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Research article

Stability and pointwise-in-time convergence analysis of a finite difference scheme for a 2D nonlinear multi-term subdiffusion equation

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Abstract: In this paper, we aim to study the stability and convergence of a finite difference scheme for solving the two-dimensional nonlinear multi-term time fractional subdiffusion equation with weakly singular solutions. We apply the L1 scheme to discretize the multi-term temporal Caputo derivatives, a standard central difference method in space, and a backward formula to approximate the nonlinear term on the uniform mesh, respectively. Stability and pointwise-in-time error estimates are obtained for the fully discrete scheme. The global convergence order is α_1, and the local convergence order is 1 in the temporal direction. The theoretical analysis is verified by some numerical results.

Keywords: multi-term time fractional; nonlinear subdiffusion; L1 scheme; pointwise-in-time error estimate

1. Introduction

The following two-dimensional nonlinear multi-term time-fractional subdiffusion equations with initial and boundary conditions are considered in this paper:

$$D_t^{\alpha} u = \Delta u + f(u),\ (x, y) \in \Omega,\ 0 < t \leq T, \tag{1.1}$$

$$u(x, y, 0) = g(x, y),\ (x, y) \in \Omega, \tag{1.2}$$

$$u(x, y, t) = 0,\ (x, y) \in \partial\Omega,\ 0 \leq t \leq T, \tag{1.3}$$

where $\Omega = (0, L_1) \times (0, L_2)$ with the boundary $\partial\Omega$, $f \in C^1(\mathbb{R})$, and g is a continuous function in Ω. The operator D_t^{α} in (1.1) is defined by

$$D_t^{\alpha} = \sum_{l=1}^{J} b_l D_t^{\alpha_l},\ 0 < \alpha_J < \alpha_{J-1} < \cdots < \alpha_1 < 1,\ \mathit{and}\ b_l > 0,$$

where $D_t^{\alpha_l}$ denotes the Caputo derivative with respect to t, i.e.,

$$D_t^{\alpha_l} u = \frac{1}{\Gamma(1-\alpha_l)} \int_0^t (t-s)^{-\alpha_l} \frac{\partial u(x,y,s)}{\partial s} ds.$$

In order to describe the physical process better, some scholars use multi-term time fractional differential equations for modeling, such as the behavior of viscoelastic fluids [1, 2], the dispersion of pollutants [3], and magnetic resonance imaging [4]. The finite difference method [5–8], the finite volume method [9, 10], the fractional predictor-corrector method [11], the finite element method [12], the collocation method [13, 14], and the spectral method [15] have been developed for solving the multi-term time fractional equations. The work [16] proposed a fast linearized finite difference method for the nonlinear multi-term time-fractional wave equation using SOE approximation. Recently, a lot of work about the time fractional nonlinear subdiffusion equation has been published. For example, the paper [17] considered a Newton linearized Galerkin finite element method to solve the problem with non-smooth solutions in the time direction and provided an optimal error estimate by using the discrete fractional Gronwall-type inequality on the graded meshes. The paper [18] provided an efficient systematic framework for solving nonlinear fractional partial differential equations on unbounded domains and obtained an error estimate. Jiang et al. [19] proposed an efficient ADI scheme for the nonlinear subdiffusion equation with a weakly singular solution and obtained the pointwise-in-time error estimate. Li et al. [20] proposed a new tool, the refined discrete fractional-type Grönwall inequality, which is used to derive a sharp pointwise-in-time error estimate of the L1 scheme for the problem. However, as far as the authors know, at present there is no work dedicated to the pointwise-in-time error estimate of the L1 scheme for the nonlinear multi-term time-fractional subdiffusion equation with weakly singular solutions. The present work is designed to fill this gap. Following [21, 22], in the remainder of our paper, we make the following assumption:

Assumption 1.1. *For all* $(x, y, t) \in \bar{\Omega} \times (0, T]$, *we assume the solution satisfies*

$$\left| \frac{\partial^l u(x,y,t)}{\partial x^p \partial y^q} \right| \le C, \ l = 0, 1, 2, 3, 4, 5, \ l = p + q, \tag{1.4}$$

$$\left| \frac{\partial^k u(x,y,t)}{\partial t^k} \right| \le C(1 + t^{\alpha_1 - k}), \ k = 0, 1, 2, \tag{1.5}$$

where C is a positive constant. C in this paper represents a constant independent of time and space step, and C in different positions represents different values.

This paper consists of the following sections. In Section 2, we construct a fully discrete scheme for the problem (1.1)–(1.3). In Section 3, some lemmas are introduced that will be used in the subsequent analysis. In Section 4, convergence and stability of the fully discrete scheme proposed in Section 2 are given, and we obtain the sharp pointwise-in-time error estimate. In Section 5, the theoretical analysis is verified by four numerical examples.

