of solutions.*

Proof. Our aim is to employ Theorem 1. We utilize the functionals Φ and Ψ as introduced in (7) and (8), respectively. Therefore, we observe that the regularity assumptions on Φ and Ψ, as required in Theorem 1, are satisfied. Hence $v\in\mathcal{V}$ is a solution of (P_λ^f) if and only if v is a critical point of the function I_λ. Put

$$r_n=\frac{(L+1)^{\frac{p(p-2)}{4}}}{p}b_n^p\quad\text{for all}\quad n\in\mathbb{N}.$$

We see that $r_n>0$ for all $n\in\mathbb{N}$. From the way Φ is defined and in light of (6), (8), and (11), for each $r_n>0$, we have

$$\begin{aligned}\Phi^{-1}(-\infty,r_n)&=\{v\in\mathcal{V}:\ \Phi(v)<r_n\}\\&\subseteq\left\{v\in\mathcal{V}:\ \|v\|^p\le\frac{pr_n}{(L+1)^{\frac{p(p-2)}{4}}}\right\}\subseteq\left\{v\in\mathcal{V}:\ \|v\|_\infty^p\le\frac{pr_n}{(L+1)^{\frac{p(p-2)}{4}}}\right\}\\&=\{v\in\mathcal{V}:\ \|v\|_\infty^p\le b_n^p\},\end{aligned}$$

which implies

$$\sup_{\Phi(v)<r_n}\Psi(v)=\sup_{\Phi(v)<r_n}\sum_{\ell=1}^{L}F(\ell,v(\ell))\le\sum_{\ell=1}^{L}\max_{|x|\le b_n}F(\ell,x).$$

Now, for each $n\in\mathbb{N}$, we define

$$\upsilon(\ell)=\begin{cases}a_n & \text{if}\quad \ell\in[1,L]_{\mathbb{N}_0},\\ 0 & \text{if}\quad \ell\in\{0,L+1\}.\end{cases}$$

Clearly, $\upsilon\in\mathcal{V}$. Since υ vanishes at the end points (that is, $\upsilon(0)=0=\upsilon(L+1)$), its Riemann and Caputo fractional differences coincide. Hence, for any $\ell\in\mathbb{N}_1\cap{}_L\mathbb{N}$, we have

$$\left({}_{L+1}\nabla_\ell^\alpha\upsilon\right)(\ell)=\left({}^{\mathrm{R}}_{L+1}\nabla_\ell^\alpha\upsilon\right)(\ell)=\left({}^{\mathrm{C}}_{L+1}\nabla_\ell^\alpha\upsilon\right)(\ell)=\frac{a_n(L+1-\ell)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}$$

and

$$\left({}_\ell\nabla_0^\alpha\upsilon\right)(\ell)=\left({}^{\mathrm{R}}_\ell\nabla_0^\alpha\upsilon\right)(\ell)=\left({}^{\mathrm{C}}_\ell\nabla_0^\alpha\upsilon\right)(\ell)=\frac{a_n(\ell)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}.$$

Thus,

$$
\begin{aligned}
\Phi(\upsilon) =& \frac{1}{2}\sum_{\ell=1}^{L}\left\{\left|\left({}_{\ell}\nabla_{0}^{\alpha}\upsilon\right)(\ell)\right|^{2} + \left|\left({}_{L+1}\nabla_{\ell}^{\alpha}\upsilon\right)(\ell)\right|^{2}\right\} + \frac{1}{p}\sum_{\ell=1}^{L}|\upsilon(\ell)|^{p}\\
=& \frac{1}{2}\sum_{\ell=1}^{L}\left\{\left|\frac{a_n(\ell)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\right|^{2} + \left|\frac{a_n(L+1-\ell)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\right|^{2}\right\} + \frac{La_n^p}{p}\\
=& \frac{a_n^2}{2(\Gamma(1-\alpha))^2}\sum_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2 + \left|(L+1-\ell)^{\overline{-\alpha}}\right|^2 + \frac{La_n^p}{p}\\
=& \frac{a_n^2}{(\Gamma(1-\alpha))^2}\sum_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2 + \frac{La_n^p}{p}.
\end{aligned}
$$

