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Research article

Nonabelian embedding tensors on 3-Lie algebras and 3-Leibniz-Lie algebras

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Abstract: The purpose of this paper is to study nonabelian embedding tensors on 3-Lie algebras, and to explore the fundamental algebraic structures, cohomology and deformations associated with them. First, we introduce the concept of nonabelian embedding tensors on 3-Lie algebras. Then, we present the concept of a 3-Leibniz-Lie algebra, which constitutes the fundamental algebraic framework for a nonabelian embedding tensor on a 3-Lie algebra. Additionally, we examine the 3-Leibniz-Lie algebras that are derived from Leibniz-Lie algebras. Finally, we develop the cohomology of nonabelian embedding tensors on 3-Lie algebras and utilize the first cohomology group to characterize infinitesimal deformations.

Keywords: 3-Lie algebra; nonabelian embedding tensor; 3-Leibniz algebra; 3-Leibniz-Lie algebra; cohomology; infinitesimal deformation

1. Introduction

The concept of embedded tensors initially emerged in the research on gauged supergravity theory [1]. Using embedding tensors, the $\mathcal{N} = 8$ supersymmetric gauge theories as well as the Bagger-Lambert theory of multiple M2-branes were investigated in [2]. See [3–5] and the references therein for a great deal of literature on embedding tensors and related tensor hierarchies. In [6], the authors first observed the mathematical essence behind the embedding tensor and proved that the embedding tensor naturally produced Leibniz algebra. In the application of physics, they observed that in the construction of the corresponding gauge theory, they focused more on Leibniz algebra than on embedding tensor.

In [7], Sheng et al. considered cohomology, deformations, and homotopy theory for embedding tensors and Lie-Leibniz triples. Later on, the deformation and cohomology theory of embedding tensors on 3-Lie algebras were extensively elaborated in [8]. Tang and Sheng [9] first proposed the concept of a nonabelian embedding tensor on Lie algebras, which is a nonabelian generalization of the embedding tensors, and gave the algebraic structures behind the nonabelian embedding tensors as Leibniz-Lie algebras. This generalization for embedding tensors on associative algebras has been previously explored in [10, 11], where they are referred to as average operators with any nonzero weights. Moreover, the nonabelian embedding tensor on Lie algebras has been extended to the Hom setting in [12].

On the other hand, Filippov [13] first introduced the concepts of 3-Lie algebras and, more generally, n-Lie algebras (also called Filippov algebras). Over recent years, the study and application of 3-Lie algebras have expanded significantly across the realms of mathematics and physics, including string theory, Nambu mechanics [14], and M2-branes [15, 16]. Further research on 3-Lie algebras could be found in [17–19] and references cited therein.

Drawing inspiration from Tang and Sheng's [9] terminology of nonabelian embedding tensors and recognizing the significance of 3-Lie algebras, cohomology, and deformation theories, this paper primarily investigates the nonabelian embedding tensors on 3-Lie algebras, along with their fundamental algebraic structures, cohomology, and deformations.

This paper is organized as follows: Section 2 first recalls some basic notions of 3-Lie algebras and 3-Leibniz algebras. Then we introduce the coherent action of a 3-Lie algebra on another 3-Lie algebra and the notion of nonabelian embedding tensors on 3-Lie algebras with respect to a coherent action. In Section 3, the concept of 3-Leibniz-Lie algebra is presented as the fundamental algebraic structure for a nonabelian embedding tensor on the 3-Lie algebra. Naturally, a 3-Leibniz-Lie algebra induces a 3-Leibniz algebra. Subsequently, we study 3-Leibniz-Lie algebras induced by Leibniz-Lie algebras. In Section 4, the cohomology theory of nonabelian embedding tensors on 3-Lie algebras is introduced. As an application, we characterize the infinitesimal deformation using the first cohomology group.

All vector spaces and algebras considered in this paper are on the field $\mathbb{K}$ with the characteristic of 0.

2. Nonabelian embedding tensors on 3-Lie algebras

This section recalls some basic notions of 3-Lie algebras and 3-Leibniz algebras. After that, we introduce the coherent action of a 3-Lie algebra on another 3-Lie algebra, and we introduce the concept of nonabelian embedding tensors on 3-Lie algebras by its coherent action as a nonabelian generalization of embedding tensors on 3-Lie algebras [8].

Definition 2.1. *(see [13]) A 3-Lie algebra is a pair $(L, [-,-,-]_L)$ consisting of a vector space L and a skew-symmetric ternary operation $[-,-,-]_L : \wedge^3 L \to L$ such that*

$$[l_1, l_2, [l_3, l_4, l_5]_L]_L = [[l_1, l_2, l_3]_L, l_4, l_5]_L + [l_3, [l_1, l_2, l_4]_L, l_5]_L + [l_3, l_4, [l_1, l_2, l_5]_L]_L, \tag{2.1}$$

for all $l_i \in L,\ 1 \le i \le 5$.

A homomorphism between two 3-Lie algebras $(L_1, [-,-,-]_{L_1})$ and $(L_2, [-,-,-]_{L_2})$ is a linear map $f : L_1 \to L_2$ that satisfies $f([l_1, l_2, l_3]_{L_1}) = [f(l_1), f(l_2), f(l_3)]_{L_2}$, for all $l_1, l_2, l_3 \in L_1$.

Definition 2.2. *1) (see [20]) A representation of a 3-Lie algebra $(L, [-,-,-]_L)$ on a vector space H is a skew-symmetric linear map $\rho : \wedge^2 L \to \mathrm{End}(H)$, such that*

$$\rho([l_1, l_2, l_3]_L, l_4) = \rho(l_2, l_3)\rho(l_1, l_4) + \rho(l_3, l_1)\rho(l_2, l_4) + \rho(l_1, l_2)\rho(l_3, l_4), \tag{2.2}$$

$$\rho(l_1,l_2)\rho(l_3,l_4)=\rho(l_3,l_4)\rho(l_1,l_2)+\rho([l_1,l_2,l_3]_L,l_4)+\rho(l_3,[l_1,l_2,l_4]_L), \tag{2.3}$$

for all $l_1,l_2,l_3,l_4\in L$. We also denote a representation of L on H by $(H;\rho)$.
2) A coherent action of a 3-Lie algebra $(L,[-,-,-]_L)$ on another 3-Lie algebra $(H,[-,-,-]_H)$ is defined by a skew-symmetric linear map $\rho:\wedge^2L\to \mathrm{Der}(H)$ that satisfies Eqs (2.2) and (2.3), along with the condition that

$$[\rho(l_1,l_2)h_1,h_2,h_3]_H=0, \tag{2.4}$$

for all $l_1,l_2\in L$ and $h_1,h_2,h_3\in H$. We denote a coherent action of L on H by $(H,[-,-,-]_H;\rho^{\dagger})$.

Note that Eq (2.4) and $\rho(l_1,l_2)\in\mathrm{Der}(H)$ imply that

$$\rho(l_1,l_2)[h_1,h_2,h_3]_H=0. \tag{2.5}$$

Example 2.3. *Let $(H,[-,-,-]_H)$ be a 3-Lie algebra. Define $ad:\wedge^2H\to\mathrm{Der}(H)$ by*

$$ad(h_1,h_2)h:=[h_1,h_2,h]_H,\ \ \text{for all}\ \ h_1,h_2,h\in H.$$

Then $(H;ad)$ is a representation of $(H,[-,-,-]_H)$, which is called the adjoint representation. Furthermore, if the ad satisfies

$$[ad(h_1,h_2)h_1',h_2',h_3']_H=0,\ \ \text{for all}\ h_1',h_2',h_3'\in H,$$

then $(H,[-,-,-]_H;ad^{\dagger})$ is a coherent adjoint action of $(H,[-,-,-]_H)$.

Definition 2.4. *(see [21]) A 3-Leibniz algebra is a vector space $\mathcal{L}$ together with a ternary operation $[-,-,-]_{\mathcal{L}}:\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}\to\mathcal{L}$ such that*

$$[l_1,l_2,[l_3,l_4,l_5]_{\mathcal{L}}]_{\mathcal{L}}=[[l_1,l_2,l_3]_{\mathcal{L}},l_4,l_5]_{\mathcal{L}}+[l_3,[l_1,l_2,l_4]_{\mathcal{L}},l_5]_{\mathcal{L}}+[l_3,l_4,[l_1,l_2,l_5]_{\mathcal{L}}]_{\mathcal{L}},$$

for all $l_i\in\mathcal{L}, 1\le i\le 5$.

Proposition 2.5. *Let $(L,[-,-,-]_L)$ and $(H,[-,-,-]_H)$ be two 3-Lie algebras, and let ρ be a coherent action of L on H. Then, $L\oplus H$ is a 3-Leibniz algebra under the following map:*

$$[l_1+h_1,l_2+h_2,l_3+h_3]_\rho:=[l_1,l_2,l_3]_L+\rho(l_1,l_2)h_3+[h_1,h_2,h_3]_H,$$

for all $l_1,l_2,l_3\in L$ and $h_1,h_2,h_3\in H$. This 3-Leibniz algebra $(L\oplus H,[-,-,-]_\rho)$ is called the nonabelian hemisemidirect product 3-Leibniz algebra, which is denoted by $L\ltimes_\rho H$.

