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Research article

Liouville-type theorem for the stationary fractional compressible MHD system in anisotropic Lebesgue spaces

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Abstract: This paper is devoted to the study of the Liouville-type theorem of the stationary fractional compressible MHD systems in anisotropic Lebesgue spaces in $\mathbb{R}^3$. We showed that the solution is trivial when certain anisotropic integrability conditions are satisfied in terms of the velocity and the magnetic field components.

Keywords: fractional compressible MHD system; Liouville-type theorem; anisotropic Lebesgue spaces

1. Introduction and main results

In this paper, we are interested in the Liouville-type theorem in anisotropic Lebesgue spaces for the following stationary fractional compressible MHD system:

$$\begin{cases} \operatorname{div}(\rho u) = 0, & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\alpha} u + \operatorname{div}(\rho u \otimes u) - (b \cdot \nabla) b + \nabla P = 0, & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta} b + (u \cdot \nabla) b - (b \cdot \nabla) u = 0, & \text{in } \mathbb{R}^3, \\ \operatorname{div} b = 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

Here, $u = (u_1(x), u_2(x), u_3(x))$, $b = (b_1(x), b_2(x), b_3(x))$ and ρ represent the velocity field, the magnetic field, and the density, respectively. $P(\rho) = a\rho^{\gamma}$ is the pressure with constant $a > 0$ and the adiabatic exponent $\gamma \geq 1$. α and β are positive constants. The fractional Laplacian $(-\Delta)^{\alpha}$ is defined at the Fourier level by the symbol $|\ \xi\ |^{2\alpha}$.

When $b = 0$, $\alpha = 1$, and ρ =constant, the above system (1.1) reduces to the classical 3D stationary Navier-Stokes system

$$\begin{cases} -\Delta u + (u \cdot \nabla) u + \nabla P = 0, & \text{in } \mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

The Liouville problem for (1.2) still remains open: Is zero the only decay solution of (1.2) that verifies the finite Dirichlet integral condition?

$$D(u) = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx < \infty. \tag{1.3}$$

There are numerous results on the Liouville problem for (1.2). One of the first results is due to Galdi [1], who proved that $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$ is sufficient to imply that $u = 0$. In [2], Chae showed that $\Delta u \in L^{\frac{6}{5}}(\mathbb{R}^3)$, which with the same scaling as (1.3), implies that $u = 0$. In [3], Seregin proved that $u = 0$ if $u \in L^6(\mathbb{R}^3) \cap BMO^{-1}$. Sufficient conditions involving the head pressure for the triviality of the solution to the Navier–Stokes equations are studied by Chae in [4–6]. In [7], Chae and Wolf proved that the solution u to (1.2) is trivial if the L^s mean oscillation of the potential function V of u has a certain growth condition near infinity. In [8], Chae and Yoneda proved that if the solution $u \in \dot{H}^1(\mathbb{R}^3)$ to (1.2) satisfies additional conditions characterized by the decays near infinity and by the oscillation, then $u = 0$. In [9, 10], Jarrín and his collaborators studied the Liouville-type theorems in Lorentz and Morrey spaces. Kozono, Terasawa, and Wakasugi proved in [11] that $u = 0$ if the vorticity $\omega = o(|x|^{-\frac{5}{3}})$ as $|x| \to \infty$ or $\|u\|_{L^{\frac{9}{2},\infty}} \le \delta D(u)^{\frac{1}{3}}$ for a small constant δ. For more studies on the Liouville problem of the stationary Navier–Stokes equations, we refer to [12–14] and references therein.

For the compressible Navier–Stokes system

$$\begin{cases} -\Delta u + \operatorname{div}(\rho u \otimes u) + \nabla P = 0, \quad \operatorname{div}(\rho u) = 0 \quad \text{in } \mathbb{R}^d \\ P = a\rho^\gamma, \gamma > 1, \end{cases} \tag{1.4}$$

Chae [15] showed that the (1.4) has only a trivial solution $u = 0, \rho =$constant, provided that

$$\|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} < \infty, \qquad \text{when } 2 \le d \le 6,$$
$$\|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} + \|u\|_{L^{\frac{3d}{d-1}}(\mathbb{R}^d)} < \infty, \qquad \text{when } d \ge 7.$$

In [16], Li and Yu proved several improved Liouville-type theorems for the d-dimensional stationary compressible Navier–Stokes system. Particularly, they showed that $\rho \in L^\infty(\mathbb{R}^d)$ and $u \in \dot{H}^1(\mathbb{R}^d)$ are sufficient to guarantee $u = 0$ and $\rho =$constant when $d \ge 4$. See [17–19] and references therein for more studies on the Liouville problem of the stationary compressible Navier–Stokes system.

When $\alpha \in (0, 1)$, $b = 0$ and $\rho =$constant, system (1.1) reduces to the following stationary fractional Navier-Stokes system:

$$\begin{cases} (-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla P = 0, & \text{in } \mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

To our knowledge, there are few results on the Liouville problem of such a system. In [20], Wang and Xiao proved that the smooth solution $u \in \dot{H}^\alpha(\mathbb{R}^3) \cap L^{\frac{9}{2}}(\mathbb{R}^3)$ of (1.5) is trivial for $\alpha \in (0, 1)$. In [21], Yang proved the same result for $\frac{5}{6} \le \alpha < 1$. Recently, Chamorro and Poggi [22] proved an almost sharp Liouville's theorem for the stationary fractional Navier–Stokes system.