We have

$$
\Psi(\upsilon) = \sum_{\ell=1}^{L} F(\ell, \upsilon(\ell)) = \sum_{\ell=1}^{L} F(\ell, a_n).
$$

In addition, from (A_1), we have $\Phi(\upsilon_n) < r_n$. Thus, for all large enough values of n, we obtain

$$
\begin{aligned}
\varphi(r_n) =& \inf_{\tilde{v}\in\Phi^{-1}(-\infty,r_n)} \frac{\sup\limits_{v\in\Phi^{-1}(-\infty,r_\ell)} \Psi(v) - \Psi(\tilde{v})}{r_n - \Phi(\tilde{v})}\\
\leq& \frac{\sup\limits_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v) - \sum\limits_{\ell=1}^{L} F(\ell, \upsilon(\ell))}{r_n - \Phi(\upsilon)}\\
=& \frac{\sup\limits_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v) - \sum\limits_{\ell=1}^{L} F(\ell, a_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p - \left(\frac{a_n^2}{(\Gamma(1-\alpha))^2}\sum\limits_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2 + \frac{La_n^p}{p}\right)}\\
\leq& \frac{\sum\limits_{\ell=1}^{L}\max\limits_{|x|\leq b_n} F(\ell, x) - \sum\limits_{\ell=1}^{L} F(\ell, a_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p - \left(\frac{a_n^2}{(\Gamma(1-\alpha))^2}\sum\limits_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2 + \frac{La_n^p}{p}\right)}.
\end{aligned} \tag{22}
$$

Hence, due to (A_2), we get

$$
\gamma \leq \lim_{n\to\infty} \varphi(r_n) \leq \mathcal{A}^{\infty} < \infty.
$$

Now, we can verify that I_λ is unbounded from below. First, assume that $\mathcal{B}^{\infty} = \infty$. Accordingly, fix N such that

$$
N < \frac{c_n^2}{(\Gamma(1-\alpha))^2}\sum_{\ell=1}^{L}\left|\ell^{\overline{-\alpha}}\right|^2 + \frac{Lc_n^p}{p}
$$

and let $c_n > 0$ for all $n \in \mathbb{N}$ such that $c_n \to \infty$ as $n \to \infty$ and

$$\sum_{\ell=1}^{L} F(\ell, c_n) > N \quad \text{for all} \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, define

$$y_n(\ell) = c_n \quad \text{for all} \quad \ell \in [1, L]_{\mathbb{N}_0}.$$

Thus, $y_n \in \mathcal{V}$ and

$$\Phi(y_n) = \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{Lc_n^p}{p}.$$

Therefore,

$$\begin{aligned} I_\lambda(y_n) =& \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{Lc_n^p}{p} - \lambda \sum_{\ell=1}^{L} F(\ell, c_n) \\ <& \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{Lc_n^p}{p} - \lambda N, \end{aligned}$$

that is, $\lim_{n\to\infty} I_\lambda(y_n) = -\infty$. Next, suppose $\mathcal{B}^\infty < \infty$. As $\lambda > \frac{1}{\mathcal{B}^\infty}$, we can find $\varepsilon > 0$ with $\varepsilon < \mathcal{B}^\infty - \frac{1}{\lambda}$. Thus, again letting $c_n > 0$ for all $n \in \mathbb{N}$ such that $c_n \to \infty$ as $n \to \infty$ and

$$\sum_{\ell=1}^{L} F(\ell, c_n) > \mathcal{B}^\infty - \varepsilon \quad \text{for all} \quad n \in \mathbb{N},$$

as argued above, and by letting $y_n \in \mathcal{V}$ as before, we obtain

$$\begin{aligned} I_\lambda(y_n) =& \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{Lc_n^p}{p} - \lambda \sum_{\ell=1}^{L} F(\ell, c_n) \\ <& \frac{c_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{Lc_n^p}{p} - \lambda(\mathcal{B}^\infty - \varepsilon). \end{aligned}$$

Therefore, $\lim_{n\to\infty} I_\lambda(y_n) = -\infty$. Hence, in either case, I_λ is not bounded from below. This completes the proof.

