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Research article

The binomial sums for four types of polynomials involving floor and ceiling functions

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Abstract: Several binomial sums are established for the Pell polynomials and the Pell-Lucas polynomials, as well as two types of the Chebyshev polynomials and the Fibonacci-Lucas numbers, which include two special cases proposed by Hideyuki Othsuka in 2024.

Keywords: the binomial theorem; Pell polynomials; Pell-Lucas polynomials; the first kind of Chebyshev polynomials; the second kind of Chebyshev polynomials

1. Introduction and outline

The Pell and Pell-Lucas polynomials have unique values and applications in various branches of mathematics [1–4]. They are defined as follows [5]:

• Recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \quad \text{and} \quad Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x).$$

• Boundary conditions

$$P_0(x) = 0,\ P_1(x) = 1 \quad \text{and} \quad Q_0(x) = 2,\ Q_1(x) = 2x.$$

• Binet formulae

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.1}$$

and

$$Q_n(x) = \alpha^n + \beta^n, \tag{1.2}$$

where $\alpha = \alpha(x) = x + \sqrt{x^2+1}$ and $\beta = \beta(x) = x - \sqrt{x^2+1}$ are the roots of an associated quadratic equation $y^2 - 2xy - 1 = 0$.

• Special values

$$P_n(1) = P_n \quad \text{and} \quad Q_n(1) = Q_n;$$

$$P_n(\tfrac{1}{2}) = F_n \quad \text{and} \quad Q_n(\tfrac{1}{2}) = L_n,$$

where $P_n,\ Q_n,\ F_n$ and L_n are the Pell, Pell-Lucas, Fibonacci, and Lucas numbers, respectively.

There are several kinds of Chebyshev polynomials, which have extensive research and application value in mathematics, engineering, and physics [6, 7]. In particular, the first kind and the second kind polynomials are defined as follows [5, 8, 9]:

• Recurrence relations

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x) \quad \text{and} \quad U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x).$$

• Boundary conditions

$$T_0(x) = 1,\ T_1(x) = x \quad \text{and} \quad U_0(x) = 1,\ U_1(x) = 2x.$$

• Binet formulae

$$T_n(x) = \frac{1}{2}(\eta^n + \gamma^n), \tag{1.3}$$

and

$$U_n(x) = \frac{\eta^{n+1} - \gamma^{n+1}}{\eta - \gamma}, \tag{1.4}$$

where $\eta = \eta(x) = x + \sqrt{x^2 - 1}$ and $\gamma = \gamma(x) = x - \sqrt{x^2 - 1}$ are zeros of the quadratic characteristic equation $y^2 - 2xy + 1 = 0$.

The binomial coefficients are hot topics in combinatorial mathematics. Recently, Othsuka [10] conjectured the following identities about Fibonacci and Lucas numbers:

$$\sum_{k=0}^{n} \binom{n}{k} F_{2\lfloor k/2 \rfloor} = \frac{1}{2}(F_{2n+1} - F_{n+2}) \tag{1.5}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} L_{2\lfloor k/2 \rfloor} = \frac{1}{2}(L_{2n+1} + L_{n+2}). \tag{1.6}$$

Inspired by Eqs (1.5) and (1.6), we shall investigate, in this paper, the following two sums:

$$\sum_{k=0}^{n} \binom{n}{k} \Phi_{2\lfloor k/2 \rfloor + \delta}(x),$$

$$\sum_{k=0}^{n} \binom{n}{k} \Phi_{2\lceil k/2 \rceil + \delta}(x),$$

where $\delta \in \mathbb{N}_0$ is a fixed integer, $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and the ceiling functions [11], respectively. $\Phi_n(x)$ is assigned to one of the four polynomials below:

$$\{P_n(x), Q_n(x), T_n(x), U_n(x)\}.$$

Then we will obtain several binomial sums for four types of polynomials involving the floor and ceiling functions.

The rest of the paper will be organized as follows: In the next four sections, we present summation formulae for the four types of polynomials, respectively. Considering the special values of the Pell and Pell-Lucas polynomials, in Section 6, we provide several identities regarding the Fibonacci-Lucas numbers. Finally, we provide a summary and further observation.

