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Research article

Revealing asymmetric homoclinic and heteroclinic orbits

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Abstract: Although scholars have proven the existence of a pair of homoclinic orbits to the origin, or a pair of heteroclinic orbits to the origin along with a pair of nontrivial equilibria in symmetric Lorenz, Chen, and Lü systems, they have rarely dealt with asymmetric ones of the corresponding asymmetric analogues, to the best of our knowledge. To clarify this subject, this work revisited an asymmetric Chen system and reveals a single/a pair of asymmetric heteroclinic/homoclinic orbits, which are justified with numerical experiments.

Keywords: asymmetric Chen system; a single/a pair of asymmetric heteroclinic/homoclinic orbits; Lyapunov function

1. Introduction

In 2006, Li et al. revisited the Chen system and revealed that there exists a pair of symmetric heteroclinic orbits to the origin and a pair of nontrivial equilibria by utilizing the tools of Lyapunov function and α-/ω-limit set [1]. Due to the advantage of concentrating solely on how to construct an appropriate Lyapunov function rather than the mutual disposition between unstable and stable manifolds of a closed orbit or saddle, such result opened the floodgates to the study of heteroclinic orbits of other Lorenz-like analogues, such as the united Lorenz-type Lorenz system [2], the complex Lorenz system [3], the 5D hyperchaotic system [4], the unified hyperchaotic Lorenz-type system [5], the four-dimensional circuit system [6], the four-dimensional chaotic system [7], the four-thirds-degree Lorenz-like system [8–11], the six-fifths-degree Lorenz-like system [12], the five-thirds-degree Lorenz-like analogue [13], the cubic Lorenz-like analogues [14, 15], the periodically forced Lorenz-like analogues [16, 17], and the quadratic one [18].

As defined in [19, 20], an orbit $x(t)$ of a dynamical system is a homoclinic (resp. heteroclinic) orbit if and only if $\lim_{t\to-\infty} x(t) = x_1$ (or L_1), $\lim_{t\to\infty} x(t) = x_2$ or L_1 (or L_1), $x_1 = x_2$ (or $L_1 = L_2$) (resp. $x_1 \neq x_2$ (or $L_1 \neq L_2$)), where $x_{1,2}$ and $L_{1,2}$ are equilibria and closed orbits, respectively. In the sense of Shilnikov, chaos occurring in 3D autonomous differential systems may be divided into Shilnikov homoclinic or heteroclinic orbit-type chaos or hybrid-type chaos [19]. In the study of homoclinic and heteroclinic orbits, various powerful tools have been developed, such as contraction map and boundary problems [19], Poincaré map [20], Melnikov method [21], tracking stable and unstable manifolds [22], and fishing principle [23], among others. Particularly, Haller and Wiggins formulated methods to study the existence of homoclinic or heteroclinic obits to periodic orbits, hyperbolic fixed points, or combinations of hyperbolic fixed points and/or periodic orbits in a class of two-degree-of-freedom-integrable Hamiltonian systems subject to arbitrary Hamiltonian perturbations [24]. Escalante-González and E. Campos created a double-scroll attractor that emerges from a heteroclinic loop [25, Figure 1, p.5] and further developed an approach to designing multistable piecewise linear systems with self-excited and hidden multiscroll attractors. Anzo-Hernández et al. also considered the itinerary synchronization between piecewise linear systems coupled with unidirectional links [26]. In the restricted circular three-body problem, planetary scientists and engineers must consider heteroclinic connections between period orbits for space missions [27–29]. For other practical applications, such as biomathematics [30–33], cell signaling [34], and fluid mechanics [35], homoclinic and heteroclinic orbits play an important role.

However, as we all know, little attention has been given to the scenario of asymmetric heteroclinic orbits in neighboring Lorenz-like systems. One cannot help but wonder whether the aforementioned method is applicable to proving the existence of a pair of asymmetric heteroclinic orbits of Lorenz-like systems (if they exist), i.e., the asymmetric Chen system [36, 37]. In this effort, we reexamine this system and present the following contributions:

(i) Proving the existence of a pair of asymmetric heteroclinic orbits and a single heteroclinic orbit using two different Lyapunov functions.

(ii) Proving the existence of a pair of asymmetric homoclinic orbits and a single homoclinic orbit using a Hamiltonian function.

The following, Section 2 summarizes the main results on the homoclinic and heteroclinic orbits of the asymmetric Chen system. Then, we provide the corresponding detail proofs and numerical simulations in Section 3. Section 4 draws conclusions and discusses future work.

2. Asymmetric Chen system and main results

In 2002, Lü et al. introduced the following asymmetric Chen system [36]:

$$\dot{x} = a(y-x), \dot{y} = (c-a)x - xz + cy + m, \dot{z} = -bz + xy, am \neq 0, b, c \in \mathbb{R}, \tag{2.1}$$

performed a mirror operation to merge together two single-scroll attractors, and ultimately found the so-called compound structure of the Chen attractor, which unfortunately cannot reflect Lyapunov exponents and Lyapunov dimensions of the Chen attractor [37]. Numerical experiments demonstrated other dynamical behaviors of system (2.1), such as symmetry with respect to $m = 0$, bifurcation, and period-doubling to chaos. In contrast, analytical work involving homoclinic and heteroclinic orbits is still absent, as far as we know.

First, based on the Cardano formula, the following assertions hold, where $\Delta = (\frac{bm}{2})^2 - (\frac{b(2c-a)}{3})^3$.

Theorem 2.1. *(i) If* $\Delta = 0$, $b(2c-a) = 0$ *(resp.* $b(2c-a) \neq 0$*) and* $bm \neq 0$*, then system (2.1) has a single equilibrium point* $D_1 = (\sqrt[3]{bm}, \sqrt[3]{bm}, \frac{\sqrt[3]{(bm)^2}}{b})$ *(resp. two ones* $S_1 = (2\sqrt[3]{\frac{bm}{2}}, 2\sqrt[3]{\frac{bm}{2}}, \frac{4}{b}\sqrt[3]{\frac{(bm)^2}{4}})$ *and* $S_2 = (\sqrt[3]{\frac{bm}{2}}, \sqrt[3]{\frac{bm}{2}}, \frac{1}{b}\sqrt[3]{\frac{(bm)^2}{4}})$*).*

(ii) If $\Delta > 0$*, then system (2.1) has a unique equilibrium point* $G_1 = (t_1, t_1, \frac{t_1^2}{b})$*, where* $t_1^3 - b(2c-a)t_1 - bm = 0$*.*

(iii) If $\Delta < 0$*, then* $E_i = (h_i, h_i, \frac{h_i^2}{b})$ *,*$i = 1, 2, 3$ *are three different equilibria of system (2.1), where* $h_i = 2\sqrt{\frac{b(2c-a)}{3}}\cos(\theta + \frac{2(i-1)\pi}{3})$ *and* $\theta = \frac{1}{3}\arccos(\frac{3\sqrt{3}m}{2(2c-a)\sqrt{b(2c-a)}})$ *.*

Second, we derive the asymmetric heteroclinic orbits to $E_{1,2}$ and E_3 and to S_1 and S_2 in system (2.1) in the following. The proofs will be outlined in Section 3.

