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Oscillation criterion for half-linear sublinear functional noncanonical dynamic equations

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Abstract: In this study, we derive new criteria that ensure the oscillation of solutions to noncanonical dynamic equations that are half-linear sublinear functional. These results not only resolve an open issue in numerous works in the literature but also emulate Ohriskatype and Hille-type criteria for canonical dynamic equations. We provide examples to demonstrate the accuracy, usefulness, and flexibility of the main results.

Keywords: time scales; oscillation; half-linear; sublinear; dynamic equations; differential equations

1. Introduction

Oscillation has drawn significant interest from researchers in engineering and science due to its essential applications in mechanical vibrations. Models may include advanced terms or delays to account for the effects of temporal contexts on solutions. Numerous studies have been conducted regarding oscillation in delay differential equations, such as [1–5], advanced differential equations, such as [6–9], and dynamic equations, such as [10, 11], Also, various models are used to study oscillation phenomena in practical applications. In biology, mathematical models have been enhanced with cross-diffusion factors to better account for delay and oscillation effects; see [12–15]. Current research focuses on differential equations essential for analyzing real-world phenomena. This paper explores their application in the turbulent flow of a polytropic gas through porous materials and non-Newtonian fluid theory. A solid understanding of the underlying mathematics is crucial for these fields; for more details, refer to [16–20]. Therefore, this work aims to investigate the oscillatory behavior of a particular class of second-order noncanonical half-linear sublinear functional dynamic equations

$$\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} + r_2(s)\Phi\left(\varkappa(\zeta(s))\right) = 0 \tag{1.1}$$

on an arbitrary unbounded above time scale $\mathbb{T}$, where $s \in [s_0, \infty)_{\mathbb{T}}$, $s_0 \geq 0$, $s_0 \in \mathbb{T}$; $\Phi(u) := |u|^{\kappa} \operatorname{sgn} u$, $0 < \kappa \leq 1$; $r_1, r_2 : \mathbb{T} \to (0, \infty)$ and $\zeta : \mathbb{T} \to \mathbb{T}$ are rd-continuous functions such that $\lim_{s\to\infty} \zeta(s) = \infty$.

By a solution of equation (1.1), we mean a nontrivial real-valued function $\varkappa \in \mathrm{C}^1_{\mathrm{rd}}[T_{\varkappa}, \infty)_{\mathbb{T}}$, $T_{\varkappa} \in [s_0, \infty)_{\mathbb{T}}$ such that $r_1\Phi\left(\varkappa^{\Delta}\right) \in \mathrm{C}^1_{\mathrm{rd}}[T_{\varkappa}, \infty)_{\mathbb{T}}$ and $\varkappa$ satisfies (1.1) on $[T_{\varkappa}, \infty)_{\mathbb{T}}$, where C_{rd} represents rd-continuous functions. We propose [21–24] as a very helpful introduction to time scale calculus. According to Trench [25], Equation (1.1) is said to be in noncanonical form when

$$\int_{s_0}^{\infty} \frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)} < \infty, \tag{1.2}$$

and canonical form when

$$\int_{s_0}^{\infty} \frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)} = \infty. \tag{1.3}$$

A solution $\varkappa$ of (1.1) is oscillatory if it is not positive or negative; otherwise, it is nonoscillatory. Solutions that vanish at infinity are excluded. We call that Eq (1.1) is oscillatory if all its solutions oscillate.

The subsequent presents oscillation results for differential equations associated with the oscillation results for (1.1). It also provides a comprehensive summary of the significant contributions of this paper, with our results demonstrating that they can be applied to consolidate specific outcomes regarding oscillation in differential and difference equations and expanded to ascertain oscillatory behavior in additional cases. When $\mathbb{T} = \mathbb{R}$, then (1.1) transforms into the half-linear sublinear differential equation.

$$\left[r_1(s)\Phi\left(\varkappa'(s)\right)\right]' + r_2(s)\Phi\left(\varkappa(\zeta(s))\right) = 0. \tag{1.4}$$

Fite [26] proved that the differential equation

$$\varkappa''(s) + r_2(s)\varkappa(s) = 0, \tag{1.5}$$

is oscillatory if

$$\int_{s_0}^{\infty} r_2(\varsigma)\mathrm{d}\varsigma = \infty. \tag{1.6}$$

Hille [27] improved criterion (1.6) and showed that if

$$\liminf_{s\to\infty} s \int_{s}^{\infty} r_2(\varsigma)\mathrm{d}\varsigma > \frac{1}{4}, \tag{1.7}$$

then Eq (1.5) is oscillatory. Erbe [28] extended (1.7) to the delay equation

$$\varkappa''(s) + r_2(s)\varkappa(\zeta(s)) = 0, \tag{1.8}$$

where $\zeta(s) \le s$ and proved that if

$$\liminf_{s\to\infty} s \int_s^\infty \frac{\zeta(\varsigma)}{\varsigma} r_2(\varsigma)\mathrm{d}\varsigma > \frac{1}{4}, \tag{1.9}$$

then Eq (1.8) is oscillatory. Ohriska [29] showed that equation (1.8) is oscillatory if

$$\limsup_{s\to\infty} s \int_s^\infty \frac{\zeta(\varsigma)}{\varsigma} r_2(\varsigma)\mathrm{d}\varsigma > 1. \tag{1.10}$$

When $\mathbb{T}=\mathbb{Z}$, then (1.1) becomes the half-linear sublinear difference equation

$$\Delta\left[r_1(s)\Phi\left(\Delta\varkappa(s)\right)\right] + r_2(s)\Phi\left(\varkappa(\zeta(s))\right) = 0.$$

