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Research article

On the inverse stability of $z^n + c$

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Abstract: Let $\phi(z) = z^d + c$ be a polynomial over a field K. We study the inverse stability of $\phi(z)$ over K. In this paper, we establish some sufficient conditions for the inverse stability of $\phi(z)$ over the field of rational numbers and a function field. Furthermore, we also provide necessary and sufficient conditions for the inverse stability of $\phi(z)$ over a finite field.

Keywords: stability; polynomial irreducibility; iterations; arithmetic dynamics

1. Introduction

Arithmetic dynamics focuses on number-theoretic and algebraic-geometric problems arising from iteration. Let K be a field. A polynomial $\phi(z) \in K[z]$ is said to be stable if all its iterates are irreducible over K. This concept originated in 1985 with R.W.K. Odoni [1], who used the Chebotarev density theorem to provide an asymptotic estimate for the number of prime divisors in the sequence defined by $a_{n+1} = \phi(a_n)$, with $a_1 = 2$ and $\phi(x) = x^2 - x + 1 \in \mathbb{Z}[x]$. The application of the Chebotarev density theorem requires studying the Galois groups of iterated polynomials, which in turn necessitates understanding the reducibility of polynomial iterates.

Subsequently, the concept of stability has been extensively developed by numerous researchers. For instance, the stability of quadratic polynomials (see, e.g., [2–5]), binomial polynomials (see, e.g., [6–8]), trinomial polynomials [9], Eisenstein polynomials [10], and the estimation of the number of stable polynomials over finite fields have all been studied extensively (see, e.g., [3,11,12]).

In 2017, R. Jones and L. Alon generalized the concept of polynomial stability to eventual stability, and they also gave some applications of eventual stability. We refer the reader to [13] for more details.

Recently, in 2024, K. Cheng [14] introduced a related concept called inversely stable polynomials (see Definition 2.1). Cheng demonstrated that a polynomial $\phi(z) = z^p + az + b \in \mathbb{F}_p[z]$ is inversely stable over $\mathbb{F}_p$ if and only if $a = -1$ and $b \neq 0$.

Moreover, it is straightforward to verify that if $\phi(z) \in K[z]$ is inversely stable over K, then $\left(\frac{1}{\phi(z)}, \infty\right)$ is eventually stable over K.

In arithmetic dynamics, the iterative behavior of binomial polynomials is an important topic. For example, it is closely related to problems such as the finiteness of primitive prime divisors in orbits generated by binomial polynomials (see [15]), height estimates for binomial polynomials (see [16]), and the existence of rational periodic points of binomial polynomials (see [17], Theorem 4).

In this paper, we establish some sufficient conditions for the binomial polynomial $\phi(z) = z^d + c$ to be inversely stable over the rational number field and function fields. Furthermore, we also provide necessary and sufficient conditions for the inverse stability of $\phi(z)$ over a finite field.

2. Main results

We employ norm maps to reduce problems over larger fields to base fields. By using properties of unique factorization domains, the abc theorem for function fields, and character sum theory over finite fields, we provide characterizations of inverse stability for binomial polynomials over three types of fields. In this section, we shall formally state these results.

Definition 2.1. *Let K be a field and $\phi(z) \in K[z]$. Define $\Phi(z) := \frac{1}{\phi(z)} \in K(z)$. For $n \in \mathbb{N}^*$, let the n-th iterate of $\Phi(z)$ be defined as $\Phi^{(n)}(z) = \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{n \text{ times}}(z)$. We express the $\Phi^{(n)}(z)$ in its reduced form as $\Phi^{(n)}(z) = \frac{f_{n,\phi}(z)}{g_{n,\phi}(z)}$, where $f_{n,\phi}(z)$ and $g_{n,\phi}(z)$ are coprime polynomials in $K[z]$. A polynomial $\phi(z) \in K[z]$ is called inversely stable over K if $g_{n,\phi}(z)$ is irreducible over K for each $n \in \mathbb{N}^*$.*

Theorem 2.2. *Let R be a unique factorization domain, and let $U(R)$ denote the unit group of R. Let $d \in \mathbb{N}^*$ with $d \geqslant 2$, and suppose $c \notin uR^p$ for all primes $p \mid d$ and $u \in U(R)$. Let K be the fraction field of R. If the polynomial $\phi(z) = z^d + c \in R[z]$ is irreducible over K, then $\phi(z)$ is inversely stable over K.*

Corollary 2.3. *Let $d \in \mathbb{N}^*$ with $d \geqslant 2$. Let $c \in \mathbb{Z}$ and $\phi(z) = z^d + c$ be irreducible over $\mathbb{Q}$. Then $\phi(z)$ is inversely stable over $\mathbb{Q}$ if*

(i) d is odd, or

(ii) d is even and c is not a square of an integer.

Remark 1. 1) *Corollary 2.3 implies that there are infinitely many inverse stable polynomials $\phi(z)$ over $\mathbb{Q}$, which thereby induce a family of eventually stable rational maps $\left(\frac{1}{\phi(z)}, \infty\right)$. Therefore, Corollary 2.3 provides data support for the "Everywhere Eventual Stability Conjecture" (refer to [13] for details).*

2) *Let S be a finite set of places of the rational number field $\mathbb{Q}$ containing the archimedean place. Let $\phi(z) = z^d + c \in \mathbb{Z}[z]$ be an irreducible polynomial over $\mathbb{Q}$ with $d \geqslant 2$. Let $\Phi(z) = \frac{1}{\phi(z)}$. Suppose that either d is odd, or d is even and c is not a square of an integer. By Theorem 3.1 of [13] and Corollary 2.3, we obtain that for every $\gamma \in \mathbb{P}^1(\mathbb{Q})$ that is not preperiodic under Φ, the set $O_{S,\gamma} \cap O_\Phi^-(\infty)$ is finite, where $O_{S,\gamma}$ is the ring of S-integers relative to γ, and $O_\Phi^-(\infty)$ denotes the backward orbit of ∞ under Φ.*

When K is a rational function field, we obtain the following results.

Theorem 2.4. *Let $K = F(t)$ be the rational function field in one variable over a field F of characteristic 0. Let $d \geqslant 3$, and let $c \in R = F[t]$ with $c \notin F$. Suppose $\phi(z) = z^d + c$ is irreducible over K. Then $\phi(z)$ is inversely stable over K.*

Before stating our results concerning inverse stability over finite fields, we recall the definition of m-free.

Definition 2.5. ([18], Definition 5.1) Let $\mathbb{F}_q$ be a finite field of q elements and let m be a positive integer such that $m \mid q-1$. We say that an elelment $\alpha \in \mathbb{F}_q^*$ is m-free if the equality $\alpha = \beta^d$ with $\beta \in \mathbb{F}_q$, for any divisor d of m, implies $d = 1$.

