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Research article

Hybrid mean value involving some two-term exponential sums and fourth Gauss sums

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Abstract: Let $q \geq 3$ be a positive integer. For any integers m, n, k, h, the two-term exponential sums $C(m,n,k,h;q)$ is defined as $C(m,n,k,h;q) = \sum\limits_{a=1}^{q} e\left(\frac{ma^k+na^h}{q}\right)$, where $k > h \geq 2$. The main purpose of this paper is to use analytic methods and the properties of classical Gauss sums to study the mean value involving two-term exponential sums and fourth Gauss sums, and to provide some asymptotic formulas and identities. Previously, only the case of $h = 1$ had been studied.

Keywords: two-term exponential sums; fourth Gauss sums; mean value; asymptotic formulas; identities

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1. Introduction

Let $q \geq 3$ be a positive integer. For any integers m, n, k, h, the two-term exponential sums $C(m,n,k,h;q)$ is defined as

$$C(m,n,k,h;q) = \sum_{a=1}^{q} e\left(\frac{ma^k+na^h}{q}\right),$$

where $k > h$. When $n = 0$, this sum reduces to the k-th Gauss sum, defined by

$$G(m,k;q) = \sum_{a=1}^{q} e\left(\frac{ma^k}{q}\right).$$

These summations are very important in additive number theory because there are close relations between the summation and the Waring's problem. About its arithmetical properties, many authors had studied it. For example, Davenport and Heilbronn [1] proved that

$$C(m,n,k,1;p^r) \ll_k p^{\theta r}(b,p^r), \qquad if\ p \nmid m. \tag{1.1}$$

where $\theta = \frac{2}{3}$ if $k = 3$, and $\theta = \frac{3}{4}$ if $k \geq 3$. Applying Weil's estimate for exponential sums over finite fields, Hua [2] proved that $\theta = \frac{1}{2}$ for all $k \geq 2$. Loxton and Smith [3], Smith [4], Loxton and Vaughan [5], Dabrowski and Fisher [6], among others, have made improvements to (1.1). Recently, Li [7], utilizing the recursive formula for the fourth Gauss sum and relevant results on two-term exponential sums, further extended the study to derive a fourth-order linear recurrence formula for the fourth Gauss sum and the two-term exponential sum

$$M_k(p) = \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right)^k \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2$$

in the case where $p \equiv 1 \bmod 4$.

We have been exploring the properties of two-term exponential sums, specifically $\left| \sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right) \right|^t$. The main challenge arises when the lower-order terms of the two-term exponential sum are 2. If a passes through a residue system modulo p while a^2 does not pass through the same residue system, direct application of properties of Gauss sums or trigonometric identity becomes complex. Yuan and Wang [8], utilizing the orthogonality and parity properties of characters modulo p, ingeniously transformed the problem and ultimately provided a concise formula. Specifically, for $p \equiv 3 \bmod 4$, the following identity holds:

$$\sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^4 = p \cdot (7p^2 - 14p - 5), \tag{1.2}$$

however, when $p \equiv 1 \bmod 4$, the research on the problem becomes relatively complex, and they did not provide relevant results.

For the sake of convenience, we have introduced some symbols

$$\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \overline{a}}{p}\right); \gamma = \sum_{a=1}^{p-1} \psi(a^2 - 1)\overline{\psi}(a^2 + 1).$$

The main purpose of this paper is to study the asymptotic properties of the t-th power mean

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^k \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^t, \tag{1.3}$$

and give some exact formulas for (1.3) with $t = 2$; for convenience, we will denote it as $T_k(p)$. That is, we shall prove the following:

Theorem 1. Let p be a prime with $p \equiv 5 \bmod 8$. Then we have the identity

$$T_2(p) = 6p^3 - 6p^{\frac{5}{2}} - 15p^2 + 6p^{\frac{3}{2}} - 3p - 4p\alpha(1 + \alpha - \sqrt{p}).$$