For the stationary fractional compressible Navier–Stokes system

$$\begin{cases} (-\Delta)^\alpha u + \operatorname{div}(\rho u \otimes u) + \nabla P = 0, & \text{in } \mathbb{R}^d \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^d \end{cases}$$

Wang and Xiao [20] proved that ρ =constant and $u = 0$ provided that

$$\|\rho\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{\dot{H}^\alpha(\mathbb{R}^d)} + \|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} < \infty, \qquad \text{when } \alpha \geq \frac{1}{2},$$
$$\|\rho\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{\dot{H}^\alpha(\mathbb{R}^d)} + \|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} + \|u\|_{L^{\frac{3d}{d-1}}(\mathbb{R}^d)} < \infty, \qquad \text{when } \alpha < \frac{1}{2}.$$

When $\alpha = \beta = 1$ and ρ =constant, system (1.1) reduces to the usual MHD system. There are also many results on the Liouville-type theorems for the stationary MHD system. In [23], Chae, Degond, and Liu proved that the solution to the stationary incompressible MHD and Hall-MHD system is trivial if $u, b \in L^{\frac{9}{2}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\nabla u, \nabla b \in L^2(\mathbb{R}^3)$. Later, Zeng [24] improved this result by removing the boundedness assumption of b and the finite Dirichlet integral assumption $\nabla u, \nabla b \in L^2(\mathbb{R}^3)$. Another interesting result of Chae and Weng [25] showed that $u = b = 0$ if $u \in L^3(\mathbb{R}^3)$ and $\nabla u, \nabla b \in L^2(\mathbb{R}^3)$. In [26], Chae and Wolf proved Liouville-type theorems for the stationary MHD and the stationary Hall-MHD systems by assuming suitable growth conditions at infinity for the mean oscillations for the potential functions. This work has been generalized in [27] by Chae et al.. In [28, 29], Wang studied the Liouville-type theorems for the planar stationary MHD equations. For more related studies, we refer to [30–35] and references therein.

Recently, many authors have been interested in the Liouville-type theorems for the stationary Navier-Stokes equations and the stationary MHD system in anisotropic Lebesgue spaces. The anisotropic Lebesgue space is defined as follows:

Definition. *Let $u = u(x_1, x_2, x_3)$ be a measurable function on $\mathbb{R}^3$ and $1 \leq p, q, r \leq \infty$. We say that u belongs to the anisotropic Lebesgue space $L^p_{x_1} L^q_{x_2} L^r_{x_3}(\mathbb{R}^3)$, provided that*

$$\|u\|_{L^p_{x_1} L^q_{x_2} L^r_{x_3}(\mathbb{R}^3)} = \left\| \left\| \|u\|_{L^p_{x_1}(\mathbb{R})} \right\|_{L^q_{x_2}(\mathbb{R})} \right\|_{L^r_{x_3}(\mathbb{R})} < \infty.$$

Here $\|\cdot\|_{L^p_{x_i}(\mathbb{R})}$ denotes the L^p norm with respect to the variable x_i.

Clearly, $L^p_{x_1} L^p_{x_2} L^p_{x_3}(\mathbb{R}^3)$ coincides with the usual Lebesgue space $L^p(\mathbb{R}^3)$. Throughout the paper, for any vector $\vec{p} = (p_1, p_2, p_3)$, we use the notation $\|\cdot\|_{L^{\vec{p}}(\mathbb{R}^3)}$ to denote $\|\cdot\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_{x_3}(\mathbb{R}^3)}$.

In [36], Luo and Yin proved that the bounded smooth solution $u \in \dot{H}^1(\mathbb{R}^3)$ to (1.2) is trivial if

$$u_i \in L^{p_i}_{x_1} L^{q_i}_{x_2} L^{r_i}_{x_3}(\mathbb{R}^3) \quad \text{with } \frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{r_i} = \frac{2}{3}, \quad i = 1, 2, 3.$$

Note that when $p_i = q_i = r_i = \frac{9}{2}$, this result recovers the classical result of Galdi [1]. Moreover, each component u_j of the velocity u may belong to different anisotropic spaces. Phan [37] proved that the solution $u \in H^1_{\text{loc}}(\mathbb{R}^3)$ to (1.2) is trivial if

$$u \in L^q_{x_1} L^q_{x_2} L^r_{x_3}(\mathbb{R}^3) \quad \text{with } \frac{2}{q} + \frac{1}{r} \geq \frac{2}{3}.$$

This result requires all components u_1, u_2 and u_3 lie in the same anisotropic space. Chae [38] proved that the solution $u \in L^6(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ to (1.2) is trivial if

$$u_j \in L^s_{x_l} L^s_{x_k} L^{\frac{q}{q-2}}_{x_j}(\mathbb{R}^3) \quad \text{with } \frac{4}{q} + \frac{2}{s} \geq 1, \quad s \in [1, \infty], \quad q \in (2, \infty), \quad \forall j = 1, 2, 3.$$

Note that a different order of integration for different components is allowed. In [39], Chae generalized this result to MHD equations. Fan and Wang [40] also studied the Liouville problem for the stationary incompressible MHD system; they proved that $u, b \in L^q_{x_1} L^q_{x_2} L^r_{x_3}(\mathbb{R}^3)$ implies that $u = b = 0$, provided that $q, r \in [3, +\infty)$ and $\frac{2}{q} + \frac{1}{r} \geq \frac{2}{3}$. They also claimed that $u = b = 0$ if $u, b \in L^p_{x_1} L^q_{x_2} L^r_{x_3}(\mathbb{R}^3)$ with $p, q, r \in [3, \infty)$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{2}{3}$. For the studies on Liouville-type theorems for the stationary compressible MHD system, we refer to Wu [41] and references therein.

