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ERA, 33(3): 1333–1336.
DOI: 10.3934/era.2025060
Received: 22 December 2024
Revised: 20 February 2025
Accepted: 21 February 2025
Published: 05 March 2025

Expository

Bohr compactification of separable locally convex spaces

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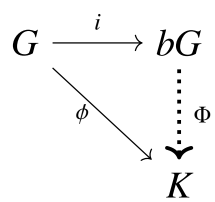
Abstract: It is not well-known that for each separable real locally convex space, its Bohr compactification is isomorphic as a topological group to the Bohr compactification of the topological group $\mathbb{R}$ of all real numbers. This is the case for each separable real Banach space. In this expository research note, we state the results explicitly and provide accessible proofs.

Keywords: Bohr compactification; abelian topological group; Banach space; Pontryagin duality; dual group; separable locally convex space

1. Introduction and preliminaries

Throughout this article, all topological groups considered are Hausdorff.

Definition 1.1. *Let G be any abelian topological group. Then, the compact abelian group bG is said to be the Bohr compactification of G if there is a continuous homomorphism i from G into bG such that for every continuous homomorphism ϕ of G into any compact abelian group K there is a unique continuous homomorphism Φ of bG into K, such that $\Phi \circ i = \phi$.*



The Bohr compactification has been well studied in the literature e.g., [1, Page 99], [2, Page 430], [3, Page 30], [4, Page 482], especially for locally compact abelian groups. The left adjoint functor theorem [5, Page 121] shows that it exists and is unique up to isomorphism. In the case of locally compact abelian groups and in the case of real locally convex spaces, the mapping i is one-to-one.

Remark 1.1. *In the literature, it is clear how to construct the Bohr compactification of an abelian topological group G. First, form the Pontryagin dual* $\widehat{G}$ *of the topological group G (See [1, Page 47]): Put the discrete topology on the underlying group of* $\widehat{G}$ *to obtain the topological group* $\widehat{G}_d$*. Then, the Bohr compactification bG of G is the Pontryagin dual group of* $\widehat{G}_d$*.*

Remark 1.2. *It follows from Remark 1.1 that if topological groups* G_1 *and* G_2 *are such that their dual groups* $\widehat{G}_1$ *and* $\widehat{G}_2$ *are algebraically isomorphic, then* bG_1 *is isomorphic as a topological group to* bG_2*.*

Example 1.1. *[4, Example 8.106(i)], [2, 25.4]. The Bohr compactification of the discrete topological group* $\mathbb{Q}$ *of all rational numbers is* $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$*. To see this, observe that* $\mathbb{Q}$ *is torsionfree, divisible, and countable of rank 1 so that its dual group is torsionfree and divisible. Indeed, the dual group algebraically is readily seen to be a vector space over* $\mathbb{Q}$ *of dimension* $\mathfrak{c}$*; that is,* $\widehat{\mathbb{Q}}$ *is algebraically isomorphic to the (restricted) direct sum* $\mathbb{Q}^{(\mathfrak{c})}$*. Thus,* $b\mathbb{Q}$ *is the direct product* $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$*.*

Example 1.2. *[4, Example 8.106(iii)]. The Bohr compactification* $b\mathbb{R}$ *of the topological group* $\mathbb{R}$ *of all real numbers is (also)* $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$*. To see this, observe that the dual group of* $\mathbb{R}$ *is* $\mathbb{R}$*. This too is a vector space over* $\mathbb{Q}$ *of dimension* $\mathfrak{c}$ *and so it also is algebraically isomorphic to* $\mathbb{Q}^{(\mathfrak{c})}$*. Thus,* $b\mathbb{R}$ *is* $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$*.*

Example 1.3. *For each positive integer n,* $\mathbb{R}^n$ *is a vector space over* $\mathbb{Q}$ *of dimension* $\mathfrak{c}$*, and so as in Example 1.2, the Bohr compactification of the topological group* $\mathbb{R}^n$ *is (also)* $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$*.*

Remark 1.3. *[4, Remark 8.103]. It is well-known that for any real topological vector space E the vector space dual E′ of E is algebraically isomorphic as a group to the Pontryagin dual group* $\widehat{E}$ *of the topological group E.*

2. Bohr compactification of separable Banach spaces and of their duals

While it is possible to give a proof of the general result for separable real locally convex spaces, we think it is instructive first to give the proof for separable Banach spaces, which more readers are familiar with.

Lemma 2.1. *Let B be a Banach space. The cardinality of the dual B′ is* $\mathfrak{c}$ *if and only if bB is isomorphic as a topological group to* $b\mathbb{R}$*.*

Proof. If the cardinality of B' is $\mathfrak{c}$, then by Remark 1.3 the Pontryagin dual group $\widehat{B}$ has cardinality $\mathfrak{c}$; that is $\widehat{B}$ is a vector space over $\mathbb{Q}$ of dimension $\mathfrak{c}$. Thus, the cardinality of $\widehat{B}_d = \mathfrak{c}$. Thus, $\widehat{B}_d = \mathbb{Q}^{(\mathfrak{c})}$. Hence, $bB = \widehat{\mathbb{Q}}^{\mathfrak{c}} = b\mathbb{R}$.