Theorem 2.2. *Set* $a > 0$, $b - 2a \geq 0$, $2c - a > 0$, $a - c > 0$, $(2c-a)m \neq 0$, $mh_1 > 0$ *and* $mh_2 < 0$*.*

(a) If $(\frac{bm}{2})^2 - (\frac{b(2c-a)}{3})^3 < 0$*, then there exists a pair of asymmetric heteroclinic orbits connecting* $E_{1,2}$ *and* E_3 *in system (2.1).*

(b) If $(\frac{bm}{2})^2 = (\frac{b(2c-a)}{3})^3$*, then there exists a single heteroclinic orbit joining* S_1 *and* S_2 *in system (2.1).*

At last, we formulate the asymmetric homoclinic orbits to E_3 and to S_2 in the following theorem and prove their existence in Section 3.

Theorem 2.3. *Set* $a = c$, $b = 2a$, $\triangle = x^2 - \frac{x^4}{4a^2} + mx$ *and* $\triangle_* = h_3^2 - \frac{h_3^4}{4a^2} + mh_3$*.*

(i) If $(am)^2 - (\frac{2a^2}{3})^3 < 0$*, then there exists a pair of asymmetric homoclinic orbits to* E_3 *in system (2.1):* $y = x \pm \sqrt{\triangle - \triangle_*}$*.*

(ii) If $(am)^2 - (\frac{2a^2}{3})^3 = 0$*, then system (2.1) has a single homoclinic orbit to* S_2*:* $-axy + \frac{ay^2}{2} + \frac{x^4}{8a} - mx = -\frac{a\sqrt[3]{(am)^2}}{2} + \frac{\sqrt[3]{(am)^4}}{8a} - m\sqrt[3]{am}$*.*

For the sake of readability, the following denotations are introduced:

(a) $\phi_t(q_3^0) = (x(t; x_3^0), y(t; y_3^0), z(t; z_3^0))$: an orbit starting from the initial condition $q_3^0 = (x_3^0, y_3^0, z_3^0)$ for system (2.1).

(b) $\gamma = \{\phi_t(q_3^0) | \phi_t(q_3^0) = (x(t; x_3^0), y(t; y_3^0), z(t; z_3^0)) \in W^u(E_3)\}$, $t \in \mathbb{R}$: any one branch of $W^u(E_3)$ for $x(t; x_3^0) < 0$, $m > 0$ (resp. $x(t; x_3^0) > 0$, $m < 0$) as $t \to -\infty$.

(c) $V_1^1(\phi_t(q_3^0)) = \frac{1}{2}[b(b-2a)(y-x)^2 + (-bz + x^2)^2 + \frac{b-2a}{2a}(-h_1^2 + x^2)^2 + \frac{bm(b-2a)}{h_1 a}(-h_1 + x)^2]$, and $V_1^2(\phi_t(q_3^0)) = \frac{1}{2}[b(b-2a)(y-x)^2 + (-bz + x^2)^2 + \frac{b-2a}{2a}(-h_2^2 + x^2)^2 - \frac{bm(b-2a)}{h_2 a}(-h_2 + x)^2]$ for $b - 2a > 0$, $V_2^1(\phi_t(q_3^0)) = \frac{1}{2}[(y-x)^2 + \frac{1}{4a^2}(-h_1^2 + x^2)^2 + \frac{m}{h_1 a}(-h_1 + x)^2]$, $V_2^2(\phi_t(q_3^0)) = \frac{1}{2}[(y-x)^2 + \frac{1}{4a^2}(-h_2^2 + x^2)^2 - \frac{m}{h_2 a}(-h_2 + x)^2]$ for $b - 2a = 0$ and $z = \frac{x^2}{2a}$: the Lyapunov functions for $a > 0$, $2c - a > 0$, $a - c > 0$, $(2c-a)m \neq 0$, $mh_1 > 0$ and $mh_2 < 0$.

3. Asymmetric homoclinic and heteroclinic orbits

In this section, first we compute the derivatives of $V_{1,2}^{1,2}$ along $\phi_t(q_3^0)$:

$$\frac{dV_1^{1,2}(\phi_t(q_3^0))}{dt}\Big|_{(2.1)} = -b(-bz + x^2)^2 - b(b-2a)(a-c)(y-x)^2 \tag{3.1}$$

and

$$\frac{dV_2^{1,2}(\phi_t(q_3^0))}{dt}\Big|_{(2.1)} = -(a-c)(y-x)^2. \tag{3.2}$$

Utilizing Lyapunov functions $V_{1,2}^{1,2}$, we give the proof of Theorem 2.2, as shown below.

Here, we only study the case of $(\frac{bm}{2})^2 - (\frac{b(2c-a)}{3})^3 < 0$. The case of $(\frac{bm}{2})^2 = (\frac{b(2c-a)}{3})^3$ is similar and is omitted.

First, one needs to derive the following statements.

Theorem 3.1. *When $b \geq 2a > 0$, $2c > a > c$, $(2c-a)m \neq 0$, $mh_1 > 0$, $mh_2 < 0$ and $(\frac{bm}{2})^2 - (\frac{b(2c-a)}{3})^3 < 0$, one comes to the following conclusions:*

(i) When $t_1 < t_2$ and $V_{1,2}^{1,2}(\phi_{t_1}(q_3^0)) = V_{1,2}^{1,2}(\phi_{t_2}(q_3^0))$, $q_3^0 \in \{E_1, E_2, E_3\}$.

(ii) When $\lim_{t\to-\infty} \phi_t(q_3^0) = E_3$ and $q_3^0 \neq E_3$, $V_{1,2}^{1,2}(E_3) > V_{1,2}^{1,2}(\phi_t(q_3^0))$.

Proof. (i) For $b \geq 2a > 0$, $2c > a > c$, $(2c-a)m \neq 0$, $mh_1 > 0$, $mh_2 < 0$ and $(\frac{bm}{2})^2 - (\frac{b(2c-a)}{3})^3 < 0$, the fact $\frac{dV_{1,2}^{1,2}(\phi_t(q_3^0))}{dt}\Big|_{(2.1)} \leq 0$ holds. According to the assumed conditions, $\forall t \in (t_1, t_2)$, we arrive at the conclusion $\frac{dV_{1,2}^{1,2}(\phi_t(q_3^0))}{dt}\Big|_{(2.1)} = 0$ and thus

$$\dot{x}(t; x_3^0) \equiv \dot{y}(t; y_3^0) \equiv \dot{z}(t; z_3^0) \equiv 0, \tag{3.3}$$

i.e., q_3^0 is a fixed point. Exactly speaking, $\dot{x}(t; x_3^0) = a(y-x) = 0$ for all $t \in \mathbb{R}$ yields $x(t) = x_3^0$ and $\dot{y}(t; y_3^0) = 0$.

For $b = 2a$, the invariant algebraic surface $Q(\phi_t(q_3^0)) = z - \frac{x^2}{2a} = 0$ together with $\phi_t(q_3^0) \in \{y - x = 0\}$ leads to (3.3).

(ii) The fact that $V_{1,2}^{1,2}(E_3) > V_{1,2}^{1,2}(\phi_t(q_3^0))$ easily follows from reductio. In fact, $\exists t_0 \in \mathbb{R}$, such that $0 < V_{1,2}^{1,2}(E_3) \leq V_{1,2}^{1,2}(\phi_{t_0}(q_3^0))$. Based on the first conclusion, $q_3^0 \in \{E_1, E_2, E_3\}$. In addition, $q_3^0 \neq E_3$, which contradicts the assumed condition $\phi_t(q_3^0) \to E_3$ for $t \to -\infty$. Hence, it follows that $V_{1,2}^{1,2}(E_3) > V_{1,2}^{1,2}(\phi_t(q_3^0))$, $\forall t \in \mathbb{R}$. □

In light of Theorem 3.1, we prove Theorem 2.2(a).