Thandapani et al. [30] considered the equation

$$\Delta^2\left(\varkappa(s)\right) + r_2(s)\varkappa(s) = 0, \tag{1.11}$$

and it was proved that Eq (1.11) is oscillatory if

$$\sum_{\varsigma=s_0}^{\infty} r_2(\varsigma) = \infty. \tag{1.12}$$

If $\mathbb{T}=\{s : s = q^n,\ n \in \mathbb{N}_0,\ q > 1\}$, then (1.1) converts the half-linear sublinear q-difference equation

$$\Delta_q\left[r_1(s)\Phi\left(\Delta_q\varkappa(s)\right)\right] + r_2(s)\Phi\left(\varkappa(\zeta(s))\right) = 0.$$

Regarding canonical dynamic equations on time scales, Karpuz [31] studied the canonical dynamic equation

$$\left[r_1(s)\varkappa^\Delta(s)\right]^\Delta + r_2(s)\varkappa(\sigma(s)) = 0, \tag{1.13}$$

and obtained that if

$$\limsup_{s\to\infty} \frac{\mu(s)}{r_1(s)} < \infty, \quad \int_{s_0}^\infty \frac{\Delta\varsigma}{r_1(\varsigma)} = \infty,$$

and

$$\liminf_{s\to\infty} \left\{\int_{s_0}^s \frac{\Delta\varsigma}{r_1(\varsigma)} \int_s^\infty r_2(\varsigma)\Delta\varsigma\right\} > \frac{1}{4},$$

then Eq (1.13) is oscillatory. Erbe et al. [32] created the Hille-type and Ohriska-type criteria for the canonical dynamic equation

$$\left(r_1(s)\left(\varkappa^\Delta(s)\right)^\kappa\right)^\Delta + r_2(s)\varkappa^\kappa(\zeta(s)) = 0, \tag{1.14}$$

where $\zeta(s) \le s$, $0 < \kappa \le 1$ is a quotient of odd positive integers,

$$r_1^\Delta(s) \ge 0, \quad \text{and} \quad \int_{s_0}^\infty \frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)} = \infty, \tag{1.15}$$

and obtained that if

$$\int_{s_0}^{\infty} \zeta^{\kappa}(\varsigma) r_2(\varsigma)\Delta\varsigma = \infty, \tag{1.16}$$

and one of the following criteria holds:

$$\liminf_{s\to\infty} \frac{s^{\kappa}}{r_1(s)} \int_{\sigma(s)}^{\infty} \left(\frac{\zeta(\varsigma)}{\sigma(\varsigma)}\right)^{\kappa} r_2(\varsigma)\Delta\varsigma > \frac{\kappa^{\kappa}}{\ell^{\kappa^2}(\kappa+1)^{\kappa+1}}; \tag{1.17}$$

$$\limsup_{s\to\infty} \frac{s^{\kappa}}{r_1(s)} \int_{s}^{\infty} \left(\frac{\zeta(\varsigma)}{\varsigma}\right)^{\gamma} r_2(\varsigma)\Delta\varsigma > 1, \tag{1.18}$$

where $\ell := \liminf_{s\to\infty} \dfrac{s}{\sigma(s)} > 0$, then Eq (1.14) is oscillatory. Hassan et al. [33] studied (1.14) and showed that if (1.15) holds and

$$\liminf_{s\to\infty} \frac{s^{\kappa}}{r_1(s)} \int_{s}^{\infty} \left(\frac{\zeta(\varsigma)}{\varsigma}\right)^{\gamma} r_2(\varsigma)\Delta\varsigma > \frac{\kappa^{\kappa}}{\ell^{\kappa|1-\kappa|}(\kappa+1)^{\kappa+1}}. \tag{1.19}$$

Then Eq (1.14) is oscillatory. By using (1.19), it is clear that the second-order Euler dynamic equations

$$s\sigma(s)\varkappa^{\Delta\Delta}(s) + \lambda\varkappa(s) = 0, \tag{1.20}$$

and

$$s\sigma(s)\varkappa^{\Delta\Delta}(s) + \lambda\varkappa(\sigma(s)) = 0, \tag{1.21}$$

are oscillatory if $\lambda > \dfrac{1}{4}$. It is well known that this is the best possible case for the second-order Euler differential equation

$$s^2\varkappa''(s) + \lambda\varkappa(s) = 0.$$

Also, we note that criterion (1.19) improves (1.17) since

$$\frac{s^{\kappa}}{r_1(s)} \int_{s}^{\infty} \left(\frac{\zeta(\varsigma)}{\varsigma}\right)^{\gamma} r_2(\varsigma)\Delta\varsigma \geq \frac{s^{\kappa}}{r_1(s)} \int_{\sigma(s)}^{\infty} \left(\frac{\zeta(\varsigma)}{\sigma(\varsigma)}\right)^{\kappa} r_2(\varsigma)\Delta\varsigma,$$

and

$$\frac{\kappa^{\kappa}}{\ell^{\kappa|1-\kappa|}(\kappa+1)^{\kappa+1}} < \frac{\kappa^{\kappa}}{\ell^{\kappa^2}(\kappa+1)^{\kappa+1}} \quad \text{for } \kappa \geq \frac{1}{2}.$$

For more Hille-type and Ohriska-type criteria, see [34–38].