Recently, Zeng [42] studied the Liouville-type theorems for the stationary fractional incompressible MHD system and proved that the solution $(u, b) \in \dot{H}^\alpha(\mathbb{R}^3) \times \dot{H}^\beta(\mathbb{R}^3)$ is trivial provided that $u = (u_1, u_2, u_3), b = (b_1, b_2, b_3)$ such that $(u_j, b_j) \in L^{\vec{p}_j}(\mathbb{R}^3) \times L^{\vec{q}_j}(\mathbb{R}^3)$ with

$$\sum_{l=1}^{3} \frac{1}{p_{j,l}} \geq \frac{2}{3}, \quad \sum_{l=1}^{3} \frac{1}{q_{j,l}} \geq \frac{2}{3}, \quad p_{j,l}, q_{j,l} \in [3, +\infty), \quad \forall j, l = 1, 2, 3.$$

Different from the above-mentioned results on the MHD system, which require all components u_1, u_2, u_3 and b_1, b_2, b_3 to lie in the same space, the result of Zeng [42] allows each component u_i and b_i to belong to different anisotropic spaces.

Inspired by the aforementioned results, this paper aims to establish a Liouville-type theorem for the stationary fractional compressible magnetohydrodynamic equations in anisotropic Lebesgue spaces. Our main result is as follows.

Theorem 1. *Let* $0 < \alpha, \beta < 1$, $(\rho, u, B) \in L^\infty(\mathbb{R}^3) \times \dot{H}^\alpha(\mathbb{R}^3) \times \dot{H}^\beta(\mathbb{R}^3)$ *be a smooth solution to* (1.1)*; then* $u = b = 0$ *provided that*

$$\begin{cases} u_i \in L^{\vec{p}_i}(\mathbb{R}^3) \quad \text{with} \quad \sum_{j=1}^{3} \frac{1}{p_{i,j}} \geq 2 \quad \text{if} \quad \frac{1}{2} \leq \alpha < 1, \\ u_i \in L^{\vec{p}_i}(\mathbb{R}^3) \cap L^{\vec{q}_i}(\mathbb{R}^3) \quad \text{with} \quad \sum_{j=1}^{3} \frac{1}{p_{i,j}} \geq 2 \quad \text{and} \quad \sum_{j=1} \frac{1}{q_{i,j}} \geq \frac{2}{3} \quad \text{if} \quad 0 < \alpha < \frac{1}{2}, \end{cases} \tag{1.6}$$

and

$$\begin{cases} b_i \in L^{\vec{\xi}_i}(\mathbb{R}^3) \quad \text{with} \quad \sum_{j=1}^{3} \frac{1}{\xi_{i,j}} \geq 2 \quad \text{if} \quad \frac{1}{2} \leq \beta < 1, \\ b_i \in L^{\vec{\xi}_i}(\mathbb{R}^3) \cap L^{\vec{\eta}_i}(\mathbb{R}^3) \quad \text{with} \quad \sum_{j=1}^{3} \frac{1}{\xi_{i,j}} \geq 2 \quad \text{and} \quad \sum_{j=1} \frac{1}{\eta_{i,j}} \geq \frac{2}{3} \quad \text{if} \quad 0 < \beta < \frac{1}{2}, \end{cases} \tag{1.7}$$

where $p_{i,j}, \xi_{i,j} \in [1, \frac{3}{2}]$ *and* $q_{i,j}, \eta_{i,j} \in [3, +\infty)$ *for* $i, j = 1, 2, 3$.

Remark 2. *The assumption* (1.7) *can be replaced by the following assumption:*

$$b_i \in L^{\vec{\xi}_i}(\mathbb{R}^3) \quad \text{with} \quad \sum_{j=1}^{3} \frac{1}{\xi_{i,j}} \geq \frac{2}{3}, \quad \xi_{i,j} \in [3, +\infty) \quad \text{for } i, j = 1, 2, 3. \tag{1.8}$$

See (3.12) *for the estimates of* I_{12} *and* I_2 *in the proof of Theorem 1 for details. Moreover, by the embedding* $\dot{H}^\beta(\mathbb{R}^3) \hookrightarrow L^{\frac{6}{3-2\beta}}(\mathbb{R}^3)$ *(see [43, Theorem 1.38, p.29] for example) and the fact that* $\frac{3-2\beta}{6} \times 3 \geq$

$\frac{2}{3}$ *when* $0 < \beta \le \frac{5}{6}$*, the additional assumption* (1.8) (*and also* (1.7)) *on b can be omitted if* $\frac{1}{2} \le \beta \le \frac{5}{6}$*. Here* $\beta \ge \frac{1}{2}$ *is needed to ensure that* $\frac{6}{3-2\alpha} \ge 3$*. To emphasize this observation, we state the following Corollary:*

Corollary 3. *Let* $0 < \alpha, \beta < 1$, $(\rho, u, B) \in L^\infty(\mathbb{R}) \times \dot{H}^\alpha(\mathbb{R}) \times \dot{H}^\beta(\mathbb{R})$ *be a smooth solution to* (1.1)*; then* $u = b = 0$ *provided that one of the following conditions is fulfilled:*