Corollary 1. Let p be an odd prime with $p \equiv 5 \bmod 8$. Then we have

$$T_2(p) = 6p^3 - 6p^{\frac{5}{2}} + O(p^2).$$

Theorem 2. Let p be a prime with $p \equiv 1 \bmod 8$. Then we have the identity

$$T_2(p) = 6p^3 + 6p^{\frac{5}{2}} - 23p^2 + 6p^{\frac{3}{2}} - 3p + 4p\alpha(3 - \alpha - \sqrt{p}) + 2p(\tau^2(\psi)\gamma + \tau^2(\overline{\psi})\overline{\gamma}),$$

where $\tau(\psi) = \sum_{a=0}^{p-1} \psi(a) e\left(\frac{a}{p}\right)$, which usually denotes the classical Gauss sums.

Corollary 2. Let p be an odd prime with $p \equiv 1 \bmod 8$. Then we have

$$T_2(p) = 6p^3 + O(p^{\frac{5}{2}}).$$

Theorem 3. Let p be a prime with $p \equiv 3 \bmod 4$. Then for any integer $k \geq 4$, we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^{2k} \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^4 = p^{k+1} \cdot (7p^2 - 14p - 5).$$

Some notes:

(1) When $p \equiv 3 \bmod 4$, we have $\left(\frac{-1}{p}\right) = -1$. In this case, for any integer m with $(m, p) = 1$, according to Theorem 7.5.4 in reference [9], we have

$$G(m, 2; p) = i \left(\frac{m}{p}\right) \sqrt{p},$$

where $i^2 = -1$. Combining with the properties of Gauss sums, we obtain

$$G(m, 4; p) = 1 + \sum_{a=1}^{p-1} \left(1 + \left(\frac{a}{p}\right)\right) e\left(\frac{ma^2}{p}\right) = \left(\frac{m}{p}\right) i \sqrt{p},$$

then

$$|G(m, 4; p)|^2 = p, \tag{1.4}$$

when $p \equiv 3 \bmod 4$, the hybrid means of $T_k(p)$ are easily obtained. Hence, in this section, we only consider the case when $p \equiv 1 \bmod 4$.

(2) From the recursive formula for $G^4(m, 4; p) = \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^4$ in [10], we can derive a fourth-order linear recurrence formula for $T_k(p)$.

When $p = 8k + 5$, the recursive formula for $G^4(m, 4; p)$ is as follows:

$$\begin{aligned} G^4(m, 4; p) &= -2pG^2(m, 4; p) + 4pG(m, 4; p) \left(\sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p}\right) \right) \\ &\quad - 9p^2 + p \left(\sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p}\right) \right)^2 \\ &= -2pG^2(m, 4; p) + 8p\alpha G(m, 4; p) - 9p^2 + 4p\alpha^2. \end{aligned}$$

Through simple calculations, we obtain the recursive formula for $T_k(p)$ as follows:

$$
\begin{aligned}
T_k(p) &= \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^k \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} G^k(m,4;p)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} G^4(m,4;p)G^{k-4}(m,4;p)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1}\left(\left(-2pG^2(m,4;p)+8p\alpha G(m,4;p)-9p^2+4p\alpha^2\right)G^{k-4}(m,4;p)\right) \\
&\quad \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= -2pT_{k-2}(p)+8p\alpha T_{k-3}(p)+(4p\alpha^2-9p^2)T_{k-4}(p).
\end{aligned}
$$

Similarly, when $p = 8k+1$, according to [10], we know

$$
\begin{aligned}
G^4(m,4;p) &= 6pG^2(m,4;p)+4pG(m,4;p)\left(\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right)\right) \\
&\quad -p^2+p\left(\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right)\right)^2 \\
&= 6pG^2(m,4;p)+8p\alpha G(m,4;p)-p^2+4p\alpha^2,
\end{aligned}
$$

then we have

$$
T_k(p) = 6pT_{k-2}(p)+8p\alpha T_{k-3}(p)+(4p\alpha^2-p^2)T_{k-4}(p).
$$

(3) When $p \equiv 1 \bmod 4$ and k is any positive integer, for the hybrid mean given by

$$
\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^{2k} \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2,
$$

as k increases, the computation becomes increasingly complex. Whether precise computational formulas or corresponding recurrence relations can be obtained remains a question worth investigating.