Remark 19. If $\{a_n\}$ and $\{b_n\}$ are real sequences such that $\lim_{n\to\infty} b_n = \infty$ and such that (A_1) from Theorem 18 is satisfied, then, assuming that $\mathcal{A}_\infty = 0$ and $\mathcal{B}^\infty = \infty$, Theorem 18 ensures that for each $\lambda > 0$, the boundary value problem (P_λ^f) admits an infinite number of solutions.

Theorem 20. *Assume that*

$$\liminf_{\xi\to\infty} \frac{\sum_{\ell=1}^{L} \max_{|x|\le\xi} F(\ell, x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p}\xi^p} < \limsup_{\xi\to\infty} \frac{\sum_{\ell=1}^{L} F(\ell, \xi)}{\frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{L\xi^p}{p}}. \qquad (A_3)$$

In this case, for all

$$\lambda \in \left(\frac{\frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\overline{\ell^{-\alpha}}\right|^2 + \frac{L\xi^p}{p}}{\sum_{\ell=1}^{L} F(\ell, \xi)}, \frac{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p}\xi^p}{\sum_{\ell=1}^{L} \max_{|x|\le\xi} F(\ell, x)} \right),$$

the boundary value problem (P_λ^f) *possesses an unbounded sequence of solutions.*

Proof. We pick $b_n > 0$ for all $n \in \mathbb{N}$ with $b_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\sum_{\ell=1}^{L} \max_{|x|\le b_n} F(\ell, x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p} = \liminf_{\xi\to\infty} \frac{\sum_{\ell=1}^{L} \max_{|x|\le\xi} F(\ell, x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \xi^p}.$$

Now, as $\Phi(0) = \Psi(0) = 0$, we may take $a_n = 0$ for all $n \in \mathbb{N}$ in (22), and then the conclusion follows from Theorem 1.

Now, we present an example that illustrates Theorem 20.

Example 21. Let $p = 4$ and $L = 3$. We consider the boundary value problem

$$\begin{cases} {}_4\nabla_\ell^{\frac{1}{2}}({}_\ell\nabla_0^{\frac{1}{2}}(v(\ell))) + {}_\ell\nabla_0^{\frac{1}{2}}({}_4\nabla_\ell^{\frac{1}{2}}(v(\ell))) + \varphi_4(v(\ell)) = \lambda f(v(\ell)), & \ell \in \{1, 2, 3\}, \\ v(0) = v(4) = 0, \end{cases} \tag{23}$$

where

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, 0], \\ 4x^3 + 4x^3 \sin(\pi e^x) + \pi x^4 e^x \cos(\pi e^x)) & \text{if} \quad x \in (0, \infty). \end{cases}$$

Some computation yields

$$F(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, 0], \\ x^4(1 + \sin(\pi e^x)) & \text{if} \quad x \in (0, \infty). \end{cases}$$

Since

$$\liminf_{\xi\to\infty} \frac{\sum_{\ell=1}^{3} \max_{|x|\le\xi} F(x)}{64\xi^4} = 0$$

and

$$\limsup_{\xi\to\infty} \frac{\sum_{\ell=1}^{3} F(\xi)}{\frac{\xi^2}{(\Gamma(1/2))^2} \sum_{\ell=1}^{3} \left|\overline{\ell^{-\frac{1}{2}}}\right|^2 + \frac{3\xi^4}{4}} = 8,$$

we clearly see that all assumptions of Theorem 20 are satisfied, and then (23), for every $\lambda \in \left(\frac{1}{8}, \infty\right)$, has an unbounded sequence of solutions in $\{v : [0, 4]_{\mathbb{N}_0} \to \mathbb{R} : v(0) = v(4) = 0\}$.

Here, we point out several simple corollaries of our main results.