In this paper, we also provide necessary and sufficient conditions for the inverse stability of $\phi(z)$ over a finite field.

Theorem 2.6. *Let $K = \mathbb{F}_q$ be a finite field of q elements. Let $\phi(z) = z^d + c \in K[z]$, where $d \geqslant 2$. Suppose that $\phi(z)$ is irreducible over K. Define the sequence:*

$$x_1 = c, \quad x_2 = (-1)^d(c^{d+1}+1), \quad x_{n+2} = (-1)^d c x_{n+1}^d + x_n^{d^2}, \quad n \in \mathbb{N}^*.$$

Then $\phi(z)$ is inversely stable over K if and only if $\frac{x_{n+1}}{x_n}$ is $\mathrm{rad}(d)$-free for every $n \in \mathbb{N}^*$, where $\mathrm{rad}(d) = \prod_{\substack{p \mid d,\\ p \text{ is a prime}}} p$.

Corollary 2.7. *Let $p = 2^{2^n}+1$ be a Fermat prime and $d = 2^{2^n-1}, n \geqslant 2$. Then there are at least $2^{2^n-3} - 2^{2^{n-1}-2}$ distinct values of $c \in \mathbb{F}_p$ such that $z^d + c$ is inversely stable over $\mathbb{F}_p$.*

Remark 2. *Constructing a family of irreducible polynomials over finite fields is an important topic in the area of finite fields* (see [14]). *Note that if $\phi(z)$ is inversely stable over $\mathbb{F}_q$, then $g_{n,\phi}(z)$ is irreducible over $\mathbb{F}_q$ for each $n \in \mathbb{N}^*$. Therefore, Corollary* 2.7 *has constructed a family of irreducible polynomials for each inversely stable polynomial x^d+c over $\mathbb{F}_p$, where $p = 2^{2^n}+1$ is a Fermat prime and $d = 2^{2^n-1}, n \geqslant 2$.*

This paper is organized as follows: In Section 3, we shall give the proof of Theorem 2.2. In Section 4, we shall give the proof of Theorem 2.4. In Section 5, we shall give the proof of Theorem 2.6 and Corollary 2.7.

3. Proofs of theorem 2.2 and corollary 2.3

In this section, we give the proofs of Theorem 2.2 and Corollary 2.3. First, we shall prove some lemmas that will be used in the proofs of our main results. Let K be a field. A rational function $\varphi(z) = \frac{f(z)}{g(z)} \in K(z)$ is a quotient of polynomials $f(z), g(z) \in K[z]$ with no common factors. The degree of φ is $\deg\varphi = \max\{\deg f, \deg g\}$. The rational function φ of degree d induces a rational map (morphism) of the projective space $\mathbb{P}^1(\overline{K})$,

$$\varphi : \mathbb{P}^1(\overline{K}) \longrightarrow \mathbb{P}^1(\overline{K}), \ \ \varphi([X:Y]) = \left[Y^d f(X/Y) : Y^d g(X/Y)\right].$$

A point $P \in \mathbb{P}^1(\overline{K})$ is said to be periodic under φ if $\varphi^{(n)}(P) = P$ for some $n \geqslant 1$.

Lemma 3.1. *Let $d \in \mathbb{N}^*$, and let K be a field such that $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > 0$ with $\mathrm{char}(K)$ prime to d. Consider the polynomial $\phi(z) = z^d + c \in K[z], c \neq 0$ and define the rational function $\Phi(z) := \frac{1}{\phi(z)} \in K(z)$. For each $n \in \mathbb{N}^*$, denote the n-th iterate of Φ by $\Phi^{(n)}(z) = \frac{f_{n,\phi}(z)}{g_{n,\phi}(z)}$, where $f_{n,\phi}(z)$ and $g_{n,\phi}(z) \in K[z]$ are coprime polynomials. If $\infty = [1:0] \in \mathbb{P}^1(K)$ is not periodic under $\Phi(z)$, then for any $n \in \mathbb{N}^*$, the degree of $g_{n,\phi}(z)$ is d^n.*

Proof. Note that the map $\Phi^{(n)} : \mathbb{P}^1(\overline{K}) \to \mathbb{P}^1(\overline{K})$ is given by

$$\Phi^{(n)}([X:Y]) = \left[Y^e f_{n,\phi}\left(\frac{X}{Y}\right) : Y^e g_{n,\phi}\left(\frac{X}{Y}\right)\right],$$

where $e = \deg \Phi^{(n)}(z)$.

It follows that $\Phi^{(n)}([\alpha : 1]) = \left[f_{n,\phi}(\alpha) : g_{n,\phi}(\alpha)\right]$, and hence

$$\Phi^{(n)}([\alpha : 1]) = \infty \quad \text{if and only if} \quad g_{n,\phi}(\alpha) = 0.$$

By assumption we have $\infty \notin (\Phi^{(n)})^{-1}(\infty)$ for any $n \in \mathbb{N}^*$. Thus, for $n \geqslant 1$,

$$(\Phi^{(n)})^{-1}(\infty) = \{[\alpha : 1] \in \mathbb{P}^1(\overline{K}) \mid \Phi^{(n)}([\alpha : 1]) = \infty\}.$$

Thus,

$$(\Phi^{(n)})^{-1}(\infty) = \{[\alpha : 1] \in \mathbb{P}^1(\overline{K}) \mid g_{n,\phi}(\alpha) = 0\}.$$

Next, we prove that $\#(\Phi^{(n)})^{-1}(\infty) = d^n$.

Since $\Phi([X : Y]) = [Y^d : X^d + cY^d]$, we have $\Phi(\infty) = [0 : 1]$ and $\Phi([0 : 1]) = [1 : c]$. Since $\infty \notin (\Phi^{(n)})^{-1}(\infty)$ for all n, neither $[0 : 1]$ nor $[1 : c]$ belongs to $(\Phi^{(n)})^{-1}(\infty)$.

For any $P = [1 : t] \in \mathbb{P}^1(\overline{K})$, we have

$$\Phi([X : Y]) = P \quad \text{if and only if} \quad \left[Y^d : X^d + cY^d\right] = [1 : t],$$

which simplifies to $\left(\frac{X}{Y}\right)^d + c - t = 0$. Therefore, if $t \neq c$, it follows that $\#\Phi^{-1}(P) = d$.