2. Preliminaries

In this section, we give some lemmas, which will be applied to prove the main results.

Lemma 1. Let p be an odd prime with $p \equiv 1 \bmod 4$. For any fourth-order character ψ modulo p, we have

$$\sum_{a=1}^{p-1} \psi(a^4 - 1) = \frac{1 + \psi(-1)}{\sqrt{p}}(\tau^2(\psi) - \sqrt{p}), \tag{2.1}$$

$$\sum_{a=1}^{p-1} \psi^2(a^4 - 1) = -2 + \frac{\psi(-1)}{\sqrt{p}}(\tau^2(\psi) + \tau^2(\overline{\psi})), \tag{2.2}$$

$$\sum_{a=1}^{p-1} \overline{\psi}(a^4 - 1) = \frac{1 + \overline{\psi}(-1)}{\sqrt{p}}(\tau^2(\overline{\psi}) - \sqrt{p}), \tag{2.3}$$

where $\psi^2 = \chi_2$ and $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre symbol.

Proof. In fact, this is Lemma 2.1 of [11], so its proof is omitted.

Lemma 2. Let p be an odd prime with $p \equiv 1 \bmod 4$. For any fourth-order character ψ modulo p, we have

$$\tau^2(\psi) + \tau^2(\overline{\psi}) = 2\sqrt{p}\alpha.$$

Proof. See Lemma 2.2 in [11].

Lemma 3. Let p be an odd prime with $p \equiv 1 \bmod 4$, and a be any integer with $(a, p) = 1$. For any fourth-order character ψ modulo p, we have

$$\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^2 = \begin{cases} 2\left(\tau(\psi) - \tau(\overline{\psi})\sqrt{p}\right) + \tau(\psi)\sqrt{p}\gamma, & \text{if } p \equiv 1 \bmod 8, \\ \tau(\psi)\sqrt{p}\gamma, & \text{if } p \equiv 5 \bmod 8. \end{cases}$$

Proof. First, using the trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{if } q \nmid n. \end{cases} \tag{2.4}$$

Then, utilizing the properties of Dirichlet characters, $\psi^4 = \overline{\psi}^4 = \chi_0$ and $\overline{\psi}\chi_2 = \psi$, we can obtain

$$\begin{aligned} &\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^2 \\ &= \sum_{m=1}^{p-1} \psi(m) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{m(a^4 - b^4) + (a^2 - b^2)}{p}\right) \\ &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{b^2(a^2 - 1)}{p}\right) \sum_{m=1}^{p-1} \psi(m) e\left(\frac{mb^4(a^4 - 1)}{p}\right) \\ &= \tau(\psi) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \overline{\psi}(b^4) \overline{\psi}(a^4 - 1) e\left(\frac{b^2(a^2 - 1)}{p}\right) \end{aligned}$$

$$= \tau(\psi)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\psi}(a^4-1)e\left(\frac{b^2(a^2-1)}{p}\right)$$
$$= \tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}(a^4-1)\left(\sum_{b=1}^{p-1}(1+\chi_2(b))e\left(\frac{b(a^2-1)}{p}\right)\right)$$
$$= \tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}(a^4-1)\sum_{b=1}^{p-1}e\left(\frac{b(a^2-1)}{p}\right)$$
$$+ \tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}(a^4-1)\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(a^2-1)}{p}\right)$$
$$= -\tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}(a^4-1)+\tau(\psi)\tau(\chi_2)\sum_{a=1}^{p-1}\overline{\psi}(a^4-1)\chi_2(a^2-1)$$
$$= -\tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}(a^4-1)+\tau(\psi)\tau(\chi_2)\sum_{a=1}^{p-1}\psi(a^2-1)\overline{\psi}(a^2+1).$$

