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Research article

A high-order Chebyshev-type method for solving nonlinear equations: local convergence and applications

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Abstract: In this paper, the local convergence of a high-order Chebyshev-type method without the second derivative is studied. We study the convergence under ω-continuity conditions based on the first derivative. The uniqueness of the solution and the radii of convergence domains are obtained. In contrast to the conditions used in previous studies, the new conditions of convergence are weaker. In addition, the attractive basins of the family with different parameters are studied, which can show the different stability of the family. Finally, in numerical experiments, the iterative method is used to solve different nonlinear models, including vertical stresses, civil engineering problem, blood rheology model, and so on. Theoretical results of convergence criteria are verified.

Keywords: high-order method; local convergence; nonlinear equation; Chebyshev-type method; radius of convergence

1. Introduction

In this paper, our main purpose is to estimate an approximate solution γ_* of the equation

$$P(s) = 0, \tag{1.1}$$

where $P : \Omega \subseteq T_1 \to T_2$ is a scalar function in an open convex interval Ω.

Solving problems of nonlinear equations is widely used in many fields, such as physics, chemistry, and biology [1]. Usually, the analytical solution of nonlinear equations is difficult to obtain in general cases. Therefore, in most situations, iterative methods are applied to find approximate solutions [2]. The most famous and fundamental iterative method is Newton's method [3]. Currently, many methods are constructed on the basis of Newton's method, and they are called Newton-type methods [4, 5]. Convergence analysis is an important part of the research of the iterative method [6]. The issue of local convergence is, based on the information surrounding a solution, to find estimates of the radii of the convergence balls [7]. At present, many scholars study the local convergence analysis of iterative methods, such as Argyros et al. studied the local convergence of a third-order iterative method [8] and Chebyshev-type method [9]. In addition, some iterative methods and their local convergence are used in the study of diffusion equations [10–13]. The domain of convergence is an important problem in the study of iterative process; see [14]. Generally, the domain of convergence is small, which limits the choice of initial points. Thus, it is crucial that the domain of convergence is expanded without additional conditions. This paper will study the local convergence of a Chebyshev-type method without the second derivative in order to broaden its applied range.

The classical Chebyshev-Halley type methods of third-order convergence [15], which improves Newton's methods are defined by

$$s_{n+1} = s_n - (1 + \frac{1}{2}(1 - \lambda K_F(s_n))^{-1} K_F(s_n) P'(s_n)^{-1} P(s_n), \tag{1.2}$$

where

$$K_F(s_n) = P'(s_n)^{-1} P''(s_n) P'(s_n)^{-1} P(s_n),$$

This method includes Halley's method [16] for $\lambda = \frac{1}{2}$, Chebyshev's method [17] for $\lambda = 0$, and the super-Halley method for $\lambda = 1$. Since these methods need to calculate the second derivative, they have an expensive computational cost. To avoid the second derivative, some scholars have proposed some variants of Chebyshev-Halley type methods free from the second derivative [18, 19]. Cordero et al. [20] proposed a high-order three-step form of the modified Chebyshev–Halley type method:

$$\begin{aligned} t_n &= s_n - P'(s_n)^{-1} P(s_n), \\ z_n &= s_n - (1 + P(t_n)(P(s_n) - 2\beta P(t_n))^{-1}) P'(s_n)^{-1} P(s_n), \\ s_{n+1} &= z_n - ([z_n, t_n; P] + 2(z_n - t_n)[z_n, t_n, s_n; P] - (z_n - t_n)[t_n, s_n, s_n; P])^{-1} P(z_n), \end{aligned} \tag{1.3}$$

where $\beta \in \mathbb{R}$ denotes a parameter, and $s_0 \in \Omega$ denotes an initial point. $[.,.;P]$ and $[.,.,.;P]$ denote divided difference of order one and two, in particular, the second-order divided difference cannot be generalized to Banach spaces. So, we study the local convergence of method (1.3) in real spaces. The order of convergence of the above method is at least six, and if $\beta = 1$, it is optimal order eight.

However, earlier proofs of the analysis of convergence required third or higher derivatives. This limits the applicability of the above method. For example, define $P(s)$ on $\Omega = [0, 1]$ by

$$P(s) = \begin{cases} s^3 \ln s^2 - s^5 + s^4, & s \neq 0; \\ 0, & s = 0. \end{cases} \tag{1.4}$$

Then, $P'''(s) = 6 \ln s^2 - 60s^2 + 24s + 22$ is unbounded on Ω. So when using the iterative method to solve the equation (1.4), the convergence order of the iterative method cannot be guaranteed. In this paper, the analysis of local convergence for method (1.3) only uses the first-order derivative.In particular, using Lipschitz continuity conditions based on the first derivative, the applicability of method (1.3) is extended.

The rest part of this paper is laid out as follows: Section 2 is devoted to the study of local convergence for method (1.3) by using assumptions based on the first derivative. Also, the uniqueness of the solution and the radii of convergence balls are analyzed. In Section 3, according to the different parameter values, the fractal graphs of the family are drawn. The convergence and stability of the iterative method are analyzed by drawing the attractive basins. In Section 4, the convergence criteria are verified by some numerical examples. Finally, conclusions appear in Section 5.