2. Fully discrete scheme

Let M_1, M_2, and N be three positive integers. Divide space and time uniformly into $M_1 \times M_2$ and N parts, respectively. Let $\{t_n | t_n = n\tau, \ 0 \le n \le N\}$ be a uniform partition of $[0, T]$ with the time step

$\tau = T/N$. Let $h_1 = L_1/M_1$ and $h_2 = L_2/M_2$ be the spatial steps. So spatial grid points consist of $x_i = ih_1$ and $y_j = jh_2$, where $i = 0, 1, \cdots, M_1$ and $j = 0, 1, \cdots, M_2$. The spatial grid is represented by $\Omega' = \{(x_i, y_j) | i = 0, 1, \cdots, M_1, j = 0, 1, \cdots, M_2\}$. Let $\Omega_h = \Omega' \cap \Omega$, and $\partial\Omega_h = \Omega' \cap \partial\Omega$. Defining $u^n_{i,j} = u(x_i, y_j, t_n)$, the previous equations (1.1)–(1.3) at the grid point $\left(x_i, y_j, t_n\right)$ can be transformed into

$$D^{\alpha}_{\tau} u^n_{i,j} = \delta^2_x u^n_{i,j} + \delta^2_y u^n_{i,j} + f(u^{n-1}_{i,j}) + \sum_{l=1}^{J} b_l (r_l)^n_{i,j} + R^n_{i,j} + (\bar{r})^n_{i,j},\ (x_i, y_j) \in \Omega_h,\ 0 < t_n \leq T, \tag{2.1}$$

$$u^0_{i,j} = g(x_i, y_j),\ (x_i, y_j) \in \Omega_h, \tag{2.2}$$

$$u^n_{i,j} = 0,\ (x_i, y_j) \in \partial\Omega_h,\ 0 \leq t_n \leq T, \tag{2.3}$$

where the notations in (2.1) will be given afterwards.

Applying an L1 formula for the multi-term time Caputo derivatives, i.e., set

$$D^{\alpha_l}_{\tau} u^n_{i,j} = d^{\alpha_l}_1 u^n_{i,j} - d^{\alpha_l}_n u^0_{i,j} - \sum_{k=1}^{n-1} (d^{\alpha_l}_k - d^{\alpha_l}_{k+1}) u^{n-k}_{i,j},$$

where $d^{\alpha_l}_k = \frac{\tau^{-\alpha_l}[k^{1-\alpha_l} - (k-1)^{1-\alpha_l}]}{\Gamma(2-\alpha_l)}$. For simplicity, defining $d_k = \sum\limits_{l=1}^{J} b_l d^{\alpha_l}_k$, it yields that

$$D^{\alpha}_{\tau} u^n_{i,j} := \sum_{l=1}^{J} b_l D^{\alpha_l}_{\tau} u^n_{i,j} = d_1 u^n_{i,j} - d_n u^0_{i,j} - \sum_{k=1}^{n-1} (d_k - d_{k+1}) u^{n-k}_{i,j}. \tag{2.4}$$

A standard second-order approximation is used to discretize $\Delta u^n_{i,j}$:

$$\begin{aligned}
\Delta u^n_{i,j} &\approx \delta^2_x u^n_{i,j} + \delta^2_y u^n_{i,j},\\
\delta^2_x u^n_{i,j} &= \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{{h_1}^2},\\
\delta^2_y u^n_{i,j} &= \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{{h_2}^2}.
\end{aligned}$$

To approximate the nonlinear term $f(u^n_{i,j})$, we utilize the backward formula, expressed as

$$f(u^n_{i,j}) = f(u^{n-1}_{i,j}) + (\bar{r})^n_{i,j}.$$

In (2.1), $(r_l)^n_{i,j}$ and $R^n_{i,j}$ are truncation errors, i.e.,

$$\begin{aligned}
(r_l)^n_{i,j} &= D^{\alpha_l}_{\tau} u^n_{i,j} - D^{\alpha_l}_{t} u^n_{i,j},\\
R^n_{i,j} &= \Delta u^n_{i,j} - (\delta^2_x u^n_{i,j} + \delta^2_y u^n_{i,j}).
\end{aligned}$$

Leaving out the truncation errors in (2.1) and replacing $u^n_{i,j}$ by $U^n_{i,j}$, the fully discrete scheme is obtained:

$$D^{\alpha}_{\tau} U^n_{i,j} = \delta^2_x U^n_{i,j} + \delta^2_y U^n_{i,j} + f(U^{n-1}_{i,j}),\ (x_i, y_j) \in \Omega_h,\ 0 < t_n \leq T, \tag{2.5}$$

$$U^0_{i,j} = g(x_i, y_j),\ (x_i, y_j) \in \Omega_h, \tag{2.6}$$

$$U^n_{i,j} = 0,\ (x_i, y_j) \in \partial\Omega_h,\ 0 \leq t_n \leq T. \tag{2.7}$$

3. Preliminary lemmas

Lemma 3.1. *For $(x_i, y_j) \in \Omega_h$, $1 \le n \le N$, it holds that*

$$\left|(\bar{r})_{i,j}^n\right| \le C\tau t_n^{\alpha_1-1}.$$

Proof. Due to the continuity of f' and the boundedness of u, $f'(\zeta)$ is bounded, where ζ is between $u_{i,j}^n$ and $u_{i,j}^{n-1}$. We obtain

$$\left|(\bar{r})_{i,j}^n\right| = \left|f(u_{i,j}^n) - f(u_{i,j}^{n-1})\right| \le C\left|u_{i,j}^n - u_{i,j}^{n-1}\right|.$$