Concerning the noncanonical form, Hassan et al. [39] established some interesting oscillation criteria for the delay noncanonical linear dynamic equation

$$\left[r_1(s)\varkappa^{\Delta}(s)\right]^{\Delta} + r_2(s)\varkappa(\zeta(s)) = 0, \tag{1.22}$$

where $\zeta(s) \leq s$ and $\int_{s_0}^{\infty} \dfrac{\Delta\varsigma}{r_1(\varsigma)} < \infty$, which are as follows:

Theorem 1.1 (see [39]). *Equation* (1.22) *is oscillatory if one of the following criteria holds:*

$$\liminf_{s\to\infty}\left\{\left(\int_s^\infty \frac{\Delta\varsigma}{r_1(\varsigma)}\right)\left(\int_T^s r_2(\varsigma)\Delta\varsigma\right)\right\} > \frac{1}{4}; \tag{1.23}$$

$$\limsup_{s\to\infty}\left\{\left(\int_s^\infty \frac{\Delta\varsigma}{r_1(\varsigma)}\right)\left(\int_T^s r_2(\varsigma)\Delta\varsigma\right)\right\} > 1, \tag{1.24}$$

for sufficiently large $T \in [s_0,\infty)_{\mathbb{T}}$.

Also, Hassan et al. [40] established, in particular, Hille-type and Ohriska-type oscillation criteria for the advanced noncanonical linear dynamic equation (1.22) where $\zeta(s) \geq s$, as shown in the following result:

Theorem 1.2 (see [40]). *Equation* (1.22) *is oscillatory if one of the following conditions is satisfied*

$$\liminf_{s\to\infty}\left\{\left(\int_s^\infty \frac{\Delta\varsigma}{r_1(\varsigma)}\right)\left(\int_T^s \frac{\int_{\zeta(\varsigma)}^\infty \frac{\Delta v}{r_1(v)}}{\int_\varsigma^\infty \frac{\Delta v}{r_1(v)}} r_2(\varsigma)\Delta\varsigma\right)\right\} > \frac{1}{4}; \tag{1.25}$$

$$\limsup_{s\to\infty}\left\{\left(\int_s^\infty \frac{\Delta\varsigma}{r_1(\varsigma)}\right)\left(\int_T^s \frac{\int_{\zeta(\varsigma)}^\infty \frac{\Delta v}{r_1(v)}}{\int_\varsigma^\infty \frac{\Delta v}{r_1(v)}} r_2(\varsigma)\Delta\varsigma\right)\right\} > 1, \tag{1.26}$$

for sufficiently large $T \in [s_0,\infty)_{\mathbb{T}}$.

It is crucial to emphasize that previous research, such as [31, 33–35, 38], primarily focuses on the canonical form, indicating that condition (1.3) holds. This study aims to expand on the findings of [39, 40] by determining the oscillatory Hille-type and Ohriska-type criteria for the noncanonical half-linear sublinear dynamic equation (1.1) in both cases $\zeta(s) \leq s$ and $\zeta(s) \geq s$. The results given in this study have successfully solved a previously unsolved problem that was discussed in several of the author's articles, such as [16, 33, 35, 39, 40].

This paper is organized as follows: After this introduction, we present the main results in Section 2 for $\zeta(s) \leq s$, the main results in Section 3 for $\zeta(s) \geq s$, and the discussion and conclusion in Section 4.

2. Oscillation criteria for (1.1) when $\zeta(s) \leq s$

In the following results, we will present Hille-type and Ohriska-type oscillation criteria for the noncanonical case of Eq (1.1) when $\zeta(s) \leq s$ on $[s_0,\infty)_{\mathbb{T}}$.

Theorem 2.1. *If for sufficiently large* $T \in [s_0,\infty)_{\mathbb{T}}$,

$$\tilde{A} := \liminf_{s\to\infty}\left\{\tilde{P}(s)\int_s^\infty \left[\frac{\tilde{P}^{1-\kappa}(\varsigma)}{r_1(\varsigma)}\right]^{1/\kappa}\Delta\varsigma\right\} > \frac{1}{4\kappa}, \tag{2.1}$$

where

$$\tilde{P}(s) := \int_T^s r_2(\varsigma)\Delta\varsigma,$$

then Eq (1.1) *is oscillatory.*

Proof. Suppose that $\varkappa$ is a nonoscillatory solution of (1.1) on $[s_0, \infty)_{\mathbb{T}}$. Without loss of generality, let $\varkappa(\zeta(s)) > 0$ on $[s_0, \infty)_{\mathbb{T}}$. By applying the same method as in the proof of Case (a) of [39, Theorem 1], we obtain

$$\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0 \text{ and } \varkappa^{\Delta}(s) < 0,$$

eventually. Then there exists an $s_1 \in [s_0, \infty)_{\mathbb{T}}$ such that for $s \in [s_1, \infty)_{\mathbb{T}}$,

$$\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0 \text{ and } \varkappa^{\Delta}(s) < 0.$$

Define

$$\varpi(s) := -\frac{\Phi(\varkappa(s))}{r_1(s)\Phi(\varkappa^{\Delta}(s))}. \tag{2.2}$$

Hence,

$$\begin{aligned}
\varpi^{\Delta}(s) &= -\frac{(\Phi(\varkappa(s)))^{\Delta}}{r_1(s)\Phi(\varkappa^{\Delta}(s))} - \left(\frac{1}{r_1(s)\Phi(\varkappa^{\Delta}(s))}\right)^{\Delta}\Phi(\varkappa^{\sigma}(s)) \\
&= -\frac{(\Phi(\varkappa(s)))^{\Delta}}{r_1(s)\Phi(\varkappa^{\Delta}(s))} + \frac{\left(r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right)^{\Delta}}{r_1(s)\Phi(\varkappa^{\Delta}(s))\ \left(r_1(s)\Phi(\varkappa^{\Delta}(s))\right)^{\sigma}}\Phi(\varkappa^{\sigma}(s)) \\
&= -\frac{(\Phi(\varkappa(s)))^{\Delta}}{r_1(s)\Phi(\varkappa^{\Delta}(s))} - r_2(s)\frac{\Phi(\varkappa(\zeta(s)))}{r_1(s)\Phi(\varkappa^{\Delta}(s))}\frac{\Phi(\varkappa^{\sigma}(s))}{\left(r_1(s)\Phi(\varkappa^{\Delta}(s))\right)^{\sigma}} \\
&= -\frac{(\Phi(\varkappa(s)))^{\Delta}}{r_1(s)\Phi(\varkappa^{\Delta}(s))} - \frac{\Phi(\varkappa(\zeta(s)))}{\Phi(\varkappa(s))}r_2(s)\varpi(s)\varpi^{\sigma}(s) \\
&\leq -\frac{\kappa}{r_1(s)}\left(\frac{\left|\varkappa^{\Delta}(s)\right|}{\varkappa(s)}\right)^{1-\kappa} - \frac{\Phi(\varkappa(\zeta(s)))}{\Phi(\varkappa(s))}r_2(s)\varpi(s)\varpi^{\sigma}(s),
\end{aligned} \tag{2.3}$$