2. Local convergence

In this Section, we study the local convergence analysis of method (1.3) under Lipschitz continuity conditions. There are some parameters and scalar functions to be used to prove local convergence of method (1.3). $\beta \in \mathbb{R}$ and $\theta \geq 0$ are parameters. Suppose the continuous function $\upsilon_0 : [0, +\infty) \to \mathbb{R}$ is nondecreasing, $\upsilon_0(0) = 0$, and

$$\upsilon_0(\xi) - 1 = 0 \tag{2.1}$$

has a smallest solution $\gamma_0 \in [0, +\infty) - \{0\}$.
Let the continuous function $\upsilon : [0, \gamma_0) \to \mathbb{R}$ be nondecreasing and $\upsilon(0) = 0$. Functions h_1 and g_1 on the interval $[0, \gamma_0)$ are defined by

$$h_1(\xi) = \frac{\int_0^1 \upsilon(|\theta - 1|\xi)d\theta}{1 - \upsilon_0(\xi)}$$

and

$$g_1(\xi) = h_1(\xi) - 1.$$

Then we obtain

$$g_1(0) = h_1(0) - 1 < 0$$

and $g_1(\xi) \to \infty$ as $\xi \to \gamma_0^-$. According to the intermediate value theorem, the equation $g_1(\xi) = 0$ has roots in $(0, \gamma_0)$. Let r_1 be the smallest root. Suppose continuous function $\omega_1 : [0, \gamma_0) \to \mathbb{R}$ is nondecreasing and $\omega_1(0) = 0$. Functions h_2 and g_2 on the interval $[0, \gamma_0)$ are defined by

$$h_2(\xi) = \int_0^1 \upsilon_0(|\theta|\xi)d\theta + 2|\beta|h_1(\xi) \int_0^1 \omega_1(\xi|\theta|h_1(\xi))d\theta$$

and

$$g_2(\xi) = h_2(\xi) - 1.$$

Then we have

$$g_2(0) = h_2(0) - 1 < 0$$

and $g_2(\xi) \to \infty$ as $\xi \to \gamma_0^-$. Similarly, the equation $g_2(\xi) = 0$ has roots in $(0, \gamma_0)$. Let r_2 be smallest root. Functions h_3 and g_3 on the interval $[0, r_2)$ are defined by

$$h_3(\xi) = h_1(\xi)[1 + \frac{\int_0^1 \omega_1(\xi|\theta|h_2(\xi))\omega_1(|\theta|\xi)d\theta}{(1 - \upsilon_0(\xi))(1 - h_2(\xi))}]$$

and

$$g_3(\xi) = h_3(\xi) - 1.$$

Then we have

$$g_3(0) = h_3(0) - 1 < 0$$

and $g_3(\xi) \to \infty$ as $\xi \to r_2^-$. Similarly, the equation $g_3(\xi) = 0$ has roots in $(0, r_2)$. Let r_3 be the smallest root. Suppose continuous functions $\omega_0, \omega_2 : [0, \gamma_0)^2 \to \mathbb{R}$ and $\omega_3 : [0, \gamma_0)^3 \to \mathbb{R}$ are nondecreasing with $\omega_0(0, 0) = 0$, $\omega_2(0, 0) = 0$, and $\omega_3(0, 0, 0) = 0$. Functions h_4 and g_4 on the interval $[0, \gamma_0)$ are defined by

$$h_4(\xi) = \omega_0(h_3(\xi)\xi, h_1(\xi)\xi) + \xi(h_1(\xi) + h_3(\xi))(\omega_2(\xi(h_3(\xi) + h_1(\xi)), \xi(h_1(\xi) + 1)) + \omega_3(h_3(\xi)\xi, h_1(\xi)\xi, \xi))$$

and

$$g_4(\xi) = h_4(\xi) - 1.$$

Then we have

$$g_4(0) = h_4(0) - 1 < 0$$

and $g_4(\xi) \to \infty$ as $\xi \to r_3^-$. Similarly, the equation $g_4(\xi) = 0$ has roots in $(0, r_3)$. Let r_4 be the smallest root. Functions h_5 and g_5 on the interval $[0, r_4)$ are defined by

$$h_5 = [1 - \frac{\int_0^1 \omega_1(\xi|\theta|h_3(\xi))d\theta}{1 - h_4(\xi)}]h_3(\xi)$$

and

$$g_5(\xi) = h_5(\xi) - 1.$$

We have

$$g_5(0) = h_5(0) - 1 < 0$$

and $g_5(\xi) \to \infty$ as $\xi \to r_4^-$. Similarly, the equation $g_5(\xi) = 0$ has roots in $(0, r_4)$. Let r_5 be the smallest root.

Set

$$r = \min\{r_1, r_3, r_5\}. \tag{2.2}$$

Then, for each $\xi \in [0, r)$, we have that

$$0 \leq h_1(\xi) < 1, \tag{2.3}$$

$$0 \leq h_2(\xi) < 1, \tag{2.4}$$

$$0 \leq h_3(\xi) < 1, \tag{2.5}$$

$$0 \leq h_4(\xi) < 1, \tag{2.6}$$

$$0 \leq h_5(\xi) < 1. \tag{2.7}$$

Applying the above conclusions, the analysis of local convergence for method (1.3) can be proved.

Theorem 2.1. Suppose $P : \Omega \subset T_1 \to T_2$ is a scalar function. $[.,.;P] : \Omega^2 \to L(T_1, T_2)$ and $[.,.,.;P] : \Omega^3 \to L(T_1, T_2)$ are divided differences of one and two. Let $\gamma_* \in \Omega$ and continuous function $\upsilon_0 : [0, +\infty) \to \mathbb{R}$ be nondecreasing with $\upsilon_0(0) = 0$ such that each $x \in \Omega$

$$P(\gamma_*) = 0,\ P'(\gamma_*)^{-1} \in L(T_1, T_2), \tag{2.8}$$

$$||P'(\gamma_*)^{-1}(P'(s) - P'(\gamma_*))|| \leq \upsilon_0(||s - \gamma_*||). \tag{2.9}$$