Hence, for any $i \in \mathbb{N}^*$, we have

$$\left|\left(\Phi^{(1)}\right)^{-1}(\infty)\right| = d \quad \text{and} \quad \left|\left(\Phi^{(i+1)}\right)^{-1}(\infty)\right| = d\left|\left(\Phi^{(i)}\right)^{-1}(\infty)\right|.$$

It follows that $\left|(\Phi^n)^{-1}(\infty)\right| = d^n$. Thus, $g_{n,\phi}$ has d^n distinct roots in $\overline{K}$. Combining this result with $\deg(g_{n,\phi}) \leqslant d^n$, we conclude that $\deg(g_{n,\phi}) = d^n$ for any $n \in \mathbb{N}^*$. □

Lemma 3.2. *Let F be a field, and let $f(z) = z^d + m \in F[z]$ be an irreducible polynomial. Denote by $\overline{F}$ the algebraic closure of F, and let $\gamma \in \overline{F}$ be a root of $f(z)$. Let $a, b, e, t \in F$ with $ae \neq 0$. We denote by $N_{F(\gamma)/F}$ the norm map associated with the field extension $F(\gamma)/F$. Then*

$$N_{F(\gamma)/F}\left(\frac{a\gamma + b}{e\gamma + t}\right) = \frac{b^d + (-1)^d m a^d}{t^d + (-1)^d m e^d}.$$

Proof. The conjugates of $\frac{a\gamma+b}{e\gamma+t}$ are $\frac{a\gamma_i+b}{e\gamma_i+t}$ for $i = 1, 2, \ldots, d$, where $\gamma_1, \gamma_2, \ldots, \gamma_d \in \overline{F}$ are the roots of $f(z) = z^d + m$. Hence, we have

$$N_{F(\gamma)/F}\left(\frac{a\gamma + b}{e\gamma + t}\right) = \prod_{i=1}^{d} \frac{a\gamma_i + b}{e\gamma_i + t}.$$

Using the fact that $f(z)=\prod_{i=1}^{d}(z-\gamma_i)$, we obtain

$$\prod_{i=1}^{d}(a\gamma_i+b)=a^d(-1)^dm+b^d,\quad \prod_{i=1}^{d}(e\gamma_i+t)=e^d(-1)^dm+t^d.$$

Therefore,

$$N_{F(\gamma)/F}\left(\frac{a\gamma+b}{e\gamma+t}\right)=\frac{b^d+(-1)^dma^d}{t^d+(-1)^dme^d}.$$

This completes the proof. □

Lemma 3.3. *Let R be a unique factorization domain and let $U(R)$ denote the unit group of R. Let $c\in R$ and $c\notin U(R)\cup\{0\}$. Let $d\in\mathbb{N}^*$ with $d\geqslant 2$. Define a sequence of matrices $\{A_j\}_{j\geqslant1}$ in $M_{2\times2}(R)$ by the following relations:*

$$A_1=\begin{bmatrix}x_1 & y_1\\ z_1 & w_1\end{bmatrix}=\begin{bmatrix}c & -1\\ 1 & 0\end{bmatrix},$$

and for $j\geqslant 1$,

$$A_{j+1}=\begin{bmatrix}x_{j+1} & y_{j+1}\\ z_{j+1} & w_{j+1}\end{bmatrix}=\begin{bmatrix}(-1)^dcx_j^d+y_j^d & (-1)^{d+1}x_j^d\\ (-1)^dcz_j^d+w_j^d & (-1)^{d+1}z_j^d\end{bmatrix}.$$

Then the following statements hold:

(i) *For all $n\geqslant1$, $x_{n+2}=(-1)^dcx_{n+1}^d+x_n^{d^2}$, $\gcd(x_{n+1},x_n)=1$, $z_{n+1}=(-1)^dx_n$.*

(ii) *For all $n\geqslant1$, $c\mid x_{2n-1}$, $c\mid(x_{2n}-(-1)^d)$, $\gcd\left(\frac{x_{2n-1}}{c},c\right)=1$.*

(iii) *If $c\notin uR^p$ for all primes $p|d$ and $u\in U(R)$, then for all $n\geqslant1$,*

$$x_{2n-1}\notin uR^p\quad\text{for all primes } p|d \text{ and } u\in U(R).$$

(iv) *∞ is not periodic under $\Phi(z)=\frac{1}{z^d+c}$ if and only if $x_n\neq0$ for all $n\geqslant1$.*

(v) *Assume that ∞ is not periodic under the map $\Phi(z)=\frac{1}{z^d+c}$. If $c\notin uR^p$ for all primes $p\mid d$ and $u\in U(R)$, then for all $n\geqslant1$, we have $\frac{x_{n+1}}{x_n}\notin\pm K^p$ for all primes $p\mid d$, where K is the fraction field of R.*

Proof. (i) and (ii) are trivial from the definition.

(iii). Assume that $x_{2n-1}=u_1r_1^{p_1}$ for some $u_1\in U(R)$, $r_1\in R$, and prime $p_1\mid d$. Then, we have

$$c\cdot\frac{x_{2n-1}}{cu_1}=r_1^{p_1}.$$

By (ii), we know that c and $\frac{x_{2n-1}}{cu_1}$ are coprime elements in R. It follows that c can be written as $c=u_2r_2^{p_1}$, where $u_2\in U(R)$ and $r_2\in R$. This contradicts the assumption.

(iv). Define the sequences $\{a_n\}_{n\in\mathbb{N}^*}$ and $\{b_n\}_{n\in\mathbb{N}^*}$ in R as follows:

$$a_1=0,\quad b_1=1,\quad a_{n+1}=b_n^d,\quad b_{n+1}=a_n^d+cb_n^d.$$

Then, $\Phi^{(n)}(\infty) = [a_n : b_n]$.

Now, observe that

$$b_{n+2} = cb_{n+1}^d + b_n^{d^2}, b_1 = 1, b_2 = c.$$

It is obvious that $x_{2n-1} = b_{2n}, x_{2n} = (-1)^d b_{2n+1}$. Hence, ∞ is not periodic under Φ if and only if $x_n \neq 0$ for all $n \geqslant 1$.

(v). Assume $\frac{x_{n+1}}{x_n} \in \pm K^p$ for some prime $p \mid d$. Since $\gcd(x_{n+1}, x_n) = 1$, it follows that there exist $u_3, u_4 \in U(R)$ such that $x_{n+1} \in u_3 R^p$ and $x_n \in u_4 R^p$. By (iii), this is impossible. □

Lemma 3.4. ([19], Theorem 8.1.6.) Let K be a field, $d \geqslant 2$ an integer, and $a \in K$. The polynomial $X^d + a$ is irreducible over K if and only if $a \notin -K^p$ for all primes p dividing d, and $a \notin 4K^4$ whenever $4 \mid d$.

With the above preparations, we can now prove Theorem 2.2.

Proof of Theorem 2.2.

Proof. Let the sequence $\{x_n\}_{n\in\mathbb{N}^*}$ be defined in Lemma 3.3.

Claim 1: ∞ is not periodic under Φ.