Then we estimate the truncation error in two cases ([19], page 6). Note the condition (1.5).
Case A: $n \ge 2$

$$\begin{aligned}
\left|(\bar{r})_{i,j}^n\right| &\le C\left|u_{i,j}^n - u_{i,j}^{n-1}\right| \\
&= C\tau\left|\frac{\partial u(ih_1, jh_2, t)}{\partial t}\right|_{t=\xi} \quad (t_{n-1} < \xi < t_n) \\
&\le C\tau(1 + \xi^{\alpha_1-1}) \\
&\le C\tau(1 + t_{n-1}^{\alpha_1-1}) \\
&\le C\tau(1 + t_n^{\alpha_1-1}) \\
&\le C\tau t_n^{\alpha_1-1},
\end{aligned}$$

where we have used $t_{n-1} \simeq t_n$ for $n \ge 2$ in the penultimate inequality.
Case B: $n = 1$

$$\begin{aligned}
\left|(\bar{r})_{i,j}^1\right| &\le C\left|\int_0^{t_1} \frac{\partial u(ih_1, jh_2, s)}{\partial s} ds\right| \\
&\le C\int_0^{t_1} \left|\frac{\partial u(ih_1, jh_2, s)}{\partial s}\right| ds \\
&\le C\int_0^{t_1} (1 + s^{\alpha_1-1}) ds \\
&= C(\tau + \frac{1}{\alpha_1}\tau^{\alpha_1}) \\
&\le C\tau^{\alpha_1} = C\tau t_1^{\alpha_1-1}.
\end{aligned}$$

The proof is completed. □

Lemma 3.2. *For $(x_i, y_j) \in \Omega_h$, $1 \le n \le N$, we have* $\left|R_{i,j}^n\right| \le C(h_1^2 + h_2^2)$.

Proof. According to the Taylor expansion, we can see that

$$\begin{aligned}\left|R_{i,j}^n\right| &= \left|\Delta u_{i,j}^n - (\delta_x^2 u_{i,j}^n + \delta_y^2 u_{i,j}^n)\right| \\ &= \left|(u_{xx}(x_i, y_j, t_n) - \delta_x^2 u_{i,j}^n) + (u_{yy}(x_i, y_j, t_n) - \delta_y^2 u_{i,j}^n)\right| \\ &\le \frac{h_1^2}{12}\left|\left.\frac{\partial^4 u(x, y_j, t_n)}{\partial x^4}\right|_{x=\xi_1}\right| + \frac{h_2^2}{12}\left|\left.\frac{\partial^4 u(x_i, y, t_n)}{\partial y^4}\right|_{y=\xi_2}\right| \\ &\le C(h_1^2 + h_2^2),\end{aligned}$$

where $x_{i-1} < \xi_1 < x_{i+1}$, and $y_{j-1} < \xi_2 < y_{j+1}$. □

Lemma 3.3. *[23, Lemma 6]* $\left|\sum\limits_{l=1}^{J} b_l(r_l)_{i,j}^n\right| \le Cn^{-\min\{2-\alpha_1,\alpha_1+1\}}$, *for* $(x_i, y_j) \in \Omega_h$, $1 \le n \le N$.

Define the positive stability multipliers $\theta_0 = \mu$, $\theta_n = \mu \sum\limits_{k=1}^{n} (d_k - d_{k+1})\theta_{n-k}$, where $\mu = d_1^{-1}$.

Lemma 3.4. *([24], Lemma 5.1) θ_n is a monotonically decreasing sequence with respect to n. For $n = 0, 1, \cdots, N$,*

$$\theta_n \le b_1^{-1}\Gamma(2-\alpha_1)\tau^{\alpha_1}(n+1)^{\alpha_1-1}.$$

Notation. θ_n here has the following relationship with σ_n^α in Literature [24]: $\theta_n = \sigma_n^\alpha$.

Lemma 3.5. *([24], Lemma 5.2, Corollary 5.1) For $n = 1, 2, \cdots, N$,*

$$\sum_{j=1}^{n} j^{-\beta}\theta_{n-j} \le b_1^{-1}\Gamma(2-\alpha_1)\tau^{\alpha_1}[\kappa_{\beta,n}(\frac{n}{2})^{\alpha_1-1} + \frac{1}{\alpha_1}(\frac{n}{2})^{\alpha_1-\beta}],$$

where $\beta \ge 0$ and

$$\kappa_{\beta,n} = \begin{cases} 1 + \frac{1-n^{1-\beta}}{\beta-1}, & for\ \beta \ne 1, \\ 1 + \ln n, & for\ \beta = 1. \end{cases}$$

If there exists $\underline{\alpha}$, which satisfies $0 < \underline{\alpha} \leqslant \alpha_1 < 1$, we have

$$\sum_{j=1}^{n} j^{-\beta}\theta_{n-j} \le \begin{cases} \hat{C} & for\ \beta = 0, \\ \hat{C}K_{\beta,n}\tau t_n^{\alpha_1-1} & for\ \beta > 1, \end{cases}$$

where constant $\hat{C}$ depends on $\underline{\alpha}$ and T.