Let $\Omega_0 = \Omega \cap B(\gamma_*, \gamma_0)$. There exist $\beta \in \mathbb{R}$, $M \geq 0$, continuous functions $\upsilon, \omega_1 : [0, \gamma_0) \to \mathbb{R}$, $\omega_0, \omega_2 : [0, \gamma_0)^2 \to \mathbb{R}$, $\omega_3 : [0, \gamma_0)^3 \to \mathbb{R}$ be nondecreasing such that for each $x, y, z \in \Omega_0$

$$||P'(\gamma_*)^{-1}(P'(s) - P'(t))|| \leq \upsilon(||s - t||) \tag{2.10}$$

$$||P'(\gamma_*)^{-1}([s, t; P] - P'(\gamma_*))|| \leq \omega_0(||s - \gamma_*||, ||t - \gamma_*||) \tag{2.11}$$

$$||P'(\gamma_*)^{-1}P'(s)|| \leq \omega_1(||s - \gamma_*||) \tag{2.12}$$

$$\|P'(\gamma_*)^{-1}([z,t,s;P]-[t,s,s;P])\| \le \omega_2(\|z-t\|,\|t-s\|) \tag{2.13}$$

$$\|P'(\gamma_*)^{-1}[z,t,s;P]\| \le \omega_3(\|z-\gamma_*\|,\|t-\gamma_*\|,\|s-\gamma_*\|) \tag{2.14}$$

and

$$\bar{U}(\gamma_*,r) \subseteq \Omega. \tag{2.15}$$

Then the sequence $\{s_n\}$ produced for $s_0 \in U(\gamma_*,r)-\{\gamma_*\}$ by method (1.3) converges to γ_* and remains in $U(\gamma_*,r)$ for each $n=0,1,2\ldots$. Furthermore, the following estimates hold:

$$\|t_n-\gamma_*\| \le h_1(\|s_n-\gamma_*\|)\|s_n-\gamma_*\| \le \|s_n-\gamma_*\| < r, \tag{2.16}$$

$$\|z_n-\gamma_*\| \le h_3(\|s_n-\gamma_*\|)\|s_n-\gamma_*\| \le \|s_n-\gamma_*\|, \tag{2.17}$$

and

$$\|s_{n+1}-\gamma_*\| \le h_5(\|s_n-\gamma_*\|)\|s_n-\gamma_*\| \le \|s_n-\gamma_*\|, \tag{2.18}$$

where functions $h_i (i=1,3,5)$ have been defined. Moreover, for $R \ge r$, if there exists that

$$\int_0^1 \upsilon_0(|\theta-1|R)d\theta < 1, \tag{2.19}$$

then, the solution $\gamma_* \in \bar{U}(\gamma_*,R) \subseteq \Omega$ of equation $P(s)=0$ is unique.

Proof Using $s_0 \in U(\gamma_*,r)$, (2.8), and the definition of r, we obtain

$$\|P'(\gamma_*)^{-1}(P'(s_0)-P'(\gamma_*))\| \le \upsilon_0(\|s_0-\gamma_*\|) < \upsilon_0(r) < 1. \tag{2.20}$$

According to the Banach lemma [2], we obtain $P'(s_0)$ is invertible and

$$\|P'(s_0)^{-1}P'(\gamma_*)\| \le \frac{1}{1-\upsilon_0(\|s_0-\gamma_*\|)} < \frac{1}{1-\upsilon_0(r)}. \tag{2.21}$$

Then, t_0 is well defined. Therefore, we can write that

$$\begin{aligned} t_0-\gamma_* &= s_0-\gamma_*-P'(s_0)^{-1}P(s_0) \\ &= -P'(s_0)^{-1}P'(\gamma_*)\int_0^1 P'(\gamma_*)^{-1}[P'(\gamma_*+\theta(s_0-\gamma_*))-P'(s_0)](s_0-\gamma_*)d\theta. \end{aligned} \tag{2.22}$$

Using (2.2), (2.3), (2.10), (2.20), and (2.21), we obtain in turn that

$$\begin{aligned} \|t_0-\gamma_*\| &\le \|P'(s_0)^{-1}P'(\gamma_*)\|\|\int_0^1 P'(\gamma_*)^{-1}[P'(\gamma_*+\theta(\|s_0-\gamma_*\|))-P'(s_0)]\|d\theta\|s_0-\gamma_*\| \\ &\le \frac{\int_0^1 \upsilon(\|\gamma_*+\theta(\|s_0-\gamma_*\|)-s_0\|)d\theta}{1-\upsilon_0(\|s_0-\gamma_*\|)}\|s_0-\gamma_*\| \\ &= \frac{\int_0^1 \upsilon(\|(\theta-1)(s_0-\gamma_*)\|)d\theta}{1-\upsilon_0(\|s_0-\gamma_*\|)}\|s_0-\gamma_*\| \\ &= h_1(\|s_0-\gamma_*\|)\|s_0-\gamma_*\| < \|s_0-\gamma_*\| < r, \end{aligned} \tag{2.23}$$

which shows the estimate (2.16) for $n=0$ and $t_0 \in U(\gamma_*,r)$.