Proof. For any $l_1,l_2,l_3,l_4,l_5\in L$ and $h_1,h_2,h_3,h_4,h_5\in H$, by Eqs (2.1)–(2.5), we have

$$\begin{aligned}
&[l_1+h_1,l_2+h_2,[l_3+h_3,l_4+h_4,l_5+h_5]_\rho]_\rho-[[l_1+h_1,l_2+h_2,l_3+h_3]_\rho,l_4+h_4,l_5+h_5]_\rho\\
&\quad-[l_3+h_3,[l_1+h_1,l_2+h_2,l_4+h_4]_\rho,l_5+h_5]_\rho-[l_3+h_3,l_4+h_4,[l_1+h_1,l_2+h_2,l_5+h_5]_\rho]_\rho\\
=&[l_1,l_2,[l_3,l_4,l_5]_L]_L+\rho(l_1,l_2)\rho(l_3,l_4)h_5+\rho(l_1,l_2)[h_3,h_4,h_5]_H+[h_1,h_2,\rho(l_3,l_4)h_5]_H\\
&\quad+[h_1,h_2,[h_3,h_4,h_5]_H]_H-[[l_1,l_2,l_3]_L,l_4,l_5]_L-\rho([l_1,l_2,l_3]_L,l_4)h_5-[\rho(l_1,l_2)h_3,h_4,h_5]_H\\
&\quad-[[h_1,h_2,h_3]_H,h_4,h_5]_H-[l_3,[l_1,l_2,l_4]_L,l_5]_L-\rho(l_3,[l_1,l_2,l_4]_L)h_5-[h_3,\rho(l_1,l_2)h_4,h_5]_H
\end{aligned}$$

$$
\begin{aligned}
&-[h_3,[h_1,h_2,h_4]_H,h_5]_H-[l_3,l_4,[l_1,l_2,l_5]_L]_L-\rho(l_3,l_4)\rho(l_1,l_2)h_5-\rho(l_3,l_4)[h_1,h_2,h_5]_H\\
&-[h_3,h_4,\rho(l_1,l_2)h_5]_H-[h_3,h_4,[h_1,h_2,h_5]_H]_H\\
=&[h_1,h_2,\rho(l_3,l_4)h_5]_H-\rho(l_3,l_4)[h_1,h_2,h_5]_H\\
=&0.
\end{aligned}
$$

Thus, $(L\oplus H,[-,-,-]_\rho)$ is a 3-Leibniz algebra. □

Definition 2.6. *1) A nonabelian embedding tensor on a 3-algebra $(L,[-,-,-]_L)$ with respect to a coherent action $(H,[-,-,-]_H;\rho^\dagger)$ is a linear map $\Lambda: H\to L$ that satisfies the following equation:*

$$[\Lambda h_1,\Lambda h_2,\Lambda h_3]_L=\Lambda(\rho(\Lambda h_1,\Lambda h_2)h_3+[h_1,h_2,h_3]_H), \tag{2.6}$$

for all $h_1,h_2,h_3\in H$.

2) A nonabelian embedding tensor 3-Lie algebra is a triple (H,L,Λ) consisting of a 3-Lie algebra $(L,[-,-,-]_L)$, a coherent action $(H,[-,-,-]_H;\rho^\dagger)$ of L and a nonabelian embedding tensor $\Lambda: H\to L$. We denote a nonabelian embedding tensor 3-Lie algebra (H,L,Λ) by the notation $H\xrightarrow{\Lambda}L$.

3) Let $H\xrightarrow{\Lambda_1}L$ and $H\xrightarrow{\Lambda_2}L$ be two nonabelian embedding tensor 3-Lie algebras. Then, a homomorphism from $H\xrightarrow{\Lambda_1}L$ to $H\xrightarrow{\Lambda_2}L$ consists of two 3-Lie algebras homomorphisms $f_L: L\to L$ and $f_H: H\to H$, which satisfy the following equations:

$$\Lambda_2\circ f_H=f_L\circ\Lambda_1, \tag{2.7}$$

$$f_H(\rho(l_1,l_2)h)=\rho(f_L(l_1),f_L(l_2))f_H(h), \tag{2.8}$$

for all $l_1,l_2\in L$ and $h\in H$. Furthermore, if f_L and f_H are nondegenerate, (f_L,f_H) is called an isomorphism from $H\xrightarrow{\Lambda_1}L$ to $H\xrightarrow{\Lambda_2}L$.

Remark 2.7. *If $(H,[-,-,-]_H)$ is an abelian 3-Lie algebra, then we can get that Λ is an embedding tensor on 3-Lie algebra (see [8]). In addition, If $\rho=0$, then Λ is a 3-Lie algebra homomorphism from H to L.*

Example 2.8. *Let H be a 4-dimensional linear space spanned by $\alpha_1,\alpha_2,\alpha_3$ and α_4. We define a skew-symmetric ternary operation $[-,-,-]_H:\wedge^3H\to H$ by*

$$[\alpha_1,\alpha_2,\alpha_3]_H=\alpha_4.$$

Then $(H,[-,-,-]_H)$ is a 3-Lie algebra. It is obvious that $(H,[-,-,-]_H;ad^\dagger)$ is a coherent adjoint action of $(H,[-,-,-]_H)$. Moreover,

$$\Lambda=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

is a nonabelian embedding tensor on $(H,[-,-,-]_H)$.

Next, we use graphs to describe nonabelian embedding tensors on 3-Lie algebras.

Theorem 2.9. *A linear map $\Lambda : H \to L$ is a nonabelian embedding tensor on a 3-Lie algebra $(L, [-,-,-]_L)$ with respect to the coherent action $(H, [-,-,-]_H; \rho^\dagger)$ if and only if the graph $Gr(\Lambda) = \{\Lambda h + h \mid h \in H\}$ forms a subalgebra of the nonabelian hemisemidirect product 3-Leibniz algebra $L \ltimes_\rho H$.*

Proof. Let $\Lambda : H \to L$ be a linear map. Then, for any $h_1, h_2, h_3 \in H$, we have

$$[\Lambda h_1 + h_1, \Lambda h_2 + h_2, \Lambda h_3 + h_3]_\rho = [\Lambda h_1, \Lambda h_2, \Lambda h_3]_L + \rho(\Lambda h_1, \Lambda h_2)h_3 + [h_1, h_2, h_3]_H,$$

Thus, the graph $Gr(\Lambda) = \{\Lambda h + h \mid h \in H\}$ is a subalgebra of the nonabelian hemisemidirect product 3-Leibniz algebra $L \ltimes_\rho H$ if and only if Λ satisfies Eq (2.6), which implies that Λ is a nonabelian embedding tensor on L with respect to the coherent action $(H, [-,-,-]_H; \rho^\dagger)$. □

Because H and $Gr(\Lambda)$ are isomorphic as linear spaces, there is an induced 3-Leibniz algebra structure on H.

Corollary 2.10. *Let $H \xrightarrow{\Lambda} L$ be a nonabelian embedding tensor 3-Lie algebra. If a linear map $[-,-,-]_\Lambda : \wedge^3 H \to H$ is given by*

$$[h_1, h_2, h_3]_\Lambda = \rho(\Lambda h_1, \Lambda h_2)h_3 + [h_1, h_2, h_3]_H, \tag{2.9}$$

for all $h_1, h_2, h_3 \in H$, then $(H, [-,-,-]_\Lambda)$ is a 3-Leibniz algebra. Moreover, Λ is a homomorphism from the 3-Leibniz algebra $(H, [-,-,-]_\Lambda)$ to the 3-Lie algebra $(L, [-,-,-]_L)$. This 3-Leibniz algebra $(H, [-,-,-]_\Lambda)$ is called the descendent 3-Leibniz algebra.

Proposition 2.11. *Let (f_L, f_H) be a homomorphism from $H \xrightarrow{\Lambda_1} L$ to $H \xrightarrow{\Lambda_2} L$. Then f_H is a homomorphism of descendent 3-Leibniz algebra from $(H, [-,-,-]_{\Lambda_1})$ to $(H, [-,-,-]_{\Lambda_2})$.*

Proof. For any $h_1, h_2, h_3 \in H$, by Eqs (2.7)–(2.9), we have

$$\begin{aligned} f_H([h_1, h_2, h_3]_{\Lambda_1}) =& f_H(\rho(\Lambda_1 h_1, \Lambda_1 h_2)h_3 + [h_1, h_2, h_3]_H) \\ =& \rho(f_L(\Lambda_1 h_1), f_L(\Lambda_1 h_2))f_H(h_3) + f_H([h_1, h_2, h_3]_H) \\ =& \rho(\Lambda_2 f_L(h_1), \Lambda_2 f_L(h_2))f_H(h_3) + [f_H(h_1), f_H(h_2), f_H(h_3)]_H \\ =& [f_H(h_1), f_H(h_2), f_H(h_3)]_{\Lambda_2}. \end{aligned}$$

The proof is finished. □

3. 3-Leibniz-Lie algebras

In this section, we present the concept of the 3-Leibniz-Lie algebra, which serves as the fundamental algebraic framework for the nonabelian embedding tensor 3-Lie algebra. Then we study 3-Leibniz-Lie algebras induced by Leibniz-Lie algebras.

Definition 3.1. *A 3-Leibniz-Lie algebra $(H, [-,-,-]_H, \{-,-,-\}_H)$ encompasses a 3-Lie algebra $(H, [-,-,-]_H)$ and a ternary operation $\{-,-,-\}_H : \wedge^3 H \to H$, which satisfies the following equations:*

$$\{h_1, h_2, h_3\}_H = -\{h_2, h_1, h_3\}_H, \tag{3.1}$$

$$\{h_1,h_2,\{h_3,h_4,h_5\}_H\}_H =\{\{h_1,h_2,h_3\}_H,h_4,h_5\}_H+\{h_3,\{h_1,h_2,h_4\}_H,h_5\}_H+ \\ \{h_3,h_4,\{h_1,h_2,h_5\}_H\}_H+\{[h_1,h_2,h_3]_H,h_4,h_5\}_H+\{h_3,[h_1,h_2,h_4]_H,h_5\}_H, \quad (3.2)$$

$$\{h_1,h_2,[h_3,h_4,h_5]_H\}_H =[\{h_1,h_2,h_3\}_H,h_4,h_5]_H=0, \quad (3.3)$$

for all $h_1,h_2,h_3,h_4,h_5\in H$.