Lemma 3.6. *([25], Lemma 5.1) If $\{y_n\}$ is a non-negative sequence, and it satisfies $y_n \le a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2} + b\tau \sum\limits_{j=1}^{n-1} t_{n-j}^{\alpha-1} y_j$, $1 \le n \le N$, we get $y_n \le C(a_1 t_n^{-\eta_1} + a_2 t_n^{-\eta_2})$, $1 \le n \le N$, where $0 \le \eta_1, \eta_2 < 1$, $a_1, a_2, b > 0$, $0 < \alpha < 1$ and N is a positive integer.*

Suppose g^j is an arbitrary mesh function; we define an integral operator B_t^α, which satisfies $B_t^\alpha(g^0) = 0$, $B_t^\alpha(g^n) = \sum\limits_{j=1}^{n} \theta_{n-j} g^j$, for $n = 1, 2, \cdots, N$.

Lemma 3.7. *([23], Lemma 2) For any mesh function $\{U^j\}_{j=0}^N$, the following formula holds.*

$$B_t^\alpha(D_\tau^\alpha U^n) = U^n - U^0,\ for\ n = 1, 2, \cdots, N.$$

4. Convergence and stability analysis

Theorem 4.1. *If $U_{i,j}^n$ is the solution of* (2.5)–(2.7) *at point (x_i, y_j, t_n), and suppose $\left|U_{i_0,j_0}^n\right| = \|U^n\|_\infty$, it holds that*

$$\left|U_{i_0,j_0}^n\right| \le \left|U_{i_0,j_0}^0\right| + \sum_{k=1}^{n} \theta_{n-k} \left|f(U_{i_0,j_0}^{k-1})\right|, \ for\ n \ge 0. \tag{4.1}$$

Proof. From (2.5), it is easy to see that $U_{i,j}^n$ satisfies

$$\begin{aligned}&(\sum_{l=1}^{J} b_l d_1^{\alpha_l} + \frac{2}{h_1^2} + \frac{2}{h_2^2}) \left|U_{i,j}^n\right| \\ =& \left|\frac{U_{i+1,j}^n + U_{i-1,j}^n}{h_1^2} + \frac{U_{i,j+1}^n + U_{i,j-1}^n}{h_2^2} + \sum_{l=1}^{J}\sum_{k=2}^{n} b_l (d_{k-1}^{\alpha_l} - d_k^{\alpha_l}) U_{i.j}^{n-k+1} + d_n U_{i,j}^0 + f(U_{i,j}^{n-1})\right|.\end{aligned}$$

Based on (2.7) and the fact that $d_{k-1}^{\alpha_l} > d_k^{\alpha_l}$, we have

$$\begin{aligned}&(\sum_{l=1}^{J} b_l d_1^{\alpha_l} + \frac{2}{h_1^2} + \frac{2}{h_2^2}) \left|U_{i_0,j_0}^n\right| \\ \le& \left|\frac{2U_{i_0,j_0}^n}{h_1^2}\right| + \left|\frac{2U_{i_0,j_0}^n}{h_2^2}\right| + \sum_{l=1}^{J}\sum_{k=2}^{n} b_l (d_{k-1}^{\alpha_l} - d_k^{\alpha_l}) \left|U_{i_0,j_0}^{n-k+1}\right| + d_n \left|U_{i_0,j_0}^0\right| + \left|f(U_{i_0,j_0}^{n-1})\right|.\end{aligned}$$

Then, we obtain

$$(\sum_{l=1}^{J} b_l d_1^{\alpha_l}) \left|U_{i_0,j_0}^n\right| \le \sum_{l=1}^{J}\sum_{k=2}^{n} b_l (d_{k-1}^{\alpha_l} - d_k^{\alpha_l}) \left|U_{i_0,j_0}^{n-k+1}\right| + d_n \left|U_{i_0,j_0}^0\right| + \left|f(U_{i_0,j_0}^{n-1})\right|,$$

which, according to the definition of D_τ^α, is equivalent to the following expression:

$$D_\tau^\alpha \left|U_{i_0,j_0}^n\right| \le \left|f(U_{i_0,j_0}^{n-1})\right|, \ 1 \le n \le N. \tag{4.2}$$

By applying the integral operator to both sides of the above formula, we obtain

$$B_t^\alpha (D_\tau^\alpha \left|U_{i_0,j_0}^n\right|) \le B_t^\alpha \left|f(U_{i_0,j_0}^{n-1})\right|.$$

Due to Lemma 3.7 and the above definition of B_t^α, (4.1) holds for $n \ge 1$. Obviously, (4.1) also holds for $n = 0$. □

Theorem 4.2. *Suppose that u is the solution of* (1.1)–(1.3) *and satisfies Assumption 1.1, and that U is the solution of* (2.5)–(2.7). *There exist positive constants τ_0 and h_0. When $\tau < \tau_0$ and $h_1, h_2 < h_0$, it holds for $m > 0$ that*

$$\|u^m - U^m\|_\infty \le C(\tau t_m^{\alpha_1 - 1} + h_1^2 + h_2^2). \tag{4.3}$$

Proof. For $m = 0$, (4.3) holds obviously. Assuming that (4.3) holds for $m = 0, 1, 2 \cdots n-1 (n \geq 1)$, then for sufficiently small τ, h_1, h_2, and $1 \leq k \leq n$, we obtain

$$\begin{aligned}\left\|U^{k-1}\right\|_\infty &\leq \left\|u^{k-1}\right\|_\infty + \left\|u^{k-1} - U^{k-1}\right\|_\infty \\ &\leq \left\|u^{k-1}\right\|_\infty + 1.\end{aligned}$$