Using (2.2), (2.4), (2.10), (2.12), (2.16), and (2.23), we obtain

$$
\begin{aligned}
&\|(P'(\gamma_*)(s_0-\gamma_*))^{-1}[P(s_0)-P(\gamma_*)-2\beta P(t_0)-P'(\gamma_*)(s_0-\gamma_*)]\|\\
\leq&\frac{1}{\|s_0-\gamma_*\|}\|\int_0^1 P'(\gamma_*)^{-1}(P'(\gamma_*+\theta(s_0-\gamma_*))-P'(\gamma_*))(s_0-\gamma_*)d\theta\|\\
&+\frac{1}{\|s_0-\gamma_*\|}\cdot 2|\beta|\|\int_0^1 P'(\gamma_*)^{-1}P'(\gamma_*+\theta(t_0-\gamma_*))d\theta\|\|t_0-\gamma_*\|\\
\leq&\frac{1}{\|s_0-\gamma_*\|}\|\int_0^1 P'(\gamma_*)^{-1}(P'(\gamma_*+\theta(s_0-\gamma_*))-P'(\gamma_*))(s_0-\gamma_*)d\theta\|\\
&+\frac{1}{\|s_0-\gamma_*\|}\cdot 2|\beta|\int_0^1 \omega_1(\|\theta(t_0-\gamma_*)\|)d\theta\|t_0-\gamma_*\|\\
\leq&\int_0^1 \upsilon_0(\|\theta(s_0-\gamma_*)\|)d\theta+2|\beta|h_1(\|s_0-\gamma_*\|)\int_0^1 \omega_1(\theta(h_1(\|s_0-\gamma_*\|)\|s_0-\gamma_*\|)r)d\theta\\
=&h_2(\|s_0-\gamma_*\|)<h_2(r)<1,
\end{aligned}
\tag{2.24}
$$

where

$$
P'(\gamma_*)^{-1}P(t_0)=P'(\gamma_*)^{-1}(P(t_0)-P(\gamma_*))=\int_0^1 P'(\gamma_*)^{-1}P'(\gamma_*+\theta(t_0-\gamma_*))(t_0-\gamma_*)d\theta, \tag{2.25}
$$

so

$$
\begin{aligned}
\|P'(\gamma_*)^{-1}P(t_0)\| &\leq \int_0^1 \omega_1(\|\theta(t_0-\gamma_*)\|)\|t_0-\gamma_*\|d\theta\\
&\leq h_1(\|s_0-\gamma_*\|)\|s_0-\gamma_*\|\int_0^1 \omega_1(\|\theta(t_0-\gamma_*)\|)d\theta
\end{aligned}
\tag{2.26}
$$

and

$$
\|\gamma_*+\theta(t_0-\gamma_*)-\gamma_*\|=\theta\|t_0-\gamma_*\|\leq\|t_0-\gamma_*\|\leq r.
$$

Thus, $(P(s_0)-2\beta P(t_0))^{-1}\in L(T_1,T_2)$ and

$$
\|(P(s_0)-2\beta P(t_0))^{-1}P'(\gamma_*)\|\leq\frac{1}{(1-h_2(\|s_0-\gamma_*\|))\|s_0-\gamma_*\|}. \tag{2.27}
$$

So, z_0 is well defined.

Using (2.2), (2.5), (2.12), (2.16), (2.21), (2.24), and (2.27), we have that

$$
\begin{aligned}
\|z_0-\gamma_*\|\leq&\|s_0-\gamma_*-P'(s_0)^{-1}P(s_0)\|\\
&+\|P'(\gamma_*)^{-1}P(t_0)\|\|P'(\gamma_*)^{-1}P(s_0)\|\|P'(s_0)^{-1}P'(\gamma_*)\|\\
&\|(P(s_0)-2\beta P(t_0))^{-1}P'(\gamma_*)\|\\
\leq&h_1(\|s_0-\gamma_*\|)\|s_0-\gamma_*\|+\frac{\int_0^1 \omega_1(\|\theta(t_0-\gamma_*)\|)\omega_1(\|\theta(s_0-\gamma_*)\|)d\theta\|t_0-\gamma_*\|}{(1-\upsilon_0(\|s_0-\gamma_*\|))(1-h_2(\|s_0-\gamma_*\|))}\\
\leq&h_1(\|s_0-\gamma_*\|)\|s_0-\gamma_*\|[1+\frac{\int_0^1 \omega_1(\|\theta(t_0-\gamma_*)\|)\omega_1(\|\theta(s_0-\gamma_*)\|)d\theta}{(1-\upsilon_0(\|s_0-\gamma_*\|))(1-h_2(\|s_0-\gamma_*\|))}]\\
=&h_3(\|s_0-\gamma_*\|)\|s_0-\gamma_*\|<\|s_0-\gamma_*\|<r,
\end{aligned}
\tag{2.28}
$$

which shows the estimate (2.17) for $n = 0$ and $z_0 \in U(\gamma_*, r)$.

Next, we shall show that

$$([z_0, t_0; P] + 2(z_0 - t_0)[z_0, t_0, s_0; P] - (z_0 - t_0)[t_0, s_0, s_0; P])^{-1} \in L(T_1, T_2). \tag{2.29}$$

Using (2.2), (2.6), (2.11), (2.13), and (2.14), we have that

$$\begin{aligned}
&\|P'(\gamma_*)^{-1}([z_0, t_0; P] + 2(z_0 - t_0)[z_0, t_0, s_0; P] - (z_0 - t_0)[t_0, s_0, s_0; P] - P'(\gamma_*))\| \\
\le&\|P'(\gamma_*)^{-1}([z_0, t_0; P] - P'(\gamma_*))\| + \|z_0 - t_0\|\|P'(\gamma_*)^{-1}([z_0, t_0, s_0; P] - [t_0, s_0, s_0; P])\| \\
&+ \|z_0 - t_0\|\|P'(\gamma_*)^{-1}[z_0, t_0, s_0; P]\| \\
\le&\omega_0(\|z_0 - \gamma_*\|, \|t_0 - \gamma_*\|) + (\|z_0 - \gamma_*\| + \|t_0 - \gamma_*\|)(\omega_2(\|z_0 - t_0\|, \|t_0 - s_0\|) \\
&+ \omega_3(\|z_0 - \gamma_*\|, \|t_0 - \gamma_*\|, \|s_0 - \gamma_*\|)) \\
=&\omega_0(h_3(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\|, h_1(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\|) + (h_3(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\| \\
&+ h_1(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\|)(\omega_2(h_3(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\| + h_1(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\|, \\
&(h_1(\|s_0 - \gamma_*\|) + 1)\|s_0 - \gamma_*\|) \\
&+ \omega_3(h_3(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\|, h_1(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\|, \|s_0 - \gamma_*\|)) \\
=&h_4(\|s_0 - \gamma_*\|) < 1.
\end{aligned} \tag{2.30}$$