Proof. (a.1) Since $\frac{dV_{1,2}^{1,2}(\phi_t(q_3^0))}{dt}\Big|_{(2.1)} \leq 0$ for $b \geq 2a > 0$, $2c > a > c$, $(2c-a)m \neq 0$, $mh_1 > 0$, $mh_2 < 0$ and $(\frac{bm}{2})^2 - (\frac{b(2c-a)}{3})^3 < 0$, we have

$$0 \leq V_{1,2}^{1,2}(\phi_t(q_3^0)) \leq V_{1,2}^{1,2}(q_3^0), \tag{3.4}$$

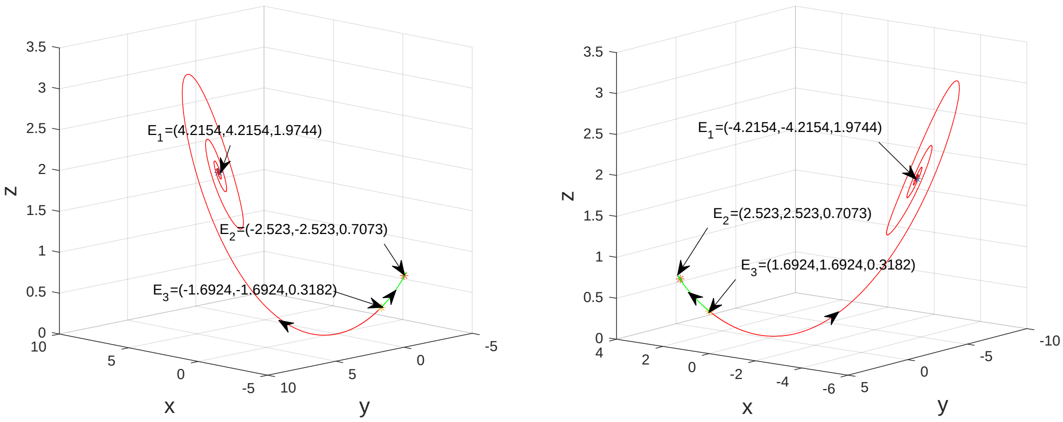
$\forall t \in \mathbb{R}$, i.e., $\lim_{t\to+\infty} V_{1,2}^{1,2}(\phi_t(q_3^0)) = (V_{1,2}^{1,2})^*(q_3^0)$ exist. Meanwhile, it also leads to the boundedness of $\phi_t(q_3^0)$ for all $t \geq 0$. Define ω-limit set of $\phi_t(q_3^0)$ by $\Omega(q_3^0)$. Namely, $\forall \hat{q} \in \Omega(q_3^0)$, $\phi_t(\hat{q}) \in \Omega(q_3^0)$. In addition, $\exists t_n \to \infty$, $n \to \infty$, we arrive at $V_{1,2}^{1,2}(\phi_t(\hat{q})) = \lim_{n\to+\infty} V_{1,2}^{1,2}(\phi_{t_n}(q_3^0)) = (V_{1,2}^{1,2})^*(q_3^0) = const.$ Therefore, $\forall t_1 < t_2$ such that $V_{1,2}^{1,2}(\phi_{t_1}(\hat{q})) = V_{1,2}^{1,2}(\phi_{t_2}(\hat{q}))$. From Theorem 3.1, $\hat{q} \in \{E_1, E_2, E_3\}$.

(a.2) Suppose $p(t, q_i^0)$ are homoclinic orbits to E_i through initial conditions $q_i^0 \notin \{E_i\}$, i.e., $\lim_{t\to\pm\infty} p(t, q_i^0) = k_i$, $k_i \in \{E_i\}$, $i = 1, 2, 3$. Because of $\frac{dV_{1,2}^{1,2}(\phi_t(q_i^0))}{dt}\Big|_{(2.1)} \leq 0$, one has

$$0 \leq V_{1,2}^{1,2}(k_i) = V_{1,2}^{1,2}(p(-\infty, q_i^0)) \leq V_{1,2}^{1,2}(p(t, q_i^0)) \leq V_{1,2}^{1,2}(p(\infty, q_i^0)) = V_{1,2}^{1,2}(k_i), \tag{3.5}$$

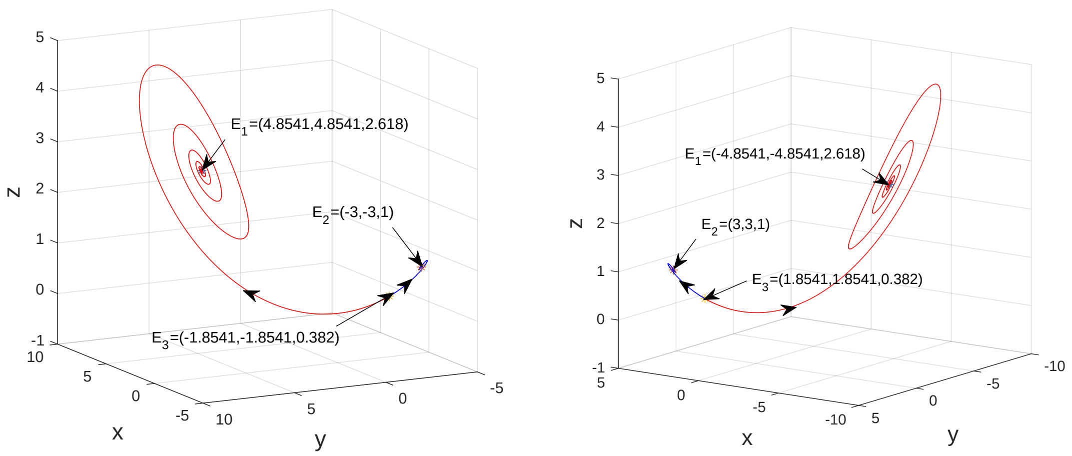
i.e., $V_{1,2}^{1,2}(p(t,q_i^0)) = V_{1,2}^{1,2}(k_i)$, $\forall t \in \mathbb{R}$, which suggests $q_i^0 \in \{E_i\}$ and a contradiction occurs. Namely, system (2.1) has no homoclinic orbits when $b \geq 2a > 0$, $2c > a > c$, $(2c-a)m \neq 0$, $mh_1 > 0$, $mh_2 < 0$.

From the first conclusion (a.1), all branches of $W^u(E_3)$ belong to the ω-limit set $\{E_1, E_2, E_3\}$.



(a) $m = 2$ (b) $m = -2$

Figure 1. For $(a, c, b) = (4.5, 3, 9)$ and (a) $m = 2$, $(x_0^1, y_0^1, z_0^1) = (-1.6925, -1.6923, 0.3182)$, $(x_0^2, y_0^2, z_0^1) = (-1.6923, -1.6925, 0.3182)$, (b) $m = -2$, $(x_0^3, y_0^3, z_0^1) = (1.6925, 1.6923, 0.3182)$, $(x_0^4, y_0^4, z_0^1) = (1.6923, 1.6925, 0.3182)$, a pair of asymmetric heteroclinic orbits to $E_{1,2}$ and E_3 of system (2.1) for $b = 2a$.