It is worth noting that certain binomial coefficients identities and theorems play significant roles in our derivation:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}, \tag{1.7}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \tag{1.8}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k+1} x^{2k+1} = \frac{(1+x)^n - (1-x)^n}{2}, \tag{1.9}$$

$$\sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} x^{2k} = \frac{(1+x)^n + (1-x)^n}{2}. \tag{1.10}$$

They will be utilized in the proof processes for each type of polynomial. Compared with the two identities of Othsuka [10], our results can not only derive them when x takes special values but also obtain some conclusions about P_n, Q_n, F_n, and L_n that have similarly elegant forms.

2. The Pell polynomials $P_n(x)$

In this section, we will explore the sums when $\Phi_n(x)$ is the Pell polynomials $P_n(x)$, that is, the following two sums:

$$\sum_{k=0}^{n} \binom{n}{k} P_{2\lfloor k/2 \rfloor + \delta}(x), \tag{2.1}$$

$$\sum_{k=0}^{n} \binom{n}{k} P_{2\lceil k/2 \rceil + \delta}(x). \tag{2.2}$$

The main results are enunciated in the following theorem.

Theorem 1.

(i) $$\sum_{k=0}^{n} \binom{n}{k} P_{2\lfloor k/2 \rfloor + \delta}(x) = \frac{1}{2(\alpha-\beta)} \Big\{ (\alpha^{n+\delta} + \beta^{\delta-1})(1-\beta)^{n+1} - (\beta^{n+\delta} + \alpha^{\delta-1})(1-\alpha)^{n+1} \Big\}.$$

(ii) $$\sum_{k=0}^{n} \binom{n}{k} P_{2\lceil k/2 \rceil + \delta}(x) = \frac{1}{2(\alpha-\beta)} \Big\{ (\alpha^{n+1+\delta} - \beta^{\delta})(1-\beta)^{n+1} - (\beta^{n+1+\delta} - \alpha^{\delta})(1-\alpha)^{n+1} \Big\}.$$

Proof of (i)**.**

For the first summation (2.1), we can evaluate it in the following manner:

$$\sum_{k=0}^{n}\binom{n}{k}P_{2\lfloor k/2\rfloor+\delta}(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left\{\binom{n}{2k}P_{2k+\delta}(x)+\binom{n}{2k+1}P_{2k+\delta}(x)\right\}.$$

By recalling equation (1.7), we can rewrite it as follows:

$$\sum_{k=0}^{n}\binom{n}{k}P_{2\lfloor k/2\rfloor+\delta}(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}P_{2k+\delta}(x). \tag{2.3}$$

Then, by making use of the Binet formula of $P_n(x)$ (1.1), it is not hard to make the following calculation:

$$\begin{aligned}&\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}P_{2k+\delta}(x)\\&=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}\frac{\alpha^{2k+\delta}-\beta^{2k+\delta}}{\alpha-\beta}\\&=\frac{1}{\alpha-\beta}\left\{\alpha^{\delta-1}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}\alpha^{2k+1}-\beta^{\delta-1}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}\beta^{2k+1}\right\}.\end{aligned}$$

Finally, using the binomial Theorem (1.9), we can reformulate the final expression below

$$\begin{aligned}&\frac{1}{\alpha-\beta}\left\{\alpha^{\delta-1}\frac{(1+\alpha)^{n+1}-(1-\alpha)^{n+1}}{2}-\beta^{\delta-1}\frac{(1+\beta)^{n+1}-(1-\beta)^{n+1}}{2}\right\}\\=&\frac{1}{2(\alpha-\beta)}\left\{(\alpha^{n+\delta}+\beta^{\delta-1})(1-\beta)^{n+1}-(\beta^{n+\delta}+\alpha^{\delta-1})(1-\alpha)^{n+1}\right\},\end{aligned}$$

where the product of α and β satisfies the relationship $\alpha\beta=-1$.