Corollary 22. *Suppose that there are real sequences* $\{a_n\}$ *and* $\{b_n\}$ *with* $b_n \to \infty$ *as* $n \to \infty$ *and such that* (A_1) *from Theorem 18 holds,* $\mathcal{A}_\infty < 1$*, and* $\mathcal{B}_\infty > 1$*. Then, the boundary value problem*

$$\begin{cases} {}_{L+1}\nabla_\ell^\alpha({}_\ell\nabla_0^\alpha(v(\ell))) + {}_\ell\nabla_0^\alpha({}_{L+1}\nabla_\ell^\alpha(v(\ell))) + \varphi_p(v(\ell)) = f(\ell, v(\ell)), \quad \ell \in [1, L]_{\mathbb{N}_0}, \\ v(0) = v(L+1) = 0 \end{cases} \tag{P^f}$$

possesses an unbounded sequence of solutions.

Corollary 23. *Suppose* $\mathcal{B}_\infty > 1$ *and*

$$\liminf_{\xi \to \infty} \frac{\sum_{\ell=1}^{L} \max_{|x| \le \xi} F(\ell, x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \xi^p} < 1.$$

Then the boundary value problem (P^f) *possesses an unbounded sequence of solutions.*

Corollary 24. *Suppose that there are real sequences* $\{a_n\}$ *and* $\{b_n\}$ *with* $b_n \to \infty$ *as* $n \to \infty$ *and such that* (A_1) *from Theorem 18 holds,* $f_1 \in \mathrm{C}([1, L]_{\mathbb{N}_0} \times \mathbb{R}, \mathbb{R})$*, and*

$$F_1(\ell, x) = \int_0^x f_1(\ell, \xi) \mathrm{d}\xi \quad \textit{for all} \quad (\ell, x) \in [0, L]_{\mathbb{N}_0} \times \mathbb{R}.$$

Moreover, assume

$$\lim_{n \to \infty} \frac{\sum_{\ell=1}^{L} \max_{|x| \le b_n} F_1(\ell, x) - \sum_{\ell=1}^{L} F_1(\ell, a_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p - \left(\frac{a_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left| \ell^{\overline{-\alpha}} \right|^2 + \frac{L a_n^p}{p} \right)} < \infty \tag{A$_4$}$$

and

$$\limsup_{\xi \to \infty} \frac{\sum_{\ell=1}^{L} F_1(\ell, \xi)}{\frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left| \ell^{\overline{-\alpha}} \right|^2 + \frac{L \xi^p}{p}} = \infty. \tag{A$_5$}$$

Then, for all $f_i \in \mathrm{C}([1, L]_{\mathbb{N}_0} \times \mathbb{R}, \mathbb{R})$*, by writing*

$$F_i(\ell, x) = \int_0^x f_i(\ell, \xi) \mathrm{d}\xi \quad \textit{for all} \quad (\ell, x) \in [1, L]_{\mathbb{N}_0} \times \mathbb{R}$$

for $2 \le i \le n$*, such that*

$$\max \left\{ \sup_{\xi \in \mathbb{R}} F_i(\ell, \xi) : \ 2 \le i \le n \right\} \le 0$$

and

$$\min \left\{ \liminf_{\xi \to \infty} \frac{F_i(\ell, \xi)}{\xi^2} : \ 2 \le i \le n \right\} > -\infty,$$

for all

$$\lambda \in \left(0, \frac{1}{\lim\limits_{n\to\infty} \dfrac{\sum\limits_{\ell=1}^{L} \max\limits_{|x|\le b_n} F_1(\ell,x) - \sum\limits_{\ell=1}^{L} F_1(\ell,a_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p - \left(\dfrac{a_n^2}{(\Gamma(1-\alpha))^2} \sum\limits_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \dfrac{La_n^p}{p} \right)}} \right),$$

the boundary value problem

$$\begin{cases} {}_{L+1}\nabla_{\ell}^{\alpha}({}_{\ell}\nabla_{0}^{\alpha}(v(\ell))) + {}_{\ell}\nabla_{0}^{\alpha}({}_{L+1}\nabla_{\ell}^{\alpha}(v(\ell))) + \varphi_p(v(\ell)) = \lambda f_i(\ell, v(\ell)), \quad \ell \in [1, L]_{\mathbb{N}_0}, \\ v(0) = v(L+1) = 0 \end{cases} \tag{24}$$

has an unbounded sequence of solutions.