A homomorphism between two 3-Leibniz-Lie algebras $(H_1,[-,-,-]_{H_1},\{-,-,-\}_{H_1})$ and $(H_2,[-,-,-]_{H_2},\{-,-,-\}_{H_2})$ is a 3-Lie algebra homomorphism $f:(H_1,[-,-,-]_{H_1})\to(H_2,[-,-,-]_{H_2})$ such that $f(\{h_1,h_2,h_3\}_{H_1})=\{f(h_1),f(h_2),f(h_3)\}_{H_2}$, for all $h_1,h_2,h_3\in H_1$.

Remark 3.2. *A 3-Lie algebra* $(H,[-,-,-]_H)$ *naturally constitutes a 3-Leibniz-Lie algebra provided that the underlying ternary operation* $\{h_1,h_2,h_3\}_H=0$*, for all* $h_1,h_2,h_3\in H$.

Example 3.3. *Let* $(H,[-,-,-]_H)$ *be a 4-dimensional 3-Lie algebra given in Example 2.8. We define a nonzero operation* $\{-,-,-\}_H:\wedge^3H\to H$ *by*

$$\{\alpha_1,\alpha_2,\alpha_3\}_H=-\{\alpha_2,\alpha_1,\alpha_3\}_H=\alpha_4.$$

Then $(H,[-,-,-]_H,\{-,-,-\}_H)$ *is a 3-Leibniz-Lie algebra.*

The subsequent theorem demonstrates that a 3-Leibniz-Lie algebra inherently gives rise to a 3-Leibniz algebra.

Theorem 3.4. *Let* $(H,[-,-,-]_H,\{-,-,-\}_H)$ *be a 3-Leibniz-Lie algebra. Then the ternary operation* $\langle-,-,-\rangle_H:\wedge^3H\to H$*, defined as*

$$\langle h_1,h_2,h_3\rangle_H:=[h_1,h_2,h_3]_H+\{h_1,h_2,h_3\}_H, \quad (3.4)$$

for all $h_1,h_2,h_3\in H$*, establishes a 3-Leibniz algebra structure on* H*. This structure is denoted by* $(H,\langle-,-,-\rangle_H)$ *and is referred to as the subadjacent 3-Leibniz algebra.*

Proof. For any $h_1,h_2,h_3,h_4,h_5\in H$, according to $(H,[-,-,-]_H)$ is a 3-Lie algebra and Eqs (3.2)–(3.4), we have

$$\begin{aligned}
&\langle h_1,h_2,\langle h_3,h_4,h_5\rangle_H\rangle_H-\langle\langle h_1,h_2,h_3\rangle_H,h_4,h_5\rangle_H-\langle h_3,\langle h_1,h_2,h_4\rangle_H,h_5\rangle_H\\
&-\langle h_3,h_4,\langle h_1,h_2,h_5\rangle_H\rangle_H\\
=&[h_1,h_2,[h_3,h_4,h_5]_H]_H+[h_1,h_2,\{h_3,h_4,h_5\}_H]_H+\{h_1,h_2,[h_3,h_4,h_5]_H\}_H\\
&+\{h_1,h_2,\{h_3,h_4,h_5\}_H\}_H-[[h_1,h_2,h_3]_H,h_4,h_5]_H-[\{h_1,h_2,h_3\}_H,h_4,h_5]_H\\
&-\{[h_1,h_2,h_3]_H,h_4,h_5\}_H-\{\{h_1,h_2,h_3\}_H,h_4,h_5\}_H-[h_3,[h_1,h_2,h_4]_H,h_5]_H\\
&-[h_3,\{h_1,h_2,h_4\}_H,h_5]_H-\{h_3,[h_1,h_2,h_4]_H,h_5\}_H-\{h_3,\{h_1,h_2,h_4\}_H,h_5\}_H\\
&-[h_3,h_4,[h_1,h_2,h_5]_H]_H-[h_3,h_4,\{h_1,h_2,h_5\}_H]_H-\{h_3,h_4,[h_1,h_2,h_5]_H\}_H\\
&-\{h_3,h_4,\{h_1,h_2,h_5\}_H\}_H\\
=&\{h_1,h_2,\{h_3,h_4,h_5\}_H\}_H-\{[h_1,h_2,h_3]_H,h_4,h_5\}_H-\{\{h_1,h_2,h_3\}_H,h_4,h_5\}_H\\
&-\{h_3,[h_1,h_2,h_4]_H,h_5\}_H-\{h_3,\{h_1,h_2,h_4\}_H,h_5\}_H-\{h_3,h_4,\{h_1,h_2,h_5\}_H\}_H\\
=&0.
\end{aligned}$$

Hence, $(H,\langle-,-,-\rangle_H)$ is a 3-Leibniz algebra. □

The following theorem shows that a nonabelian embedding tensor 3-Lie algebra induces a 3-Leibniz-Lie algebra.

Theorem 3.5. *Let $H \xrightarrow{\Lambda} L$ be a nonabelian embedding tensor 3-Lie algebra. Then $(H, [-,-,-]_H, \{-,-,-\}_\Lambda)$ is a 3-Leibniz-Lie algebra, where*

$$\{h_1, h_2, h_3\}_\Lambda := \rho(\Lambda h_1, \Lambda h_2)h_3, \tag{3.5}$$

for all $h_1, h_2, h_3 \in H$.

Proof. For any $h_1, h_2, h_3, h_4, h_5 \in H$, by Eqs (2.3), (2.6), and (3.5), we have

$$\begin{aligned}
&\{h_1, h_2, h_3\}_\Lambda = \rho(\Lambda h_1, \Lambda h_2)h_3 = -\rho(\Lambda h_2, \Lambda h_1)h_3 = -\{h_2, h_1, h_3\}_\Lambda,\\
&\{\{h_1, h_2, h_3\}_\Lambda, h_4, h_5\}_\Lambda + \{h_3, \{h_1, h_2, h_4\}_\Lambda, h_5\}_\Lambda + \{h_3, h_4, \{h_1, h_2, h_5\}_\Lambda\}_\Lambda\\
&\quad + \{[h_1, h_2, h_3]_H, h_4, h_5\}_\Lambda + \{h_3, [h_1, h_2, h_4]_H, h_5\}_\Lambda - \{h_1, h_2, \{h_3, h_4, h_5\}_\Lambda\}_\Lambda\\
=&\rho(\Lambda\rho(\Lambda h_1, \Lambda h_2)h_3, \Lambda h_4)h_5 + \rho(\Lambda h_3, \Lambda\rho(\Lambda h_1, \Lambda h_2)h_4)h_5 + \rho(\Lambda h_3, \Lambda h_4)\rho(\Lambda h_1, \Lambda h_2)h_5\\
&\quad + \rho(\Lambda[h_1, h_2, h_3]_H, \Lambda h_4)h_5 + \rho(\Lambda h_3, \Lambda[h_1, h_2, h_4]_H)h_5 - \rho(\Lambda h_1, \Lambda h_2)\rho(\Lambda h_3, \Lambda h_4)h_5\\
=&\rho(\Lambda\rho(\Lambda h_1, \Lambda h_2)h_3, \Lambda h_4)h_5 + \rho(\Lambda h_3, \Lambda\rho(\Lambda h_1, \Lambda h_2)h_4)h_5 + \rho(\Lambda h_3, \Lambda h_4)\rho(\Lambda h_1, \Lambda h_2)h_5\\
&\quad + \rho\big([\Lambda h_1, \Lambda h_2, \Lambda h_3]_L - \Lambda\rho(\Lambda h_1, \Lambda h_2)h_3, \Lambda h_4\big)h_5 + \rho\big(\Lambda h_3, [\Lambda h_1, \Lambda h_2, \Lambda h_4]_L\\
&\quad - \Lambda\rho(\Lambda h_1, \Lambda h_2)h_4\big)h_5 - \rho(\Lambda h_1, \Lambda h_2)\rho(\Lambda h_3, \Lambda h_4)h_5\\
=&\rho(\Lambda h_3, \Lambda h_4)\rho(\Lambda h_1, \Lambda h_2)h_5 + \rho([\Lambda h_1, \Lambda h_2, \Lambda h_3]_L, \Lambda h_4)h_5 + \rho(\Lambda h_3, [\Lambda h_1, \Lambda h_2, \Lambda h_4]_L)h_5\\
&\quad - \rho(\Lambda h_1, \Lambda h_2)\rho(\Lambda h_3, \Lambda h_4)h_5\\
=&0.
\end{aligned}$$

Furthermore, by Eqs (2.4), (2.5), and (3.5), we have

$$\begin{aligned}
[\{h_1, h_2, h_3\}_\Lambda, h_4, h_5]_H &= [\rho(\Lambda h_1, \Lambda h_2)h_3, h_4, h_5]_H = 0,\\
\{h_1, h_2, [h_3, h_4, h_5]_H\}_\Lambda &= \rho(\Lambda h_1, \Lambda h_2)[h_3, h_4, h_5]_H = 0.
\end{aligned}$$

Thus, $(H, [-,-,-]_H, \{-,-,-\}_\Lambda)$ is a 3-Leibniz-Lie algebra. □

Proposition 3.6. *Let (f_L, f_H) be a homomorphism from $H \xrightarrow{\Lambda_1} L$ to $H \xrightarrow{\Lambda_2} L$. Then f_H is a homomorphism of 3-Leibniz-Lie algebras from $(H, [-,-,-]_H, \{-,-,-\}_{\Lambda_1})$ to $(H, [-,-,-]_H, \{-,-,-\}_{\Lambda_2})$.*

Proof. For any $h_1, h_2, h_3 \in H$, by Eqs (2.7), (2.8), and (3.5), we have

$$\begin{aligned}
f_H(\{h_1, h_2, h_3\}_{\Lambda_1}) =& f_H(\rho(\Lambda_1 h_1, \Lambda_1 h_2)h_3)\\
=&\rho(f_L(\Lambda_1 h_1), f_L(\Lambda_1 h_2))f_H(h_3)\\
=&\rho(\Lambda_2 f_H(h_1), \Lambda_2 f_H(h_2))f_H(h_3)\\
=&\{f_H(h_1), f_H(h_2), f_H(h_3)\}_{\Lambda_2}.
\end{aligned}$$

The proof is finished. □

Motivated by the construction of 3-Lie algebras from Lie algebras [17], at the end of this section, we investigate 3-Leibniz-Lie algebras induced by Leibniz-Lie algebras.