This completes the proof of (i). □

Proof of (ii)**.**

Analogous to the derivation of (2.3), the second sum (2.2) can be transformed into

$$\sum_{k=0}^{n}\binom{n}{k}P_{2\lceil k/2\rceil+\delta}(x)=\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}P_{2k+\delta}(x).$$

By making use of Binet's formula of $P_n(x)$ (1.1), we obtain

$$\begin{aligned}&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}P_{2k+\delta}(x)\\&=\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\frac{\alpha^{2k+\delta}-\beta^{2k+\delta}}{\alpha-\beta}\end{aligned}$$

$$=\frac{1}{\alpha-\beta}\left\{\alpha^{\delta}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\alpha^{2k}-\beta^{\delta}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\beta^{2k}\right\}.$$

In conjunction with (1.10),
the last expression can be rewritten as

$$\begin{aligned}&\frac{1}{\alpha-\beta}\left\{\alpha^{\delta}\frac{(1+\alpha)^{n+1}+(1-\alpha)^{n+1}}{2}-\beta^{\delta}\frac{(1+\beta)^{n+1}+(1-\beta)^{n+1}}{2}\right\}\\=&\frac{1}{2(\alpha-\beta)}\left\{(\alpha^{n+1+\delta}-\beta^{\delta})(1-\beta)^{n+1}-(\beta^{n+1+\delta}-\alpha^{\delta})(1-\alpha)^{n+1}\right\}.\end{aligned}$$

This completes the proof of (ii). □

Taking $x=\frac{1}{2}$ and $x=1$, the polynomials $P_n(x)$ correspond to the Fibonacci numbers F_n and the Pell numbers P_n. Considering the Binet formulae of F_n , P_n, and Q_n, we derive the following corollaries from the two equations in Theorem 1.

Corollary 2.

(i) $\boxed{x=1}$ $$\sum_{k=0}^{n}\binom{n}{k}P_{2\lfloor k/2\rfloor+\delta}=\begin{cases}\sqrt{2}^{\,n-1}(P_{n+\delta}-P_{\delta-1}), & n\equiv_2 1;\\ \sqrt{2}^{\,n-4}(Q_{n+\delta}+Q_{\delta-1}), & n\equiv_2 0.\end{cases}$$

(ii) $\boxed{x=\frac{1}{2}}$ $$\sum_{k=0}^{n}\binom{n}{k}F_{2\lfloor k/2\rfloor+\delta}=\frac{1}{2}\left\{F_{2n+1+\delta}+(-1)^{\delta-1}F_{n+2-\delta}\right\}.$$

Corollary 3.

(i) $\boxed{x=1}$ $$\sum_{k=0}^{n}\binom{n}{k}P_{2\lceil k/2\rceil+\delta}=\begin{cases}\sqrt{2}^{\,n-1}(P_{n+1+\delta}+P_{\delta}), & n\equiv_2 1;\\ \sqrt{2}^{\,n-4}(Q_{n+1+\delta}-Q_{\delta}), & n\equiv_2 0.\end{cases}$$

(ii) $\boxed{x=\frac{1}{2}}$ $$\sum_{k=0}^{n}\binom{n}{k}F_{2\lceil k/2\rceil+\delta}=\frac{1}{2}\left\{F_{2n+2+\delta}-(-1)^{\delta}F_{n+1-\delta}\right\}.$$

When $\delta=0$ in Corollary 2(ii), we obtain the identity (1.5) proposed by Othsuka [10].

We need to pay attention that there are some cases where P_n, Q_n, and F_n have negative subscripts in corollaries 2 and 3. For example, when $\delta=0$, $P_{\delta-1}=P_{-1}$, $Q_{\delta-1}=Q_{-1}$. Here we point that when P_n, Q_n, F_n, and L_n have negative subscripts, they can be extended through recursive relations below [12, 13]: $(n\geq 0)$

$$P_{-n}=(-1)^{n+1}P_n,\ Q_{-n}=(-1)^nQ_n;$$

$$F_{-n}=(-1)^{n+1}F_n,\ L_{-n}=(-1)^nL_n.$$

3. The Pell-Lucas polynomials $Q_n(x)$

In this section, we will explore the sums when $\Phi_n(x)$ is the Pell-Lucas polynomial $Q_n(x)$, that is, the following two sums:

$$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lfloor k/2\rfloor+\delta}(x), \tag{3.1}$$

$$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lceil k/2\rceil+\delta}(x). \tag{3.2}$$

The main results are enunciated in the following theorem.

Theorem 4.