due to Pötzsche chain rule (see [23, Theorem 1.90]),

$$\begin{aligned}
(\Phi(\varkappa(s)))^{\Delta} &= \kappa\varkappa^{\Delta}(s)\int_0^1 \left[(1-h)\varkappa(s) + h\varkappa^{\sigma}(s)\right]^{\kappa-1} \mathrm{d}h \\
&\leq \kappa\varkappa^{\kappa-1}(s)\varkappa^{\Delta}(s) = -\kappa\varkappa^{\kappa-1}(s)\left|\varkappa^{\Delta}(s)\right|.
\end{aligned}$$

Also, by using the fact that $\varkappa^{\Delta}(s) < 0$, we obtain

$$\varpi^{\Delta}(s) \leq -\frac{\kappa}{r_1(s)}\left(\frac{\left|\varkappa^{\Delta}(s)\right|}{\varkappa(s)}\right)^{1-\kappa} - r_2(s)\varpi(s)\varpi^{\sigma}(s). \tag{2.4}$$

Integrating (1.1) and using the fact that $\varkappa^{\Delta}(s) < 0$, we obtain

$$-r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right) > -r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right) + r_1(s_1)\Phi\left(\varkappa^{\Delta}(s_1)\right) \geq \Phi(\varkappa(s))\int_{s_1}^{s} r_2(\varsigma)\Delta\varsigma, \tag{2.5}$$

that is,

$$r_1^{1/\kappa}(s)\left|\varkappa^{\Delta}(s)\right| \geq \varkappa(s)\left(\int_{s_1}^{s} r_2(\varsigma)\Delta\varsigma\right)^{1/\kappa}.$$

In this case, we have

$$\frac{\kappa}{r_1(s)}\left(\frac{\left|\varkappa^{\Delta}(s)\right|}{\varkappa(s)}\right)^{1-\kappa} \geq \kappa\left[\frac{1}{r_1(s)}\left(\int_{s_1}^{s} r_2(\varsigma)\,\Delta\varsigma\right)^{1-\kappa}\right]^{1/\kappa} = \kappa\left[\frac{\tilde{P}^{1-\kappa}(s)}{r_1(s)}\right]^{1/\kappa}. \tag{2.6}$$

Substituting (2.6) into (2.4), we obtain

$$\varpi^{\Delta}(s) \leq -\kappa\left[\frac{\tilde{P}^{1-\kappa}(s)}{r_1(s)}\right]^{1/\kappa} - r_2(s)\varpi(s)\varpi^{\sigma}(s). \tag{2.7}$$

Integrating (2.7) from s to v, we obtain

$$\varpi(v) - \varpi(s) \leq -\kappa\int_{s}^{v}\left[\frac{\tilde{P}^{1-\kappa}(\varsigma)}{r_1(\varsigma)}\right]^{1/\kappa}\Delta\varsigma - \int_{s}^{v} r_2(\varsigma)\varpi(\varsigma)\varpi^{\sigma}(\varsigma)\Delta\varsigma.$$

Due to $\varpi > 0$ and $\varpi^{\Delta} < 0$ and letting $v \to \infty$, we obtain

$$\kappa\int_{s}^{\infty}\left[\frac{\tilde{P}^{1-\kappa}(\varsigma)}{r_1(\varsigma)}\right]^{1/\kappa}\Delta\varsigma \leq \varpi(s) - \int_{s}^{\infty} r_2(\varsigma)\varpi(\varsigma)\varpi^{\sigma}(\varsigma)\,\Delta\varsigma. \tag{2.8}$$

By multiplying each side of (2.8) by $\tilde{P}(s)$, we obtain

$$\kappa\tilde{P}(s)\int_{s}^{\infty}\left[\frac{\tilde{P}^{1-\kappa}(\varsigma)}{r_1(\varsigma)}\right]^{1/\kappa}\Delta\varsigma \leq \tilde{P}(s)\varpi(s) - \tilde{P}(s)\int_{s}^{\infty} r_2(\varsigma)\varpi(\varsigma)\varpi^{\sigma}(\varsigma)\,\Delta\varsigma. \tag{2.9}$$