(a) $\frac{1}{2} \le \alpha < 1$, $\beta > \frac{5}{6}$ *or* $0 < \beta < \frac{1}{2}$, $u_i \in L^{\vec{p}_i}(\mathbb{R}^3)$, $b_i \in L^{\vec{\xi}_i}(\mathbb{R}^3)$ *with*

$$\sum_{j=1}^{3} \frac{1}{p_{i,j}} \ge 2, \quad \sum_{j=1}^{3} \frac{1}{\xi_{i,j}} \ge \frac{2}{3}, \quad p_{i,j} \in [1, \frac{3}{2}], \quad \xi_{i,j} \in [3, +\infty)$$

for $i, j = 1, 2, 3$*; or*

(b) $\frac{1}{2} \le \alpha < 1$, $\frac{1}{2} \le \beta \le \frac{5}{6}$, $u_i \in L^{\vec{p}_i}(\mathbb{R}^3)$ *with*

$$\sum_{j=1}^{3} \frac{1}{p_{i,j}} \ge 2, \quad p_{i,j} \in [1, \frac{3}{2}]$$

for $i, j = 1, 2, 3$*; or*

(c) $0 < \alpha < \frac{1}{2}$, $\beta > \frac{5}{6}$ *or* $0 < \beta < \frac{1}{2}$, $u_i \in L^{\vec{p}_i}(\mathbb{R}^3) \cap L^{\vec{q}_i}(\mathbb{R}^3)$, $b_i \in L^{\vec{\xi}_i}(\mathbb{R}^3)$ *with*

$$\sum_{j=1}^{3} \frac{1}{p_{i,j}} \ge 2, \quad \sum_{j=1}^{3} \frac{1}{q_{i,j}} \ge \frac{2}{3}, \quad \sum_{j=1}^{3} \frac{1}{\xi_{i,j}} \ge \frac{2}{3}, \quad p_{i,j} \in [1, \frac{3}{2}], \quad q_{i,j}, \xi_{i,j} \in [3, +\infty)$$

for $i, j = 1, 2, 3$*; or*

(d) $0 < \alpha < \frac{1}{2}$, $\frac{1}{2} \le \beta \le \frac{5}{6}$, $u_i \in L^{\vec{p}_i}(\mathbb{R}^3) \cap L^{\vec{q}_i}(\mathbb{R}^3)$ *with*

$$\sum_{j=1}^{3} \frac{1}{p_{i,j}} \ge 2, \quad \sum_{j=1}^{3} \frac{1}{q_{i,j}} \ge \frac{2}{3}, \quad p_{i,j} \in [1, \frac{3}{2}], \quad q_{i,j}, \xi_{i,j} \in [3, +\infty)$$

for $i, j = 1, 2, 3$.

Remark 4. *When* $b = 0$*, Theorem 1 improves the result of Wang and Xiao [20] for* $d = 3$*, since* $u \in L^{\frac{3}{2}}(\mathbb{R}^3)$ *and* $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$ *satisfy* $\frac{2}{3} \times 3 = 2$ *and* $\frac{2}{9} \times 3 = \frac{2}{3}$*, respectively. Indeed, our result strictly covered the result of [20] for* $d = 3$, $\alpha < \frac{1}{2}$*, since their result requires* $u \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^{\frac{9}{2}}(\mathbb{R}^3)$*, but our result (case (d) with* $b = 0$ *in Corollary 3) shows that* $u \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ *is sufficient.*

2. Preliminaries

2.1. Caffarelli–Silvestre extension

We first recall the well-known Caffarelli–Silvestre extension for the fractional Laplacian operator $(-\Delta)^\alpha$ with $\alpha \in (0, 1)$ in [44]. Throughout this paper, we use $\bar{\nabla}$ and $\overline{\text{div}}$ to denote the gradient and divergence operators on $\mathbb{R}^4_+$, respectively. We say a distribution $u \in \dot{H}^\alpha(\mathbb{R}^3)$ if $|\xi|^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^3)$, where $\hat{u}(\xi)$ denotes the Fourier transform of u. Let $u \in \dot{H}^a\left(\mathbb{R}^3\right)$ and set $\lambda = 1 - 2\alpha$, according to [44], there is an extension in $\mathbb{R}^4_+$, denoted by u^* such that

$$\begin{cases}\overline{\operatorname{div}}\left(y^{\lambda}\bar{\nabla}u^{*}\right)=0, & (x,y)\in\mathbb{R}^{4}_{+},\\ u^{*}(x,0)=u(x), & x\in\mathbb{R}^{3}.\end{cases} \tag{2.1}$$

Furthermore, it holds that

$$-C_{\alpha}\lim_{y\to0^{+}} y^{\lambda}\partial_{y}u^{*}=(-\Delta)^{\alpha}u(x), \quad x\in\mathbb{R}^{3}, \tag{2.2}$$

and

$$\|u\|^{2}_{\dot{H}^{\alpha}(\mathbb{R}^{3})}=\iint_{\mathbb{R}^{4}_{+}} y^{\lambda}\left|\bar{\nabla}u^{*}\right|^{2}dxdy, \tag{2.3}$$

where C_{α} is a constant depending only on α. This u^{*} is called the α-extension of u. The following L^{p} integrability of such u^{*} plays a crucial role in our proof.