(i) $$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lfloor k/2\rfloor+\delta}(x)=\frac{1}{2}\Big\{(\alpha^{n+\delta}-\beta^{\delta-1})(1-\beta)^{n+1}+(\beta^{n+\delta}-\alpha^{\delta-1})(1-\alpha)^{n+1}\Big\},$$

(ii) $$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lceil k/2\rceil+\delta}(x)=\frac{1}{2}\Big\{(\beta^{n+1+\delta}+\alpha^{\delta})(1-\alpha)^{n+1}+(\alpha^{n+1+\delta}+\beta^{\delta})(1-\beta)^{n+1}\Big\}.$$

Proof of (i)**.**
For the first summation (3.1), referring to (2.3), we can rewrite it as

$$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lfloor k/2\rfloor+\delta}(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}Q_{2k+\delta}(x).$$

Then, by making use of the Binet formula of $Q_n(x)$ (1.2), we obtain

$$\begin{aligned}&\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}Q_{2k+\delta}(x)\\ =&\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}(\alpha^{2k+\delta}+\beta^{2k+\delta})\\ =&\alpha^{\delta-1}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}\alpha^{2k+1}+\beta^{\delta-1}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}\beta^{2k+1}.\end{aligned}$$

Finally, applying the identity (1.9), the last expression can be rewritten as

$$\begin{aligned}&\alpha^{\delta-1}\frac{(1+\alpha)^{n+1}-(1-\alpha)^{n+1}}{2}+\beta^{\delta-1}\frac{(1+\beta)^{n+1}-(1-\beta)^{n+1}}{2}\\ =&\frac{1}{2}\Big\{(\alpha^{n+\delta}-\beta^{\delta-1})(1-\beta)^{n+1}+(\beta^{n+\delta}-\alpha^{\delta-1})(1-\alpha)^{n+1}\Big\}.\end{aligned}$$

This completes the proof of (i). □

Proof of (ii)**.**

Similar to (2.3), summation (3.2) can be transformed into the following form:

$$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lceil k/2\rceil+\delta}(x)=\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}Q_{2k+\delta}(x).$$

By making use of Binet's formula of $Q_n(x)$ (1.2), it is not hard to make the following calculation:

$$\begin{aligned}&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}Q_{2k+\delta}(x)\\=&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}(\alpha^{2k+\delta}+\beta^{2k+\delta})\\=&\alpha^{\delta}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\alpha^{2k}+\beta^{\delta}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\beta^{2k}.\end{aligned}$$

Applying the identity (1.10), the last expression can be rewritten as

$$\begin{aligned}&\alpha^{\delta}\frac{(1+\alpha)^{n+1}+(1-\alpha)^{n+1}}{2}+\beta^{\delta}\frac{(1+\beta)^{n+1}+(1-\beta)^{n+1}}{2}\\=&\frac{1}{2}\Big\{(\beta^{n+1+\delta}+\alpha^{\delta})(1-\alpha)^{n+1}+(\alpha^{n+1+\delta}+\beta^{\delta})(1-\beta)^{n+1}\Big\}.\end{aligned}$$

This completes the proof of (ii). □

Taking $x=\frac{1}{2},\ x=1$, respectively, and considering the Binet formulae of F_n, P_n, and Q_n, we derive the following corollaries from the two equations in Theorem 4.

Corollary 5.

(i) $\boxed{x=1}$
$$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lfloor k/2\rfloor+\delta}=\begin{cases}\sqrt{2}^{\,n-1}(Q_{n+\delta}-Q_{\delta-1}), & n\equiv_2 1;\\ \sqrt{2}^{\,n+2}(P_{n+\delta}+P_{\delta-1}), & n\equiv_2 0.\end{cases}$$

(ii) $\boxed{x=\frac{1}{2}}$
$$\sum_{k=0}^{n}\binom{n}{k}L_{2\lfloor k/2\rfloor+\delta}=\frac{1}{2}\Big\{L_{2n+1+\delta}-(-1)^{\delta-1}L_{n+2-\delta}\Big\}.$$

Corollary 6.

(i) $\boxed{x=1}$
$$\sum_{k=0}^{n}\binom{n}{k}Q_{2\lceil k/2\rceil+\delta}=\begin{cases}\sqrt{2}^{\,n-1}(Q_{n+1+\delta}+Q_{\delta}), & n\equiv_2 1;\\ \sqrt{2}^{\,n+2}(P_{n+1+\delta}-P_{\delta}), & n\equiv_2 0.\end{cases}$$

(ii) $\boxed{x=\frac{1}{2}}$
$$\sum_{k=0}^{n}\binom{n}{k}L_{2\lceil k/2\rceil+\delta}=\frac{1}{2}\Big\{L_{2n+2+\delta}+(-1)^{\delta}L_{n+1-\delta}\Big\}.$$

When $\delta=0$ in Corollary 5(ii), it's just the identities (1.6) proposed by Othsuka [10].