Definition 3.7. *(see [9]) A Leibniz-Lie algebra* $(H,[-,-]_H,\rhd)$ *encompasses a Lie algebra* $(H,[-,-]_H)$ *and a binary operation* $\rhd : H\otimes H\to H$*, ensuring that*

$$\begin{aligned}&h_1\rhd(h_2\rhd h_3)=(h_1\rhd h_2)\rhd h_3+h_2\rhd(h_1\rhd h_3)+[h_1,h_2]_H\rhd h_3,\\&h_1\rhd[h_2,h_3]_H=[h_1\rhd h_2,h_3]_H=0,\end{aligned}$$

for all $h_1,h_2,h_3\in H$.

Theorem 3.8. *Let* $(H,[-,-]_H,\rhd)$ *be a Leibniz-Lie algebra, and let* $\varsigma\in H^*$ *be a trace map, which is a linear map that satisfies the following conditions:*

$$\varsigma([h_1,h_2]_H)=0,\quad \varsigma(h_1\rhd h_2)=0,\quad \textit{for all}\ \ h_1,h_2\in H.$$

Define two ternary operations by

$$\begin{aligned}&[h_1,h_2,h_3]_{H_\varsigma}=\varsigma(h_1)[h_2,h_3]_H+\varsigma(h_2)[h_3,h_1]_H+\varsigma(h_3)[h_1,h_2]_H,\\&\{h_1,h_2,h_3\}_{H_\varsigma}=\varsigma(h_1)h_2\rhd h_3-\varsigma(h_2)h_1\rhd h_3,\quad \textit{for all}\ \ h_1,h_2,h_3\in H.\end{aligned}$$

Then $(H,[-,-,-]_{H_\varsigma},\{-,-,-\}_{H_\varsigma})$ *is a 3-Leibniz-Lie algebra.*

Proof. First, we know from [17] that $(H,[-,-,-]_{H_\varsigma})$ is a 3-Lie algebra. Next, for any $h_1,h_2,h_3,h_4,h_5\in H$, we have

$$\{h_1,h_2,h_3\}_{H_\varsigma}=\varsigma(h_1)h_2\rhd h_3-\varsigma(h_2)h_1\rhd h_3=-\big(\varsigma(h_2)h_1\rhd h_3-\varsigma(h_1)h_2\rhd h_3\big)=-\{h_2,h_1,h_3\}_{H_\varsigma}$$

and

$$\begin{aligned}&\{\{h_1,h_2,h_3\}_{H_\varsigma},h_4,h_5\}_{H_\varsigma}+\{h_3,\{h_1,h_2,h_4\}_{H_\varsigma},h_5\}_{H_\varsigma}+\{h_3,h_4,\{h_1,h_2,h_5\}_{H_\varsigma}\}_{H_\varsigma}\\&\quad+\{[h_1,h_2,h_3]_{H_\varsigma},h_4,h_5\}_{H_\varsigma}+\{h_3,[h_1,h_2,h_4]_{H_\varsigma},h_5\}_{H_\varsigma}-\{h_1,h_2,\{h_3,h_4,h_5\}_{H_\varsigma}\}_{H_\varsigma}\\=&\varsigma(h_1)\varsigma(h_2\rhd h_3)h_4\rhd h_5-\varsigma(h_4)\varsigma(h_1)(h_2\rhd h_3)\rhd h_5-\varsigma(h_2)\varsigma(h_1\rhd h_3)h_4\rhd h_5\\&\quad+\varsigma(h_4)\varsigma(h_2)(h_1\rhd h_3)\rhd h_5+\varsigma(h_3)\varsigma(h_1)(h_2\rhd h_4)\rhd h_5-\varsigma(h_1)\varsigma(h_2\rhd h_4)h_3\rhd h_5\\&\quad-\varsigma(h_3)\varsigma(h_2)(h_1\rhd h_4)\rhd h_5+\varsigma(h_2)\varsigma(h_1\rhd h_4)h_3\rhd h_5+\varsigma(h_1)\varsigma(h_3)h_4\rhd(h_2\rhd h_5)\\&\quad-\varsigma(h_1)\varsigma(h_4)h_3\rhd(h_2\rhd h_5)-\varsigma(h_2)\varsigma(h_3)h_4\rhd(h_1\rhd h_5)+\varsigma(h_2)\varsigma(h_4)h_3\rhd(h_1\rhd h_5)\\&\quad+\varsigma(h_1)\varsigma([h_2,h_3]_H)h_4\rhd h_5-\varsigma(h_4)\varsigma(h_1)[h_2,h_3]_H\rhd h_5+\varsigma(h_2)\varsigma([h_3,h_1]_H)h_4\rhd h_5\\&\quad-\varsigma(h_4)\varsigma(h_2)[h_3,h_1]_H\rhd h_5+\varsigma(h_3)\varsigma([h_1,h_2]_H)h_4\rhd h_5-\varsigma(h_4)\varsigma(h_3)[h_1,h_2]_H\rhd h_5\\&\quad+\varsigma(h_3)\varsigma(h_1)[h_2,h_4]_H\rhd h_5-\varsigma(h_1)\varsigma([h_2,h_4]_H)h_3\rhd h_5+\varsigma(h_3)\varsigma(h_2)[h_4,h_1]_H\rhd h_5\\&\quad-\varsigma(h_2)\varsigma([h_4,h_1]_H)h_3\rhd h_5+\varsigma(h_3)\varsigma(h_4)[h_1,h_2]_H\rhd h_5-\varsigma(h_4)\varsigma([h_1,h_2]_H)h_3\rhd h_5\\&\quad-\varsigma(h_1)\varsigma(h_3)h_2\rhd(h_4\rhd h_5)+\varsigma(h_2)\varsigma(h_3)h_1\rhd(h_4\rhd h_5)+\varsigma(h_1)\varsigma(h_4)h_2\rhd(h_3\rhd h_5)\\&\quad-\varsigma(h_2)\varsigma(h_4)h_1\rhd(h_3\rhd h_5)\\=&-\varsigma(h_4)\varsigma(h_1)(h_2\rhd h_3)\rhd h_5+\varsigma(h_4)\varsigma(h_2)(h_1\rhd h_3)\rhd h_5+\varsigma(h_3)\varsigma(h_1)(h_2\rhd h_4)\rhd h_5\\&\quad-\varsigma(h_3)\varsigma(h_2)(h_1\rhd h_4)\rhd h_5+\varsigma(h_1)\varsigma(h_3)h_4\rhd(h_2\rhd h_5)-\varsigma(h_1)\varsigma(h_4)h_3\rhd(h_2\rhd h_5)\end{aligned}$$

$$
\begin{aligned}
&- \varsigma(h_2)\varsigma(h_3)h_4 \rhd (h_1 \rhd h_5) + \varsigma(h_2)\varsigma(h_4)h_3 \rhd (h_1 \rhd h_5) - \varsigma(h_4)\varsigma(h_1)[h_2, h_3]_H \rhd h_5\\
&- \varsigma(h_4)\varsigma(h_2)[h_3, h_1]_H \rhd h_5 + \varsigma(h_3)\varsigma(h_1)[h_2, h_4]_H \rhd h_5 + \varsigma(h_3)\varsigma(h_2)[h_4, h_1]_H \rhd h_5\\
&- \varsigma(h_1)\varsigma(h_3)h_2 \rhd (h_4 \rhd h_5) + \varsigma(h_2)\varsigma(h_3)h_1 \rhd (h_4 \rhd h_5) + \varsigma(h_1)\varsigma(h_4)h_2 \rhd (h_3 \rhd h_5)\\
&- \varsigma(h_2)\varsigma(h_4)h_1 \rhd (h_3 \rhd h_5)\\
=&0.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\{h_1, h_2, [h_3, h_4, h_5]_{H_\varsigma}\}_{H_\varsigma}\\
=&\varsigma(h_1)\varsigma(h_3)h_2 \rhd [h_4, h_5]_H - \varsigma(h_2)\varsigma(h_3)h_1 \rhd [h_4, h_5]_H + \varsigma(h_1)\varsigma(h_4)h_2 \rhd [h_5, h_3]_H\\
&- \varsigma(h_2)\varsigma(h_4)h_1 \rhd [h_5, h_3]_H + \varsigma(h_1)\varsigma(h_5)h_2 \rhd [h_3, h_4]_H - \varsigma(h_2)\varsigma(h_5)h_1 \rhd [h_3, h_4]_H\\
=&0
\end{aligned}
$$

and

$$
\begin{aligned}
&[\{h_1, h_2, h_3\}_{H_\varsigma}, h_4, h_5]_{H_\varsigma}\\
=&\varsigma(h_1)\varsigma(h_2 \rhd h_3)[h_4, h_5]_H + \varsigma(h_4)\varsigma(h_1)[h_5, h_2 \rhd h_3]_H + \varsigma(h_5)\varsigma(h_1)[h_2 \rhd h_3, h_4]_H\\
&- \varsigma(h_2)\varsigma(h_1 \rhd h_3)[h_4, h_5]_H - \varsigma(h_4)\varsigma(h_2)[h_5, h_1 \rhd h_3]_H - \varsigma(h_5)\varsigma(h_2)[h_1 \rhd h_3, h_4]_H\\
=&0.
\end{aligned}
$$

Hence Eqs (3.1)–(3.3) hold and we complete the proof. □

4. Cohomology and infinitesimal deformations of nonabelian embedding tensors on 3-Lie algebras

In this section, we revisit fundamental results pertaining to the representations and cohomologies of 3-Leibniz algebras. We construct a representation of the descendent 3-Leibniz algebra $(H, [-, -, -]_\Lambda)$ on the vector space L and define the cohomologies of a nonabelian embedding tensor on 3-Lie algebras. As an application, we characterize the infinitesimal deformation using the first cohomology group.