Proof. Put $F(\ell,\xi) = \sum\limits_{i=1}^{n} F_i(\ell,\xi)$ for $(\ell,\xi) \in [1,L]_{\mathbb{N}_0} \times \mathbb{R}$. ($A_4$), along with

$$\min\left\{ \liminf_{\xi\to\infty} \frac{F_i(\ell,\xi)}{\xi^2} : \ 2 \le i \le n \right\} > -\infty,$$

ensures

$$\limsup_{\xi\to\infty} \frac{\sum\limits_{\ell=1}^{L} F(\ell,\xi)}{\dfrac{\xi^2}{(\Gamma(1-\alpha))^2} \sum\limits_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \dfrac{L\xi^p}{p}} = \limsup_{\xi\to\infty} \frac{\sum\limits_{i=1}^{n} \sum\limits_{\ell=1}^{L} F_i(\ell,\xi)}{\dfrac{\xi^2}{(\Gamma(1-\alpha))^2} \sum\limits_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \dfrac{L\xi^p}{p}} = \infty.$$

Moreover, Assumption (A_4), together with the condition

$$\max\left(\sup_{\xi\in\mathbb{R}} F_i(\ell,\xi) : \ 2 \le i \le n \right) \le 0,$$

implies

$$\lim_{n\to\infty} \frac{\sum_{\ell=1}^{L} \max_{|x|\le b_n} F(\ell,x) - \sum_{\ell=1}^{L} F(\ell,\sigma_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p - \left(\frac{\sigma_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{L\sigma_n^p}{p} \right)}$$

$$\le \lim_{n\to\infty} \frac{\sum_{\ell=1}^{L} \max_{|x|\le b_n} F_1(\ell,x) - \sum_{\ell=1}^{L} F_1(\ell,a_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} b_n^p - \left(\frac{a_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{La_n^p}{p} \right)} < \infty.$$

Hence, an application of Theorem 18 completes the proof.

Corollary 25. *Let* $f_1 \in \mathrm{C}([1,L]_{\mathbb{N}_0} \times \mathbb{R}, \mathbb{R})$ *and put*

$$F_1(\ell,x) = \int_0^x f_1(\ell,\xi)\mathrm{d}\xi \quad \textit{for all} \quad (\ell,x) \in [1,L]_{\mathbb{N}_0} \times \mathbb{R}.$$

Assume that

$$\liminf_{\xi\to\infty} \frac{\sum_{\ell=1}^{L} \max_{|x|\le\sigma(\xi)} F_1(\ell,x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \xi^p} < \infty$$

and

$$\limsup_{\xi\to\infty} \frac{\sum_{\ell=1}^{L} F_1(\ell,\xi)}{\frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left|\ell^{\overline{-\alpha}}\right|^2 + \frac{L\xi^p}{p}} = \infty.$$

Then, for all $f_i \in \mathrm{C}([1,L]_{\mathbb{N}_0} \times \mathbb{R}, \mathbb{R})$*, by writing*

$$F_i(\ell,x) = \int_0^x f_i(\ell,\xi)\mathrm{d}\xi \quad \textit{for all} \quad (\ell,x) \in [1,L]_{\mathbb{N}_0} \times \mathbb{R}$$

for $2 \le i \le n$*, such that*

$$\max\left\{ \sup_{\xi\in\mathbb{R}} F_i(\ell,\xi) : \ 2 \le i \le n \right\} \le 0$$

and

$$\min\left\{ \liminf_{\xi\to\infty} \frac{F_i(\ell,\xi)}{\xi^p} : \ 2 \le i \le n \right\} > -\infty,$$

for all

$$\lambda \in \left] 0, \frac{1}{\liminf\limits_{\xi \to \infty} \frac{\sum\limits_{\ell=1}^{L} \max\limits_{|x| \le \xi} F(\ell, x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \xi^p}} \right[,$$

the boundary value problem (24) *has an unbounded sequence of solutions.*

Now put

$$\mathcal{B}^0 = \limsup_{\xi \to 0} \frac{\sum\limits_{\ell=1}^{L} F(\ell, \xi)}{\frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum\limits_{\ell=1}^{L} \left| \ell^{\overline{-\alpha}} \right|^2 + \frac{L\xi^p}{p}}.$$