Let us discuss whether the inequality (4.3) holds when $m = n$. Let $e^m_{i,j} = u^m_{i,j} - U^m_{i,j}$ for $0 \leq m \leq n$. Subtracting (2.5)–(2.7) from (2.1)–(2.3), we obtain

$$\begin{aligned}&D^\alpha_\tau e^m_{i,j} = \delta^2_x e^m_{i,j} + \delta^2_y e^m_{i,j} + (R_f)^m_{i,j} + \sum_{l=1}^{J} b_l(r_l)^m_{i,j} + R^m_{i,j} + (\bar{r})^m_{i,j},\ (x_i, y_j) \in \Omega_h,\ 0 < t_m \leq T,\\ &e^0_{i,j} = 0,\ (x_i, y_j) \in \Omega_h,\\ &e^m_{i,j} = 0,\ (x_i, y_j) \in \partial\Omega_h,\ 0 \leq t_m \leq T,\end{aligned}$$

where $(R_f)^m_{i,j} = f(u^{m-1}_{i,j}) - f(U^{m-1}_{i,j})$. Considering the continuity of f', the boundedness of $u^{m-1}_{i,j}$ and $U^{m-1}_{i,j}$, we have

$$\left|(R_f)^m_{i,j}\right| = \left|f(u^{m-1}_{i,j}) - f(U^{m-1}_{i,j})\right| \leq C\left|e^{m-1}_{i,j}\right|. \tag{4.4}$$

Let $\left|e^m_{i_0,j_0}\right| = \|e^m\|_\infty$. Using Lemmas 3.1–3.3 and inequality (4.4), similar to (4.2), it can be obtained that for $1 \leq m \leq n$,

$$D^\alpha_\tau \left|e^m_{i_0,j_0}\right| \leq C_1\left|e^{m-1}_{i_0,j_0}\right| + C_2 m^{-\min\{2-\alpha_1,\alpha_1+1\}} + C_3(h_1^2 + h_2^2) + C_4\tau t_m^{\alpha_1-1}. \tag{4.5}$$

Applying the definition of B^α_t again, and Lemmas 3.4 and 3.5, we can further obtain

$$\begin{aligned}\left|e^m_{i_0,j_0}\right| &\leq C_1\sum_{k=1}^{m}\theta_{m-k}\left|e^{k-1}_{i_0,j_0}\right| + C_2\sum_{k=1}^{m}\theta_{m-k}k^{-\min\{2-\alpha_1,1+\alpha_1\}}\\ &+ C_3\sum_{k=1}^{m}\theta_{m-k}(h_1^2+h_2^2) + C_4\tau^{\alpha_1}\sum_{k=1}^{m}\theta_{m-k}k^{\alpha_1-1}\\ &= C_1\sum_{k=1}^{m-1}\theta_{m-k-1}\left|e^{k}_{i_0,j_0}\right| + C_2\sum_{k=1}^{m}\theta_{m-k}k^{-\min\{2-\alpha_1,1+\alpha_1\}}\\ &+ C_3\hat{C}(h_1^2+h_2^2) + C_4\tau^{\alpha_1}\sum_{k=1}^{m}\theta_{m-k}k^{\alpha_1-1}\\ &\leq C_1 b_1^{-1}\Gamma(2-\alpha_1)\tau^{\alpha_1}\sum_{k=1}^{m-1}(m-k)^{\alpha_1-1}\left|e^{k}_{i_0,j_0}\right|\\ &+ C_2\hat{C}\kappa_{\sigma_1,m}\tau t_m^{\alpha_1-1} + C_3\hat{C}(h_1^2+h_2^2)\\ &+ C_4\tau^{2\alpha_1}b_1^{-1}\Gamma(2-\alpha_1)\left[\left(1-\frac{1}{\alpha_1}\right)\left(\frac{m}{2}\right)^{\alpha_1-1} + \frac{m^{2\alpha_1-1}}{\alpha_1 2^{\alpha_1-1}} + \frac{1}{\alpha_1}\left(\frac{m}{2}\right)^{2\alpha_1-1}\right]\\ &\leq C\left(\tau^{\alpha_1}\sum_{k=1}^{m-1}(m-k)^{\alpha_1-1}\left|e^{k}_{i_0,j_0}\right| + \tau t_m^{\alpha_1-1} + (h_1^2+h_2^2)\right),\end{aligned}$$

where $\sigma_1 = \min\{2-\alpha_1, \alpha_1+1\}$. From Lemma 3.6, we have

$$\left|e_{i_0,j_0}^m\right| \le C(\tau t_m^{\alpha_1-1} + h_1^2 + h_2^2),\ for\ 1 \le m \le n. \tag{4.6}$$

In summary, the inequality (4.3) holds for $m = n$, which finishes the mathematical induction. The proof is complete. □

Remark 4.1. *The global maximum error of the numerical solution is*

$$\max_{1\le n\le N} \|u^n - U^n\|_\infty \le C(\tau^{\alpha_1} + h_1^2 + h_2^2).$$