For any $\varepsilon \in (0,1)$, there exists an $s_2 \in [s_1,\infty)_{\mathbb{T}}$ such that, for $s \in [s_2,\infty)_{\mathbb{T}}$,

$$\tilde{P}(s)\int_{s}^{\infty}\left[\frac{\tilde{P}^{1-\kappa}(\varsigma)}{r_1(\varsigma)}\right]^{1/\kappa}\Delta\varsigma \geq \varepsilon\tilde{A} \quad \text{and} \quad \tilde{P}(s)\varpi(s) \geq \varepsilon W, \tag{2.10}$$

where

$$0 \leq W := \liminf_{s\to\infty}\tilde{P}(s)\varpi(s) \leq 1,$$

in view of (2.2) and (2.5). It follows from (2.9) and (2.10) that

$$\begin{aligned}
\varepsilon\kappa\tilde{A} &\leq \tilde{P}(s)\varpi(s) \\
&\quad -\tilde{P}(s)\int_{s}^{\infty}\frac{r_2(\varsigma)}{\tilde{P}(\varsigma)\tilde{P}^{\sigma}(\varsigma)}\tilde{P}(\varsigma)\varpi(\varsigma)\tilde{P}^{\sigma}(\varsigma)\varpi^{\sigma}(\varsigma)\;\Delta\varsigma \\
&\leq \tilde{P}(s)\varpi(s) \\
&\quad -\tilde{P}(s)(\varepsilon W)^2\int_{s}^{\infty}\frac{r_2(\varsigma)}{\tilde{P}(\varsigma)\tilde{P}^{\sigma}(\varsigma)}\,\Delta\varsigma \\
&= \tilde{P}(s)\varpi(s) - \tilde{P}(s)(\varepsilon W)^2\int_{s}^{\infty}\left(\frac{-1}{\tilde{P}(\varsigma)}\right)^{\Delta}\Delta\varsigma \\
&= \tilde{P}(s)\varpi(s) - (\varepsilon W)^2,
\end{aligned}$$

due to $\tilde{P}(s) \to \infty$ as $s \to \infty$. Taking the lim inf of each side of the last inequality as $s \to \infty$, we obtain

$$\varepsilon\kappa\tilde{A} \leq W - (\varepsilon W)^2.$$

Since $\varepsilon > 0$ is arbitrary, we achieve

$$\kappa\tilde{A} \leq W - W^2 \leq \frac{1}{4},$$

which is a contradiction to (2.1)

Example 2.1. Consider the second-order half-linear sublinear delay dynamic equation

$$\left[\tilde{P}(s)\sqrt{\sqrt{s}\tilde{P}^{\sigma}(s)\varkappa^{\Delta}(s)}\operatorname{sgn}\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} + \frac{1}{\tilde{\beta}\sqrt{s}}\sqrt{\varkappa(\zeta(s))}\operatorname{sgn}(\varkappa(\zeta(s))) = 0, \tag{2.11}$$

where $\tilde{\beta} > 0$ is a constant. Here,

$$\kappa = \frac{1}{2},\ r_1(s) = \tilde{P}(s)\sqrt{\sqrt{s}\tilde{P}^{\sigma}(s)},\ \text{and } r_2(s) := \frac{1}{\tilde{\beta}\sqrt{s}}.$$

It is easy to see that

$$\tilde{P}(s) = \frac{1}{\tilde{\beta}}\int_T^s \frac{\Delta\varsigma}{\sqrt{\varsigma}} \to \infty \text{ as } s \to \infty,$$

by [24, Example 5.60]. Also,

$$\begin{aligned}
&\liminf_{s\to\infty}\left\{\tilde{P}(s)\int_s^{\infty}\left[\frac{\tilde{P}^{1-\kappa}(\varsigma)}{r_1(\varsigma)}\right]^{1/\kappa}\Delta\varsigma\right\}\\
&= \liminf_{s\to\infty}\left\{\tilde{P}(s)\int_s^{\infty}\frac{1}{\sqrt{\varsigma}\tilde{P}(\varsigma)\tilde{P}^{\sigma}(\varsigma)}\Delta\varsigma\right\}\\
&= \tilde{\beta}\liminf_{s\to\infty}\left\{\tilde{P}(s)\int_s^{\infty}\left(\frac{-1}{\tilde{P}(\varsigma)}\right)^{\Delta}\Delta\varsigma\right\} = \tilde{\beta}.
\end{aligned}$$

In view of Theorem 2.1, Equation (2.11) is oscillatory if $\tilde{\beta} > \frac{1}{2}$.

Theorem 2.2. *If for sufficiently large* $T \in [s_0, \infty)_{\mathbb{T}}$,

$$\limsup_{s\to\infty}\left\{\int_s^{\infty}\frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)}\left(\int_T^s r_2(\varsigma)\Delta\varsigma\right)^{1/\kappa}\right\} > 1, \tag{2.12}$$

then Eq (1.1) *is oscillatory.*

Proof. Suppose that $\varkappa$ is a nonoscillatory solution of (1.1) on $[s_0, \infty)_{\mathbb{T}}$. Without loss of generality, let $\varkappa(\zeta(s)) > 0$ on $[s_0, \infty)_{\mathbb{T}}$. By applying the same method as in the proof of Case (a) of [39, Theorem 1], we obtain

$$\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0 \text{ and } \varkappa^{\Delta}(s) < 0,$$

eventually. Then there exists an $s_1 \in [s_0, \infty)_{\mathbb{T}}$ such that for $s \in [s_1, \infty)_{\mathbb{T}}$,

$$\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0 \text{ and } \varkappa^{\Delta}(s) < 0.$$

In accordance with the proof of Theorem 2.1, Case (b), we conclude that

$$-r_1^{1/\kappa}(s)\varkappa^{\Delta}(s) \geq \varkappa(s)\left(\int_{s_1}^{s} r_2(\varsigma)\Delta\varsigma\right)^{1/\kappa}.$$

Since $\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0$, we obtain

$$\varkappa(s) > -\int_s^{\infty} \frac{r_1^{1/\kappa}(\varsigma)\varkappa^{\Delta}(\varsigma)}{r_1^{1/\kappa}(\varsigma)}\Delta\varsigma \geq -r_1^{1/\kappa}(s)\varkappa^{\Delta}(s)\int_s^{\infty}\frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)}.$$

Therefore,

$$-r_1^{1/\kappa}(s)\varkappa^{\Delta}(s) \geq -r_1^{1/\kappa}(s)\varkappa^{\Delta}(s)\left\{\int_s^{\infty}\frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)}\left(\int_{s_1}^{s} r_2(\varsigma)\Delta\varsigma\right)^{1/\kappa}\right\}.$$

Consequently, we have

$$\limsup_{s\to\infty}\left\{\int_s^{\infty}\frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)}\left(\int_{s_1}^{s} r_2(\varsigma)\Delta\varsigma\right)^{1/\kappa}\right\} \leq 1,$$

which contradicts (2.12)

3. Oscillation criteria for (1.1) when $\zeta(s) \geq s$

In this section, we will introduce Hille-type and Ohriska-type oscillation criteria for the noncanonical case of Eq (1.1) when $\zeta(s) \geq s$ on $[s_0,\infty)_{\mathbb{T}}$.