By the Banach lemma, we have that $([z_0, t_0; P] + 2(z_0 - t_0)[z_0, t_0, s_0; P] - (z_0 - t_0)[t_0, s_0, s_0; P])$ is invertible and

$$\|([z_0, t_0; P] + 2(z_0 - t_0)[z_0, t_0, s_0; P] - (z_0 - t_0)[t_0, s_0, s_0; P])^{-1} P'(\gamma_*)\| \le \frac{1}{1 - h_4(\|s_0 - \gamma_*\|)}. \tag{2.31}$$

Denote $\triangle = [z_0, t_0; P] + 2(z_0 - t_0)[z_0, t_0, s_0; P] - (z_0 - t_0)[t_0, s_0, s_0; P]$. Thus, x_1 is well defined. Using $x_1 \in U(\gamma_*, r)$, (2.2), (2.8), (2.12), (2.28), and (2.31), we have that

$$\begin{aligned}
\|s_1 - \gamma_*\| \le&\|z_0 - \gamma_*\| - \|\triangle^{-1} P'(\gamma_*) P'(\gamma_*)^{-1} P(z_0)\| \\
\le&\|z_0 - \gamma_*\| - \frac{\int_0^1 \omega_1(\theta\|z_0 - \gamma_*\|)d\theta}{1 - h_4(\|s_0 - \gamma_*\|)}\|z_0 - \gamma_*\| \\
\le&[1 - \frac{\int_0^1 \omega_1(\theta\|z_0 - \gamma_*\|)d\theta}{1 - h_4(\|s_0 - \gamma_*\|)}]\|z_0 - \gamma_*\| \\
\le&[1 - \frac{\int_0^1 \omega_1(\theta\|z_0 - \gamma_*\|)d\theta}{1 - h_4(\|s_0 - \gamma_*\|)}]h_3(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\| \\
=&h_5(\|s_0 - \gamma_*\|)\|s_0 - \gamma_*\| < \|s_0 - \gamma_*\| < r,
\end{aligned} \tag{2.32}$$

which shows the estimate (2.18) for $n = 0$ and $s_1 \in U(\gamma_*, r)$. By substituting s_0, t_0, z_0, s_1 in the previous estimates with s_k, t_k, z_k, s_{k+1}, we get (2.16)–(2.18). Using the estimates

$$\|s_{k+1} - \gamma_*\| < \|s_k - \gamma_*\| < r,$$

we derive that $s_{k+1} \in U(\gamma_*, r)$ and $\lim_{k\to\infty} s_k = \gamma_*$.

Finally, in order to prove the uniqueness of the solution γ_*, suppose there exists a second solution $y_* \in \bar{B}(\gamma_*, R)$, then $P(y_*) = 0$. Denote $T = \int_0^1 P'(y_* + \theta(\gamma_* - y_*))d\theta$. Since $T(y_* - \gamma_*) = P(y_*) - P(\gamma_*) = 0$, if T is invertible then $y_* = \gamma_*$. In fact, by (2.19), we obtain

$$\begin{aligned} \|P'(\gamma_*)^{-1}(T - P'(\gamma_*))\| &\leq \int_0^1 \upsilon_0(\|y_* + \theta(\gamma_* - y_*) - \gamma_*\|)d\theta \\ &\leq \int_0^1 \upsilon_0(\|(\theta - 1)(\gamma_* - y_*)\|)d\theta \\ &< \int_0^1 \upsilon_0(|\theta - 1|R)d\theta < 1. \end{aligned} \tag{2.33}$$

Thus, according to the Banach lemma, T is invertible. Since $0 = P(y_*) - P(\gamma_*) = T(y_* - \gamma_*)$, we conclude that $\gamma_* = y_*$. The proof is over.

3. Attractive basins

In this section, we study some dynamical properties of the family of the iterative methods (1.3), which are based on their attractive basins on the complex polynomial $f(z)$. The convergence and stability of the iterative methods are compared by studying the structure of attractive basins.

There are some dynamical concepts and basic results to be used later. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function on the Riemann sphere $\hat{\mathbb{C}}$. The orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as