When $p \equiv 1 \bmod 4$, $\tau(\chi_2) = \sqrt{p}$, combined with (2.3) in Lemma 1, we obtain

$$\sum_{m=1}^{p-1}\psi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^4+a^2}{p}\right)\right|^2$$
$$= -\tau(\psi)\left(\frac{1+\overline{\psi}(-1)}{\sqrt{p}}(\tau^2(\overline{\psi})-\sqrt{p})\right)+\tau(\psi)\sqrt{p}\gamma$$
$$= (1+\overline{\psi}(-1))\left(\tau(\psi)-\psi(-1)\tau(\overline{\psi})\sqrt{p}\right)+\tau(\psi)\sqrt{p}\gamma.$$

When $p \equiv 1 \bmod 8$, $\psi(-1) = \overline{\psi}(-1) = 1$; when $p \equiv 5 \bmod 8$, $\psi(-1) = \overline{\psi}(-1) = -1$. Therefore, we have

$$\sum_{m=1}^{p-1}\psi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^4+a^2}{p}\right)\right|^2$$
$$= \begin{cases} 2\left(\tau(\psi)-\tau(\overline{\psi})\sqrt{p}\right)+\tau(\psi)\sqrt{p}\gamma, & \text{if } p \equiv 1 \bmod 8, \\ \tau(\psi)\sqrt{p}\gamma, & \text{if } p \equiv 5 \bmod 8. \end{cases}$$

This proves Lemma 3.

Lemma 4. Let p be an odd prime with $p \equiv 1 \bmod 4$, then

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^4+a^2}{p}\right)\right|^2 = \begin{cases} 2\sqrt{p}(1-\alpha-\sqrt{p}), & \text{if } p \equiv 1 \bmod 8, \\ 2\sqrt{p}(1+\alpha-\sqrt{p}), & \text{if } p \equiv 5 \bmod 8. \end{cases}$$

Proof. From the proof of Lemma 3, we obtain

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^4+a^2}{p}\right)\right|^2$$

$$
\begin{aligned}
&= \sum_{m=1}^{p-1}\chi_2(m)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{m(a^4-b^4)+(a^2-b^2)}{p}\right)\\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{b^2(a^2-1)}{p}\right)\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{mb^4(a^4-1)}{p}\right)\\
&= \tau(\chi_2)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(b^4)\chi_2(a^4-1)e\left(\frac{b^2(a^2-1)}{p}\right)\\
&= \tau(\chi_2)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(a^4-1)e\left(\frac{b^2(a^2-1)}{p}\right)\\
&= \tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a^4-1)\left(\sum_{b=1}^{p-1}(1+\chi_2(b))e\left(\frac{b(a^2-1)}{p}\right)\right)\\
&= \tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a^4-1)\sum_{b=1}^{p-1} e\left(\frac{b(a^2-1)}{p}\right)\\
&\quad + \tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a^4-1)\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(a^2-1)}{p}\right)\\
&= -\tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a^4-1)+\tau^2(\chi_2)\sum_{a=1}^{p-1}\chi_2(a^4-1)\chi_2(a^2-1)\\
&= -\tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a^4-1)+\tau^2(\chi_2)\sum_{a=1}^{p-1}\chi_2(a^2+1).
\end{aligned}
$$

Utilizing the properties of the Legendre symbol, then

$$
\sum_{a=0}^{p-1}\chi_2(a^2+n) = \begin{cases} -1, & \text{if } (p,n)=1,\\ p-1, & \text{if } (p,n)=p. \end{cases} \tag{2.5}
$$

Combining (2.5) with (2.2) from Lemma 1, we obtain

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2\\
&\qquad = -\tau(\chi_2)(-2+\frac{\psi(-1)}{\sqrt{p}}(\tau^2(\psi)+\tau^2(\overline{\psi}))) - 2p,
\end{aligned}
$$

from Lemma 2, then

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2\\
&\qquad = -\tau(\chi_2)(-2+\psi(-1)2\alpha) - 2p\\
&\qquad = \sqrt{p}(2-\psi(-1)2\alpha) - 2p
\end{aligned}
$$

$$= \begin{cases} 2\sqrt{p}(1-\alpha-\sqrt{p}), & \text{if } p \equiv 1 \bmod 8, \\ 2\sqrt{p}(1+\alpha-\sqrt{p}), & \text{if } p \equiv 5 \bmod 8. \end{cases}$$

This proves Lemma 4.