If $-c \in K^d$, then $z^d + c = z^d - (-c)$ is reducible over K. Therefore, $-c \notin K^d$.

It is clear that $x_1 = c \neq 0$ and $x_2 = (-1)^d(c^{d+1} + 1) \neq 0$; otherwise, we would have $-c = \left(\frac{1}{c}\right)^d \in K^d$, which contradicts the assumption $-c \notin K^d$.

Assume $x_n \neq 0$ and $x_{n+1} \neq 0$. If $x_{n+2} = 0$, then $-c = \left(-\frac{x_n^d}{x_{n+1}}\right)^d \in K^d$. This contradicts the assumption $-c \notin K^d$. Hence, $x_n \neq 0$ for all $n \in \mathbb{N}^*$.

By Lemma 3.3 (iv), ∞ is not periodic under Φ. This completes the proof of Claim 1.

Let $\{Q_i\}_{i\geqslant 1}$ be a sequence in $\mathbb{P}^1(\overline{K})$ such that $\Phi(Q_1) = \infty$ and $\Phi(Q_{i+1}) = Q_i$ for all $i \geqslant 1$. Since ∞ is not periodic under Φ, and $\Phi(\infty) = [0 : 1]$, we can express each Q_i as $Q_i = [\beta_i : 1]$, where $\beta_i \in \overline{K}$ and $\beta_i \neq 0$ for all $i \in \mathbb{N}^*$. Thus, we have

$$\phi(\beta_1) = 0, \phi(\beta_{i+1}) = \frac{1}{\beta_i}, i \geqslant 1.$$

It is obvious that β_n is a root of the polynomial $g_{n,\phi}(z)$.

Claim 2: $z^d + c - \frac{1}{\beta_n}$ is irreducible over $K(\beta_n)$ for every $n \geqslant 1$.

We shall prove this claim by induction on n.

By Lemma 3.2, we have

$$N_{K(\beta_1)/K}\left(\frac{c\beta_1 - 1}{\beta_1}\right) = \frac{c^{d+1} + 1}{c} = (-1)^d \frac{x_2}{x_1},$$

where x_1 and x_2 are as defined in Lemma 3.3.

By Lemma 3.3 (v), we deduce that $(-1)^d \frac{x_2}{x_1} \notin \pm K^p$ for all primes $p \mid d$, and hence

$$\frac{c\beta_1 - 1}{\beta_1} \notin -K(\beta_1)^p \quad \text{for all primes } p \mid d.$$

Obviously, if $4|d$ and $\frac{c\beta_1-1}{\beta_1} \in 4K(\beta_1)^4$, then $\frac{c\beta_1-1}{\beta_1} \in K(\beta_1)^2$ and $(-1)^d \frac{x_2}{x_1} \in K^2$, which also contradicts Lemma 3.3 (v). So $\frac{c\beta_1-1}{\beta_1} \notin 4K(\beta_1)^4$, when $4|d$. By Lemma 3.4, Claim 2 holds for $n = 1$.

Therefore, $[K(\beta_2):K(\beta_1)] = d$.

Assume that $[K(\beta_i):K(\beta_{i-1})] = d$ for each $2 \leqslant i \leqslant n$. We will prove that $[K(\beta_{n+1}):K(\beta_n)] = d$. This means that we will prove that $z^d + c - \frac{1}{\beta_n}$ is irreducible over the field $K(\beta_n)$.

Based on the inductive hypothesis, we know that $z^d + c - \frac{1}{\beta_j}$ is irreducible over the field $K(\beta_j)$ for each j with $1 \leqslant j \leqslant n-1$. Given a fixed j with $1 \leqslant j \leqslant n-1$. Set $F = K(\beta_j), \gamma = \beta_{j+1}, f(z) = z^d + c - \frac{1}{\beta_j}$. Then $K(\beta_{j+1}) = K(\beta_j)(\beta_{j+1}) = F(\beta_{j+1})$. Therefore, for any $l \geqslant 1$, we have

$$\begin{aligned} N_{K(\beta_{j+1})/K(\beta_j)}\left(\frac{x_l\beta_{j+1}+y_l}{z_l\beta_{j+1}+w_l}\right) &= N_{F(\gamma)/F}\left(\frac{x_l\gamma+y_l}{z_l\gamma+w_l}\right) \\ &\overset{\text{Lemma 3.2}}{=\!=\!=\!=} \frac{y_l^d+(-1)^d(c-\frac{1}{\beta_j})x_l^d}{w_l^d+(-1)^d(c-\frac{1}{\beta_j})z_l^d} \\ &= \frac{x_{l+1}\beta_j+y_{l+1}}{z_{l+1}\beta_j+w_{l+1}}. \end{aligned}$$

where x_l, y_l, z_l, and w_l are defined in Lemma 3.3.

This implies that

$$\begin{aligned} N_{K(\beta_n)/K(\beta_{n-1})}\left(\frac{x_1\beta_n + y_1}{z_1\beta_n + w_1}\right) &= \frac{x_2\beta_{n-1} + y_2}{z_2\beta_{n-1} + w_2}, \\ N_{K(\beta_{n-1})/K(\beta_{n-2})}\left(\frac{x_2\beta_{n-1} + y_2}{z_2\beta_{n-1} + w_2}\right) &= \frac{x_3\beta_{n-2} + y_3}{z_3\beta_{n-2} + w_3}, \\ &\vdots \\ N_{K(\beta_2)/K(\beta_1)}\left(\frac{x_{n-1}\beta_2 + y_{n-1}}{z_{n-1}\beta_2 + w_{n-1}}\right) &= \frac{x_n\beta_1 + y_n}{z_n\beta_1 + w_n}. \end{aligned}$$

By Lemma 3.2 and the fact that $\phi(z) = z^d + c$ is irreducible over K, we obtain

$$N_{K(\beta_1)/K}\left(\frac{x_n\beta_1 + y_n}{z_n\beta_1 + w_n}\right) = \frac{x_{n+1}}{z_{n+1}} = (-1)^d\frac{x_{n+1}}{x_n}.$$

So,

$$N_{K(\beta_n)/K}\left(\frac{x_1\beta_n + y_1}{z_1\beta_n + w_1}\right) = (-1)^d\frac{x_{n+1}}{x_n}.$$

Finally, by Lemma 3.3 (v), we conclude that

$$c - \frac{1}{\beta_n} = \frac{x_1\beta_n + y_1}{z_1\beta_n + w_1} \notin -K(\beta_n)^p \quad \text{for all primes} \quad p|d,$$

and

$$c - \frac{1}{\beta_n} \notin 4K(\beta_n)^4 \quad \text{whenever} \quad 4|d.$$

From Lemma 3.4, we deduce that $z^d + c - \frac{1}{\beta_n}$ is irreducible over the field $K(\beta_n)$. This completes the proof of Claim 2.