$$\{z_0, f(z_0), f^2(z_0), \cdots, f^n(z_0), \cdots\}.$$

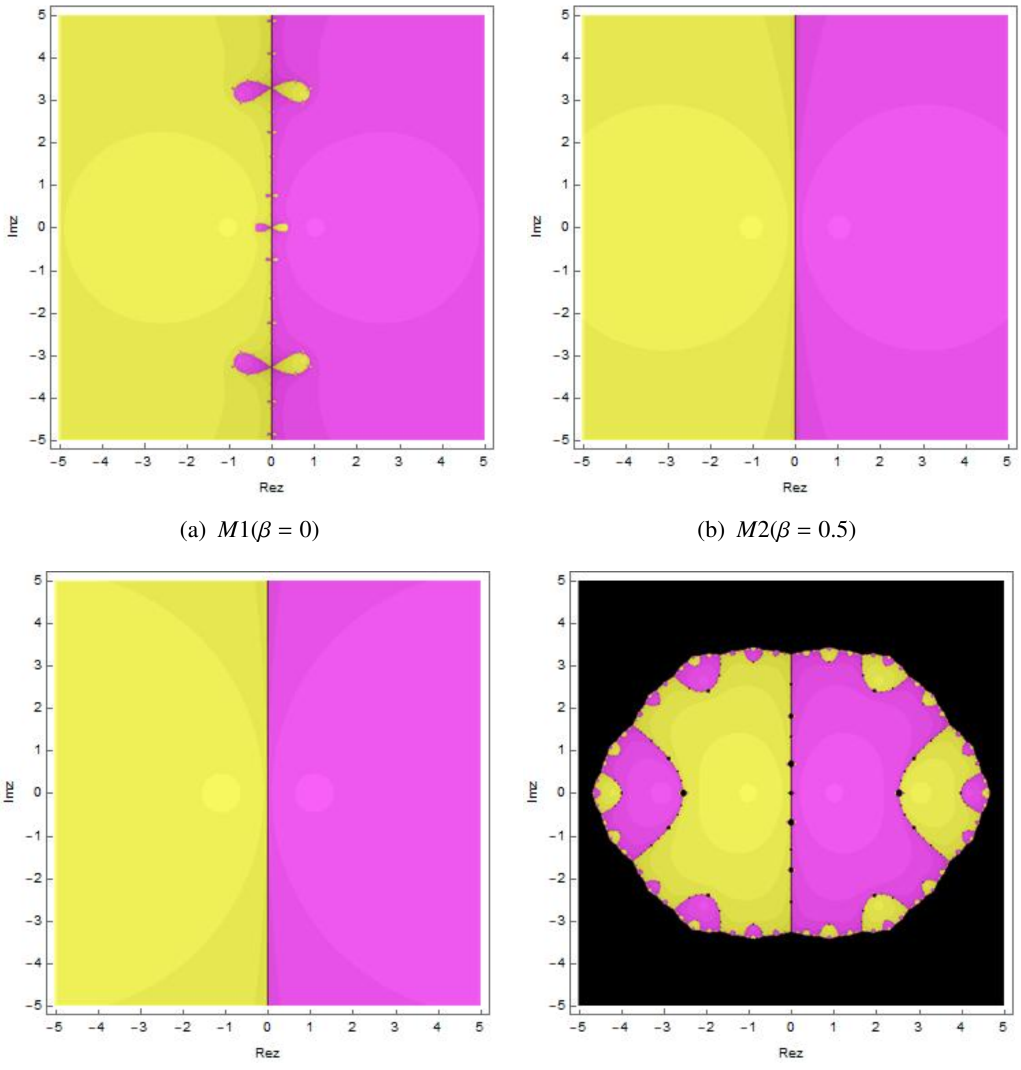
In addition, if $f(z_0) = z_0$, z_0 is a fixed point. There are the following four cases:
- If $|f'(z_0)| < 1$, z_0 is an attractive point;
- If $|f'(z_0)| = 1$, z_0 is a neutral point;
- If $|f'(z_0)| > 1$, z_0 is a repulsive point;
- If $|f'(z_0)| = 0$, z_0 is an super-attractive point.

The basin of attraction of an attractor z_* is defined by

$$\mathcal{A}(z_*) = \{z_0 \in \hat{\mathbb{C}} : f^n(z_0) \to z_*, n \to \infty\}.$$

Consider the following four members of the family (1.3): $M_1(\beta = 0)$, $M_2(\beta = 0.5)$, $M_3(\beta = 1)$, $M_4(\beta = 2)$. In this study, the complex plane is $\Omega = [-5, 5] \times [-5, 5]$ with 500×500 points. If the sequence converges to roots, it is represented in pink, yellow, and blue. Otherwise, black represents other cases, including non-convergence. When the family (1.3) is applied to the complex polynomials $f(z) = z^2 - 1$ and $f(z) = z^3 - 1$, their attractive basins are shown in Figures 1 and 2.

In Figures 1 and 2, the fractal graphs of the methods $M1$ and $M4$ have some black zones. The black zones indicates non-convergence, and the initial value of the black area causes the iteration to fail; relatively speaking, the method without a black region is better. However, the fractal graphs of the methods $M2$ and $M3$ have a black zone. As a result, the convergence of the methods $M2$ and $M3$ is better than that of the methods $M1$ and $M4$. In addition, the method $M3$ has the largest basins of attraction compared to the other three methods. Thus, the stable parameters are $\beta = 0.5, 1$.



(a) $M1(\beta = 0)$ (b) $M2(\beta = 0.5)$

(c) $M3(\beta = 1)$ (d) $M4(\beta = 2)$

Figure 1. Basins of attraction of the methods $M_i (i = 1, 2, 3, 4)$, for $f(z) = z^2 - 1$.