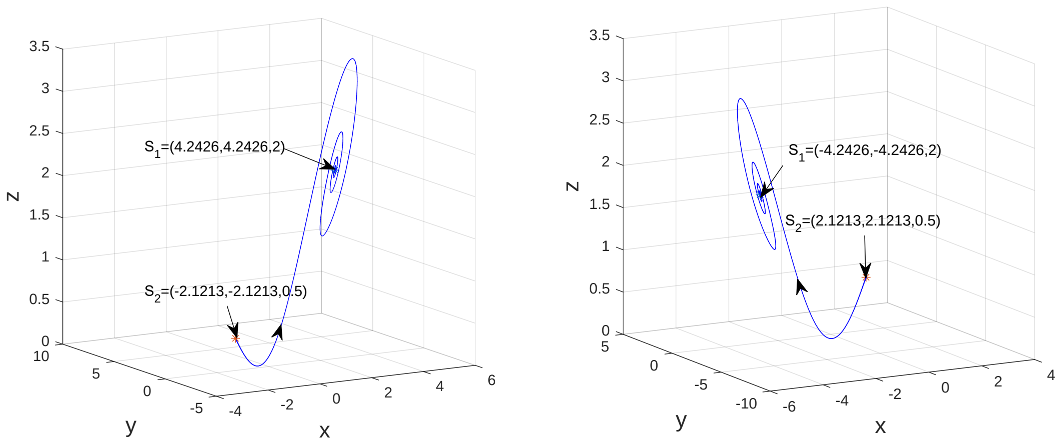


(a) $m = 3$ (b) $m = -3$

Figure 2. For $(a, c, b) = (4, 3, 9)$ and (a) $m = 3$, $(x_0^5, y_0^5, z_0^2) = (-1.8542, -1.8540, 0.382)$, $(x_0^6, y_0^6, z_0^2) = (-1.8540, -1.8542, 0.382)$, (b) $m = -3$, $(x_0^7, y_0^7, z_0^2) = (1.8542, 1.8540, 0.382)$, $(x_0^8, y_0^8, z_0^2) = (1.8540, 1.8542, 0.382)$, a pair of asymmetric heteroclinic orbits to $E_{1,2}$ and E_3 of system (2.1) for $b > 2a$.

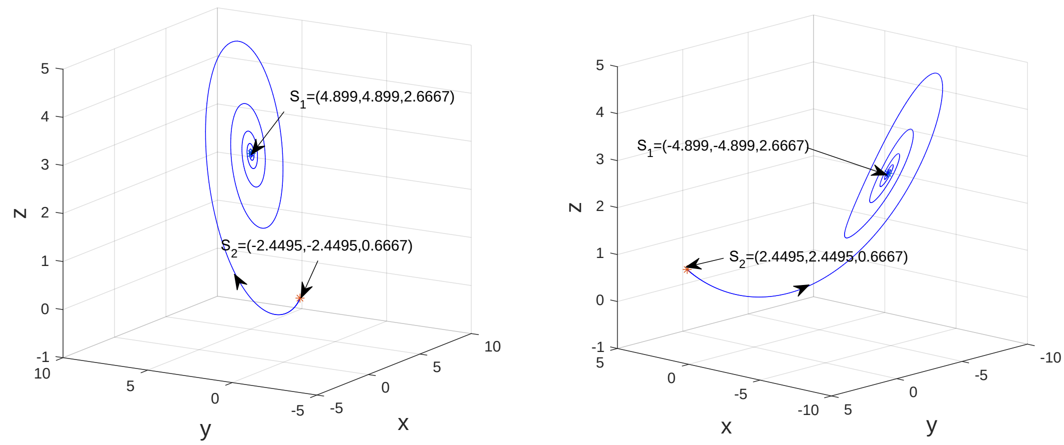
Due to $V_{1,2}^{1,2}(E_{1,2}) < V_{1,2}^{1,2}(E_3)$ with $b \geq 2a > 0$, $2c > a > c$, $(2c-a)m \neq 0$, $mh_1 > 0$, $mh_2 < 0$ and $(\frac{bm}{2})^2 - (\frac{b(2c-a)}{3})^3 < 0$, $\hat{q}$ has to be either E_1 or E_2. Because of the asymmetry of system (2.1), $p(t, q_3^0)$

tending to $E_{1,2}$ creates a pair of asymmetric heteroclinic orbits connecting $E_{1,2}$ and E_3, as depicted in Figures 1–4. □



(a) $m = 3\sqrt{2}/2$ (b) $m = -3\sqrt{2}/2$

Figure 3. For $(a, c, b) = (4.5, 3, 9)$ and (a) $m = 3\sqrt{2}/2$, $(x_0^9, y_0^9, z_0^3) = (-2.1214, -2.1212, 0.5)$, (b) $m = -3\sqrt{2}/2$, $(x_0^{10}, y_0^{10}, z_0^3) = (2.1214, 2.1212, 0.5)$, a single heteroclinic orbit to S_1 and S_2 of system (2.1).



(a) $m = 4\sqrt{6}/3$ (b) $m = -4\sqrt{6}/3$

Figure 4. For $(a, c, b) = (4, 3, 9)$ and (a) $m = 4\sqrt{6}/3$, $(x_0^{11}, y_0^{11}, z_0^4) = (-2.4496, -2.4494, 0.6667)$, (b) $m = -4\sqrt{6}/3$, $(x_0^{12}, y_0^{12}, z_0^4) = (2.4496, 2.4494, 0.6667)$, a single heteroclinic orbit to S_1 and S_2 of system (2.1).

Lastly, we prove Theorem 2.3, i.e., the existence of a pair of asymmetric homoclinic orbits to E_3 and a single homoclinic orbit to S_2. At this time, system (2.1) reduces into the following two-dimensional one:

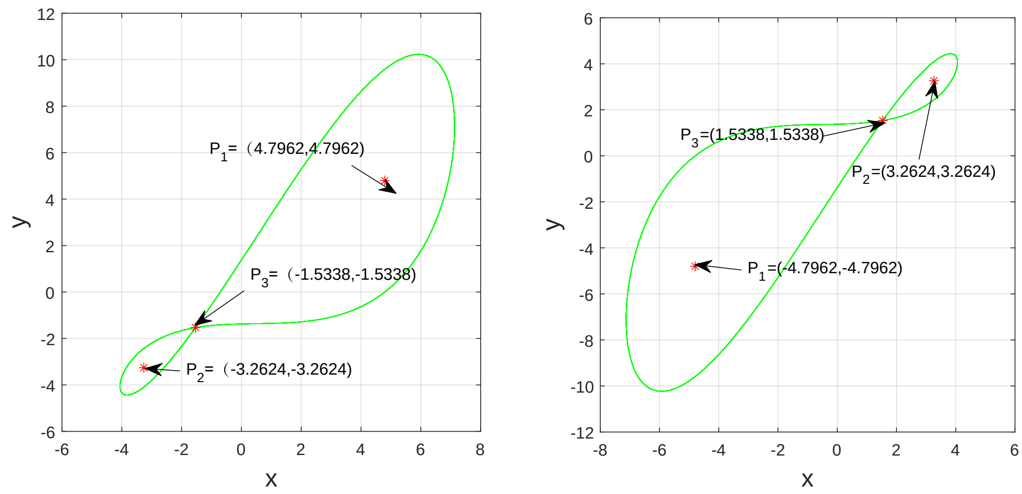
$$\dot{x} = a(y - x), \dot{y} = (c - a)x - \frac{x^3}{2a} + cy + m, \tag{3.6}$$

which is a Hamiltonian function for $a = c$:

$$H(x, y) = \frac{x^4}{8a} - mx - axy + \frac{ay^2}{2}. \tag{3.7}$$

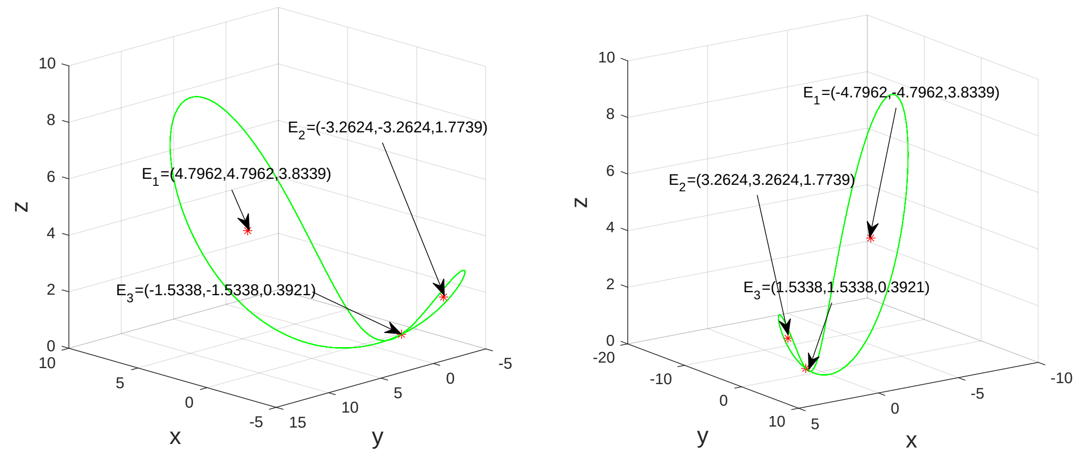
From Eq (3.7), we can easily obtain Theorem 2.3 and omit it here.

Evidently, system (3.6) has the following equilibrium points: $P_i = (x_i, x_i)$, $x_i^3 - 2a^2 x_i - 2am = 0$, $i = 1, 2, 3$.



(a) $m = 4$ (b) $m = -4$

Figure 5. When $(a, c) = (3, 3)$, (a) $m = 4$, (b) $m = -4$, a pair of asymmetric homoclinic orbits to P_3 of system (3.6).



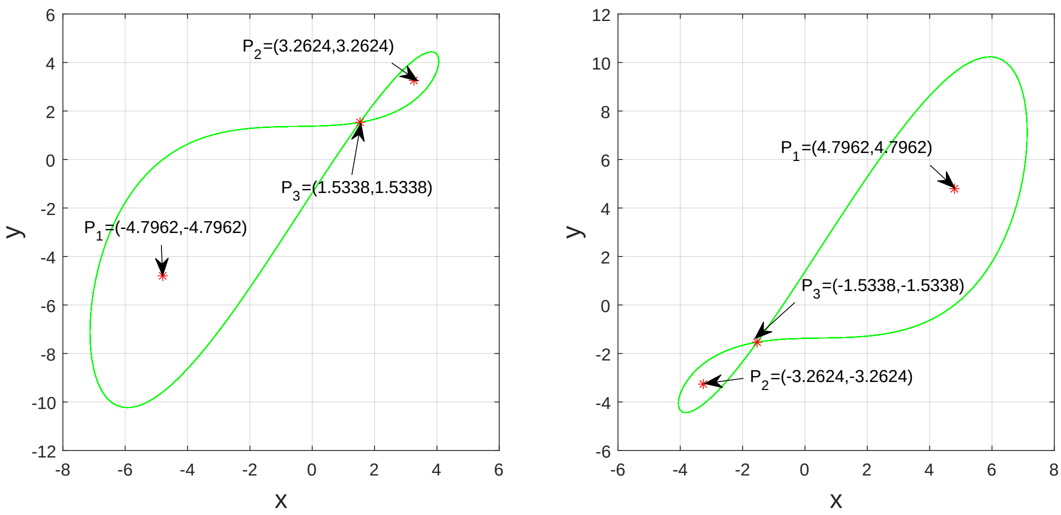
(a) $m = 4$ (b) $m = -4$

Figure 6. When $(a, c, b) = (3, 3, 6)$, (a) $m = 4$, (b) $m = -4$, a pair of asymmetric homoclinic orbits to E_3 of system (2.1).

1) $(a, m) = (3, \pm 4)$. System (3.6) (resp. (2.1)) has three equilibria: $P_1 = (\pm 4.7962, \pm 4.7962)$, $P_2 = (\mp 3.2624, \mp 3.26242)$, $P_3 = (\mp 1.5338, \mp 1.5338)$ (resp. $E_1 = (\pm 4.7962, \pm 4.7962, 3.8339)$, $E_2 =$

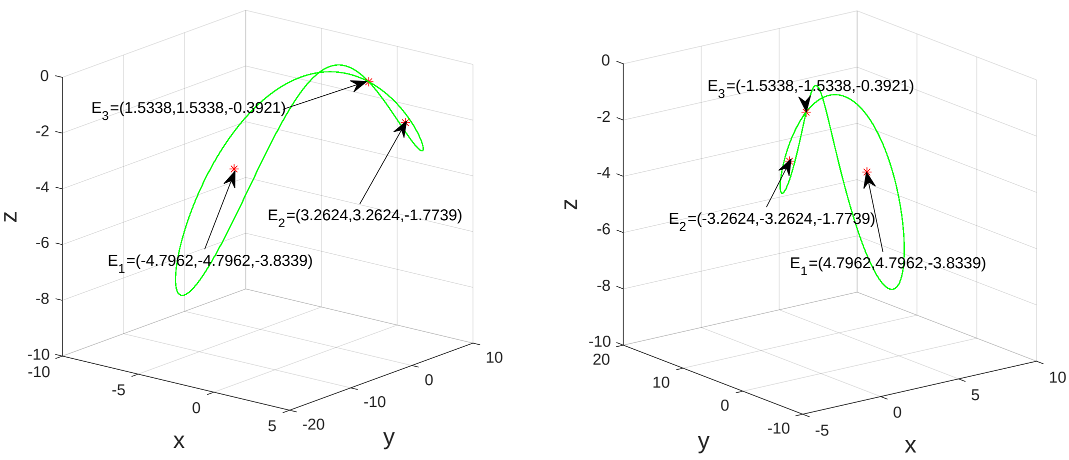
$(\mp 3.2624, \mp 3.26242, 1.7739)$, $E_3 = (\mp 1.5338, \mp 1.5338, 0.3921)$) with the corresponding eigenvalues: $\lambda_{1,2} = \pm 5.0503i, \pm 2.6391i, \pm 2.3391$ (resp. $(\lambda_{1,2}, \lambda_3) = (\pm 5.0503i, -6), (\pm 2.6391i, -6), (\pm 2.3391, -6)$). Figure 5 (resp. Figure 6) depicts a pair of asymmetric homoclinic orbits to P_3 (resp. E_3) of system (3.6).

2) $(a, m) = (-3, \pm 4)$. System (3.6) (resp. (2.1)) has three equilibria: $P_1 = (\mp 4.7962, \mp 4.7962)$, $P_2 = (\pm 3.2624, \pm 3.2624)$, $P_3 = (\pm 1.5338, \pm 1.5338)$ (resp. $E_1 = (\mp 4.7962, \mp 4.7962, -3.8339)$, $E_2 = (\pm 3.2624, \pm 3.26242, -1.7739)$, $E_3 = (\pm 1.5338, \pm 1.5338, -0.3921)$) with the corresponding eigenvalues: $\lambda_{1,2} = \pm 5.0503i, \pm 2.6391i, \pm 2.3391$ (resp. $(\lambda_{1,2}, \lambda_3) = (\pm 5.0503i, 6), (\pm 2.6391i, 6), (\pm 2.3391, 6)$). Figure 7 (resp. Figure 8) illustrates a pair of asymmetric homoclinic orbits to P_3 (resp. E_3).