4. The first kind of Chebyshev polynomials $T_n(x)$

In this section, we will explore the sums when $\Phi_n(x)$ is the first kind of Chebyshev polynomials $T_n(x)$, that is, the following two sums:

$$\sum_{k=0}^{n}\binom{n}{k}T_{2\lfloor k/2\rfloor+\delta}(x), \tag{4.1}$$

$$\sum_{k=0}^{n}\binom{n}{k}T_{2\lceil k/2\rceil+\delta}(x). \tag{4.2}$$

The main results are enunciated in the following theorem.

Theorem 7.

(i) $$\sum_{k=0}^{n}\binom{n}{k}T_{2\lfloor k/2\rfloor+\delta}(x)=\frac{1}{4}\Big\{(\eta^{n+\delta}+\gamma^{\delta-1})(1+\gamma)^{n+1}-[(-1)^{n+1}\eta^{n+\delta}+\gamma^{\delta-1}](1-\gamma)^{n+1}\Big\}.$$

(ii) $$\sum_{k=0}^{n}\binom{n}{k}T_{2\lceil k/2\rceil+\delta}(x)=\frac{1}{4}\Big\{(\eta^{n+1+\delta}+\gamma^{\delta})(1+\gamma)^{n+1}+[(-1)^{n+1}\eta^{n+1+\delta}+\gamma^{\delta}](1-\gamma)^{n+1}\Big\}.$$

Proof of (i)**.**
For the first summation (4.1), referring to (2.3), we have

$$\sum_{k=0}^{n}\binom{n}{k}T_{2\lfloor k/2\rfloor+\delta}(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}T_{2k+\delta}(x).$$

Then, by making use of the Binet formula of $T_n(x)$ (1.3), there has

$$\begin{aligned}&\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}T_{2k+\delta}(x)\\ =&\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}(\eta^{2k+\delta}+\gamma^{2k+\delta})\\ =&\frac{1}{2}\Big\{\eta^{\delta-1}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}\eta^{2k+1}+\gamma^{\delta-1}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}\gamma^{2k+1}\Big\}.\end{aligned}$$

Finally, applying (1.9), the last expression can be rewritten as

$$\begin{aligned}&\frac{1}{2}\Big\{\eta^{\delta-1}\frac{(1+\eta)^{n+1}-(1-\eta)^{n+1}}{2}+\gamma^{\delta-1}\frac{(1+\gamma)^{n+1}-(1-\gamma)^{n+1}}{2}\Big\}\\ =&\frac{1}{4}\Big\{(\eta^{n+\delta}+\gamma^{\delta-1})(1+\gamma)^{n+1}-[(-1)^{n+1}\eta^{n+\delta}+\gamma^{\delta-1}](1-\gamma)^{n+1}\Big\},\end{aligned}$$

where the product of η and γ satisfies the relationship $\eta\gamma = 1$.

This completes the proof of (i).

Proof of (ii)**.**

Analogous to the derivation of (2.3), the second sum (4.2) can be transformed into

$$\sum_{k=0}^{n}\binom{n}{k}T_{2\lceil k/2\rceil+\delta}(x) = \sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}T_{2k+\delta}(x).$$

By making use of Binet's formula of $T_n(x)$ (1.3), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}T_{2k+\delta}(x)\\
=&\frac{1}{2}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}(\eta^{2k+\delta}+\gamma^{2k+\delta})\\
=&\frac{1}{2}\left\{\eta^{\delta}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\eta^{2k}+\gamma^{\delta}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\gamma^{2k}\right\}.
\end{aligned}$$

Applying (1.10), the last expression can be rewritten as

$$\begin{aligned}
&\frac{1}{2}\left\{\eta^{\delta}\frac{(1+\eta)^{n+1}+(1-\eta)^{n+1}}{2}+\gamma^{\delta}\frac{(1+\gamma)^{n+1}+(1-\gamma)^{n+1}}{2}\right\}\\
=&\frac{1}{4}\left\{(\eta^{n+1+\delta}+\gamma^{\delta})(1+\gamma)^{n+1}+[(-1)^{n+1}\eta^{n+1+\delta}+\gamma^{\delta}](1-\gamma)^{n+1}\right\}.
\end{aligned}$$

This completes the proof of (ii).