Lemma 5. Let p be an odd prime with $p \equiv 1 \bmod 4$, then

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 = \begin{cases} 2p^2+2p^{\frac{3}{2}}-5p+2p^{\frac{1}{2}}-1, & \text{if } p \equiv 1 \bmod 8, \\ 2p^2-2p^{\frac{3}{2}}-5p+2p^{\frac{1}{2}}-1, & \text{if } p \equiv 5 \bmod 8. \end{cases}$$

Proof. From the properties of quadratic Gauss sums, we obtain

$$\left|\sum_{a=1}^{p-1} e\left(\frac{a^2}{p}\right)\right|^2 = \left|\sum_{a=1}^{p} e\left(\frac{a^2}{p}\right)-1\right|^2 = \begin{cases} p-2\sqrt{p}+1, & p \equiv 1 \bmod 4, \\ p+1, & p \equiv 3 \bmod 4. \end{cases} \tag{2.6}$$

Further calculations yield

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= \sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 - \left|\sum_{a=1}^{p-1} e\left(\frac{a^2}{p}\right)\right|^2 \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{mb^4(a^4-1)+b^2(a^2-1)}{p}\right) - \left|\sum_{a=1}^{p-1} e\left(\frac{a^2}{p}\right)\right|^2 \\
&= p \sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{b^2(a^2-1)}{p}\right) - (p-2\sqrt{p}+1) \\
&= p \sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} (1+\chi_2(b)) e\left(\frac{b(a^2-1)}{p}\right) - (p-2\sqrt{p}+1) \\
&= p \sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{b(a^2-1)}{p}\right) \\
&\quad + p \sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b(a^2-1)}{p}\right) - (p-2\sqrt{p}+1)
\end{aligned}$$

$$= p \sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p \\ a^2\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} 1 + p \sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p \\ a^2\not\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{b(a^2-1)}{p}\right)$$

$$+ p\tau(\chi_2) \sum_{\substack{a=1 \\ a^4\equiv 1 \bmod p}}^{p-1} \chi_2(a^2-1) - (p - 2\sqrt{p} + 1)$$

$$= 2p(p-1) - 2p + 2p\tau(\chi_2)\left(\frac{-2}{p}\right) - (p - 2\sqrt{p} + 1)$$

$$= 2p^2 - 5p + 2p\sqrt{p}\left(\frac{-2}{p}\right) + 2\sqrt{p} - 1,$$

using the properties of the Legendre symbol, then

$$\begin{aligned} \left(\frac{-2}{p}\right) &= \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) \\ &= (-1)^{\frac{p-1}{2}}(-1)^{\frac{p^2-1}{8}} \\ &= \begin{cases} 1, & \text{if } \ p \equiv 1 \text{ or } 3 \bmod 8, \\ -1, & \text{if } \ p \equiv 5 \text{ or } 7 \bmod 8. \end{cases} \end{aligned} \tag{2.7}$$

According to (2.7), we can obtain

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right) \right|^2 = \begin{cases} 2p^2 + 2p^{\frac{3}{2}} - 5p + 2p^{\frac{1}{2}} - 1, & \text{if } \ p \equiv 1 \bmod 8, \\ 2p^2 - 2p^{\frac{3}{2}} - 5p + 2p^{\frac{1}{2}} - 1, & \text{if } \ p \equiv 5 \bmod 8. \end{cases}$$

This proves Lemma 5.

Lemma 6. Let p be an odd prime with $p \equiv 1 \bmod 4$, then

$$\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \psi(a^2-1)\overline{\psi}(a^2+1) \right|^2 = 2(p-1)(p-3).$$

Proof. According to the properties of the character modulo p, we have

$$\sum_{\chi \bmod p} \chi(a)\overline{\chi}(n) = \begin{cases} \phi(p), & a \equiv n \bmod p, \\ 0, & a \not\equiv n \bmod p. \end{cases} \tag{2.8}$$