Lemma 5. *(Lemma 2.2 in [20]). Let $\alpha\in(0,1)$ and u^{*} be the α-extension of $u\in L^{p}\left(\mathbb{R}^{3}\right)$ given by (2.1); it holds that*

$$\left(\iint_{\mathbb{R}^{4}_{+}} y^{1-2\alpha}|u^{*}|^{\frac{(5-2\alpha)p}{3}}dxdy\right)^{\frac{(5-2\alpha)p}{3}}\le C\|u\|_{L^{p}(\mathbb{R}^{3})}. \tag{2.4}$$

By the embedding theorem $\dot{H}^{\alpha}\left(\mathbb{R}^{3}\right)\hookrightarrow L^{\frac{6}{3-2\alpha}}\left(\mathbb{R}^{3}\right)$, if we choose $p=\frac{6}{3-2\alpha}$ in Lemma 2.1, it holds that

$$\left(\iint_{\mathbb{R}^{4}_{+}} y^{1-2\alpha}|u^{*}|^{\frac{2(5-2\alpha)}{3-2\alpha}}dxdy\right)^{\frac{3-2\alpha}{2(5-2\alpha)}}\le C\|u\|_{\dot{H}^{\alpha}(\mathbb{R}^{3})}. \tag{2.5}$$

2.2. Hölder's inequality and interpolation inequality in anisotropic Lebesgue spaces.

The following Hölder's inequality in anisotropic Lebesgue space (see [45] for example) are frequently referred to in the sequel.

Lemma 6. *For $\vec{p}=(p_1,p_2,p_3)$, $\vec{q}=(q_1,q_2,q_3)$ and $\vec{r}=(r_1,r_2,r_3)$ with*

$$\frac{1}{p_i}+\frac{1}{q_i}=\frac{1}{r_i}, \quad 1\le p_i,q_i,r_i\le\infty, \quad i=1,2,3,$$

and $f\in L^{\vec{p}}\left(\mathbb{R}^{3}\right)$, $g\in L^{\vec{q}}\left(\mathbb{R}^{3}\right)$, it holds that

$$\|fg\|_{L^{\vec{r}}(\mathbb{R}^{3})}\le\|f\|_{L^{\vec{p}}(\mathbb{R}^{3})}\|g\|_{L^{\vec{q}}(\mathbb{R}^{3})}$$

We can also prove the following interpolation inequality in anisotropic Lebesgue space.

Lemma 7. *For $\vec{p}=(p_1,p_2,p_3)$, $\vec{q}=(q_1,q_2,q_3)$, $\vec{r}=(r_1,r_2,r_3)$ and $\theta\in[0,1]$ with*

$$\frac{\theta}{p_i}+\frac{1-\theta}{q_i}=\frac{1}{r_i}, \quad 1\le p_i,q_i,r_i\le\infty, \quad i=1,2,3,$$

and $f\in L^{\vec{p}}\left(\mathbb{R}^{3}\right)\cap L^{\vec{q}}\left(\mathbb{R}^{3}\right)$, it holds that

$$\|f\|_{L^{\vec{r}}(\mathbb{R}^{3})}\le\|f\|^{\theta}_{L^{\vec{p}}(\mathbb{R}^{3})}\|f\|^{1-\theta}_{L^{\vec{q}}(\mathbb{R}^{3})}.$$

Proof. By successively using the classical interpolation inequality and Hölder's inequality, we have

$$\begin{aligned}
\|f\|_{L^{\vec{r}}(\mathbb{R}^3)} &= \Big\|\Big\|\|f\|_{L^{r_1}(\mathbb{R})}\Big\|_{L^{r_2}(\mathbb{R})}\Big\|_{L^{r_3}(\mathbb{R})} \\
&\leq \Big\|\Big\|\|f\|^{\theta}_{L^{p_1}(\mathbb{R})}\|f\|^{1-\theta}_{L^{q_1}(\mathbb{R})}\Big\|_{L^{r_2}(\mathbb{R})}\Big\|_{L^{r_3}(\mathbb{R})} \\
&\leq \Big\|\Big\|\|f\|^{\theta}_{L^{p_1}(\mathbb{R})}\Big\|_{L^{\frac{p_2}{\theta}}(\mathbb{R})}\Big\|\|f\|^{1-\theta}_{L^{q_1}(\mathbb{R})}\Big\|_{L^{\frac{q_2}{1-\theta}}(\mathbb{R})}\Big\|_{L^{r_3}(\mathbb{R})} \\
&= \Big\|\Big\|\|f\|_{L^{p_1}(\mathbb{R})}\Big\|^{\theta}_{L^{p_2}(\mathbb{R})}\|\|f\|_{L^{q_1}(\mathbb{R})}\|^{1-\theta}_{L^{q_2}(\mathbb{R})}\Big\|_{L^{r_3}(\mathbb{R})} \\
&\leq \Big\|\Big\|\|f\|_{L^{p_1}(\mathbb{R})}\Big\|^{\theta}_{L^{p_2}(\mathbb{R})}\Big\|_{L^{\frac{p_3}{\theta}}}\Big\|\|f\|_{L^{q_1}(\mathbb{R})}\|^{1-\theta}_{L^{q_2}(\mathbb{R})}\Big\|_{L^{\frac{q_3}{1-\theta}}(\mathbb{R})} \\
&= \|f\|^{\theta}_{L^{\vec{p}}(\mathbb{R}^3)}\|f\|^{1-\theta}_{L^{\vec{q}}(\mathbb{R}^3)}.
\end{aligned}$$

□

Though the above inequalities are stated for $\mathbb{R}^3$, they hold for any domain $\Omega \subset \mathbb{R}^3$ by a simple zero extension argument.