When $x = 1$, the first kind of Chebyshev polynomials $T_n(1)$ reduce to the constant sequence $\{1\}$. Thus, Theorem 7 reduces to the familiar binomial identity

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n.$$

5. The second kind of Chebyshev polynomials $U_n(x)$

In this section, we will explore the sums when $\Phi_n(x)$ is the second kind of Chebyshev polynomials $U_n(x)$, that is, the following two sums:

$$\sum_{k=0}^{n}\binom{n}{k}U_{2\lfloor k/2\rfloor+\delta}(x), \tag{5.1}$$

$$\sum_{k=0}^{n}\binom{n}{k}U_{2\lceil k/2\rceil+\delta}(x). \tag{5.2}$$

The main results are enunciated in the following theorem.

Theorem 8.

$$\text{(i)} \quad \sum_{k=0}^{n} \binom{n}{k} U_{2\lfloor k/2\rfloor+\delta}(x) = \frac{1}{2(\eta-\gamma)} \Big\{ (\eta^{n+1+\delta} - \gamma^{\delta})(1+\gamma)^{n+1} - [(-1)^{n+1}\eta^{n+1+\delta} - \gamma^{\delta}](1-\gamma)^{n+1} \Big\}.$$

$$\text{(ii)} \quad \sum_{k=0}^{n} \binom{n}{k} U_{2\lceil k/2\rceil+\delta}(x) = \frac{1}{2(\eta-\gamma)} \Big\{ (\eta^{n+2+\delta} - \gamma^{\delta+1})(1+\gamma)^{n+1} + [(-1)^{n+1}\eta^{n+2+\delta} - \gamma^{\delta+1}](1-\gamma)^{n+1} \Big\},$$

Proof of (i)**.**
Similarly to (2.3), we can rewrite (5.1) as

$$\sum_{k=0}^{n} \binom{n}{k} U_{2\lfloor k/2\rfloor+\delta}(x) = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n+1}{2k+1} U_{2k+\delta}(x).$$

Then, by making use of the Binet formula of $U_n(x)$ (1.4), it is not hard to make the following calculation:

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n+1}{2k+1} T_{2k+\delta}(x) \\
=&\frac{1}{\eta-\gamma} \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n+1}{2k+1} (\eta^{2k+\delta+1} - \gamma^{2k+\delta+1}) \\
=&\frac{1}{\eta-\gamma} \Big\{ \eta^{\delta} \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n+1}{2k+1} \eta^{2k+1} - \gamma^{\delta} \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n+1}{2k+1} \gamma^{2k+1} \Big\}.
\end{aligned}$$

Finally, applying the identity (1.9), the last expression can be rewritten as

$$\begin{aligned}
&\frac{1}{\eta-\gamma} \Big\{ \eta^{\delta} \frac{(1+\eta)^{n+1} - (1-\eta)^{n+1}}{2} - \gamma^{\delta} \frac{(1+\gamma)^{n+1} - (1-\gamma)^{n+1}}{2} \Big\} \\
=&\frac{1}{2(\eta-\gamma)} \Big\{ (\eta^{n+1+\delta} - \gamma^{\delta})(1+\gamma)^{n+1} - [(-1)^{n+1}\eta^{n+1+\delta} - \gamma^{\delta}](1-\gamma)^{n+1} \Big\}.
\end{aligned}$$

This completes the proof of (i). □

Proof of (ii)**.**
Similar to (2.3), summation (5.2) can be transformed into the following form:

$$\sum_{k=0}^{n} \binom{n}{k} U_{2\lceil k/2\rceil+\delta}(x) = \sum_{k=0}^{\lceil \frac{n}{2}\rceil} \binom{n+1}{2k} U_{2k+\delta}(x).$$

By making use of Binet's formula of $U_n(x)$ (1.4), there has

$$\sum_{k=0}^{\lceil \frac{n}{2}\rceil} \binom{n+1}{2k} U_{2k+\delta}(x)$$

$$=\frac{1}{\eta-\gamma}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}(\eta^{2k+\delta+1}-\gamma^{2k+\delta+1})$$

$$=\frac{1}{\eta-\gamma}\Big\{\eta^{\delta+1}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\eta^{2k}-\gamma^{\delta+1}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n+1}{2k}\gamma^{2k}\Big\}.$$