Noting that $p \nmid (a^2+1)$, $p \mid (a^2-1)$, and using (2.8), we obtain

$$\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \psi(a^2-1)\overline{\psi}(a^2+1) \right|^2$$

$$= (p-1) \sum_{\substack{a=2 \\ (a^2-1)\overline{(a^2+1)} \equiv (b^2-1)\overline{(b^2+1)} \bmod p}}^{p-2} \sum_{b=2}^{p-2} 1$$

$$= (p-1) \sum_{\substack{a=2 \\ (b^2+1)(a^2-1) \equiv (a^2+1)(b^2-1) \bmod p}}^{p-2} \sum_{b=2}^{p-2} 1$$

$$= (p-1) \sum_{\substack{a=2 \\ a^2 \equiv b^2 \bmod p}}^{p-2} \sum_{b=2}^{p-2} 1 = 2(p-1)(p-3).$$

This proves Lemma 6.

3. Proofs of the Theorems

For any integer m satisfying $(m, p) = 1$, based on the classical properties of Gauss sums and the properties of the fourth-order character ψ, combined with (2.4), we have

$$\begin{aligned} G(m, 4; p) &= \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \\ &= 1 + \sum_{a=1}^{p-1} (1 + \psi(a) + \chi_2(a) + \overline{\psi}(a)) e\left(\frac{ma}{p}\right) \\ &= \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1} \psi(a) e\left(\frac{ma}{p}\right) \\ &\quad + \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1} \overline{\psi}(a) e\left(\frac{ma}{p}\right) \\ &= \chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi}). \end{aligned} \tag{3.1}$$

Let us start by proving Theorem 1.

When $p \equiv 5 \bmod 8$, we have $\psi(-1) = -1$. Utilizing the property $\overline{\tau(\psi)} = \psi(-1)\tau(\overline{\psi}) = -\tau(\overline{\psi})$, it follows that in this scenario, $\overline{\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})} = -(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi}))$. Additionally, since $\chi_2(m)\sqrt{p}$ is a real number; based on (3.1), we obtain

$$\overline{G(m, 4; p)} = \chi_2(m)\sqrt{p} - \overline{\psi}(m)\tau(\psi) - \psi(m)\tau(\overline{\psi}). \tag{3.2}$$

In this particular case, according to (3.1) and (3.2), we derive

$$\begin{aligned} |G(m, 4; p)|^2 = G(m, 4; p) \cdot \overline{G(m, 4; p)} &= p - \left(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right)^2 \\ &= 3p - \chi_2(m)(\tau^2(\psi) + \tau^2(\overline{\psi})). \end{aligned} \tag{3.3}$$

Now, considering (3.3) and incorporating Lemmas 2, 4, and 5, we obtain

$$\begin{aligned}
T_2(p) &= \sum_{m=1}^{p-1} |G(m,4;p)|^2 \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} \left(3p - \chi_2(m)(\tau^2(\psi)+\tau^2(\overline{\psi}))\right) \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= 3p \sum_{m=1}^{p-1} \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 - (\tau^2(\psi)+\tau^2(\overline{\psi})) \sum_{m=1}^{p-1} \chi_2(m) \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= 3p(2p^2 - 2p^{\frac{3}{2}} - 5p + 2p^{\frac{1}{2}} - 1) - 2\sqrt{p}\alpha\,(2\sqrt{p}(1+\alpha-\sqrt{p})) \\
&= 6p^3 - 6p^{\frac{5}{2}} - 15p^2 + 6p^{\frac{3}{2}} - 3p - 4p\alpha(1+\alpha-\sqrt{p}).
\end{aligned}$$

This proves Theorem 1.