Definition 4.1. *(see [22]) A representation of the 3-Leibniz algebra $(\mathcal{H}, [-, -, -]_{\mathcal{H}})$ is a vector space V equipped with 3 actions*

$$
\begin{aligned}
\mathfrak{l} &: \mathcal{H} \otimes \mathcal{H} \otimes V \to V,\\
\mathfrak{m} &: \mathcal{H} \otimes V \otimes \mathcal{H} \to V,\\
\mathfrak{r} &: V \otimes \mathcal{H} \otimes \mathcal{H} \to V,
\end{aligned}
$$

satisfying for any $a_1, a_2, a_3, a_4, a_5 \in \mathcal{H}$ and $u \in V$

$$\mathfrak{l}(a_1, a_2, \mathfrak{l}(a_3, a_4, u)) = \mathfrak{l}([a_1, a_2, a_3]_{\mathcal{H}}, a_4, u) + \mathfrak{l}(a_3, [a_1, a_2, a_4]_{\mathcal{H}}, u) + \mathfrak{l}(a_3, a_4, \mathfrak{l}(a_1, a_2, u)), \tag{4.1}$$

$$\mathfrak{l}(a_1, a_2, \mathfrak{m}(a_3, u, a_5)) = \mathfrak{m}([a_1, a_2, a_3]_{\mathcal{H}}, u, a_5) + \mathfrak{m}(a_3, l(a_1, a_2, u), a_5) + \mathfrak{m}(a_3, u, [a_1, a_2, a_5]_{\mathcal{H}}), \tag{4.2}$$

$$\mathfrak{l}(a_1, a_2, \mathfrak{r}(u, a_4, a_5)) = \mathfrak{r}(\mathfrak{l}(a_1, a_2, u), a_4, a_5) + \mathfrak{r}(u, [a_1, a_2, a_4]_{\mathcal{H}}, a_5) + \mathfrak{r}(u, a_4, [a_1, a_2, a_5]_{\mathcal{H}}), \tag{4.3}$$

$$\mathfrak{m}(a_1, u, [a_3, a_4, a_5]_{\mathcal{H}}) = \mathfrak{r}(\mathfrak{m}(a_1, u, a_3), a_4, a_5) + \mathfrak{m}(a_3, \mathfrak{m}(a_1, u, a_4), a_5) + \mathfrak{l}(a_3, a_4, \mathfrak{m}(a_1, u, a_5)), \tag{4.4}$$

$$\mathfrak{r}(u, a_2, [a_3, a_4, a_5]_{\mathcal{H}}) = \mathfrak{r}(\mathfrak{r}(u, a_2, a_3), a_4, a_5) + \mathfrak{m}(a_3, \mathfrak{r}(u, a_2, a_4), a_5) + \mathfrak{l}(a_3, a_4, \mathfrak{r}(u, a_2, a_5)). \tag{4.5}$$

For $n \geq 1$, denote the n-cochains of 3-Leibniz algebra $(\mathcal{H}, [-,-,-]_{\mathcal{H}})$ with coefficients in a representation $(V; \mathfrak{l}, \mathfrak{m}, \mathfrak{r})$ by

$$C^n_{3\mathrm{Leib}}(\mathcal{H}, V) = \mathrm{Hom}(\overbrace{\wedge^2\mathcal{H} \otimes \cdots \otimes \wedge^2\mathcal{H}}^{n-1} \otimes \mathcal{H}, V).$$

The coboundary map $\delta : C^n_{3\mathrm{Leib}}(\mathcal{H}, V) \to C^{n+1}_{3\mathrm{Leib}}(\mathcal{H}, V)$, for $A_i = a_i \wedge b_i \in \wedge^2\mathcal{H}, 1 \leq i \leq n$ and $c \in \mathcal{H}$, as

$$\begin{aligned}
&(\delta\varphi)(A_1, A_2, \ldots, A_n, c)\\
= &\sum_{1\leq j<k\leq n} (-1)^j \varphi(A_1, \ldots, \widehat{A_j}, \ldots, A_{k-1}, a_k \wedge [a_j, b_j, b_k]_{\mathcal{H}} + [a_j, b_j, a_k]_{\mathcal{H}} \wedge b_k, \ldots, A_n, c)\\
&+ \sum_{j=1}^{n} (-1)^j \varphi(A_1, \ldots, \widehat{A_j}, \ldots, A_n, [a_j, b_j, c]_{\mathcal{H}}) + \sum_{j=1}^{n} (-1)^{j+1} \mathfrak{l}(A_j, \varphi(A_1, \ldots, \widehat{A_j}, \ldots, A_n, c))\\
&+ (-1)^{n+1} (\mathfrak{m}(a_n, \varphi(A_1, \ldots, A_{n-1}, b_n), c) + \mathfrak{r}(\varphi(A_1, \ldots, A_{n-1}, a_n), b_n, c)).
\end{aligned}$$

It was proved in [23, 24] that $\delta^2 = 0$. Therefore, $(\oplus_{n=1}^{+\infty} C^n_{3\mathrm{Leib}}(\mathcal{H}, V), \delta)$ is a cochain complex.

Let $H \stackrel{\Lambda}{\longrightarrow} L$ be a nonabelian embedding tensor 3-Lie algebra. By Corollary 2.10, $(H, [-,-,-]_\Lambda)$ is a 3-Leibniz algebra. Next we give a representation of $(H, [-,-,-]_\Lambda)$ on L.

Lemma 4.2. *With the above notations. Define 3 actions*

$$\begin{aligned}
\mathfrak{l}_\Lambda &: H \otimes H \otimes L \to L,\\
\mathfrak{m}_\Lambda &: H \otimes L \otimes H \to L,\\
\mathfrak{r}_\Lambda &: L \otimes H \otimes H \to L,
\end{aligned}$$

by

$$\begin{aligned}
\mathfrak{l}_\Lambda(h_1, h_2, l) &= [\Lambda h_1, \Lambda h_2, l]_L,\\
\mathfrak{m}_\Lambda(h_1, l, h_2) &= [\Lambda h_1, l, \Lambda h_2]_L - \Lambda\rho(\Lambda h_1, l)h_2,\\
\mathfrak{r}_\Lambda(l, h_1, h_2) &= [l, \Lambda h_1, \Lambda h_2]_L - \Lambda\rho(l, \Lambda h_1)h_2,
\end{aligned}$$

for all $h_1, h_2 \in H, l \in L$. Then $(L; \mathfrak{l}_\Lambda, \mathfrak{m}_\Lambda, \mathfrak{r}_\Lambda)$ is a representation of the descendent 3-Leibniz algebra $(H, [-,-,-]_\Lambda)$.

Proof. For any $h_1, h_2, h_3, h_4, h_5 \in H$ and $l \in L$, by Eqs (2.1), (2.3)–(2.6), and (2.9), we have

$$\begin{aligned}
&\mathfrak{l}_\Lambda(h_1, h_2, \mathfrak{l}_\Lambda(h_3, h_4, l)) - \mathfrak{l}_\Lambda([h_1, h_2, h_3]_\Lambda, h_4, l) - \mathfrak{l}_\Lambda(h_3, [h_1, h_2, h_4]_\Lambda, l) - \mathfrak{l}_\Lambda(h_3, h_4, \mathfrak{l}_\Lambda(h_1, h_2, l))\\
=&[\Lambda h_1, \Lambda h_2, [\Lambda h_3, \Lambda h_4, l]_L]_L - [[\Lambda h_1, \Lambda h_2, \Lambda h_3]_L, \Lambda h_4, l]_L - [\Lambda h_3, [\Lambda h_1, \Lambda h_2, \Lambda h_4]_L, l]_L\\
&- [\Lambda h_3, \Lambda h_4, [\Lambda h_1, \Lambda h_2, l]_L]_L\\
=&0
\end{aligned}$$

and

$$\mathfrak{l}_\Lambda(h_1, h_2, \mathfrak{m}_\Lambda(h_3, l, h_5)) - \mathfrak{m}_\Lambda([h_1, h_2, h_3]_\Lambda, l, h_5) - \mathfrak{m}_\Lambda(h_3, \mathfrak{l}_\Lambda(h_1, h_2, l), h_5) - \mathfrak{m}_\Lambda(h_3, l, [h_1, h_2, h_5]_\Lambda)$$