Therefore, $[K(\beta_n):K] = d^n$ for any $n \in \mathbb{N}^*$. Since β_n is a root of $g_{n,\phi}$ and, by Lemma 3.1, we have $\deg(g_{n,\phi}) = d^n$, it follows that $g_{n,\phi}$ is irreducible over K for all $n \in \mathbb{N}^*$. □

Proof of Corollary 2.3.

Proof. Let $R = \mathbb{Z}$. Since $\phi(z) = z^d + c \in R[z]$ is irreducible, it follows that $c \notin -R^p$ for any prime p dividing d. When p is odd, it is clear that $-R^p = R^p$. Note that $U(R) = \{\pm 1\}$. We obtain Corollary 2.3 by Theorem 2.2. □

4. Proof of Theorem 2.4.

The following important results will be used in the proof of Theorem 2.4.

Lemma 4.1. ([20], Theorem 7.1) Let K be a field with characteristic 0, and let $\overline{K}$ be its algebraic closure. For a polynomial $f(t) \in K[t]$, define $n_0(f)$ to be the number of distinct roots of f in $\overline{K}$. Let $a(t), b(t), c(t) \in K[t]$ be polynomials that are relatively prime, such that $a(t) + b(t) = c(t)$, and not all of them have vanishing derivatives. Then, we have the inequality

$$\max\{\deg(a), \deg(b), \deg(c)\} \leqslant n_0(a(t)b(t)c(t)) - 1.$$

Proof of Theorem 2.4.

Proof. Define the sequence $\{x_n\}_{n\in\mathbb{N}^*}$ by:

$$x_1 = c,\ \ x_2 = (-1)^d(c^{d+1} + 1),\ \ x_{n+2} = (-1)^d c x_{n+1}^d + x_n^{d^2},\ \ n \geqslant 1. \tag{4.1}$$

This sequence is consistent with the sequence $\{x_n\}_{n\in\mathbb{N}^*}$ described in Lemma 3.3.

For any $n \in \mathbb{N}^*$, we have the following degree formula:

$$\deg(x_n) = \frac{d^n - 1}{d - 1}\deg(c), \tag{4.2}$$

where $\deg(c)$ denotes the degree of $c(t)$ viewed as a polynomial in t.

We first prove the following claim.

Claim: $x_{2n} \notin uR^p$ for any $n \geqslant 1$, $u \in U(R)$ and any prime p dividing d.

We first prove the claim holds for x_2. Suppose that there exist $u \in U(R)$, $z \in R$, and a prime $p \mid d$ such that

$$(-1)^d(c^{d+1} + 1) = uz^p. \tag{4.3}$$

Since $c \notin F$, c is a non-constant polynomial in $F[t]$. Taking degrees on both sides of (4.3), we obtain

$$(d + 1)\deg(c) = p\deg(z). \tag{4.4}$$

Next, we define

$$g = \frac{1}{u}(-1)^d c^{d+1} \quad \text{and} \quad h = \frac{1}{u}(-1)^d. \tag{4.5}$$

By (4.3) and (4.5), we have

$$g + h = z^p. \tag{4.6}$$

Note that g, h, and z are pairwise coprime, and not all of them have vanishing derivatives. Applying Lemma 4.1 to (4.6), we obtain the inequality

$$(d + 1)\deg(c) = \max\{\deg(g), \deg(h), \deg(z^p)\} \leqslant n_0(ghz^p) - 1.$$

By (4.4), we have

$$\begin{aligned} n_0(ghz^p) &= n_0(g) + n_0(h) + n_0(z^p) \\ &= n_0(c^{d+1}) + 0 + n_0(z) \\ &\leqslant \deg(c) + \deg(z) \\ &\leqslant \deg(c) + \tfrac{d+1}{p}\deg(c). \end{aligned}$$

Hence, we obtain

$$(d+1)\deg(c) \leqslant n_0(ghz^p) - 1 < \deg(c) + \frac{d+1}{p}\deg(c).$$

Thus, we obtain $d < \frac{d+1}{p}$. This contradicts to $p \geq 2$. Therefore, the claim holds for x_2.

Next, we prove the claim holds for each $x_{2n}, n \geqslant 2$. Suppose that there exist $k \in \mathbb{N}^*$, $u \in U(R)$, $z_k \in R$, and a prime $p \mid d$ such that

$$x_{2k+2} = uz_k^p. \tag{4.7}$$

We then define

$$g_k = \frac{1}{u}(-1)^d c x_{2k+1}^d \quad \text{and} \quad h_k = \frac{1}{u} x_{2k}^{d^2}. \tag{4.8}$$

From (4.1), (4.2), (4.7) and (4.8), we have $\deg(g_k) \geqslant 1$, $\deg(h_k) \geqslant 1$, $\deg(z_k) \geqslant 1$, and

$$g_k + h_k = z_k^p. \tag{4.9}$$

By Lemma 3.3 (i) and (ii), we have

$$\gcd(x_{2k}, c) = 1 \quad \text{and} \quad \gcd(x_{2k}, x_{2k+1}) = 1. \tag{4.10}$$

In light of (4.8)–(4.10), we obtain that g_k, h_k, and z_k are pairwise coprime. Applying Lemma 4.1 on (4.9), we obtain the following inequality:

$$\begin{aligned} \deg(g_k) + \deg(h_k) + \deg(z_k^p) &\leqslant 3\left(n_0(g_k h_k z_k^p) - 1\right) \\ &= 3\left(n_0(g_k) + n_0(h_k) + n_0(z_k) - 1\right). \end{aligned} \tag{4.11}$$

By Lemma 3.3 (ii), we have $\gcd\left(c, \frac{x_{2k+1}}{c}\right) = 1$. Thus,

$$\begin{aligned} n_0(g_k) &= n_0(c^{d+1}) + n_0\left(\left(\frac{x_{2k+1}}{c}\right)^d\right) \\ &= n_0(c) + n_0\left(\frac{x_{2k+1}}{c}\right) \leqslant \deg(c) + \deg\left(\frac{x_{2k+1}}{c}\right) \\ &= \deg(x_{2k+1}), \end{aligned} \tag{4.12}$$

Additionally, we have

$$n_0(h_k) = n_0(x_{2k}) = \deg(x_{2k}), \text{ and } n_0(z_k) \leqslant \deg(z_k). \tag{4.13}$$

Combining (4.11)–(4.13) yields

$$(d-3)\deg(x_{2k+1}) + (d^2-3)\deg(x_{2k}) + (p-3)\deg(z_k) + 4 \leqslant 0. \tag{4.14}$$

If d is odd, this inequality leads to a contradiction since $p|d$. Therefore, it is sufficient to consider the case where d is even and $p = 2$.