When $p \equiv 1 \bmod 8$, it holds that $\psi(-1) = 1$. For any integer m with $(m, p) = 1$, as stated in (3.1), we establish that $\overline{G(m,4;p)} = G(m,4;p)$, implying that $G(m,4;p)$ is a real number, then

$$\begin{aligned}
T_2(p) &= \sum_{m=1}^{p-1} |G(m,4;p)|^2 \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} \left(\chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right)^2 \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} \Big(p + \overline{\psi}^2(m)\tau^2(\psi) + \psi^2(m)\tau^2(\overline{\psi}) + 2\tau(\psi)\tau(\overline{\psi}) \\
&\qquad + 2\chi_2(m)\sqrt{p}(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi}))\Big) \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&= 3p \sum_{m=1}^{p-1} \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&\qquad + (\tau^2(\psi)+\tau^2(\overline{\psi})) \sum_{m=1}^{p-1} \chi_2(m) \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 \\
&\qquad + 2\sqrt{p} \sum_{m=1}^{p-1} (\psi(m)\tau(\psi) + \overline{\psi}(m)\tau(\overline{\psi})) \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2 .
\end{aligned}$$

Since $\tau(\psi)\tau(\overline{\psi}) = \psi(-1)p = p$, combining with Lemmas 2, 3, 4, and 5, we obtain

$$T_2(p) = \sum_{m=1}^{p-1} |G(m,4;p)|^2 \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4+a^2}{p}\right)\right|^2$$

$$
\begin{aligned}
&= 3p(2p^2 + 2p^{\frac{3}{2}} - 5p + 2p^{\frac{1}{2}} - 1) + 2\sqrt{p}\alpha(2\sqrt{p}(1 - \alpha - \sqrt{p}))\\
&\quad + 2\sqrt{p}\tau(\psi)\sum_{m=1}^{p-1}\psi(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right)\right|^2\\
&\quad + 2\sqrt{p}\tau(\overline{\psi})\sum_{m=1}^{p-1}\overline{\psi}(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right)\right|^2\\
&= 3p(2p^2 + 2p^{\frac{3}{2}} - 5p + 2p^{\frac{1}{2}} - 1) + 2\sqrt{p}\alpha(2\sqrt{p}(1 - \alpha - \sqrt{p}))\\
&\quad + 2\sqrt{p}\tau(\psi)\left(2\left(\tau(\psi) - \tau(\overline{\psi})\sqrt{p}\right) + \tau(\psi)\sqrt{p}\gamma\right)\\
&\quad + 2\sqrt{p}\tau(\overline{\psi})\left(2\left(\tau(\overline{\psi}) - \tau(\psi)\sqrt{p}\right) + \tau(\overline{\psi})\sqrt{p}\overline{\gamma}\right)\\
&= 3p(2p^2 + 2p^{\frac{3}{2}} - 5p + 2p^{\frac{1}{2}} - 1) + 2\sqrt{p}\alpha(2\sqrt{p}(1 - \alpha - \sqrt{p}))\\
&\quad + 4\sqrt{p}\tau^2(\psi) - 4p\tau(\psi)\tau(\overline{\psi}) + 2p\tau^2(\psi)\gamma + 4\sqrt{p}\tau^2(\overline{\psi}) - 4p\tau(\psi)\tau(\overline{\psi}) + 2p\tau^2(\overline{\psi})\overline{\gamma}\\
&= 6p^3 + 6p^{\frac{5}{2}} - 23p^2 + 6p^{\frac{3}{2}} - 3p + 4p\alpha(3 - \alpha - \sqrt{p}) + 2p(\tau^2(\psi)\gamma + \tau^2(\overline{\psi})\overline{\gamma}).
\end{aligned}
$$

This proves Theorem 2.

When $p \equiv 3 \bmod 4$, as stated in (1.2) and (1.4), we obtain

$$
\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^{2k} \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right)\right|^4 = p^{k+1} \cdot (7p^2 - 14p - 5).
$$

This completes the proof of Theorem 3.

For $p \equiv 5 \bmod 8$, considering $|\alpha| \leq \sqrt{p}$ and Theorem 1, we obtain

$$
T_2(p) = \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right)\right|^2 = 6p^3 - 6p^{\frac{5}{2}} + O(p^2).
$$

This completes the proof of Corollary 1.

In the case of $p \equiv 1 \bmod 8$, where $\left|\tau^2(\psi)\right| = p$, combining with Lemma 6 and Theorem 2, we have

$$
T_2(p) = \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma^4 + a^2}{p}\right)\right|^2 = 6p^3 + O(p^{\frac{5}{2}}).
$$

This completes the proof of all our theorems and corollaries.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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