$$
\begin{aligned}
=&[\Lambda h_1, \Lambda h_2, [\Lambda h_3, l, \Lambda h_5]_L]_L - [\Lambda h_1, \Lambda h_2, \Lambda\rho(\Lambda h_3, l)h_5]_L - [[\Lambda h_1, \Lambda h_2, \Lambda h_3]_L, l, \Lambda h_5]_L \\
&+ \Lambda\rho([\Lambda h_1, \Lambda h_2, \Lambda h_3]_L, l)h_5 - [\Lambda h_3, [\Lambda h_1, \Lambda h_2, l]_L, \Lambda h_5]_L + \Lambda\rho(\Lambda h_3, [\Lambda h_1, \Lambda h_2, l]_L)h_5 \\
&- [\Lambda h_3, l, [\Lambda h_1, \Lambda h_2, \Lambda h_5]_L]_L + \Lambda\rho(\Lambda h_3, l)\rho(\Lambda h_1, \Lambda h_2)h_5 + \Lambda\rho(\Lambda h_3, l)[h_1, h_2, h_5]_H \\
=& - [\Lambda h_1, \Lambda h_2, \Lambda\rho(\Lambda h_3, l)h_5]_L + \Lambda\rho([\Lambda h_1, \Lambda h_2, \Lambda h_3]_L, l)h_5 + \Lambda\rho(\Lambda h_3, [\Lambda h_1, \Lambda h_2, l]_L)h_5 \\
&+ \Lambda\rho(\Lambda h_3, l)\rho(\Lambda h_1, \Lambda h_2)h_5 + \Lambda\rho(\Lambda h_3, l)[h_1, h_2, h_5]_H \\
=& - \Lambda(\rho(\Lambda h_1, \Lambda h_2)\rho(\Lambda h_3, l)h_5 + [h_1, h_2, \rho(\Lambda h_3, l)h_5]_H) + \Lambda\rho(\Lambda h_1, \Lambda h_2)\rho(\Lambda h_3, l)h_5 \\
&+ \Lambda\rho(\Lambda h_3, l)[h_1, h_2, h_5]_H \\
=& - \Lambda[h_1, h_2, \rho(\Lambda h_3, l)h_5]_H + \Lambda\rho(\Lambda h_3, l)[h_1, h_2, h_5]_H \\
=&0,
\end{aligned}
$$

which imply that Eqs (4.1) and (4.2) hold. Similarly, we can prove that Eqs (4.3)–(4.5) are true. The proof is finished. □

Proposition 4.3. *Let $H \xrightarrow{\Lambda_1} L$ and $H \xrightarrow{\Lambda_2} L$ be two nonabelian embedding tensor 3-Lie algebras and (f_L, f_H) a homomorphism from $H \xrightarrow{\Lambda_1} L$ to $H \xrightarrow{\Lambda_2} L$. Then the induced representation $(L; \mathfrak{l}_{\Lambda_1}, \mathfrak{m}_{\Lambda_1}, \mathfrak{r}_{\Lambda_1})$ of the descendent 3-Leibniz algebra $(H, [-,-,-]_{\Lambda_1})$ and the induced representation $(L; \mathfrak{l}_{\Lambda_2}, \mathfrak{m}_{\Lambda_2}, \mathfrak{r}_{\Lambda_2})$ of the descendent 3-Leibniz algebra $(H, [-,-,-]_{\Lambda_2})$ satisfying the following equations:*

$$
\begin{aligned}
f_L(\mathfrak{l}_{\Lambda_1}(h_1, h_2, l)) =&\mathfrak{l}_{\Lambda_2}(f_H(h_1), f_H(h_2), f_L(l)), &(4.6)\\
f_L(\mathfrak{m}_{\Lambda_1}(h_1, l, h_2)) =&\mathfrak{m}_{\Lambda_2}(f_H(h_1), f_L(l), f_H(h_2)), &(4.7)\\
f_L(\mathfrak{r}_{\Lambda_1}(l, h_1, h_2)) =&\mathfrak{r}_{\Lambda_2}(f_L(l), f_H(h_1), f_H(h_2)), &(4.8)
\end{aligned}
$$

for all $h_1, h_2 \in H, l \in L$. In other words, the following diagrams commute:

$$
\begin{array}{ccc}
L & \xrightarrow{f_L} & L \\
\downarrow{\scriptstyle \mathfrak{l}_{\Lambda_1}(h_1,h_2,-)} & & \downarrow{\scriptstyle \mathfrak{l}_{\Lambda_2}(f_H(h_1),f_H(h_2),-)} \\
L & \xrightarrow{f_L} & L,
\end{array}
\quad
\begin{array}{ccc}
L & \xrightarrow{f_L} & L \\
\downarrow{\scriptstyle \mathfrak{m}_{\Lambda_1}(h_1,-,h_2)} & & \downarrow{\scriptstyle \mathfrak{m}_{\Lambda_2}(f_H(h_1),-,f_H(h_2))} \\
L & \xrightarrow{f_L} & L,
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
L & \xrightarrow{f_L} & L \\
\downarrow{\scriptstyle \mathfrak{r}_{\Lambda_1}(-,h_1,h_2)} & & \downarrow{\scriptstyle \mathfrak{r}_{\Lambda_2}(-,f_H(h_1),f_H(h_2))} \\
L & \xrightarrow{f_L} & L.
\end{array}
$$

Proof. For any $h_1, h_2 \in H, l \in L$, by Eqs (2.7) and (2.8), we have

$$
\begin{aligned}
f_L(\mathfrak{l}_{\Lambda_1}(h_1, h_2, l)) =&f_L([\Lambda_1 h_1, \Lambda_1 h_2, l]_L) = [f_L(\Lambda_1 h_1), f_L(\Lambda_1 h_2), f_L(l)]_L \\
=&[\Lambda_2 f_H(h_1), \Lambda_2 f_H(h_2), f_L(l)]_L \\
=&\mathfrak{l}_{\Lambda_2}(f_H(h_1), f_H(h_2), f_L(l)), \\
f_L(\mathfrak{m}_{\Lambda_1}(h_1, l, h_2)) =&f_L([\Lambda_1 h_1, l, \Lambda_1 h_2]_L - \Lambda_1\rho(\Lambda_1 h_1, l)h_2) \\
=&[f_L(\Lambda_1 h_1), f_L(l), f_L(\Lambda_1 h_2)]_L - f_L(\Lambda_1\rho(\Lambda_1 h_1, l)h_2) \\
=&[\Lambda_2 f_H(h_1), f_L(l), \Lambda_2 f_H(h_2)]_L - \Lambda_2 f_H(\rho(\Lambda_1 h_1, l)h_2) \\
=&[\Lambda_2 f_H(h_1), f_L(l), \Lambda_2 f_H(h_2)]_L - \Lambda_2\rho(\Lambda_2 f_H(h_1), f_L(l))f_H(h_2) \\
=&\mathfrak{m}_{\Lambda_2}(f_H(h_1), f_L(l), f_H(h_2)).
\end{aligned}
$$

And the other equation is similar to provable. □

For $n \geq 1$, let $\delta_\Lambda : C^n_{\mathrm{3Leib}}(H, L) \to C^{n+1}_{\mathrm{3Leib}}(H, L)$ be the coboundary operator of the 3-Leibniz algebra $(H, [-,-,-]_\Lambda)$ with coefficients in the representation $(L; \mathfrak{l}_\Lambda, \mathfrak{m}_\Lambda, \mathfrak{r}_\Lambda)$. More precisely, for all $\phi \in C^n_{\mathrm{3Leib}}(H, L)$, $\mathfrak{H}_i = u_i \wedge v_i \in \wedge^2 H, 1 \leq i \leq n$ and $w \in H$, we have

$$\begin{aligned}
&(\delta_\Lambda \phi)(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n, w)\\
=& \sum_{1\leq j<k\leq n} (-1)^j \phi(\mathfrak{H}_1, \ldots, \widehat{\mathfrak{H}_j}, \ldots, \mathfrak{H}_{k-1}, u_k \wedge [u_j, v_j, v_k]_\Lambda + [u_j, v_j, u_k]_\Lambda \wedge v_k, \ldots, \mathfrak{H}_n, w)\\
&+ \sum_{j=1}^n (-1)^j \phi(\mathfrak{H}_1, \ldots, \widehat{\mathfrak{H}_j}, \ldots, \mathfrak{H}_n, [u_j, v_j, w]_\Lambda) + \sum_{j=1}^n (-1)^{j+1} \mathfrak{l}_\Lambda(\mathfrak{H}_j, \phi(\mathfrak{H}_1, \ldots, \widehat{\mathfrak{H}_j}, \ldots, \mathfrak{H}_n, w))\\
&+ (-1)^{n+1} (\mathfrak{m}_\Lambda(u_n, \phi(\mathfrak{H}_1, \ldots, \mathfrak{H}_{n-1}, v_n), w) + \mathfrak{r}_\Lambda(\phi(\mathfrak{H}_1, \ldots, \mathfrak{H}_{n-1}, u_n), v_n, w)).
\end{aligned}$$

In particular, for $\phi \in C^1_{\mathrm{3Leib}}(H, L) := \mathrm{Hom}(H, L)$ and $u_1, v_1, w \in H$, we have

$$\begin{aligned}
(\delta_\Lambda \phi)(u_1, v_1, w) =& - \phi([u_1, v_1, w]_\Lambda) + \mathfrak{l}_\Lambda(u_1, v_1, \phi(w)) + \mathfrak{m}_\Lambda(u_1, \phi(v_1), w) + \mathfrak{r}_\Lambda(\phi(u_1), v_1, w)\\
=& - \phi([u_1, v_1, w]_\Lambda) + [\Lambda u_1, \Lambda v_1, \phi(w)]_L + [\Lambda u_1, \phi(v_1), \Lambda w]_L\\
&- \Lambda\rho(\Lambda u_1, \phi(v_1))w + [\phi(u_1), \Lambda v_1, \Lambda w]_L - \Lambda\rho(\phi(u_1), \Lambda v_1)w.
\end{aligned}$$