When t_n is away from 0, the local maximum error is

$$\|u^n - U^n\|_\infty \le C(\tau + h_1^2 + h_2^2).$$

Theorem 4.3. *The fully discrete scheme* (2.5)–(2.7) *is stable with respect to the initial value. If $\hat{U}_{i,j}^n$ satisfies the following equations,*

$$\sum_{l=1}^{J} b_l D_\tau^{\alpha_l} \hat{U}_{i,j}^n = \delta_x^2 \hat{U}_{i,j}^n + \delta_y^2 \hat{U}_{i,j}^n + f(\hat{U}_{i,j}^{n-1}),\ (x_i, y_j) \in \Omega_h,\ 0 < t_n \le T, \tag{4.7}$$

$$\hat{U}_{i,j}^0 = \hat{g}(x_i, y_j),\ (x_i, y_j) \in \Omega_h, \tag{4.8}$$

$$\hat{U}_{i,j}^n = 0,\ (x_i, y_j) \in \partial\Omega_h,\ 0 \le t_n \le T, \tag{4.9}$$

and $\|g - \hat{g}\|_\infty$ is sufficiently small, then

$$\|\hat{e}^n\|_\infty \le C\left\|\hat{e}^0\right\|_\infty,\ for\ n \ge 0, \tag{4.10}$$

where $\hat{e}_{i,j}^n = U_{i,j}^n - \hat{U}_{i,j}^n$.

Proof. Subtracting (4.7)–(4.9) from (2.5)–(2.7) and according to Theorem 4.1, we have

$$\left|\hat{e}_{i_0,j_0}^n\right| \le \left|\hat{e}_{i_0,j_0}^0\right| + \sum_{k=1}^{n} \theta_{n-k} \left|f(U_{i_0,j_0}^{k-1}) - f(\hat{U}_{i_0,j_0}^{k-1})\right|,$$

where $\hat{e}_{i_0,j_0}^n = \|\hat{e}^n\|_\infty$. (4.10) holds for $n = 0$ obviously. Suppose that (4.10) holds when $n = 0, 1, 2, \cdots, m-1 (m \ge 1)$, we have $\left\|\hat{U}^r\right\|_\infty \le \|\hat{e}^r\|_\infty + \|U^r\|_\infty \le C\left\|\hat{e}^0\right\|_\infty + \|U^r\|_\infty \le 1 + \|U^r\|_\infty$ on the condition that $\left\|\hat{e}^0\right\|_\infty$ is small, for $r \le m-1$. According to (4.3), $U_{i,j}^r$ is bounded. Then, combining the continuity of f' and the boundedness of $\hat{U}_{i,j}^r$, it holds that

$$\begin{aligned}
\left|\hat{e}_{i_0,j_0}^m\right| &\le \left|\hat{e}_{i_0,j_0}^0\right| + \sum_{k=1}^{m} \theta_{m-k} \left|f(U_{i_0,j_0}^{k-1}) - f(\hat{U}_{i_0,j_0}^{k-1})\right| \\
&\le \left|\hat{e}_{i_0,j_0}^0\right| + C\sum_{k=1}^{m} \theta_{m-k} \left|\hat{e}_{i_0,j_0}^{k-1}\right| \\
&\le C\left|\hat{e}_{i_0,j_0}^0\right| + C\sum_{k=1}^{m-1} \theta_{m-k-1} \left|\hat{e}_{i_0,j_0}^{k}\right| \\
&\le C\left|\hat{e}_{i_0,j_0}^0\right| + C\tau\sum_{k=1}^{m-1} t_{m-k}^{\alpha_1-1} \left|\hat{e}_{i_0,j_0}^{k}\right|.
\end{aligned}$$

Due to Lemma 3.6, we have

$$\|\hat{e}^m\|_\infty \le C\left\|\hat{e}^0\right\|_\infty.$$

Therefore, the mathematical induction ends, and the proof is complete. □

5. Numerical experiments

Example 5.1. *We first consider the following two-term time-fractional nonlinear subdiffusion equation with $b_1 = b_2 = 1$.*

$$D_t^{\alpha_1}u + D_t^{\alpha_2}u = \Delta u + u(1-u^2) + h(x,y,t),\ (x,y)\in\Omega,\ 0<t\le T, \tag{5.1}$$

$$u(x,y,0) = g(x,y),\ (x,y)\in\Omega, \tag{5.2}$$

$$u(x,y,t) = 0,\ (x,y)\in\partial\Omega,\ 0\le t\le T, \tag{5.3}$$

where $h(x,y,t)$ and $g(x,y)$ are up to the exact solution. We set the exact solution to $t^{\alpha_1} sin(\pi x) sin(\pi y)$, which satisfies Assumption 1.1. We consider the spatial domain $\Omega = (0,1)\times(0,1)$ and set $T = 1$.

Above all, we compute the convergence order in spatial direction. We set $N = 1000$ so that the influence of errors in temporal direction can be ignored compared with errors in spatial direction. The maximum errors at $t_n = 1$ and rates, when $\alpha_1 = 0.4$ and $\alpha_2 = 0.3$, are presented in Table 5.1. Numerical results show that the spatial accuracy is $O(h_1^2 + h_2^2)$. For studying temporal convergence rates, we define global errors E_G and local errors E_L by

$$E_G = \max_{1\le n\le N}\|U^n - u^n\|_\infty,\ E_L = \left\|U^N - u^N\right\|_\infty.$$

Global errors and rates in Table 5.2 show that the global temporal convergence order is α_1. When t_n is far away from 0, results are shown in Table 5.3, and we get the local temporal accuracy $O(\tau)$. All in all, numerical results are consistent with the theoretical analysis in Remark 4.1.