3. Proof of Theorem 1

This section is devoted to proving Theorem 1.

For each $R > 0$, we denote the cube in $\mathbb{R}^3$ centered at the origin with radius R by $Q_R = [-R, R]^3$. Let $\psi \in C_0^\infty(\mathbb{R})$ be a standard one-dimensional cut-off function such that

$$\psi(x) = \begin{cases} 1, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| \geq 2 \end{cases}.$$

For any $R > 0$, we define

$$\psi_R(x) = \psi(\frac{x_1}{R})\psi(\frac{x_2}{R})\psi(\frac{x_3}{R}), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Then we have

$$\psi_R(x) = \begin{cases} 1, & \text{if } x \in Q_R \\ 0, & \text{if } x \in \mathbb{R}^3 \backslash Q_{2R}. \end{cases}$$

We also denote $\chi_R(y)$ by a real nonincreasing smooth function in $\mathbb{R}$ such that

$$\chi_R(y) = \begin{cases} 0, & \text{if } y \geq 2R \\ 1, & \text{if } y \leq R \end{cases},$$

and $\left|\chi_R'(y)\right| \leq \frac{C}{R}$ for some constant C independent of $y \in \mathbb{R}$ and R.

Multiplying $(1.1)_2$ by $\phi_R u$, integrating by parts, and using the divergence-free property of u, we have

$$\int_{\mathbb{R}^3} (-\Delta)^\alpha u \cdot \psi_R u dx = \frac{1}{2}\int_{\mathbb{R}^3} (u \cdot \nabla\psi_R)\,\rho|u|^2 dx + \int_{\mathbb{R}^3} (b \cdot \nabla) b \cdot \psi_R u dx - \int_{\mathbb{R}^3} \psi_R u \cdot \nabla P\, dx. \tag{3.1}$$

Similarly, by testing $(1.1)_3$ with $\psi_R b$, we have

$$\int_{\mathbb{R}^3}(-\Delta)^\beta b\cdot\psi_R b dx=\frac{1}{2}\int_{\mathbb{R}^3}(u\cdot\nabla\psi_R)\,|b|^2dx-\int_{\mathbb{R}^3}(b\cdot\nabla\psi_R)\,(u\cdot b)dx-\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\psi_R u dx. \tag{3.2}$$

On the other hand, by (2.1), we have

$$\begin{aligned}0&=C_\alpha\iint_{\mathbb{R}^4_+}\overline{\mathrm{div}}(y^\lambda\bar\nabla u^*)\cdot u^*(\psi_R(x)\chi_R(y))dxdy\\&=C_\alpha\iint_{\mathbb{R}^4_+}\overline{\mathrm{div}}(y^\lambda\bar\nabla u^*\cdot u^*(\psi_R(x)\chi_R(y)))dxdy-C_\alpha\iint_{\mathbb{R}^4_+}y^\lambda|\bar\nabla u^*|^2(\psi_R(x)\chi_R(y))dxdy\\&\quad-C_\alpha\iint_{\mathbb{R}^4_+}y^\lambda\bar\nabla u^*\cdot u^*\bar\nabla(\psi_R(x)\chi_R(y))dxdy.\end{aligned} \tag{3.3}$$

Since $\psi_R(x)$ is supported in Q_{2R} and $\chi_R(y)=1$ in $[0,R]$, the divergence theorem gives

$$\iint_{\mathbb{R}^4_+}\overline{\mathrm{div}}(y^\lambda\bar\nabla u^*\cdot u^*(\psi_R(x)\chi_R(y)))\,dxdy=-\int_{\mathbb{R}^3}\lim_{y\to0}(y^\lambda\bar\nabla u^*)\cdot u\psi_R(x)\,dx. \tag{3.4}$$

Combining (3.3), (3.4) and (2.2), we obtain

$$\begin{aligned}&C_\alpha\iint_{\mathbb{R}^4_+}y^\lambda\left|\bar\nabla u^*\right|^2\psi_R(x)\chi_R(y)dxdy\\&=\int_{\mathbb{R}^3}(-\Delta)^\alpha u\cdot\psi_R u dx-C_\alpha\iint_{\mathbb{R}^4_+}y^\lambda\bar\nabla u^*\cdot u^*\bar\nabla\left(\psi_R(x)\chi_R(y)\right)dxdy.\end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned}&C_\beta\iint_{\mathbb{R}^4_+}y^\mu\left|\bar\nabla b^*\right|^2\psi_R(x)\chi_R(y)dxdy\\&=\int_{\mathbb{R}^3}(-\Delta)^\beta b\cdot\psi_R b dx-C_\beta\iint_{\mathbb{R}^4_+}y^\mu\bar\nabla b^*\cdot b^*\bar\nabla\left(\psi_R(x)\chi_R(y)\right)dxdy,\end{aligned} \tag{3.6}$$