With a proof similar to the proof of Theorem 18, but this time using Theorem 1 (c) instead of Theorem 1 (b), we can establish the next result. The proof is similar to the proof of Theorem 18, but here we have a real sequence $\{e_n\}$ which tends to zero at ∞ constructing r, because in Theorem 1 (c) for δ, it requires $r \to (\inf_X \Phi)^+$ instead of in Theorem 1 (b) for θ, it requires $r \to \infty$.

Theorem 26. *Suppose that there are real sequences* $\{d_n\}$ *and* $\{e_n\}$ *with* $\lim_{n\to\infty} e_n = 0$ *such that*

$$\frac{d_n^2}{(\Gamma(1-\alpha))^2} \sum_{\ell=1}^{L} \left| \ell^{\overline{-\alpha}} \right|^2 + \frac{L d_n^p}{p} < \frac{(L+1)^{\frac{p(p-2)}{4}}}{p} e_n^p \quad \textit{for every} \quad n \in \mathbb{N}, \tag{A_6}$$

$$\mathcal{A}^0 = \lim_{n \to \infty} \frac{\sum\limits_{\ell=1}^{L} \max\limits_{|x| \le e_n} F(\ell, x) - \sum\limits_{\ell=1}^{L} F(\ell, d_n)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} e_n^p - \frac{d_n^2}{(\Gamma(1-\alpha))^2} \sum\limits_{\ell=1}^{L} \left| \ell^{\overline{-\alpha}} \right|^2 + \frac{L d_n^p}{p}} < \mathcal{B}^0. \tag{A_7}$$

Then, for each

$$\lambda \in (\lambda_3, \lambda_4) \quad \textit{with} \quad \lambda_3 := \frac{1}{\mathcal{B}^0} \quad \textit{and} \quad \lambda_4 := \frac{1}{\mathcal{A}_0},$$

the boundary value problem (P_λ^f) *possesses a sequence of pairwise different solutions that strongly converges to* 0 *in* $\mathcal{V}$.

Theorem 27. *Suppose*

$$\liminf_{\xi \to 0^+} \frac{\sum\limits_{\ell=1}^{L} \max\limits_{|x| \le \xi} F(\ell, x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \xi^p} < \limsup_{\xi \to 0} \frac{\sum\limits_{\ell=1}^{L} F(\ell, \xi)}{\frac{\xi^2}{(\Gamma(1-\alpha))^2} \sum\limits_{\ell=1}^{L} \left| \ell^{\overline{-\alpha}} \right|^2 + \frac{L\xi^p}{p}}. \tag{A_8}$$

Then, for each

$$\lambda \in \left(\frac{1}{\limsup\limits_{\xi \to 0} \dfrac{\sum\limits_{\ell=1}^{L} F(\ell, \xi)}{\dfrac{\xi^2}{(\Gamma(1-\alpha))^2} \sum\limits_{\ell=1}^{L} \left| \overline{\ell^{-\alpha}} \right|^2 + \dfrac{L\xi^p}{p}}}, \frac{1}{\liminf\limits_{\xi \to 0^+} \dfrac{\sum\limits_{\ell=1}^{L} \max\limits_{|x| \le \xi} F(\ell, x)}{\frac{(L+1)^{\frac{p(p-2)}{4}}}{p} \xi^p}} \right),$$

the boundary value problem (P_λ^f) *has a sequence of pairwise different solutions that converges strongly to* 0 *in* $\mathcal{V}$.

Remark 28. By employing Theorem 26, we may obtain results that are similar to Remark 19 and Corollaries 22–25.