Theorem 3.1. *If for sufficiently large $T \in [s_0,\infty)_{\mathbb{T}}$,*

$$\tilde{B} := \liminf_{s\to\infty}\left\{\tilde{Q}(s)\int_s^{\infty}\left[\frac{\tilde{Q}^{1-\kappa}(\varsigma)}{r_1(\varsigma)}\right]^{1/\kappa}\Delta\varsigma\right\} > \frac{1}{4\kappa}, \tag{3.1}$$

where

$$\tilde{Q}(s) := \int_T^{s}\left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)}\right)^{\kappa} r_2(\varsigma)\Delta\varsigma,$$

with

$$\xi(s) := \int_s^{\infty}\frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)},$$

then Eq (1.1) *is oscillatory.*

Proof. Suppose that $\varkappa$ is a nonoscillatory solution of (1.1) on $[s_0,\infty)_{\mathbb{T}}$. Without loss of generality, let $\varkappa(s) > 0$ on $[s_0,\infty)_{\mathbb{T}}$. As in the proof of Case (a) of [39, Theorem 1], we obtain

$$\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0 \text{ and } \varkappa^{\Delta}(s) < 0,$$

eventually. Then there exists an $s_1 \in [s_0,\infty)_{\mathbb{T}}$ such that for $s \in [s_1,\infty)_{\mathbb{T}}$,

$$\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0 \text{ and } \varkappa^{\Delta}(s) < 0.$$

In accordance with the proof of Theorem 2.1, Case (b), we achieve that

$$\varpi^{\Delta}(s) \leq -\frac{\kappa}{r_1(s)}\left(\frac{\left|\varkappa^{\Delta}(s)\right|}{\varkappa(s)}\right)^{1-\kappa} - \frac{\Phi(\varkappa(\zeta(s)))}{\Phi(\varkappa(s))} r_2(s)\varpi(s)\varpi^{\sigma}(s). \tag{3.2}$$

Since $\left[r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right)\right]^{\Delta} < 0$, we obtain

$$-\varkappa(s) \leq r_1^{1/\kappa}(s)\,\varkappa^{\Delta}(s)\int_s^{\infty}\frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)} = r_1^{1/\kappa}(s)\,\varkappa^{\Delta}(s)\xi(s).$$

Hence,

$$\begin{aligned}\left(\frac{\varkappa(s)}{\xi(s)}\right)^{\Delta} &= \frac{\xi(s)\varkappa^{\Delta}(s) + r_1^{-1/\kappa}(s)\varkappa(s)}{\xi(s)\xi^{\sigma}(s)} \\ &= \frac{r_1^{1/\kappa}(s)\xi(s)\varkappa^{\Delta}(s) + \varkappa(s)}{r_1^{1/\kappa}(s)\xi(s)\xi^{\sigma}(s)} > 0,\end{aligned} \tag{3.3}$$

which implies

$$\frac{\Phi(\varkappa(\zeta(s)))}{\Phi(\varkappa(s))} \geq \left(\frac{\xi(\zeta(s))}{\xi(s)}\right)^{\kappa}. \tag{3.4}$$

Therefore, (3.2) becomes

$$\varpi^{\Delta}(s) \leq -\frac{\kappa}{r_1(s)}\left(\frac{\left|\varkappa^{\Delta}(s)\right|}{\varkappa(s)}\right)^{1-\kappa} - \left(\frac{\xi(\zeta(s))}{\xi(s)}\right)^{\kappa} r_2(s)\varpi(s)\varpi^{\sigma}(s). \tag{3.5}$$

By integrating (1.1) and by the facts that

$$\varkappa^{\Delta}(s) < 0 \text{ and } \left(\frac{\varkappa(s)}{\xi(s)}\right)^{\Delta} > 0,$$

we obtain

$$\begin{aligned}-r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right) &> -r_1(s)\Phi\left(\varkappa^{\Delta}(s)\right) + r_1(s_1)\Phi\left(\varkappa^{\Delta}(s_1)\right) \\ &\geq \int_{s_1}^{s}\left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)}\right)^{\kappa} r_2(\varsigma)\,\Phi(\varkappa(\varsigma))\,\Delta\varsigma \\ &\geq \Phi(\varkappa(s))\int_{s_1}^{s}\left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)}\right)^{\kappa} r_2(\varsigma)\,\Delta\varsigma,\end{aligned} \tag{3.6}$$

that is,

$$r_1^{1/\kappa}(s)\left|\varkappa^{\Delta}(s)\right| \geq \varkappa(s)\left(\int_{s_1}^{s}\left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)}\right)^{\kappa} r_2(\varsigma)\,\Delta\varsigma\right)^{1/\kappa}.$$