where $\mu=1-2\beta$. Combining (3.1), (3.2), (3.5) and (3.6), we obtain that

$$\begin{aligned}&C_\alpha\iint_{\mathbb{R}^4_+}y^\lambda\left|\bar\nabla u^*\right|^2\psi_R(x)\chi_R(y)dxdy+C_\beta\iint_{\mathbb{R}^4_+}y^\mu\left|\vec\nabla b^*\right|^2\psi_R(x)\chi_R(y)dxdy\\&=\int_{\mathbb{R}^3}(u\cdot\nabla\psi_R)\left(\frac{1}{2}\rho|u|^2+\frac{1}{2}|b|^2\right)dx-\int_{\mathbb{R}^3}(b\cdot\nabla\psi_R)\,(u\cdot b)dx\\&\quad-C_\alpha\iint_{\mathbb{R}^4_+}y^\lambda\bar\nabla u^*\cdot u^*\bar\nabla\left(\psi_R(x)\chi_R(y)\right)dxdy-C_\beta\iint_{\mathbb{R}^4_+}y^\mu\bar\nabla b^*\cdot b^*\bar\nabla\left(\psi_R(x)\chi_R(y)\right)dxdy\\&\quad-\int_{\mathbb{R}^3}\psi_R u\cdot\nabla P\,dx\\&=I_1+I_2+I_3+I_4+I_5.\end{aligned} \tag{3.7}$$

Now we estimate I_1. Applying Young's inequality, we have

$$\begin{aligned}|I_1|&\le C\int_{\mathbb{R}^3}|\nabla\psi_R|\,|u|^3\,dx+C\int_{\mathbb{R}^3}|\nabla\psi_R|\,|b|^3dx\\&=I_{11}+I_{12}.\end{aligned} \tag{3.8}$$

The estimate of I_{11} is divided into the following three cases:

Case 1: $\frac{5}{6} \leq \alpha < 1$.

Since $\frac{5}{6} \leq \alpha < 1$, we have $\frac{3-2\alpha}{6} \leq \frac{2}{9}$. On the other hand, for $p_{i,j} \in [1, \frac{3}{2}]$, we have $\frac{1}{p_{i,j}} \geq \frac{2}{3}$ and $\frac{1}{3p_{i,j}} \geq \frac{2}{9}$. Hence,

$$0 \leq \frac{\frac{1}{3p_{i,j}} - \frac{3-2\alpha}{6}}{\frac{1}{p_{i,j}} - \frac{3-2\alpha}{6}} \leq \frac{\frac{1}{3} - \frac{3-2\alpha}{6}}{\frac{1}{p_{i,j}} - \frac{3-2\alpha}{6}} < 1.$$

It is easily checked that $f_1(x) = \frac{\frac{1}{3x} - \frac{3-2\alpha}{6}}{\frac{1}{x} - \frac{3-2\alpha}{6}}$ is decreasing in $[1, \frac{3}{2}]$ and $f_2(x) = \frac{\frac{1}{3} - \frac{3-2\alpha}{6}}{\frac{1}{x} - \frac{3-2\alpha}{6}}$ is increasing in $[1, \frac{3}{2}]$. Therefore, for $p_{i,j} \in [1, \frac{3}{2}]$, we have

$$f_1(p_{i,j}) \leq f_1(1) = \frac{2\alpha - 1}{3 + 2\alpha} = f_2(1) \leq f_2(p_{i,j}) \leq f_2\left(\frac{3}{2}\right),$$

which is exactly

$$0 \leq \frac{\frac{1}{3p_{i,j}} - \frac{3-2\alpha}{6}}{\frac{1}{p_{i,j}} - \frac{3-2\alpha}{6}} \leq \frac{2\alpha - 1}{3 + 2\alpha} \leq \frac{\frac{1}{3} - \frac{3-2\alpha}{6}}{\frac{1}{p_{i,j}} - \frac{3-2\alpha}{6}} < 1. \tag{3.9}$$

Therefore, by choosing $\theta = \frac{2\alpha-1}{3+2\alpha} \in (0, 1)$ and defining $r_{i,j}$ such that

$$\frac{1}{r_{i,j}} = \frac{\theta}{p_{i,j}} + \frac{3 - 2\alpha}{6}(1 - \theta) = \left(\frac{1}{p_{i,j}} - \frac{3 - 2\alpha}{6}\right)\theta + \frac{3 - 2\alpha}{6},$$

we have

$$\frac{1}{r_{i,j}} \in \left[\frac{1}{3p_{i,j}}, \frac{1}{3}\right]$$

by observing (3.9). Therefore,

$$3 \leq r_{i,j} \leq 3p_{i,j} \quad \text{and thus} \quad \sum_{j=1}^{3} \frac{1}{r_{i,j}} \geq \sum_{j=1}^{3} \frac{1}{3p_{i,j}} \geq \frac{2}{3}. \tag{3.10}$$

Moreover, by using Lemma 7, we have

$$\|u_i\|_{L^{\vec{r}_i}} \leq \|u_i\|_{L^{\vec{p}_i}}^{\theta} \|u_i\|_{L^{\frac{6}{3-2\alpha}}}^{1-\theta}.$$

Thus, by letting $s_{i,j}$ be such chat

$$\frac{1}{r_{i,j}} + \frac{1}{s_{i,j}} = \frac{1}{3},$$

and

$$C_l(R) = \{R \leq |x_l| \leq 2R, |x_m| \leq 2R, |x_n| \leq 2R\}, \quad \{l, m, n\} = \{1, 2, 3\},$$