(a) $m = 4$ (b) $m = -4$

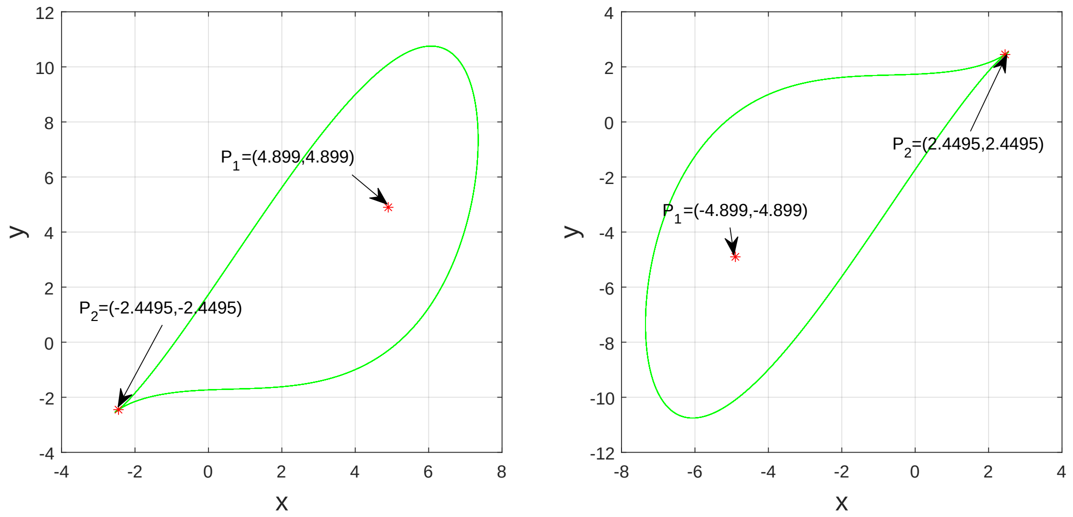
Figure 7. When $(a, c) = (-3, -3)$, (a) $m = 4$, (b) $m = -4$, a pair of asymmetric homoclinic orbits to P_3 of system (3.6).



(a) $m = 4$ (b) $m = -4$

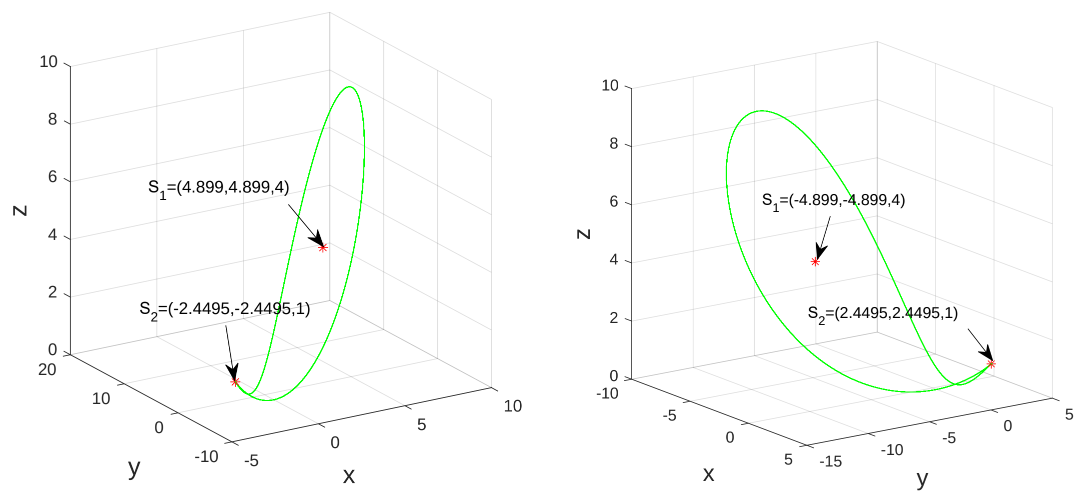
Figure 8. When $(a, c, b) = (-3, -3, -6)$, (a) $m = 4$, (b) $m = -4$, a pair of asymmetric homoclinic orbits to E_3 of system (2.1).

3) $(a, m) = (3, \pm 2\sqrt{6})$. System (3.6) (resp. (2.1)) has two equilibria: $P_1 = (\pm 4.899, \pm 4.899)$, $P_2 = (\mp 2.4495, \mp 2.4495)$ (resp. $S_1 = (\pm 4.899, \pm 4.899, 4)$, $S_2 = (\mp 2.4495, \mp 2.4495, 1)$) with the corresponding eigenvalues: $\lambda_{1,2} = \pm 5.1962i, 0, 0$ (resp. $(\lambda_{1,2}, \lambda_3) = (\pm 5.1962i, -6), (0, 0, -6)$). Figure 9 (resp. Figure 10) shows a single homoclinic orbit to P_2 (resp. S_2).



(a) $m = 2\sqrt{6}$ (b) $m = -2\sqrt{6}$

Figure 9. When $(a, c) = (3, 3)$, (a) $m = 2\sqrt{6}$, (b) $m = -2\sqrt{6}$, a single homoclinic orbit to P_2 of system (3.6).

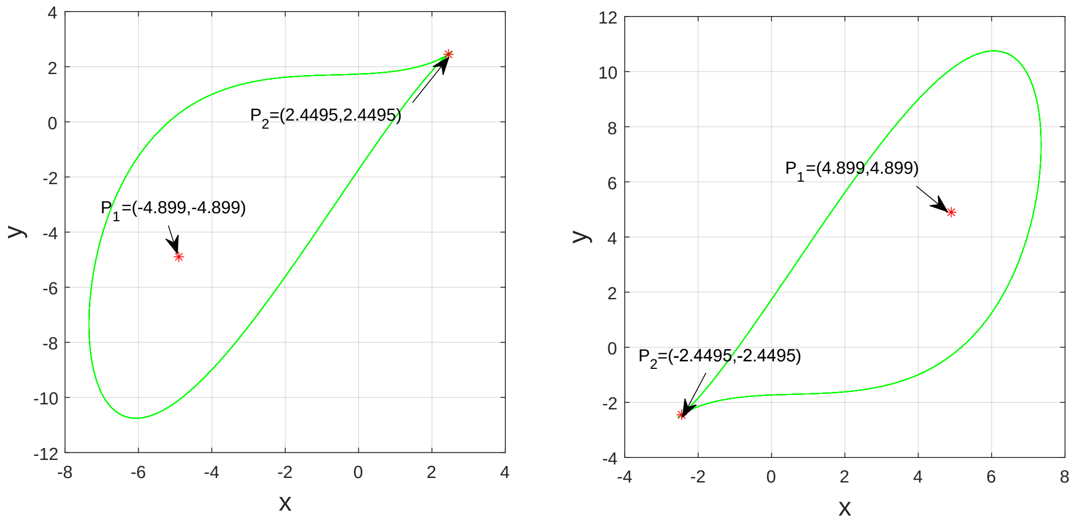


(a) $m = 2\sqrt{6}$ (b) $m = -2\sqrt{6}$

Figure 10. When $(a, c, b) = (3, 3, 6)$, (a) $m = 2\sqrt{6}$, (b) $m = -2\sqrt{6}$, a single homoclinic orbit to S_2 of system (2.1).