In this case, we have

$$\frac{\kappa}{r_1(s)}\left(\frac{\left|\varkappa^{\Delta}(s)\right|}{\varkappa(s)}\right)^{1-\kappa} \geq \kappa\left[\frac{1}{r_1(s)}\left(\int_{s_1}^{s}\left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)}\right)^{\kappa} r_2(\varsigma)\,\Delta\varsigma\right)^{1-\kappa}\right]^{1/\kappa}$$

$$= \kappa \left[\frac{\tilde{Q}^{1-\kappa}(s)}{r_1(s)} \right]^{1/\kappa}. \tag{3.7}$$

Substituting (3.7) into (3.5), we conclude that

$$\varpi^{\Delta}(s) \leq -\kappa \left[\frac{\tilde{Q}^{1-\kappa}(s)}{r_1(s)} \right]^{1/\kappa} - \left(\frac{\xi(\zeta(s))}{\xi(s)} \right)^{\kappa} r_2(s)\varpi(s)\varpi^{\sigma}(s). \tag{3.8}$$

Integrating (3.8) from s to v, we obtain

$$\varpi(v) - \varpi(s) \leq -\kappa \int_s^v \left[\frac{\tilde{Q}^{1-\kappa}(\varsigma)}{r_1(\varsigma)} \right]^{1/\kappa} \Delta\varsigma - \int_s^v \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^{\kappa} r_2(\varsigma)\varpi(\varsigma)\varpi^{\sigma}(\varsigma)\Delta\varsigma.$$

As a result of $\varpi > 0$ and $\varpi^{\Delta} < 0$ and assuming $v \to \infty$, we obtain

$$\kappa \int_s^{\infty} \left[\frac{\tilde{Q}^{1-\kappa}(\varsigma)}{r_1(\varsigma)} \right]^{1/\kappa} \Delta\varsigma \leq \varpi(s) - \int_s^{\infty} \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^{\kappa} r_2(\varsigma)\varpi(\varsigma)\varpi^{\sigma}(\varsigma)\Delta\varsigma. \tag{3.9}$$

By multiplying each side of (3.9) by $\tilde{Q}(s)$, we obtain

$$\begin{aligned} \kappa\tilde{Q}(s) \int_s^{\infty} \left[\frac{\tilde{Q}^{1-\kappa}(\varsigma)}{r_1(\varsigma)} \right]^{1/\kappa} \Delta\varsigma \quad &\leq \quad \tilde{Q}(s)\varpi(s) \\ &\quad -\tilde{Q}(s) \int_s^{\infty} \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^{\kappa} r_2(\varsigma)\varpi(\varsigma)\varpi^{\sigma}(\varsigma)\Delta\varsigma. \end{aligned} \tag{3.10}$$

For any $\varepsilon \in (0,1)$, there is an $s_2 \in [s_1, \infty)_{\mathbb{T}}$ such that, for $s \in [s_2, \infty)_{\mathbb{T}}$,

$$\tilde{Q}(s) \int_s^{\infty} \left[\frac{\tilde{Q}^{1-\kappa}(\varsigma)}{r_1(\varsigma)} \right]^{1/\kappa} \Delta\varsigma \geq \varepsilon\tilde{B} \quad \text{and} \quad \tilde{Q}(s)\varpi(s) \geq \varepsilon\overline{W}, \tag{3.11}$$

where

$$0 \leq \overline{W} := \liminf_{s\to\infty} \tilde{Q}(s)\varpi(s) \leq 1,$$

in view of (2.2) and (3.6). From (3.10) and (3.11), we infer that

$$\begin{aligned} \varepsilon\kappa\tilde{B} \quad &\leq \quad \tilde{Q}(s)\varpi(s) \\ &\quad -\tilde{Q}(s) \int_s^{\infty} \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^{\kappa} \frac{r_2(\varsigma)}{\tilde{Q}(\varsigma)\tilde{Q}^{\sigma}(\varsigma)} \tilde{Q}(\varsigma)\varpi(\varsigma)\tilde{Q}^{\sigma}(\varsigma)\varpi^{\sigma}(\varsigma)\ \Delta\varsigma \\ &\leq \quad \tilde{Q}(s)\varpi(s) \\ &\quad -\tilde{Q}(s)\left(\varepsilon\overline{W}\right)^2 \int_s^{\infty} \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^{\kappa} \frac{r_2(\varsigma)}{\tilde{Q}(\varsigma)\tilde{Q}^{\sigma}(\varsigma)} \Delta\varsigma \\ &= \quad \tilde{Q}(s)\varpi(s) - \tilde{Q}(s)\left(\varepsilon\overline{W}\right)^2 \int_s^{\infty} \left(\frac{-1}{\tilde{Q}(\varsigma)} \right)^{\Delta} \Delta\varsigma \\ &= \quad \tilde{Q}(s)\varpi(s) - \left(\varepsilon\overline{W}\right)^2, \end{aligned} \tag{3.12}$$

due to $\tilde{Q}(s) \to \infty$ as $s \to \infty$. Taking the lim inf of (3.12) as $s \to \infty$, we obtain

$$\varepsilon\kappa\tilde{B} \leq \overline{W} - \left(\varepsilon\overline{W}\right)^2.$$

Since $\varepsilon > 0$ is arbitrary, we see that

$$\kappa\tilde{B} \leq \overline{W} - \overline{W}^2 \leq \frac{1}{4},$$

which is a contradiction to (3.1).