Applying Lemma 4.1 again yields the inequality:

$$p\deg(z_k) = \deg(z_k^p) \leqslant \deg(x_{2k+1}) + \deg(x_{2k}) + \deg(z_k) - 1. \tag{4.15}$$

Combining inequalities (4.14) and (4.15) with $p = 2$, we obtain:

$$(d-4)\deg(x_{2k+1}) + (d^2-4)\deg(x_{2k}) + 5 \leqslant 0,$$

which is a contradiction. Thus, the claim is proved.

We can similarly prove that ∞ is not periodic under $\Phi(z) = \frac{1}{\phi(z)}$ as in claim 1 of the proof of the Theorem 2.2.

Let $\{Q_n\}_{n\geq 1}$ be a sequence in $\mathbb{P}^1(\overline{K})$ such that $\Phi(Q_1) = \infty$ and $\Phi(Q_{n+1}) = Q_n$ for all $n \geqslant 1$. Since ∞ is not periodic under Φ, and $\Phi(\infty) = [0 : 1]$, we can express each Q_n as $Q_n = [\beta_n : 1]$, where $\beta_n \in \overline{K}$ and $\beta_n \neq 0$ for all $n \geqslant 1$. Thus, we have $\phi(\beta_1) = 0, \phi(\beta_{n+1}) = \frac{1}{\beta_n}$ and β_n is a root of the polynomial $g_{n,\phi}(z), n \geqslant 1$.

Since $F[t]$ is a UFD and $\gcd(x_n, x_{n+1}) = 1$, applying the above claim, one can show that $(-1)^d \frac{x_{n+1}}{x_n} \notin \pm K^p$ for all $n \in \mathbb{N}^*$ and any prime $p \mid d$. Similarly, we can show that $z^d + c - \frac{1}{\beta_n}$ is irreducible over $K(\beta_n)$ for every $n \geq 1$ as in Claim 2 of the proof of Theorem 2.2.

Therefore, $[K(\beta_n) : K] = d^n$ for any $n \in \mathbb{N}^*$. Since β_n is a root of $g_{n,\phi}$ and, by Lemma 3.1, we have $\deg(g_{n,\phi}) = d^n$, it follows that $g_{n,\phi}$ is irreducible over K for all $n \geqslant 1$. Hence, $\phi(z)$ is inversely stable over K. □

5. Proofs of theorem 2.6 and corollary 2.7

The following two lemmas will be used in the proof of Theorem 2.6.

Lemma 5.1. ([7], Proposition 2.3) Let $d \geqslant 2$ be an integer and $b \in \mathbb{F}_q$. Then the binomial $x^d - b$ is irreducible in $\mathbb{F}_q[x]$ if and only if the following conditions are satisfied:

(i) $\mathrm{rad}(d) | (q-1)$;

(ii) b is $\mathrm{rad}(d)$-free;

(iii) $q \equiv 1 \pmod 4$ if $d \equiv 0 \pmod 4$.

Lemma 5.2. ([7], Corollary 2.8) Let $d \geqslant 2$ be an integer such that $\mathrm{rad}(d)|(q-1)$. Then an element $\alpha \in \mathbb{F}_{q^n}$ is $\mathrm{rad}(d)$-free if and only if $N_{q^n/q}(\alpha)$ is $\mathrm{rad}(d)$-free in $\mathbb{F}_q$.

Proof of Theorem 2.6.

Proof. Since $\phi(z)$ is irreducible over $\mathbb{F}_q$, by Lemma 5.1, we have

(i) $\mathrm{rad}(d) \mid q - 1$;

(ii) $-c$ is $\mathrm{rad}(d)$-free;

(iii) $q \equiv 1 \pmod 4$ if $d \equiv 0 \pmod 4$.

We now define a matrix sequence $\{A_j\}_{j\geq 1}$ in the finite field $\mathbb{F}_q$, analogous to that in Lemma 3.3, with the same recurrence relation and initial values. We can similarly prove that ∞ is not periodic under $\Phi(z) = \frac{1}{\phi(z)}$ as in claim 1 of the proof of Theorem 2.2.

Let $\{Q_n\}_{n\geq 1}$ be a sequence in $\mathbb{P}^1(\overline{K})$ such that $\Phi(Q_1) = \infty$ and $\Phi(Q_{n+1}) = Q_n$ for all $n \geqslant 1$. Since ∞ is not periodic under Φ, and $\Phi(\infty) = [0 : 1]$, we can express each Q_n as $Q_n = [\beta_n : 1]$, where $\beta_n \in \overline{K}$ and $\beta_n \neq 0$ for all $n \geqslant 1$. Thus, we have $\phi(\beta_1) = 0, \phi(\beta_{n+1}) = \frac{1}{\beta_n}$ and β_n is a root of the polynomial $g_{n,\phi}(z), n \geqslant 1$.

Proof of necessity. Assume that $\phi(z)$ is inversely stable over K. Hence $g_{n,\phi}$ is irreducible and so $[K(\beta_n) : K] = d^n$, $[K(\beta_{n+1}) : K(\beta_n)] = d$ for all $n \geqslant 1$, and $[K(\beta_1) : K] = d$.

It is easy to see that $z^d + c - \frac{1}{\beta_n}$ is irreducible over $K(\beta_n)$ for all $n \geqslant 1$,

By Lemma 3.2, we have $N_{K(\beta_1)/K}\left(-\frac{c\beta_1 - 1}{\beta_1}\right) = (-1)^d \frac{c^{d+1}+1}{c} = \frac{x_2}{x_1}$. Hence $\frac{x_2}{x_1}$ is rad(d)-free by Lemmas 5.1 and 5.2.

We can similarly prove that

$$N_{K(\beta_n)/K}\left(\frac{x_1\beta_n + y_1}{z_1\beta_n + w_1}\right) = (-1)^d \frac{x_{n+1}}{x_n}$$

as that in the proof of Theorem 2.2. Note that

$$\frac{1}{\beta_n} - c = -\frac{x_1\beta_n + y_1}{z_1\beta_n + w_1}.$$

Hence

$$N_{K(\beta_n)/K}\left(\frac{1}{\beta_n} - c\right) = N_{K(\beta_n)/K}\left(-\frac{x_1\beta_n + y_1}{z_1\beta_n + w_1}\right) = (-1)^{d^n}(-1)^d \frac{x_{n+1}}{x_n} = \frac{x_{n+1}}{x_n}.$$

By Lemmas 5.1 and 5.2, we obtain that $\frac{x_{n+1}}{x_n}$ is rad(d)-free.

Proof of sufficiency. Assume that $\frac{x_{n+1}}{x_n}$ is rad(d)-free for every $n \in \mathbb{N}^*$.