5. Conclusions

In this paper, we investigated the existence of one and of infinitely many solutions for a class of discrete fractional boundary value problems. As a matter of fact, by demanding an algebraic condition on the nonlinear term for small values of the parameter and requiring an additional asymptotical behavior of the potential at zero, we obtain the existence of at least one nontrivial solution for the problem. Moreover, under suitable assumptions on the oscillatory behavior at infinity of the nonlinearity, for exact collections of the parameter, we get the existence of a sequence of solutions for the problem. The main results improve and extend recent results from the literature. We also presented some examples that illustrate the applicability of the main results. The main technique of the proofs involves variational methods and critical point theorems for smooth functionals.

Conflict of interest

There is no conflict of interest.

Use of AI tools declaration

No AI tools have been used in the preparation of this study.

References

1. G. Bonanno, G. M. Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Boundary Value Probl.*, **2009** (2009). https://doi.org/10.1155/2009/670675

2. T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, *Discrete Dyn. Nat. Soc.*, **2013** (2013). https://doi.org/10.1155/2013/406910
3. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010, https://doi.org/10.1007/978-3-642-14574-2
4. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., 2000. https://doi.org/10.1142/9789812817747
5. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
6. Y. M. Chu, S. Bekiros, E. Zambrano-Serrano, O. Orozco-López, S. Lahmiri, H. Jahanshahi, et al., Artificial macro-economics: A chaotic discrete-time fractional-order laboratory model, *Chaos Solitons Fractals*, **145** (2021), 1–9. https://doi.org/10.1016/j.chaos.2021.110776
7. H. Ming, J. Wang, M. Fečkan, The application of fractional calculus in Chinese economic growth models, *Mathematics*, **7** (2019), 1–6. https://www.mdpi.com/2227-7390/7/8/665
8. M. D. Johansyah, A. K. Supriatna, E. Rusyaman, J. Saputra, Application of fractional differential equation in economic growth model: A systematic review approach, *AIMS Math.*, **6** (2021), 10266–10280, https://doi.org/10.3934/math.2021594
9. M. Galewski, G. M. Bisci, Existence results for one-dimensional fractional equations, *Math. Methods Appl. Sci.*, **39** (2016), 1480–1492. https://doi.org/10.1002/mma.3582
10. S. Heidarkhani, Multiple solutions for a nonlinear perturbed fractional boundary value problem, *Dynam. Systems Appl.*, **23** (2014), 317–331.
11. L. Kong, Existence of solutions to boundary value problems arising from the fractional advection dispersion equation, *Electron. J. Differ. Equ.*, **2013** (2013), 1–15.
12. A. Cabada, A. Iannizzotto, S. Tersian, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.*, **356** (2009), 418–428. https://doi.org/10.1016/j.jmaa.2009.02.038
13. M. Galewski, S. Głąb, On the discrete boundary value problem for anisotropic equation, *J. Math. Anal. Appl.*, **386** (2012), 956–965. https://doi.org/10.1016/j.jmaa.2011.08.053
14. M. Galewski, R. Wieteska, On the system of anisotropic discrete BVPs, *J. Differ. Equ. Appl.*, **19** (2013), 1065–1081. https://doi.org/10.1080/10236198.2012.709508
15. J. Henderson, H. B. Thompson, Existence of multiple solutions for second-order discrete boundary value problems, *Comput. Math. Appl.*, **43** (2002), 1239–1248. https://doi.org/10.1016/S0898-1221(02)00095-0
16. F. Faraci, Multiple solutions for two nonlinear problems involving the p-Laplacian, *Nonlinear Anal.*, **63** (2005), e1017–e1029. https://doi.org/10.1016/j.na.2005.02.066
17. W. Li, Z. He, The applications of sums of ranges of accretive operators to nonlinear equations involving the p-Laplacian operator, *Nonlinear Anal.*, **24** (1995), 185–193. https://doi.org/10.1016/0362-546X(94)E0051-H
18. L. Wei, R. P. Agarwal, Existence of solutions to nonlinear Neumann boundary value problems with generalized p-Laplacian operator, *Comput. Math. Appl.*, **56** (2008), 530–541. https://doi.org/10.1016/j.camwa.2008.01.013

19. G. Chai, Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator, *Boundary Value Probl.*, **2012** (2012). https://doi.org/10.1186/1687-2770-2012-18

20. J. Wang, H. Xiang, Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian operator, *Abstr. Appl. Anal.*, **2010** (2010). https://doi.org/10.1155/2010/971824

21. J. Wang, H. Xiang, Z. Liu, Positive solutions for three-point boundary value problems of nonlinear fractional differential equations with p-Laplacian, *Far East J. Appl. Math.*, **37** (2009), 33–47.