Conversely, if $bB = b\mathbb{R}$, then $bB = \widehat{\mathbb{Q}}^{\mathfrak{c}}$. Thus, $\widehat{B}_d = \widehat{\mathbb{Q}}^{(\mathfrak{c})}$. Thus, we see that $\widehat{B}$ has cardinality $\mathfrak{c}$. Hence, B' too has cardinality $\mathfrak{c}$ by Remark 1.3. □

Theorem 2.1. *Let B be a separable Banach space. Then, the Bohr compactification bB of B is isomorphic as a topological group to* $b\mathbb{R}$*, which is isomorphic as a topological group to* $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$*.*

Proof. It is a standard result in Banach space theory that every separable Banach space B is a quotient Banach space of the Banach space ℓ_1 (see, for example [6], [7, Proof of Theorem 1.5] and [8, Theorem

2.20]). Thus, the dual Banach space of B is a subspace of the dual space of ℓ_1, that is of ℓ_∞. It follows that the cardinality of the dual Banach space of B is $\mathfrak{c}$. It immediately follows from Lemma 2.1 that the Bohr compactification bB of B is isomorphic as a topological group to $b\mathbb{R}$, which is isomorphic as a topological group to $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$ by Example 1.2. □

Remark 2.1. *It was proved by Odell and Rosenthal in [9] (see also [10]) that the separable Banach space B contains an isomorphic copy of ℓ_1 if and only if the double dual Banach space of B has cardinality strictly greater than $\mathfrak{c}$. Thus, we obtain the following theorem.*

Theorem 2.2. *Let B be any separable Banach space and B' its dual space. Then, the Bohr compactification bB' of B' is isomorphic as a topological group to $b\mathbb{R}$ if and only if B does not contain an isomorphic copy of ℓ_1. In particular, $b\ell_\infty$ is not isomorphic as a topological group to $b\mathbb{R}$.*

Proof. If B does not contain ℓ_1, then the dual of B' has cardinality $\mathfrak{c}$. Thus, by Lemma 2.1, bB' is isomorphic as a topological group to $b\mathbb{R}$.

If B does contain an isomorphic copy of ℓ_1, then the dual of B' has cardinality $> \mathfrak{c}$ by Remark 2.1. Thus, bB' is not isomorphic as a topological group to $b\mathbb{R}$. □

Remark 2.2. *We have seen that $b\mathbb{R}$ is isomorphic as a topological group to $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$. It is amusing to note, therefore, that $b\mathbb{R}$ is isomorphic as a topological group to $(b\mathbb{R})^{\aleph}$, for any cardinal $\aleph \leq \mathfrak{c}$.*

3. Bohr compactification of locally convex spaces

Theorem 3.1. *Let E be a separable real locally convex space. Then, the Bohr compactification bE of E is isomorphic as a topological group to $b\mathbb{R}$, which is isomorphic as a topological group to $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$.*

Proof. As E is a separable topological space, the number of distinct continuous maps of E into $\mathbb{R}$ is not greater than $\mathfrak{c}$. Thus, the dual topological vector space E' has cardinality $\mathfrak{c}$. Therefore, $\widehat{E}$ is a vector space of cardinality $\mathfrak{c}$ over $\mathbb{Q}$. Thus, $\widehat{E_d}$ with the discrete topology has dual group $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$. Hence, bE is isomorphic as a topological group to $b\mathbb{R}$. □

Remark 3.1. *Let E be a real locally convex vector space of dimension $m \geq 2^{\mathfrak{c}}$. Thus, the cardinality of $E = m$. As E is locally convex, there are enough continuous linear functionals on E to separate points. Therefore, the cardinality of $E' \geq m$. As the dual group of $\mathbb{Q}_d^{(m)} = \widehat{\mathbb{Q}}^m$, the cardinality of bE $\geq \mathfrak{c}^m \geq 2^m > 2^{\mathfrak{c}}$, which is the cardinality of $b\mathbb{R}$. Thus, bE is not isomorphic to $b\mathbb{R}$.*

4. Conclusions

In this short expository note, we have identified the Bohr compactification $b\mathbb{R}$ of $\mathbb{R}$ as $(\widehat{\mathbb{Q}})^{\mathfrak{c}}$ and the Bohr compactification of each separable real locally convex space as isomorphic as a topological group to $b\mathbb{R}$. We have characterized the duals of separable Banach spaces, which have Bohr compactifications isomorphic to $b\mathbb{R}$. We have shown that, as expected, real locally convex spaces of dimension $\geq 2^{\mathfrak{c}}$ do not have Bohr compactifications isomorphic as topological groups to $b\mathbb{R}$.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author thanks the reviewers for helpful criticisms.

Conflict of interest

The author declares there is no conflicts of interest.

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