Applying Lemmas 5.1 and 5.2, we can similarly prove that $z^d - (\frac{1}{\beta_n} - c)$ is irreducible over $K(\beta_n)$ as that in the proof of Theorem 2.2. Therefore, $[K(\beta_n) : K(\beta_{n-1})] = d$, and so $[K(\beta_n) : K] = d^n$.

Since β_n is a root of $g_{n,\phi}$, by Lemma 3.1, we have $\deg(g_{n,\phi}) = d^n$; it follows that $g_{n,\phi}$ is irreducible over K for all $n \in \mathbb{N}^*$. Hence, $\phi(z)$ is inverse stable. □

The following lemma will be used in our proof of Corollary 2.7.

Lemma 5.3. *Let $\mathbb{F}_q$ be a finite field of odd characteristic, and let*

$$\chi : \mathbb{F}_q^* \longrightarrow \{\pm 1\}$$

be the unique nontrivial character of order 2*, i.e.,* $\chi(t) = 1$ *if and only if t is a square in* $\mathbb{F}_q^*$*. Extend χ to* $\mathbb{F}_q$ *by setting* $\chi(0) = 0$.

(1) ([21], Application 1.3, page 139) Let $f(x) = ax^3 + bx^2 + cx + d \in \mathbb{F}_q[x]$ be a cubic polynomial with distinct roots in $\overline{\mathbb{F}}_q$. Then

$$\left|\sum_{x\in\mathbb{F}_q} \chi(f(x))\right| \leqslant 2\sqrt{q}.$$

(2) ([22], Theorem 5.48) Let $f(x) = ax^2 + bx + c \in \mathbb{F}_q[x]$. Then

$$\sum_{x\in\mathbb{F}_q} \chi\left(f(x)\right) = \begin{cases} -\chi(a) & \text{if } b^2 - 4ac \neq 0, \\ \chi(a)(q-1) & \text{if } b^2 - 4ac = 0. \end{cases}$$

Proof of Corollary 2.7.

Proof. Let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol. We first prove the following claim.

Claim: If $\left(\frac{c-1}{p}\right) = 1$ and $\left(\frac{c}{p}\right) = \left(\frac{c+1}{p}\right) = -1$, then $\phi(z) = z^d + c$ is inversely stable over $\mathbb{F}_p$.

It is obvious that $\mathrm{rad}(d) = 2$ and $\left(\frac{-c}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{c}{p}\right) = -1$. Hence, $-c$ is $\mathrm{rad}(d)$-free. By Lemma 5.1, we obtain that $z^d + c$ is irreducible over $K = \mathbb{F}_p$. This implies that $-c \notin K^d$. Let $\{x_n\}$ be the sequence defined in Theorem 2.6. It is easy to see that $x_n \neq 0$ for any $n \geqslant 1$.

Since $(p-1)|d^2$, by Fermat's Little Theorem, for any integer a with $p \nmid a$, we have $a^{d^2} \equiv 1 \pmod{p}$. Hence $x_n^{d^2} = 1$ for $n \geqslant 1$. Thus, we obtain that

$$x_1 = c,\ x_2 = (-1)^d(c^{d+1} + 1),\ x_{n+2} = (-1)^d c x_{n+1}^d + 1,\ n \geqslant 1.$$

From Euler's Criterion, for any integer a and an odd prime p, we have

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}.$$

Hence,

$$(c-1)^d = 1, \quad c^d = -1, \quad (c+1)^d = -1.$$

So we have $x_2 = 1 - c$, $x_3 = c + 1$, $x_4 = 1 - c$, and $x_5 = c + 1$. Thus, the sequence $\{x_n\}_{n\geqslant1}$ follows the pattern:

$$\{x_n\}_{n\geqslant1} : c, 1-c, c+1, 1-c, c+1, \ldots$$

The sequence $\left\{\frac{x_{n+1}}{x_n}\right\}_{n\geqslant1}$ is given by:

$$\left\{\frac{x_{n+1}}{x_n}\right\}_{n\geqslant1} : \frac{1-c}{c}, \frac{1+c}{1-c}, \frac{1-c}{1+c}, \frac{1+c}{1-c}, \frac{1-c}{1+c}, \ldots$$

From $\left(\frac{c-1}{p}\right) = 1$ and $\left(\frac{c}{p}\right) = \left(\frac{c+1}{p}\right) = -1$, we know that $\frac{1-c}{c}$, $\frac{1+c}{1-c}$, and $\frac{1-c}{1+c}$ are all 2-free. By Theorem 2.6, we conclude that $\phi(z)$ is inversely stable over $\mathbb{F}_p$. This completes the proof of the claim.

It is easy to calculate that $\lfloor \frac{1}{8}\left(\sqrt{p} - 1\right)^2 \rfloor = 2^{2^n-3} - 2^{2^{n-1}-2}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Therefore, it is sufficient to show that there are at least $\lfloor \frac{1}{8}\left(\sqrt{p} - 1\right)^2 \rfloor$ distinct values of $c \in \mathbb{F}_p$ satisfying $\left(\frac{c-1}{p}\right) = 1$ and $\left(\frac{c}{p}\right) = \left(\frac{c+1}{p}\right) = -1$. Define

$$t(x) = \frac{(\chi(x-1)+1)}{2} \cdot \frac{(1-\chi(x))}{2} \cdot \frac{(1-\chi(x+1))}{2},\ x \in \mathbb{F}_p,$$

where $\chi(x) = \left(\frac{x}{p}\right)$. It is obvious that

$$t(x) = \begin{cases} 1, & \text{if } \chi(x-1) = 1, \chi(x) = \chi(x+1) = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the sum $S = \sum_{x=1}^{p} t(x)$ counts the number of $x \in \mathbb{F}_p$ satisfying the conditions $\chi(x-1) = 1, \chi(x) = \chi(x+1) = -1$.

Expanding $t(x)$, we obtain

$$t(x) = \frac{1}{8}\left[1 + \chi(x-1) - \chi(x) - \chi(x+1) - \chi(x^2-x) - \chi(x^2-1) + \chi(x^2+x) + \chi(x^3-x)\right].$$

Note that the discriminants of x^2-1, x^2-x, and x^2+x are nonzero in $\mathbb{F}_p$. By Lemma 5.3 and $\sum_{x=1}^{p} \chi(x) = 0$, we obtain that

$$S = \frac{1}{8}\left(p + 1 + \sum_{x=1}^{p} \chi(x^3 - x)\right) \geqslant \frac{1}{8}(\sqrt{p} - 1)^2 \geqslant \lfloor \frac{1}{8}(\sqrt{p} - 1)^2 \rfloor = 2^{2^n-3} - 2^{2^{n-1}-2}.$$

This completes the proof of Corollary 2.7. □

6. Conclusions

In 2024, K. Cheng introduced the concept of inverse stable polynomials over finite fields and investigated their properties for Artin-Schreier polynomials. In this paper, we first extend this notion to arbitrary fields and establish connections between inverse stability and the eventual stability proposed by R. Jones and L. Alone in 2017. Following the methodology in several references that reduce problems in larger fields to base fields via norm maps of field extensions, we systematically study inverse stability for binomial polynomials and present three directions of applications.