4. Numerical examples

4.1. Radius of convergence ball

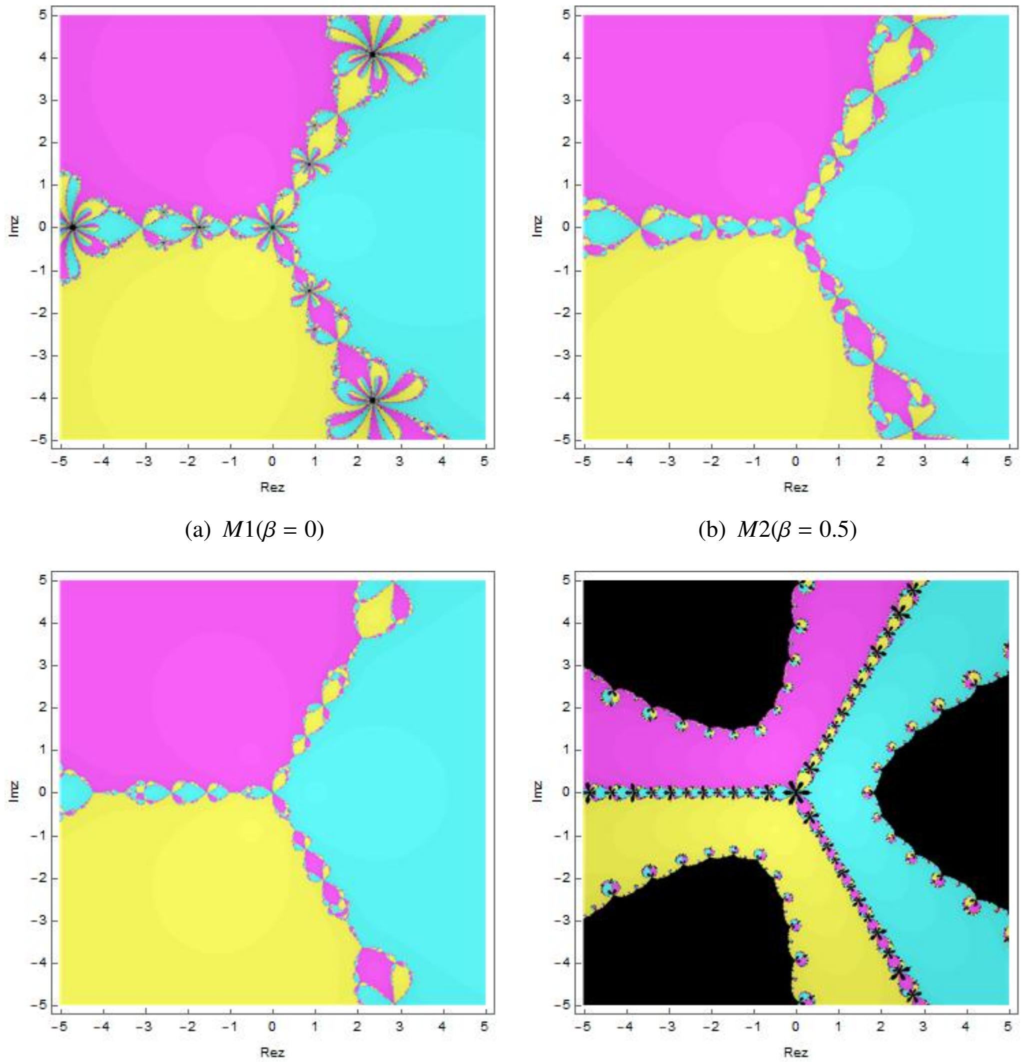
In this section, we apply the following two numerical examples to compute the above results of convergence for method (1.3).

Example 4.1. Let $\Omega = (0, 2)$; define the function $P : \Omega \to \mathbb{R}$ by

$$P(x) = x^3 - 1. \tag{4.1}$$

Thus, a root of $P(x) = 0$ is $\gamma_* = 1$. Then,

$$P'(x) = 3x^2$$



(a) $M1(\beta = 0)$ (b) $M2(\beta = 0.5)$

(c) $M3(\beta = 1)$ (d) $M4(\beta = 2)$

Figure 2. Basins of attraction of the methods $M_i(i = 1, 2, 3, 4)$, for $f(z) = z^3 - 1$.

and

$$[x, y; P] = x^2 + xy + y^2.$$

Notice that using conditions (2.9)–(2.15), $\beta = 0$, we obtain

$$\upsilon_0(\xi) = 3t, \upsilon(t) = \frac{8}{3}t,$$

$$\gamma_0 = \frac{1}{3},\ \Omega_0 = (\frac{2}{3}, \frac{4}{3}),$$

$$\omega_0(t, s) = \frac{10}{9}t + \frac{11}{9}s,\ \omega_1(t) = \frac{16}{3}t,$$

and

$$\omega_2(t, s) = \frac{1}{3}t + \frac{1}{3}s,\ \omega_3(t, s, u) = \frac{1}{3}t + \frac{1}{3}s + \frac{1}{3}u + 1.$$

Then, according to the above definition of functions $h_i(i = 1, 2, 3, 4, 5)$, one have that

$$r_1 \approx 0.230769,\ r_3 \approx 0.221531,\ r_5 \approx 0.130342 = r.$$

Example 4.2. Let $\Omega = (-1, 1)$, define the function $P : \Omega \to \mathbb{R}$ by

$$P(x) = e^x - 1. \tag{4.2}$$

Thus, a root of $P(x) = 0$ is $\gamma_* = 0$. Then,

$$P'(x) = e^x$$

and

$$[x, y; P] = \frac{1}{y - x}(e^y - e^x).$$

Notice that using conditions (2.9)–(2.15), $\beta = 1$, we obtain

$$\upsilon_0(t) = e^t - 1, \upsilon(t) = e^t - 1,$$

$$\gamma_0 = \ln 2,\ \Omega_0 = (-\ln 2, \ln 2),$$

$$\omega_0(t, s) = \frac{1}{t + s}(e^t - e^s) - 1,\ \omega_1(t) = e^t,$$

$$\omega_2(t + s, s + u) = (\frac{1}{(s + u)(u + t)} + \frac{1}{(s + u)^2})(e^u - e^s) - \frac{1}{(s + t)(u + t)}(e^s - e^t),$$

and

$$\omega_3(t, s, u) = \frac{1}{(t + u)(s + u)}(e^u - e^t) - \frac{1}{(t + s)(s + u)}(e^t - e^s).$$

Then, according to the above definition of functions $h_i(i = 1, 2, 3, 4, 5)$, one obtains

$$r_1 \approx 0.511083,\ r_3 \approx 0.270027,\ r_5 \approx 0.210013 = r.$$

4.2. Application

In this section, the iterative method (1.3) is applied to the following six practical models. For the nonlinear equations obtained from the six models, we can find the solutions of the equations and the data results, such as iterative errors. Therefore, our research is valuable for practical models in various fields.