Table 5.1. maximum errors at $t_n = 1$ and spatial convergence rates for Example 5.1.

$M_1 = M_2$	4	8	16	32	64
E	4.5780e-02	1.1335e-02	2.8363e-03	7.1957e-04	1.9089e-04
rate	2.0140	1.9987	1.9788	1.9144	*

Table 5.2. global maximum errors and temporal convergence rates for Example 5.1.

$N = M_1^2 = M_2^2$	$\alpha_1 = 0.3, \alpha_2 = 0.1$		$\alpha_1 = 0.5, \alpha_2 = 0.3$		$\alpha_1 = 0.7, \alpha_2 = 0.5$	
	E_G	*rate*	E_G	*rate*	E_G	*rate*
128	1.6333e-02	0.1690	1.0008e-02	0.3361	5.2012e-03	0.8184
256	1.4528e-02	0.1931	7.9278e-03	0.3660	2.9494e-03	0.6130
512	1.2708e-02	0.1926	6.1514e-03	0.3744	1.9284e-03	0.6260
1024	1.1120e-02	0.2038	4.7453e-03	0.4003	1.2496e-03	0.6518
2048	9.6547e-03	*	3.5956e-03	*	7.9532e-04	*

Table 5.3. local maximum errors at $t_n = 1$ and temporal convergence rates for Example 5.1.

$N = M_1^2 = M_2^2$	$\alpha_1 = 0.3, \alpha_2 = 0.1$		$\alpha_1 = 0.5, \alpha_2 = 0.3$		$\alpha_1 = 0.7, \alpha_2 = 0.5$	
	E_L	*rate*	E_L	*rate*	E_L	*rate*
128	5.0943e-03	0.8339	5.1615e-03	0.8359	5.2012e-03	0.8374
256	2.8579e-03	1.0536	2.8916e-03	1.0534	2.9109e-03	1.0531
512	1.3768e-03	0.9473	1.3933e-03	0.9477	1.4028e-03	0.9480
1024	7.1403e-04	1.0464	7.2235e-04	1.0460	7.2716e-04	1.0456
2048	3.4571e-04	*	3.4984e-04	*	3.5227e-04	*

Example 5.2. *Secondly, we consider a two-dimensional three-term time-fractional nonlinear subdiffusion equation with* $b_1 = b_2 = b_3 = 1$.

$$D_t^{\alpha_1}u + D_t^{\alpha_2}u + D_t^{\alpha_3}u = \Delta u + u(1-u^2) + h(x,y,t),\ (x,y) \in \Omega,\ 0 < t \le T, \tag{5.4}$$

$$u(x,y,0) = g(x,y),\ (x,y) \in \Omega, \tag{5.5}$$

$$u(x,y,t) = 0,\ (x,y) \in \partial\Omega,\ 0 \le t \le T, \tag{5.6}$$

where $\Omega = (0,1) \times (0,1)$ *and* $T = 1$. *We calculate* $h(x,y,t)$ *and* $g(x,y)$ *based on the exact solution*

$$t^{\alpha_1} sin(\pi x) sin(\pi y).$$

Numerical results are shown in Tables 5.4 and 5.5, which verify the theoretical analysis as well.

Table 5.4. global maximum errors and temporal convergence rates for Example 5.2.

$N = M_1^2 = M_2^2$	$\alpha_1 = 0.3, \alpha_2 = 0.2, \alpha_3 = 0.1$		$\alpha_1 = 0.5, \alpha_2 = 0.4, \alpha_3 = 0.3$		$\alpha_1 = 0.7, \alpha_2 = 0.6, \alpha_2 = 0.5$	
	E_G	*rate*	E_G	*rate*	E_G	*rate*
128	1.7505e-02	0.1822	1.0842e-02	0.3657	4.9545e-03	0.7240
256	1.5428e-02	0.2052	8.4150e-03	0.3942	2.9994e-03	0.6375
512	1.3382e-02	0.2040	6.4030e-03	0.4000	1.9282e-03	0.6436
1024	1.1618e-02	0.2159	4.8524e-03	0.4227	1.2342e-03	0.6639
2048	1.0003e-02	*	3.6200e-03	*	7.7897e-04	*

Table 5.5. local maximum errors at $t_n = 1$ and temporal convergence rates for Example 5.2.