Theorem 3.2. *If for sufficiently large $T \in [s_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{s\to\infty} \left\{ \xi(s) \left(\int_T^s \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^\kappa r_2(\varsigma)\Delta\varsigma \right)^{1/\kappa} \right\} > 1, \tag{3.13}$$

where

$$\xi(s) := \int_s^\infty \frac{\Delta\varsigma}{r_1^{1/\kappa}(\varsigma)},$$

then Eq (1.1) *is oscillatory.*

Proof. Suppose that $\varkappa$ is a nonoscillatory solution of (1.1) on $[s_0, \infty)_{\mathbb{T}}$. Without loss of generality, let $\varkappa(s) > 0$ on $[s_0, \infty)_{\mathbb{T}}$. By applying the same method as in the proof of Case (a) of [39, Theorem 1], we obtain

$$\left[r_1(s)\Phi\left(\varkappa^\Delta(s)\right)\right]^\Delta < 0 \text{ and } \varkappa^\Delta(s) < 0,$$

eventually. Then there exists an $s_1 \in [s_0, \infty)_{\mathbb{T}}$ such that for $s \in [s_1, \infty)_{\mathbb{T}}$,

$$\left[r_1(s)\Phi\left(\varkappa^\Delta(s)\right)\right]^\Delta < 0 \text{ and } \varkappa^\Delta(s) < 0.$$

In accordance with the proof of Theorem 3.1, Case (b), we conclude that

$$\varkappa(s) \geq -r_1^{1/\kappa}(s)\varkappa^\Delta(s)\xi(s),$$

and

$$-r_1^{1/\kappa}(s)\varkappa^\Delta(s) \geq \varkappa(s) \left(\int_{s_1}^s \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^\kappa r_2(\varsigma)\Delta\varsigma \right)^{1/\kappa}.$$

Therefore,

$$-r_1^{1/\kappa}(s)\varkappa^\Delta(s) \geq -r_1^{1/\kappa}(s)\varkappa^\Delta(s) \left\{ \xi(s) \left(\int_{s_1}^s \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^\kappa r_2(\varsigma)\Delta\varsigma \right)^{1/\kappa} \right\}.$$

Consequently,

$$\limsup_{s\to\infty} \left\{ \xi(s) \left(\int_{s_1}^s \left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)} \right)^\kappa r_2(\varsigma)\Delta\varsigma \right)^{1/\kappa} \right\} \leq 1,$$

which contradicts (3.13). This completes the proof.

Example 3.1. Consider the half-linear sublinear advanced dynamic equation

$$\left[s\sqrt[3]{\sigma(s)\varkappa^{\Delta}(s)}\right]^{\Delta}+\tilde{\beta}\xi(s)\sqrt[3]{\xi(s)\varkappa(\zeta(s))}=0, \tag{3.14}$$

where $\tilde{\beta}>0$ is a constant. Here,

$$\kappa=\frac{1}{3},\ r_1(s)=s\sqrt[3]{\sigma(s)},\ \text{and}\ r_2(s)=\tilde{\beta}\xi(s)\sqrt[3]{\xi(s)}.$$

Thus,

$$\begin{aligned}
&\limsup_{s\to\infty}\left\{\xi(s)\left(\int_T^s\left(\frac{\xi(\zeta(\varsigma))}{\xi(\varsigma)}\right)^{\kappa}r_2(\varsigma)\Delta\varsigma\right)^{1/\kappa}\right\}\\
&\geq\tilde{\beta}\limsup_{s\to\infty}\left\{\int_s^{\infty}\frac{\Delta\varsigma}{\varsigma^3\sigma(\varsigma)}\left(\int_T^s\left(\xi(\varsigma)\sqrt[3]{\int_{\zeta(\varsigma)}^{\infty}\frac{\Delta\omega}{\omega^3\sigma(\omega)}}\right)\Delta\varsigma\right)^3\right\}\\
&\geq\tilde{\beta}\limsup_{s\to\infty}\left\{\int_s^{\infty}\left(\frac{-1}{\varsigma^3}\right)^{\Delta}\Delta\varsigma\left(\int_T^s\left(\xi(\varsigma)\sqrt[3]{\int_{\zeta(\varsigma)}^{\infty}\left(\frac{-1}{\omega^3}\right)^{\Delta}\Delta\omega}\right)\Delta\varsigma\right)^3\right\}\\
&=\tilde{\beta}.
\end{aligned}$$

By application of Theorem 3.2, if $\tilde{\beta}>1$, then Eq (3.14) is oscillatory.

4. Discussion and Conclusions

(1) The results in this paper presented are applicable across all time scales without any restrictive conditions, including $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, and $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^n : n\in\mathbb{N}_0 \text{ for } q>1\}$.
(2) These results, unlike previous findings [2,26–29,31,33–38], do not require a condition (1.3) (the canonical case), thereby addressing an open problem noted in several papers [16,33,35,39,40].
(3) Our results extend related contributions to the second-order dynamic equations for both cases $\zeta(s)\leq s$ and $\zeta(s)\geq s$ on $[s_0,\infty)_{\mathbb{T}}$; see the following details:
(i) Criterion (2.1) reduces to (1.23) in the case where $\kappa=1$ and $\zeta(s)\leq s$;
(ii) Criterion (2.12) becomes (1.24) in the case when $\kappa=1$ and $\zeta(s)\leq s$;
(iii) Criterion (3.1) reduces to (1.25) assuming that $\kappa=1$ and $\zeta(s)\geq s$;
(iv) Criterion (3.13) becomes (1.26) under the assumption that $\kappa=1$ and $\zeta(s)\geq s$.
(4) It would be of interest to establish Hille-type and Ohriska-type oscillation criteria for the second-order half-linear noncanonical dynamic equation (1.1) when $\kappa>0$.

Author contributions

Taher S. Hassan: writing—original draft, Investigation, writing—review editing, and Supervision; Amir Abdel Menaem: Formal analysis, Resources; Mouataz Billah Mesmouli: Formal analysis, Resources; Wael W Mohammed: Formal analysis, Resources; Ismoil Odinaev: Formal analysis, Resources; Bassant M. ElMatary: writing—review editing, Validation, and Investigation. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare there are no conflicts of interest.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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