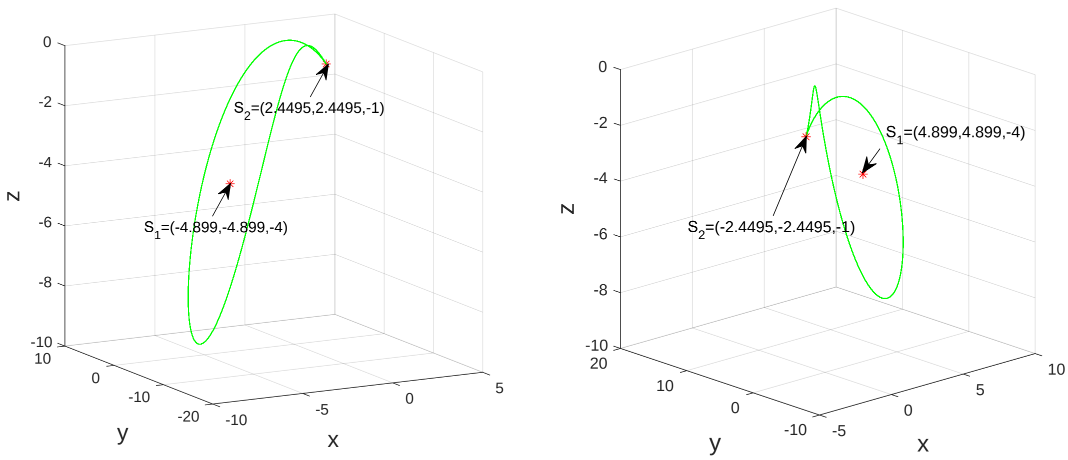
4) $(a, m) = (-3, \pm 2\sqrt{6})$. System (3.6) (resp. (2.1)) has three equilibria: $P_1 = (\mp 4.899, \mp 4.899)$, $P_{2,3} = (\pm 2.4495, \pm 2.4495)$ (resp. $S_1 = (\mp 4.899, \mp 4.899, -4)$, $S_2 = (\pm 2.4495, \pm 2.4495, -1)$) with the

corresponding eigenvalues: $\lambda_{1,2} = \pm 5.1962i, 0, 0$ (resp. $(\lambda_{1,2}, \lambda_3) = (\pm 5.1962i, 6), (0, 0, 6)$). Figure 11 (resp. Figure 12) displays a single homoclinic orbit to P_2 (resp. S_2) of system (3.6) (resp. (2.1)).



(a) $m = 2\sqrt{6}$ (b) $m = -2\sqrt{6}$

Figure 11. When $(a, c) = (-3, -3)$, (a) $m = 2\sqrt{6}$, (b) $m = -2\sqrt{6}$, a single homoclinic orbit to P_2 of system (3.6).

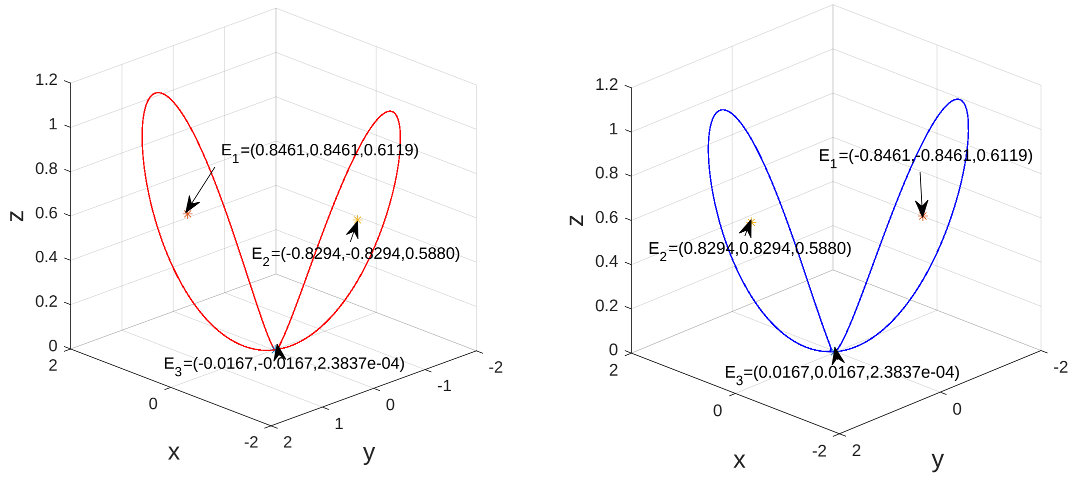


(a) $m = 2\sqrt{6}$ (b) $m = -2\sqrt{6}$

Figure 12. When $(a, c, b) = (-3, -3, -6)$, (a) $m = 2\sqrt{6}$, (b) $m = -2\sqrt{6}$, a single homoclinic orbit to S_2 of system (2.1).

Remark 1. *When $m = 0$, the results on the symmetric heteroclinic/homoclinic orbits [1, 38] are not derived from Theorems 2.2 and 2.3.*

Remark 2. *For $(a, c, m, b) = (1, 0.8, \pm 0.01, 1.17)$, Figure 13 shows a pair of asymmetric homoclinic orbits to E_3 in the non-Hamiltonian scenario of system (2.1).*



(a) $m = 0.01$ (b) $m = -0.01$

Figure 13. When $(a, c, b) = (1, 0.8, 1.17)$, (a) $m = 0.01$, (b) $m = -0.01$, a pair of asymmetric homoclinic orbits to E_3 of system (2.1).

4. Conclusions

To the best of our knowledge, whether the asymmetric Chen system has heteroclinic orbits is unknown. If they exist, is the method of Lyapunov function and α-/ω-limit set applicable to it, as the symmetric Lorenz system family? In this endeavor, we revisited an asymmetric perturbation of the Chen system and proved the existence of a single/a pair of asymmetric heteroclinic orbits by constructing suitable Lyapunov functions. With the help of a Hamiltonian function, we also proved that there exists a single/a pair of asymmetric homoclinic orbits. These homoclinic and heteroclinic orbits cast mirror images from the parameter $-m$ to m.