For any $(a_1, a_2) \in C^0_{\mathrm{3Leib}}(H, L) := \wedge^2 L$, we define $\delta_\Lambda : C^0_{\mathrm{3Leib}}(H, L) \to C^1_{\mathrm{3Leib}}(H, L), (a_1, a_2) \mapsto \delta_\Lambda(a_1, a_2)$ by

$$\delta_\Lambda(a_1, a_2)u = \Lambda\rho(a_1, a_2)u - [a_1, a_2, \Lambda u]_L, \forall u \in H.$$

Proposition 4.4. *Let $H \xrightarrow{\Lambda} L$ be a nonabelian embedding tensor 3-Lie algebra. Then $\delta_\Lambda(\delta_\Lambda(a_1, a_2)) = 0$, that is, the composition $C^0_{\mathrm{3Leib}}(H, L) \xrightarrow{\delta_\Lambda} C^1_{\mathrm{3Leib}}(H, L) \xrightarrow{\delta_\Lambda} C^2_{\mathrm{3Leib}}(H, L)$ is the zero map.*

Proof. For any $u_1, v_1, w \in V$, by Eqs (2.1)–(2.6) and (2.9) we have

$$\begin{aligned}
&\delta_\Lambda(\delta_\Lambda(a_1, a_2))(u_1, v_1, w)\\
=& - \delta_\Lambda(a_1, a_2)([u_1, v_1, w]_\Lambda) + [\Lambda u_1, \Lambda v_1, \delta_\Lambda(a_1, a_2)(w)]_L + [\Lambda u_1, \delta_\Lambda(a_1, a_2)(v_1), \Lambda w]_L\\
&- \Lambda\rho(\Lambda u_1, \delta_\Lambda(a_1, a_2)(v_1))w + [\delta_\Lambda(a_1, a_2)(u_1), \Lambda v_1, \Lambda w]_L - \Lambda\rho(\delta_\Lambda(a_1, a_2)(u_1), \Lambda v_1)w\\
=& - \Lambda\rho(a_1, a_2)[u_1, v_1, w]_\Lambda + [a_1, a_2, [\Lambda u_1, \Lambda v_1, \Lambda w]_L]_L + [\Lambda u_1, \Lambda v_1, \Lambda\rho(a_1, a_2)w]_L\\
&- [\Lambda u_1, \Lambda v_1, [a_1, a_2, \Lambda w]_L]_L + [\Lambda u_1, \Lambda\rho(a_1, a_2)v_1, \Lambda w]_L - [\Lambda u_1, [a_1, a_2, \Lambda v_1]_L, \Lambda w]_L\\
&- \Lambda\rho(\Lambda u_1, \Lambda\rho(a_1, a_2)v_1)w + \Lambda\rho(\Lambda u_1, [a_1, a_2, \Lambda v_1]_L)w + [\Lambda\rho(a_1, a_2)u_1, \Lambda v_1, \Lambda w]_L\\
&- [[a_1, a_2, \Lambda u_1]_L, \Lambda v_1, \Lambda w]_L - \Lambda\rho(\Lambda\rho(a_1, a_2)u_1, \Lambda v_1)w + \Lambda\rho([a_1, a_2, \Lambda u_1]_L, \Lambda v_1)w\\
=& - \Lambda\rho(a_1, a_2)\rho(\Lambda u_1, \Lambda v_1)w - \Lambda\rho(a_1, a_2)[u_1, v_1, w]_H + \Lambda\rho(\Lambda u_1, \Lambda v_1)\rho(a_1, a_2)w\\
&+ \Lambda[u_1, v_1, \rho(a_1, a_2)w]_H + \Lambda\rho(\Lambda u_1, \Lambda\rho(a_1, a_2)v_1)w + \Lambda[u_1, \rho(a_1, a_2)v_1, w]_H\\
&- \Lambda\rho(\Lambda u_1, \Lambda\rho(a_1, a_2)v_1)w + \Lambda\rho(\Lambda u_1, [a_1, a_2, \Lambda v_1]_L)w + \Lambda(\Lambda\rho(a_1, a_2)u_1, \Lambda v_1)w\\
&+ \Lambda[\rho(a_1, a_2)u_1, v_1, w]_H - \Lambda\rho(\Lambda\rho(a_1, a_2)u_1, \Lambda v_1)w + \Lambda\rho([a_1, a_2, \Lambda u_1]_L, \Lambda v_1)w\\
=& - \Lambda\rho(a_1, a_2)\rho(\Lambda u_1, \Lambda v_1)w + \Lambda\rho(\Lambda u_1, \Lambda v_1)\rho(a_1, a_2)w + \Lambda\rho(\Lambda u_1, \Lambda\rho(a_1, a_2)v_1)w\\
&- \Lambda\rho(\Lambda u_1, \Lambda\rho(a_1, a_2)v_1)w + \Lambda\rho(\Lambda u_1, [a_1, a_2, \Lambda v_1]_L)w + \Lambda(\Lambda\rho(a_1, a_2)u_1, \Lambda v_1)w\\
&- \Lambda\rho(\Lambda\rho(a_1, a_2)u_1, \Lambda v_1)w + \Lambda\rho([a_1, a_2, \Lambda u_1]_L, \Lambda v_1)w\\
=& - \Lambda\rho(a_1, a_2)\rho(\Lambda u_1, \Lambda v_1)w + \Lambda\rho(\Lambda u_1, \Lambda v_1)\rho(a_1, a_2)w + \Lambda\rho(\Lambda u_1, [a_1, a_2, \Lambda v_1]_L)w
\end{aligned}$$

$$+\Lambda\rho([a_1,a_2,\Lambda u_1]_L,\Lambda v_1)w$$
$$=0.$$

Therefore, we deduce that $\delta_\Lambda(\delta_\Lambda(a_1,a_2))=0$. □

Now we develop the cohomology theory of a nonabelian embedding tensor Λ on the 3-Lie algebra $(L,[-,-,-]_L)$ with respect to the coherent action $(H,[-,-,-]_H;\rho^\dagger)$.

For $n\geq 0$, define the set of n-cochains of Λ by $C^n_\Lambda(H,L):=C^n_{3\mathrm{Leib}}(H,L)$. Then $(\oplus_{n=0}^\infty C^n_\Lambda(H,L),\delta_\Lambda)$ is a cochain complex.

For $n\geq 1$, we denote the set of n-cocycles by $\mathbf{Z}^n_\Lambda(H,L)$, the set of n-coboundaries by $\mathbf{B}^n_\Lambda(H,L)$, and the n-th cohomology group of the nonabelian embedding tensor Λ by

$$\mathrm{HH}^n_\Lambda(H,L)=\frac{\mathbf{Z}^n_\Lambda(H,L)}{\mathbf{B}^n_\Lambda(H,L)}.$$

Proposition 4.5. *Let $H\xrightarrow{\Lambda_1}L$ and $H\xrightarrow{\Lambda_2}L$ be two nonabelian embedding tensor 3-Lie algebras and let (f_L,f_H) be a homomorphism from $H\xrightarrow{\Lambda_1}L$ to $H\xrightarrow{\Lambda_2}L$ in which f_H is invertible. We define a map $\Psi:C^n_{\Lambda_1}(H,L)\to C^n_{\Lambda_2}(H,L)$ by*

$$\Psi(\phi)(\mathfrak{H}_1,\mathfrak{H}_2,\ldots,\mathfrak{H}_{n-1},w)=f_L(\phi(f_H^{-1}(u_1)\wedge f_H^{-1}(v_1),\ldots,f_H^{-1}(u_{n-1})\wedge f_H^{-1}(v_{n-1}),f_H^{-1}(w))),$$

for all $\phi\in C^n_{\Lambda_1}(H,L),\mathfrak{H}_i=u_i\wedge v_i\in\wedge^2H,1\leq i\leq n-1$, and $w\in H$. Then $\Psi:(C^{n+1}_{\Lambda_1}(H,L),\delta_{\Lambda_1})\to(C^{n+1}_{\Lambda_2}(H,L),\delta_{\Lambda_2})$ is a cochain map.

That is, the following diagram commutes:

$$\begin{array}{ccc} C^n_{\Lambda_1}(H,L) & \xrightarrow{\delta_{\Lambda_1}} & C^{n+1}_{\Lambda_1}(H,L)\\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi}\\ C^n_{\Lambda_2}(H,L) & \xrightarrow{\delta_{\Lambda_2}} & C^{n+1}_{\Lambda_2}(H,L). \end{array}$$

Consequently, it induces a homomorphism Ψ^* from the cohomology group $\mathrm{HH}^{n+1}_{\Lambda_1}(H,L)$ to $\mathrm{HH}^{n+1}_{\Lambda_2}(H,L)$.