Two natural open problems emerge from this work:

- Characterizing inverse stability for other polynomial types (e.g., trinomials)
- Developing deeper estimates using advanced tools (e.g., character sums) to bound the number of inverse stable polynomials over finite fields

The stability of polynomials possessing special significance and applications merits more focus. (see, e.g., [23–27]). Our work combines field theory and number theory to address iteration-related questions, representing a characteristic approach in arithmetic dynamics. Notably, inverse stability exhibits potential applications in Arboreal–Galois representations (see [13]), warranting further exploration.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. R. W. K. Odoni, On the prime divisors of the sequence $w_{n+1} = 1 + w_1 \cdots w_n$, *J. London Math. Soc.*, **32** (1985), 1–11. https://doi.org/10.1112/jlms/s2-32.1.1
2. M. Ayad, D. L. McQuillan, Irreducibility of the iterates of a quadratic polynomial over a field, *Acta Arith.*, **93** (2000), 87–97. https://doi.org/10.4064/aa-93-1-87-97
3. D. Gómez, A. P. Nicolás, An estimate on the number of stable quadratic polynomials, *Finite Fields Appl.*, **16** (2010), 401–405. https://doi.org/10.1016/j.ffa.2010.06.005
4. R. Jones, N. Boston, Settled polynomials over finite fields, *Proc. Am. Math. Soc.*, **140** (2012), 1849–1863. https://doi.org/10.1090/S0002-9939-2011-11054-2
5. P. Illig, R. Jones, E. Orvis, Y. Segawa, N. Spinale, Newly reducible polynomial iterates, *Int. J. Number Theory*, **17** (2021), 1405–1427. https://doi.org/10.1142/S1793042121500433
6. L. Danielson, B. Fein, On the irreducibility of the iterates of $x^n - b$, *Proc. Am. Math. Soc.*, **130** (2002), 1589–1596. https://doi.org/10.1090/S0002-9939-01-06258-X
7. A. Fernandes, D. Panario, L. Reis, Stable binomials over finite fields, *Finite Fields Appl.*, **101** (2025), 102520. https://doi.org/10.1016/j.ffa.2024.102520
8. L. Shanta, S. Ritumoni, S. Himanshu, Stability of certain higher degree polynomials, *Int. J. Number Theory*, **20** (2024), 229–240. https://doi.org/10.1142/S1793042124500118
9. L. Tong, W. Qiang, On the stable polynomials of degrees $2, 3, 4$, *Finite Fields Appl.*, **99** (2024), 102474. https://doi.org/10.1016/j.ffa.2024.102474
10. N. Ali, Stabilité des polynômes, *Acta Arith.*, **119** (2005), 53–63. https://doi.org/10.4064/aa119-1-4
11. L. Mrai, A. Ostafe, Dynamical irreducibility of polynomials modulo primes, *Math. Z.*, **298** (2021), 1187–1199. https://doi.org/10.1007/s00209-020-02630-5
12. R. Lucas, On the factorization of iterated polynomials, *Rev. Mat. Iberoam.*, **36** (2020), 1957–1978. https://doi.org/10.4171/rmi/1187
13. R. Jones, L. Alon, Eventually stable rational functions, *Int. J. Number Theory.*, **13** (2017), 2299–2318. https://doi.org/10.1142/S1793042117501263
14. K. Cheng, A new direction on constructing irreducible polynomials over finite fields, *Finite Fields Appl.*, **95** (2024), 102368. https://doi.org/10.1016/j.ffa.2024.102368
15. H. Kriger, Primitive prime divisors in the critical orbit of $z^d + c$, *Int. Math. Res. Not.*, **23** (2013), 5498–5525. https://doi.org/10.1093/imrn/rns213
16. P. Ingram, Lower bounds on the canonical height associated to the morphism $z^d + c$, *Monatsh. Math.*, **157** (2007), 69–89. https://doi.org/10.1007/s00605-008-0018-6
17. P. Ingram, Canonical heights and preperiodic points for certain weighted homogeneous families of polynomials, *Int. Math. Res. Not.*, **15** (2019), 4859–4879. https://doi.org/10.1093/imrn/rnx291
18. S. Huczynska, G. L. Mullen, D. Panario, D. Thomson, Existence and properties of k-normal elements over finite fields, *Finite Fields Appl.*, **24** (2013), 170–183. https://doi.org/10.1016/j.ffa.2013.07.004

19. G. Karpilovsky, *Topics in Field Theory*, North-Holland Mathematics Studies, North-Holland Publishing Company, Amsterdam, 1989.

20. S. Lang, *Algebra, Graduate Texts in Mathematics*, Springer, Verlag, 2002.

21. J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2^{nd} edition, Graduate Texts in Mathematics, Springer, Dordrecht, 2009.

22. R. Lidl, H. Niederreiter, *Finite Fields, Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1996.

23. G. Dattoli, C. Cesarano, On a new family of Hermite polynomials associated to parabolic cylinder functions, *Appl. Math. Comput.*, **141** (2003), 143–149. https://doi.org/10.1016/S0096-3003(02)00328-4

24. G. Dattoli, P. E. Ricci, C. Cesarano, The Lagrange polynomials, the associated generalizations, and the umbral calculus, *Integr. Transforms Spec. Funct.*, **14** (2003), 181–186. https://doi.org/10.1080/1065246031000098186

25. M. M. Bhatti, M. Marin, R. Ellahi, I. M. Fudulu, Insight into the dynamics of EMHD hybrid nanofluid (ZnO/CuO-SA) flow through a pipe for geothermal energy applications, *J. Therm. Anal. Calorim.*, **148** (2023), 14261–14273. https://doi.org/10.1007/s10973-023-12565-8

26. A. K. Yadav, E. Carrera, M. Marin, M. I. A. Othman, Reflection of hygrothermal waves in a Nonlocal Theory of coupled thermo-elasticity, *Mech. Adv. Mater. Struct.*, **31** (2022), 1083–1096. https://doi.org/10.1080/15376494.2022.2130484

27. M. Marin, I. Abbas, R. Kumar, Relaxed Saint-Venant principle for thermoelastic micropolar diffusion, *Struct. Eng. Mech.*, **51** (2014), 651–662. https://doi.org/10.12989/sem.2014.51.4.651

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