$N = M_1^2 = M_2^2$	$\alpha_1 = 0.3, \alpha_2 = 0.2, \alpha_3 = 0.1$		$\alpha_1 = 0.5, \alpha_2 = 0.4, \alpha_3 = 0.3$		$\alpha_1 = 0.7, \alpha_2 = 0.6, \alpha_2 = 0.5$	
	E_L	*rate*	E_L	*rate*	E_L	*rate*
128	4.8786e-03	0.8333	4.9371e-03	0.8351	4.9545e-03	0.8362
256	2.7380e-03	1.0534	2.7675e-03	1.0532	2.7751e-03	1.0529
512	1.3193e-03	0.9470	1.3337e-03	0.9474	1.3376e-03	0.9476
1024	6.8430e-04	1.0463	6.9158e-04	1.0460	6.9355e-04	1.0455
2048	3.3134e-04	*	3.3495e-04	*	3.3600e-04	*

Example 5.3. *Thirdly, we investigate the scenario in which the nonlinear term is represented by* $f(u) = sin(u)$, *which satisfies Lipschitz condition. The corresponding equation is formulated as follows:*

$$D_t^{\alpha_1}u + D_t^{\alpha_2}u + D_t^{\alpha_3}u = \Delta u + sin(u) + h(x,y,t),\ (x,y) \in \Omega,\ 0 < t \le T, \tag{5.7}$$

$$u(x, y, 0) = g(x, y),\ (x, y) \in \Omega, \tag{5.8}$$
$$u(x, y, t) = 0,\ (x, y) \in \partial\Omega,\ 0 \le t \le T, \tag{5.9}$$

where $\Omega = (0, 1) \times (0, 1)$ *and* $T = 1$. *Similarly, we derive* $h(x, y, t)$ *and* $g(x, y)$ *based on the exact solution*

$$t^{\alpha_1} sin(\pi x) sin(\pi y).$$

The corresponding numerical results are presented in Tables 5.6 and 5.7. The global convergence order is α_1, and the local convergence order is 1 in the temporal direction.

Table 5.6. global maximum errors and temporal convergence rates for Example 5.3.

$N = M_1^2 = M_2^2$	$\alpha_1 = 0.3, \alpha_2 = 0.2, \alpha_3 = 0.1$		$\alpha_1 = 0.5, \alpha_2 = 0.4, \alpha_3 = 0.3$		$\alpha_1 = 0.7, \alpha_2 = 0.6, \alpha_2 = 0.5$	
	E_G	*rate*	E_G	*rate*	E_G	*rate*
128	1.7753e-02	0.1908	1.0852e-02	0.3664	4.8924e-03	0.7058
256	1.5554e-02	0.2101	8.4178e-03	0.3945	2.9995e-03	0.6375
1024	1.1650e-02	0.2175	4.8527e-03	0.4227	1.2342e-03	0.6639
2048	1.0019e-02	*	3.6201e-03	*	7.7897e-04	*

Table 5.7. local maximum errors at $t_n = 1$ and temporal convergence rates for Example 5.3.

$N = M_1^2 = M_2^2$	$\alpha_1 = 0.3, \alpha_2 = 0.2, \alpha_3 = 0.1$		$\alpha_1 = 0.5, \alpha_2 = 0.4, \alpha_3 = 0.3$		$\alpha_1 = 0.7, \alpha_2 = 0.6, \alpha_2 = 0.5$	
	E_L	*rate*	E_L	*rate*	E_L	*rate*
128	5.0640e-03	0.8278	4.9983e-03	0.8256	4.8924e-03	0.8229
256	2.8530e-03	1.0552	2.8204e-03	1.0557	2.7656e-03	1.0561
512	1.3730e-03	0.9454	1.3568e-03	0.9449	1.3301e-03	0.9441
1024	7.1295e-04	1.0478	7.0484e-04	1.0484	6.9132e-04	1.0489
2048	3.4485e-04	*	3.4079e-04	*	3.3415e-04	*

Example 5.4. *Finally, we consider a two-dimensional two-term time-fractional nonlinear subdiffusion equation with* $b_1 = b_2 = 1$, *whose exact solution is unknown.*

$$D_t^{\alpha_1} u + D_t^{\alpha_2} u = \Delta u + u(1 - u),\ (x, y) \in \Omega,\ 0 < t \le T, \tag{5.10}$$
$$u(x, y, 0) = \frac{1}{2} sin(\pi x) sin(\pi y),\ (x, y) \in \Omega, \tag{5.11}$$
$$u(x, y, t) = 0,\ (x, y) \in \partial\Omega,\ 0 \le t \le T, \tag{5.12}$$

where $\Omega = (0, 1) \times (0, 1)$ *and* $T = 1$.

The two-mesh method [26] is applied to compute errors and convergence rates. We take $M_1 = M_2 = 60$. E_L is redefined by

$$E_L = \left\| U^N - W^{2N} \right\|_\infty,$$

in which W^n is the numerical solution of Example 5.4 with $\tau = T/2N$. The local errors are shown in Table 5.8. The local temporal convergence rate $O(\tau)$ is consistent with Remark 4.1.

Table 5.8. local maximum errors at $t_n = 1$ and temporal convergence rates for Example 5.4.

N	$\alpha_1 = 0.3, \alpha_2 = 0.1$		$\alpha_1 = 0.5, \alpha_2 = 0.3$		$\alpha_1 = 0.7, \alpha_2 = 0.5$	
	E_L	*rate*	E_L	*rate*	E_L	*rate*
32	6.8318e-05	1.0203	1.2027e-04	1.0285	1.4785e-04	1.0455
64	3.3682e-05	1.0103	5.8957e-05	1.0154	7.1633e-05	1.0270
128	1.6722e-05	1.0053	2.9166e-05	1.0086	3.5152e-05	1.0173
256	8.3302e-06	1.0027	1.4496e-05	1.0050	1.7367e-05	1.0117
512	4.1572e-06	*	7.2231e-06	*	8.6132e-06	*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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