In the future, we hope to deal with other important issues, i.e., the rigorous proof of homoclinic orbits to E_3 in the non-Hamiltonian scenario, asymmetric hidden attractors and practical applications. Meanwhile, it is expected that the obtained results will shed light on the study of homoclinic and heteroclinic orbits of other asymmetric Lorenz-type systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. T. Li, G. Chen, G. Chen, On homoclinic and heteroclinic orbits of the Chen's system, *Int. J. Bifurcation Chaos*, **16** (2006), 3035–3041. https://doi.org/10.1142/S021812740601663X
2. H. Wang, C. Li, X. Li, New heteroclinic orbits coined, *Int. J. Bifurcation Chaos*, **26** (2016), 1650194. https://doi.org/10.1142/S0218127416501947
3. H. Wang, X. Li, Infinitely many heteroclinic orbits of a complex Lorenz system, *Int. J. Bifurcation Chaos*, **27** (2017), 1750110. https://doi.org/10.1142/S0218127417501103
4. H. Wang, X. Li, A novel hyperchaotic system with infinitely many heteroclinic orbits coined, *Chaos, Solitons Fractals*, **106** (2018), 5–15. https://doi.org/10.1016/j.chaos.2017.10.029
5. H. Wang, F. Zhang, Bifurcations, ultimate boundedness and singular orbits in a unified hyperchaotic Lorenz-type system, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 1791–1820. https://doi.org/10.3934/dcdsb.2020003
6. H. Wang, H. Fan, J. Pan, Complex dynamics of a four-dimensional circuit system, *Int. J. Bifurcation Chaos*, **31** (2021), 2150208. https://doi.org/10.1142/S0218127421502084
7. H. Wang, G. Ke, J. Pan, Q. Su, G. Dong, H. Fan, Revealing the true and pseudo-singularly degenerate heteroclinic cycles, *Indian J. Phys.*, **97** (2023), 3601–3615. https://doi.org/10.1007/s12648-023-02689-w
8. H. Wang, G. Ke, J. Pan, F. Hu, H. Fan, Multitudinous potential hidden Lorenz-like attractors coined, *Eur. Phys. J. Spec. Top.*, **231** (2022), 359–368. https://doi.org/10.1140/epjs/s11734-021-00423-3
9. H. Wang, G. Ke, J. Pan, F. Hu, H. Fan, Q. Su, Two pairs of heteroclinic orbits coined in a new sub-quadratic Lorenz-like system, *Eur. Phys. J. B*, **96** (2023), 1–9. https://doi.org/10.1140/epjb/s10051-023-00491-5
10. Z. Li, G. Ke, H. Wang, J. Pan, F. Hu, Q. Su, Complex dynamics of a sub-quadratic Lorenz-like system, *Open Phys.*, **21** (2023), 20220251. https://doi.org/10.1515/phys-2022-0251
11. G. Ke, J. Pan, F. Hu, H. Wang, Dynamics of a new four-thirds-degree sub-quadratic Lorenz-like system, *Axioms*, **13** (2024), 625. https://doi.org/10.3390/axioms13090625
12. H. Wang, J. Pan, G. Ke, Revealing more hidden attractors from a new sub-quadratic Lorenz-like system of degree $\frac{6}{5}$, *Int. J. Bifurcation Chaos*, **34** (2024), 2450071. https://doi.org/10.1142/S0218127424500718
13. H. Wang, J. Pan, G. Ke, F. Hu, A pair of centro-symmetric heteroclinic orbits coined, *Adv. Contin. Discrete Models*, **2024** (2024). https://doi.org/10.1186/s13662-024-03809-4
14. H. Wang, G. Ke, F. Hu, J. Pan, G. Dong, G. Chen, Pseudo and true singularly degenerate heteroclinic cycles of a new 3D cubic Lorenz-like system, *Results Phys.*, **56** (2024), 107243. https://doi.org/10.1016/j.rinp.2023.107243

15. H. Wang, G. Ke, J. Pan, Q. Su, Modeling, dynamical analysis and numerical simulation of a new 3D cubic Lorenz-like system, *Sci. Rep.*, **13** (2023), 6671. https://doi.org/10.1038/s41598-023-33826-4

16. H. Wang, G. Ke, J. Pan, Q. Su, Conjoined Lorenz-like attractors coined, *Miskolc Math. Notes*, **13** (2023).

17. H. Wang, J. Pan, G. Ke, Multitudinous potential homoclinic and heteroclinic orbits seized, *Electron. Res. Arch.*, **32** (2024), 1003–1016. https://doi.org/10.3934/era.2024049

18. H. Wang, J. Pan, F. Hu, G. Ke, Asymmetric singularly degenerate heteroclinic cycles, *Int. J. Bifurcation Chaos*, 2025, In press.

19. L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics, Part II*, World Scientific, Singapore, 2001. https://doi.org/10.1142/4221

20. S. Wiggins, *Introduction to Applied Nonlinear Dynamical System and Chaos*, 2^{nd} edition, Springer, New York, 2003. https://doi.org/10.1007/978-1-4757-4067-7

21. J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, 3^{rd} edition, Springer, Berlin, 1983. https://doi.org/10.1007/978-1-4612-1140-2

22. G. Tigan, D. Turaev, Analytical search for homoclinic bifurcations in the Shimizu-Morioka model, *Phys. D*, **240** (2011), 985–989. https://doi.org/10.1016/j.physd.2011.02.013

23. G. A. Leonov, Fishing principle for homoclinic and heteroclinic trajectories, *Nonlinear Dyn.*, **78** (2014), 2751–2758. https://doi.org/10.1007/s11071-014-1622-8

24. G. Haller, S. Wiggins, Orbits homoclinic to resonances: the Hamiltonian case, *Phys. D*, **66** (1993), 298–346. https://doi.org/10.1016/0167-2789(93)90071-8

25. R. J. Escalante-González, E. Campos, Multistable systems with hidden and self-excited scroll attractors generated via piecewise linear systems, *Complexity*, **2020** (2020), 7832489. https://doi.org/10.1155/2020/7832489

26. A. Anzo-Hernández, E. Campos-Cantón, M. Nicol, Itinerary synchronization between PWL systems coupled with unidirectional links, *Commun. Nonlinear Sci. Numer. Simul.*, **70** (2019), 102-124. https://doi.org/10.1016/j.cnsns.2018.10.020

27. W. S. Koon, M. W. Lo, J. E. Marsden, S. D. Ross, Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics, *Chaos*, **10** (2000), 427–469. https://doi.org/10.1063/1.166509

28. D. Wilczak, P. Zgliczyński, Heteroclinic connections between periodic orbits in planar restricted circular three body problem-A computer assisted proof, *Commun. Math. Phys.*, **234** (2003), 37–75. https://doi.org/10.1007/s00220-002-0709-0

29. D. Wilczak, P. Zgliczyński, Heteroclinic connections between periodic orbits in planar restricted circular three body problem. part II, *Commun. Math. Phys.*, **259** (2005), 561–576. https://doi.org/10.1007/s00220-005-1471-x

30. B. Aulbach, D. Flockerzi, The past in short hypercycles, *J. Math. Biol.*, **27** (1989), 223–231. https://doi.org/10.1007/BF00276104

31. R. M. May, W. Leonard, Nonlinear aspect of competition between three species, *SIAM J. Appl. Math.*, **29** (1975), 243–253. https://doi.org/10.1137/0129022

32. J. Hofbauer, K. Sigmund, On the stabilizing effect of predator and competitors on ecological communities, *J. Math. Biol.*, **27** (1989), 537–548. https://doi.org/10.1007/BF00288433

33. B. Y. Feng, The heteroclinic cycle in the model of competition between n species and its stability, *Acta Math. Appl. Sin.*, **14** (1998), 404–413. https://doi.org/10.1007/BF02683825

34. G. W. Hunt, M. A. Peletier, A. R. Champneys, P. D. Woods, M. Ahmerwaddee, C. J., Budd, et al., Cellular buckling in long structures, *Nonlinear Dyn.*, **21** (2000), 3–29. https://doi.org/10.1023/A:1008398006403

35. N. J. Balmforth, Solitary waves and homoclinic orbits, *Annu. Rev. Fluid Mech.*, **27** (1995), 335–373. https://doi.org/10.1146/annurev.fl.27.010195.002003

36. J. Lü, T. Zhou, G. Chen, S. Zhang, The compound structure of Chen's attractor, *Int. J. Bifurcation Chaos*, **12** (2002), 855–858. https://doi.org/10.1142/S0218127402004735

37. J. Liu, J. Lu, X. Wu, Dynamical analysis for the compound structure of Chen's system, in *ICARCV 2004 8th Control, Automation, Robotics and Vision Conference*, **3** (2004), 2250–2253. https://doi.org/10.1109/ICARCV.2004.1469781

38. Q. Yang, Y. Chen, Complex dynamics in the unified Lorenz-type system, *Int. J. Bifurcation Chaos*, **24** (2014), 1450055. https://doi.org/10.1142/S0218127414500552