Applying the identity (1.10), the last expression can be rewritten as

$$\frac{1}{\eta-\gamma}\Big\{\eta^{\delta+1}\frac{(1+\eta)^{n+1}+(1-\eta)^{n+1}}{2}-\gamma^{\delta+1}\frac{(1+\gamma)^{n+1}+(1-\gamma)^{n+1}}{2}\Big\}$$
$$=\frac{1}{2(\eta-\gamma)}\Big\{(\eta^{n+2+\delta}-\gamma^{\delta+1})(1+\gamma)^{n+1}+[(-1)^{n+1}\eta^{n+2+\delta}-\gamma^{\delta+1}](1-\gamma)^{n+1}\Big\}.$$

This completes the proof of (ii). □

Up to now, for the above four types of polynomials, we found that the results obtained from their summation formulae are not straightforward. As we all know, the extended Fibonacci-Lucas numbers play important roles in combinatorial mathematics [2,4]. Therefore, we carried out similar calculations for the extended Fibonacci-Lucas numbers. Surprisingly, we obtained relatively concise results.

6. The extended Fibonacci-Lucas numbers G_n

For two complex numbers a, b, and a natural number $n \in \mathbb{N}_0$, define the following extended Fibonacci-Lucas numbers $\{G_n(a,b)\}_{n\geq 0}$ by the recurrence relation [14]:

$$G_{n+2}(a,b) = G_{n+1}(a,b) + G_n(a,b)$$

with the initial values being given by $G_0(a,b) = a$ and $G_1(a,b) = b$.

By means of the usual series manipulation [5], it is not hard to show the following explicit formula:

$$G_n(a,b) = \frac{u\hat{\alpha}^n - v\hat{\beta}^n}{\hat{\alpha}-\hat{\beta}}, \tag{6.1}$$

where

$$\hat{\alpha} = \alpha(\tfrac{1}{2}) = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \hat{\beta} = \beta(\tfrac{1}{2}) = \frac{1-\sqrt{5}}{2},$$

as well as

$$u = b - a + a\hat{\alpha} \quad \text{and} \quad v = b - a + a\hat{\beta}.$$

The well-known Fibonacci and Lucas numbers are the following particular cases: $G_n(0,1) = F_n$ and $G_n(2,1) = L_n$. For the sake of brevity, they will be shortened as $G_n = G_n(a,b)$.

In this section, we consider the sums that Φ_n is G_n, that is, the following two sums:

$$\sum_{k=0}^{n}\binom{n}{k}G_{2\lfloor k/2\rfloor+\delta}, \tag{6.2}$$

$$\sum_{k=0}^{n}\binom{n}{k}G_{2\lceil k/2\rceil+\delta}. \tag{6.3}$$

In fact, we can easily derive the results of these two summations using a method similar to that in the previous sections. Consequently, we will omit the derivation process and present the results directly in the following theorems.

Theorem 9.

$$\text{(i)} \quad \sum_{k=0}^{n} \binom{n}{k} G_{2\lfloor k/2 \rfloor+\delta} = \frac{1}{2}\Big\{G_{2n+1+\delta} - (b-a)(-1)^{\delta} F_{n+2-\delta} + a(-1)^{\delta} F_{n+1-\delta}\Big\}.$$

$$\text{(ii)} \quad \sum_{k=0}^{n} \binom{n}{k} G_{2\lceil k/2 \rceil+\delta} = \frac{1}{2}\Big\{G_{2n+2+\delta} - (b-a)(-1)^{\delta} F_{n+1-\delta} + a(-1)^{\delta} F_{n-\delta}\Big\}.$$

Now, we consider the recurrence relation of Fibonacci numbers and the next identity [13]

$$L_n = F_{n+1} + F_{n-1},$$

for $n \geq 1$. It is easy for us to deduce the identity

$$F_{m+1} + 2F_m = L_{m+1}, \tag{6.4}$$

which will be used to deduce the identities of Corollaries 10 and 11.

Taking $a = 0,\ b = 1$, and $a = 2,\ b = 1$ in Theorem 9, respectively, we get the following corollaries.