22. W. Lv, Existence of solutions for discrete fractional boundary value problems with a p-Laplacian operator, *Adv. Differ. Equ.*, **2012** (2012). https://doi.org/10.1186/1687-1847-2012-163

23. V. Ambrosio, D. Repovš, Multiplicity and concentration results for a (p,q)-Laplacian problem in $\mathbb{R}^N$, *Z. Angew. Math. Phys.*, **72** (2021), 1–33. https://doi.org/10.1007/s00033-020-01466-7

24. L. Baldelli, R. Filippucci, Existence of solutions for critical (p,q)-Laplacian equations in $\mathbb{R}^N$, *Commun. Contemp. Math.*, **25** (2023), 1–26. https://doi.org/10.1142/S0219199721501091

25. B. Ricceri, A general variational principle and some of its applications, **113** (2000), 401–410. https://doi.org/10.1016/S0377-0427(99)00269-1

26. D. Barilla, M. Bohner, S. Heidarkhani, S. Moradi, Existence results for dynamic Sturm-Liouville boundary value problems via variational methods, *Appl. Math. Comput.*, **409** (2021), 1–10. https://doi.org/10.1016/j.amc.2020.125614

27. J. R. Graef, S. Heidarkhani, L. Kong, S. Moradi, On an anisotropic discrete boundary value problem of Kirchhoff type, *J. Differ. Equ. Appl.*, **27** (2021), 1103–1119. https://doi.org/10.1080/10236198.2021.1968847

28. S. Heidarkhani, G. A. Afrouzi, S. Moradi, G. Caristi, B. Ge, Existence of one weak solution for $p(x)$-biharmonic equations with Navier boundary conditions, *Z. Angew. Math. Phys.*, **67** (2016), 1–13. https://doi.org/10.1007/s00033-016-0668-5

29. G. Bonanno, P. Candito, Infinitely many solutions for a class of discrete non-linear boundary value problems, *Appl. Anal.*, **88** (2009), 605–616. https://doi.org/10.1080/00036810902942242

30. J. R. Graef, S. Heidarkhani, L. Kong, Infinitely many solutions for systems of Sturm-Liouville boundary value problems, *Results Math.*, **66** (2014), 327–341. https://doi.org/10.1007/s00025-014-0379-1

31. S. Heidarkhani, Infinitely many solutions for systems of n two-point Kirchhoff-type boundary value problems, *Ann. Polon. Math.*, **107** (2013), 133–152. https://doi.org/10.4064/ap107-2-3

32. S. Heidarkhani, Y. Zhao, G. Caristi, G. A. Afrouzi, S. Moradi, Infinitely many solutions for perturbed impulsive fractional differential systems, *Appl. Anal.*, **96** (2017), 1401–1424. https://doi.org/10.1080/00036811.2016.1192147

33. B. Ricceri, Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian, *Bull. London Math. Soc.*, **33** (2001), 331–340. https://doi.org/10.1017/S0024609301008001

34. F. M. Atıcı, P. W. Eloe, Discrete fractional calculus with the nabla operator, *Electron. J. Qual. Theory Differ. Equ.*, **2009** (2009), 1–12. https://doi.org/10.14232/ejqtde.2009.4.3

35. T. Abdeljawad, F. M. Atıcı, On the definitions of nabla fractional operators, *Abstr. Appl. Anal.*, **2012** (2012), 1–13. https://doi.org/10.1155/2012/406757

36. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, American Mathematical Society, 1986. https://doi.org/10.1090/cbms/065