Example 4.3. Vertical stresses [21]: At uniform pressure t, the Boussinesq's formula is used to calculate the vertical stress y caused by a specific point within the elastic material under the edge of the rectangular strip footing. The following formula is obtained:

$$\sigma_y = \frac{t}{\pi}x + cos(x)sin(x). \tag{4.3}$$

If the value of y is determined, we can find the value of x where the vertical stress y equals 25 percent of the applied footing stress t. When $x = 0.4$, the following nonlinear equation is obtained:

$$P_1(s) = \frac{s}{\pi} + \frac{1}{\pi}cos(s)sin(s) - \frac{1}{4}. \tag{4.4}$$

Example 4.4. Civil Engineering Problem [22]: Some horizontal construction projects, such as the topmost portion of civil engineering beams, are used in the mathematical modeling of the beams. In order to describe the exact position of the beam in this particular case, some mathematical models based on nonlinear equations have been established. The following model is given in [22]:

$$P_2(s) = s^4 + 4s^3 - 24s^2 + 16s + 16. \tag{4.5}$$

Example 4.5. The trajectory of an electron moving between two parallel plates is defined by

$$y(l) = s_0 + (v_0 + e\frac{E_0}{m\omega}sin(\omega l_0 + \alpha)) + e\frac{E_0}{m\omega^2}(cos(\omega l + \alpha) + sin(\omega + \alpha)), \tag{4.6}$$

where m and e denote the mass and the charge of the electron at rest, v_0 and s_0 denote the velocity and position of the electron at time l_0, and $E_0 sin(\omega l_0 + \alpha)$ denotes the RF electric field between the plates. By selecting specific values, one obtains

$$P_3(s) = \frac{\pi}{4} + s - \frac{1}{2}cos(s). \tag{4.7}$$

Example 4.6. Blood rheology model [23]: Medical research that concerns the physical and flow characteristics of blood is called blood rheology. Since blood is a non-Newtonian fluid, it is often referred to as a Caisson fluid. Based on the caisson flow characteristics, when the basic fluid, such as water or blood, passes through the tube, it usually maintains its primary structure. When we observe the plug flow of Caisson fluid flow, the following nonlinear equation is considered:

$$P_4(s) = \frac{s^8}{441} - \frac{8s^5}{63} + \frac{16s^2}{9} - 0.05714285714s^4 - 3.624489796s + 0.36, \tag{4.8}$$

where s is the plug flow of Caisson fluid flow.

Example 4.7. Law of population growth [24]: Population dynamics are tested by first-order linear ordinary differential equations in the following way:

$$P'(u) = sP(u) + c, \tag{4.9}$$

where s denotes the population's constant birth rate and c denotes its constant immigration rate. $P(u)$ stands for the population at time u. Then, according to solve the above linear differential equation (4.9), the following equation is obtained:

$$P(u) = (P_0 + \frac{c}{s})e^{su} - \frac{c}{s}, \tag{4.10}$$

where P_0 represents the initial population. According to the different values of the parameter and the initial conditions in [25], a nonlinear equation for calculating the birth rate is obtained:

$$P_5(s) = -e^s(\frac{435}{s} + 1000) + \frac{435}{s} + 1564. \tag{4.11}$$

Example 4.8. The non-smooth function (1.4) is defined on $\Omega = [0, 1]$ by

$$P_6(s) = \begin{cases} s^3 \ln s^2 - s^5 + s^4, & s \neq 0; \\ 0, & s = 0. \end{cases} \tag{4.12}$$

Table 1. Numerical results for the above six models.

Fun	k	s_0	$\lvert P(s_n)-P(s_{n-1})\rvert$	$\lvert P(s_n)\rvert$	$ACOC$	γ_*
P_1	5	2.5	2.21248e-101	1.17865e-101	8.0	0.415856
P_2	5	2.5	0.0000158022	7.82769e-9	8.0	2.000018
P_3	5	4.5	1.07326e-2387	9.10019e-2388	8.0	-0.309093
P_4	5	4.5	3.56215e-517	1.9089e-516	8.0	1.570111
P_5	5	4.5	3.87571e-1076	5.18954e-1073	8.0	0.100998
P_6	5	0.8	6.64779e-258	6.64779e-258	8.0	1.000000

The parameter $\beta = 1$ is selected, and the iterative method (1.3) is applied to the above six practical application examples. Table 1 gives the specific results. k denotes the number of iterations. Fun denotes the function $P_i(1 = 1, 2, 3, 4, 5)$. $|P(s_n) - P(s_{n-1})|$ denotes the error values. $|P(s_n)|$ denotes function values at the last step. Approximated computation order of convergence denotes $ACOC$. γ_* denotes the root of equation $P_i(s) = 0$ $(i = 1, 2, 3, 4, 5)$. The stopping criteria is that if the significant digits of the error precision exceed 5, the output will be made. Approximated computation order of convergence (ACOC) is defined by [26]

$$ACOC \approx \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}. \tag{4.13}$$

In Table 1, for six models, the error accuracy is from 10^{-10} to 10^{-2387}, and the computational order of convergence is the optimal order 8. When the initial point is 2.5, the error and precision of function P_1 are higher than those of function P_2. When the initial point is 4.5, the error and precision of function P_3 are higher than those of functions P_4 and P_5. At the same time, solutions to six decimal places are obtained to improve the accuracy of solutions.

5. Conclusions

In this paper, local convergence analysis of a high-order Chebyshev-type method free from second derivatives is studied under ω-continuity assumptions. In contrast to the conditions used in previous studies, the new conditions of convergence are weaker. This study extends the applicability of method (1.3). Also, the radii of convergence balls and uniqueness of the solution are also discussed. By drawing the basins of attraction, four methods with different parameter values are compared with each other. Thus, we can find that when the parameter $\beta = 1$ of method (1.3), the method $M3$ is relatively more stable. Then, two numerical examples are used to prove the criteria of convergence. Finally, we apply the method (1.3) to six concrete models. In Table 1, the numerical results such as iterative errors, ACOC, and so on are obtained. Therefore, our research is valuable for practical models in various fields.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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