Corollary 10.

$$\text{(i)} \ \boxed{a = 0,\ b = 1} \quad \sum_{k=0}^{n} \binom{n}{k} F_{2\lfloor k/2 \rfloor+\delta} = \frac{1}{2}\Big\{F_{2n+1+\delta} - (-1)^{\delta} F_{n+2-\delta}\Big\}.$$

$$\text{(ii)} \ \boxed{a = 2,\ b = 1} \quad \sum_{k=0}^{n} \binom{n}{k} L_{2\lfloor k/2 \rfloor+\delta} = \frac{1}{2}\Big\{L_{2n+1+\delta} + (-1)^{\delta} L_{n+2-\delta}\Big\}.$$

When $\delta = 0$ in the last corollary, the identities reduced to the results of (1.5) and (1.6) proposed by Othsuka [10].

Corollary 11.

$$\text{(i)} \ \boxed{a = 0,\ b = 1} \quad \sum_{k=0}^{n} \binom{n}{k} F_{2\lceil k/2 \rceil+\delta} = \frac{1}{2}\Big\{F_{2n+2+\delta} - (-1)^{\delta} F_{n+1-\delta}\Big\}.$$

$$\text{(ii)} \ \boxed{a = 2,\ b = 1} \quad \sum_{k=0}^{n} \binom{n}{k} L_{2\lceil k/2 \rceil+\delta} = \frac{1}{2}\Big\{L_{2n+2+\delta} + (-1)^{\delta} L_{n+1-\delta}\Big\}.$$

7. Conclusions and further observations

In this paper, we successfully generalized the two identities conjectured by Othsuka [10] by introducing four types of polynomials. Additionally, we also obtained some conclusions about the Pell numbers P_n, the Pell-Lucas numbers Q_n, and the extended Fibonacci-Lucas numbers G_n that have similarly elegant forms when $x = 1$. Interested readers are encouraged to explore the results of similar sums of other polynomials by the method used in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. M. Bai, W. Chu, D. Guo, Reciprocal formulae among Pell and Lucas polynomials, *Mathematics*, **10** (2022), 2691. https://doi.org/10.3390/math10152691
2. Z. Cerin, G. M. Gianella, On sums of squares of Pell-Lucas numbers, *Integers: Electron. J. Comb. Number Theory*, **6** (2006), 1–16.
3. D. Guo, W. Chu, Sums of Pell/Lucas polynomials and Fibonacci/Lucas numbers, *Mathematics*, **10** (2022), 2667. https://doi.org/10.3390/math10152667
4. D. Guo, W. Chu, Inverse tangent series Involving Pell and Pell-Lucas polynomials, *Math. Slovaca*, **72** (2022), 869–884. https://doi.org/10.1515/ms-2022-0059
5. W. Chu, V. Vincenti, Funzione generatrice e polinomi incompleti di Fibonacci e Lucas, *Boll. Un. Mat. Ital. Ser. VIII*, **6** (2003), 289–308.
6. K. Adegoke, R. Frontczak, T. Goy, Binomial Fibonacci sums from Chebyshev polynomials, *J. Integer Sequences*, **26** (2023), 1–26.
7. R. Frontczak, T. Goy, Chebyshev-Fibonacci polynomial relations using generating functions, preprint, arXiv:2103.08015.
8. Z. Fan, W. Chu, Convolutions involving Chebyshev polynomials, *Electron. J. Math.*, **3** (2022), 38–64. https://doi.org/10.47443/ejm.2022.012
9. J. C. Mason, D. C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall/CRC, New York, 2002. https://doi.org/10.1201/9781420036114
10. H. Kwong, Elementary problems and solutions (B-1345 Proposed by Hideyuki Othsuka, Saitama, Japan), *Fibonacci Q.*, **62** (2024), 84–89.
11. D. Shah, M. Sahni, R. Sahni, E. León-Castro, M. Olazabal-Lugo, Series of floor and ceiling function-Part I: Partial summations, *Mathematics*, **10** (2022), 1178. https://doi.org/10.3390/math10071178
12. A. F. Horadam, B. J. M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Q.*, **23** (1985), 7–20. https://doi.org/10.1080/00150517.1985.12429849

13. V. E. Hoggatt, *Fibonacci and Lucas Numbers*, Houghton Mifflin Company, Boston, 1969.

14. W. Chu, N. Li, Power Sums of Fibonacci and Lucas Numbers, *Quaestiones Math.*, **34** (2011), 75–83. https://doi.org/10.2989/16073606.2011.570298