Proof. For any $\phi\in C^n_{\Lambda_1}(H,L),\mathfrak{H}_i=u_i\wedge v_i\in\wedge^2H,1\leq i\leq n$, and $w\in H$, by Eqs (4.6)–(4.8) and Proposition 2.11, we have

$$\begin{aligned}
&(\delta_{\Lambda_2}\Psi(\phi))(\mathfrak{H}_1,\mathfrak{H}_2,\ldots,\mathfrak{H}_n,w)\\
=&\sum_{1\leq j<k\leq n}(-1)^j\Psi(\phi)(\mathfrak{H}_1,\ldots,\widehat{\mathfrak{H}_j},\ldots,\mathfrak{H}_{k-1},u_k\wedge[u_j,v_j,v_k]_{\Lambda_2}+[u_j,v_j,u_k]_{\Lambda_2}\wedge v_k,\ldots,\mathfrak{H}_n,w)\\
&+\sum_{j=1}^n(-1)^j\Psi(\phi)(\mathfrak{H}_1,\ldots,\widehat{\mathfrak{H}_j},\ldots,\mathfrak{H}_n,[u_j,v_j,w]_{\Lambda_2})+\sum_{j=1}^n(-1)^{j+1}\mathfrak{l}_{\Lambda_2}(\mathfrak{H}_j,\Psi(\phi)(\mathfrak{H}_1,\ldots,\widehat{\mathfrak{H}_j},\ldots,\mathfrak{H}_n,w))\\
&+(-1)^{n+1}\mathfrak{m}_{\Lambda_2}(u_n,\Psi(\phi)(\mathfrak{H}_1,\ldots,\mathfrak{H}_{n-1},v_n),w)+(-1)^{n+1}\mathfrak{r}_{\Lambda_2}(\Psi(\phi)(\mathfrak{H}_1,\ldots,\mathfrak{H}_{n-1},u_n),v_n,w)\\
=&\sum_{1\leq j<k\leq n}(-1)^jf_L(\phi(f_H^{-1}(u_1)\wedge f_H^{-1}(v_1),\ldots,\widehat{\mathfrak{H}_j},\ldots,f_H^{-1}(u_{k-1})\wedge f_H^{-1}(v_{k-1}),
\end{aligned}$$

$$f_H^{-1}(u_k) \wedge f_H^{-1}([u_j, v_j, v_k]_{\Lambda_2}) + f_H^{-1}([u_j, v_j, u_k]_{\Lambda_2}) \wedge f_H^{-1}(v_k), \ldots, f_H^{-1}(u_n) \wedge f_H^{-1}(v_n), f_H^{-1}(w)))$$
$$+ \sum_{j=1}^{n} (-1)^j f_L(\phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots, \widehat{\mathfrak{H}_j}, \ldots, f_H^{-1}(u_n) \wedge f_H^{-1}(v_n), f_H^{-1}([u_j, v_j, w]_{\Lambda_2})))$$
$$+ \sum_{j=1}^{n} (-1)^{j+1} \mathfrak{l}_{\Lambda_2}(\mathfrak{H}_j, f_L(\phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots, \widehat{\mathfrak{H}_j}, \ldots, f_H^{-1}(u_n) \wedge f_H^{-1}(v_n), f_H^{-1}(w))))$$
$$+ (-1)^{n+1} \mathfrak{m}_{\Lambda_2}(u_n, f_L(\phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots, f_H^{-1}(u_{n-1}) \wedge f_H^{-1}(v_{n-1}), f_H^{-1}(v_n))), w)$$
$$+ (-1)^{n+1} \mathfrak{r}_{\Lambda_2}(f_L(\phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots, f_H^{-1}(u_{n-1}) \wedge f_H^{-1}(v_{n-1}), f_H^{-1}(u_n))), v_n, w)$$
$$= f_L\Big(\sum_{1 \le j < k \le n} (-1)^j \phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots, \widehat{\mathfrak{H}_j}, \ldots, f_H^{-1}(u_{k-1}) \wedge f_H^{-1}(v_{k-1}),$$
$$f_H^{-1}(u_k) \wedge [f_H^{-1}(u_j), f_H^{-1}(v_j), f_H^{-1}(v_k)]_{\Lambda_1} + [f_H^{-1}(u_j), f_H^{-1}(v_j), f_H^{-1}(u_k)]_{\Lambda_1} \wedge f_H^{-1}(v_k), \ldots,$$
$$f_H^{-1}(u_n) \wedge f_H^{-1}(v_n), f_H^{-1}(w)) + \sum_{j=1}^{n} (-1)^j \phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots, \widehat{\mathfrak{H}_j}, \ldots, f_H^{-1}(u_n) \wedge f_H^{-1}(v_n),$$
$$[f_H^{-1}(u_j), f_H^{-1}(v_j), f_H^{-1}(w)]_{\Lambda_1}) + \sum_{j=1}^{n} (-1)^{j+1} \mathfrak{l}_{\Lambda_1}(f_H^{-1}(u_j), f_H^{-1}(v_j), \phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots,$$
$$\widehat{\mathfrak{H}_j}, \ldots, f_H^{-1}(u_n) \wedge f_H^{-1}(v_n), f_H^{-1}(w))) + (-1)^{n+1} \mathfrak{m}_{\Lambda_1}(f_H^{-1}(u_n), \phi(f_H^{-1}(u_1), f_H^{-1}(v_1), \ldots,$$
$$f_H^{-1}(u_{n-1}) \wedge f_H^{-1}(v_{n-1}), f_H^{-1}(v_n)), f_H^{-1}(w)) + (-1)^{n+1} \mathfrak{r}_{\Lambda_1}(\phi(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots,$$
$$f_H^{-1}(u_{n-1}) \wedge f_H^{-1}(v_{n-1}), f_H^{-1}(u_n)), f_H^{-1}(v_n), f_H^{-1}(w))\Big)$$
$$= f_L(\delta_{\Lambda_1}\phi)(f_H^{-1}(u_1) \wedge f_H^{-1}(v_1), \ldots, f_H^{-1}(u_n) \wedge f_H^{-1}(v_n), f_H^{-1}(w))$$
$$= \Psi(\delta_{\Lambda_1}\phi)(\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n, w).$$

Hence, Ψ is a cochain map and induces a cohomology group homomorphism $\Psi^* : \mathrm{HH}^{n+1}_{\Lambda_1}(H, L) \to \mathrm{HH}^{n+1}_{\Lambda_2}(H, L)$. □

At the conclusion of this section, we employ the well-established cohomology theory to describe the infinitesimal deformations of nonabelian embedding tensors on 3-Lie algebras.

Definition 4.6. *Let $\Lambda : H \to L$ be a nonabelian embedding tensor on a 3-Lie algebra $(L, [-,-,-]_L)$ with respect to a coherent action $(H, [-,-,-]_H; \rho^\dagger)$. An infinitesimal deformation of Λ is a nonabelian embedding tensor of the form $\Lambda_t = \Lambda + t\Lambda_1$, where t is a parameter with $t^2 = 0$.*

Let $\Lambda_t = \Lambda + t\Lambda_1$ be an infinitesimal deformation of Λ, then we have

$$[\Lambda_t u_1, \Lambda_t u_2, \Lambda_t u_3]_L = \Lambda_t \rho(\Lambda_t u_1, \Lambda_t u_2) u_3 + \Lambda_t [u_1, u_2, u_3]_H,$$

for all $u_1, u_2, u_3 \in H$. Therefore, we obtain the following equation:

$$\begin{aligned} &[\Lambda_1 u_1, \Lambda u_2, \Lambda u_3]_L + [\Lambda u_1, \Lambda_1 u_2, \Lambda u_3]_L + [\Lambda u_1, \Lambda u_2, \Lambda_1 u_3]_L \\ &= \Lambda_1 \rho(\Lambda u_1, \Lambda u_2) u_3 + \Lambda \rho(\Lambda_1 u_1, \Lambda u_2) u_3 + \Lambda \rho(\Lambda u_1, \Lambda_1 u_2) u_3 + \Lambda_1 [u_1, u_2, u_3]_H. \end{aligned} \tag{4.9}$$

It follows from Eq (4.9) that $\Lambda_1 \in C^1_\Lambda(H, L)$ is a 1-cocycle in the cohomology complex of Λ. Thus the cohomology class of Λ_1 defines an element in $\mathrm{HH}^1_\Lambda(H, L)$.

Let $\Lambda_t = \Lambda + t\Lambda_1$ and $\Lambda'_t = \Lambda + t\Lambda'_1$ be two infinitesimal deformations of Λ. They are said to be equivalent if there exists $a_1 \wedge a_2 \in \wedge^2 L$ such that the pair $(id_L + tad(a_1, a_2), id_H + t\rho(a_1, a_2))$ is a homomorphism from $H \xrightarrow{\Lambda_t} L$ to $H \xrightarrow{\Lambda'_t} L$. That is, the following conditions must hold:

1) The maps $id_L + tad(a_1, a_2) : L \to L$ and $id_H + t\rho(a_1, a_2) : H \to H$ are two 3-Lie algebra homomorphisms,

2) The pair $(id_L + tad(a_1, a_2), id_H + t\rho(a_1, a_2))$ satisfies:

$$(id_H + t\rho(a_1, a_2))(\rho(a, b)u) = \rho((id_L + tad(a_1, a_2))a, (id_L + tad(a_1, a_2))b)(id_H + t\rho(a_1, a_2))(u),$$

$$(\Lambda + t\Lambda'_1)(id_H + t\rho(a_1, a_2))(u) = (id_L + tad(a_1, a_2))((\Lambda + t\Lambda_1)u), \tag{4.10}$$

for all $a, b \in L, u \in H$. It is easy to see that Eq (4.10) gives rise to

$$\Lambda_1 u - \Lambda'_1 u = \Lambda\rho(a_1, a_2)u - [a_1, a_2, \Lambda u] = \delta_\Lambda(a_1, a_2)u \in C^1_\Lambda(H, L).$$

This shows that Λ_1 and Λ'_1 are cohomologous. Thus, their cohomology classes are the same in $\mathrm{HH}^1_\Lambda(H, L)$.

Conversely, any 1-cocycle Λ_1 gives rise to the infinitesimal deformation $\Lambda + t\Lambda_1$. Furthermore, we have arrived at the following result.

Theorem 4.7. *Let $\Lambda : H \to L$ be a nonabelian embedding tensor on $(L, [-, -, -]_L)$ with respect to $(H, [-, -, -]_H; \rho^\dagger)$. Then, there exists a bijection between the set of all equivalence classes of infinitesimal deformations of Λ and the first cohomology group $\mathrm{HH}^1_\Lambda(H, L)$.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare there is no conflicts of interest.

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