and using Lemma 6, we have

$$
\begin{aligned}
I_{11} \leq & \frac{C}{R}\sum_{l=1}^{3}\int_{C_l(R)}|u|^3\phi'\left(\frac{x_l}{R}\right)\phi\left(\frac{x_m}{R}\right)\phi\left(\frac{x_n}{R}\right)dx \leq \frac{C}{R}\sum_{l,i=1}^{3}\int_{C_l(R)}|u_i|^3\,dx \\
\leq & \frac{C}{R}\sum_{l,i=1}^{3}\|u_i\|^3_{L^{\vec{r}_i}(C_l(R))}\|1\|^3_{L^{\vec{s}_i}(C_l(R))} \leq \frac{C}{R}\sum_{l,i=1}^{3}\|u_i\|^{3\theta}_{L^{\vec{p}_i}(C_l(R))}\|u_i\|^{3-3\theta}_{L^{\frac{6}{3-2\alpha}}(C_l(R))}\|1\|^3_{L^{\vec{s}_i}(C_l(R))} \qquad (3.11)\\
\leq & C\sum_{l,i=1}^{3}R^{2-3\sum_{j=1}^{3}\frac{1}{r_{i,j}}}\|u_i\|^{3\theta}_{L^{\vec{p}_i}(C_l(R))}\|u_i\|^{3-3\theta}_{L^{\frac{6}{3-2\alpha}}(C_l(R))}.
\end{aligned}
$$

Here we used the fact that

$$
\begin{aligned}
\|1\|_{L^{\vec{s}_i}(C_l(R))} \leq & \left(\int_{-2R}^{2R}\left(\int_{-2R}^{2R}\left(\int_{-2R}^{2R}1^{s_1}\,dx_1\right)^{\frac{s_2}{s_1}}dx_2\right)^{\frac{s_3}{s_2}}dx_3\right)^{\frac{1}{s_3}} \\
= & \left(\left((4R)^{\frac{s_2}{s_1}}\cdot 4R\right)^{\frac{s_3}{s_2}}\cdot 4R\right)^{\frac{1}{s_3}} = (4R)^{\sum_{j=1}^{3}\frac{1}{s_{i,j}}} = (4R)^{1-\sum_{j=1}^{3}\frac{1}{r_{i,j}}}.
\end{aligned}
$$

Hence, by (1.6) and (3.10), we have

$$
|I_{11}| \to 0 \quad \text{as} \quad R\to\infty.
$$

Case 2: $\frac{1}{2} \leq \alpha < \frac{5}{6}$. By using Lemma 6 and the fractional Sobolev inequality, we have

$$
\begin{aligned}
I_{11} \leq & \frac{C}{R}\sum_{l,i=1}^{3}\int_{C_l(R)}|u_i|^3\,dx \\
\leq & \frac{C}{R}\sum_{l,i=1}^{3}\left(\int_{C_l(R)}|u_i|^{\frac{6}{3-2\alpha}}\,dx\right)^{\frac{3(3-2\alpha)}{6}}\left(\int_{C_l(R)}dx\right)^{\frac{6\alpha-3}{6}} \\
\leq & \sum_{l,i=1}^{3}R^{\frac{6\alpha-5}{2}}\|u_i\|^3_{L^{\frac{6}{3-2\alpha}}(C_l(R))} \\
\leq & \sum_{l,i=1}^{3}R^{\frac{6\alpha-5}{2}}\|u_i\|^3_{\dot{H}^{\alpha}(C_l(R))} \to 0 \quad \text{as} \quad R\to\infty.
\end{aligned}
$$

Case 3: $\alpha < \frac{1}{2}$. From Lemma 6, it follows that

$$
I_{11} \leq \frac{C}{R}\sum_{l,i=1}^{3}\int_{C_l(R)}|u_i|^3\,dx \leq \frac{C}{R}\sum_{l,i=1}^{3}\|u_i\|^3_{L^{\vec{q}_i}(C_l(R))}\|1\|^3_{L^{\vec{z}_i}(C_l(R))},
$$

where

$$
\frac{1}{q_{i,j}}+\frac{1}{z_{i,j}}=\frac{1}{3}, \quad \forall i,j=1,2,3.
$$

Thus, by (1.6) we have

$$I_{11} \leq C \sum_{l,i=1}^{3} R^{2-3\sum_{j=1}^{3} \frac{1}{q_{i,j}}} \|u_i\|^3_{L^{\vec{q}_i}(C_l(R))} \to 0 \quad \text{as} \quad R \to \infty.$$

This completes the estimate of I_{11}. Similarly, we have $I_{12} \to 0$ as $R \to \infty$. Hence, $I_1 \to 0$ as $R \to \infty$. The estimate of I_2 follows from the estimates of I_{11}, I_{12}, and the use of Young's inequality,

$$|I_2| \leq \int_{\mathbb{R}^3} |\nabla\psi_R||u||b|^2\, dx \leq \int_{\mathbb{R}^3} |\nabla\psi_R||u|^3\, dx + \int_{\mathbb{R}^3} |\nabla\psi_R||b|^3\, dx = I_{11} + I_{12} \to 0 \quad \text{as} \quad R \to \infty.$$

We remark here that we can also get the estimate of I_{12} and then I_2 under assumption (1.8) instead of (1.7). Indeed,

$$\begin{aligned} |I_{12}| \leq & \frac{C}{R} \sum_{l=1}^{3} \int_{C_l(R)} |b|^3\, dx \leq \frac{C}{R} \sum_{l,i=1}^{3} \int_{C_l(R)} |b_i|^3\, dx \leq \frac{C}{R} \sum_{l,i=1}^{3} \|b_i\|^3_{L^{\vec{\xi}_i}(C_l(R))} \|1\|^3_{L^{\vec{\tau}_i}(C_l(R))} \\ \leq & C \sum_{l,i=1}^{3} R^{2-3\sum_{j=1}^{3} \frac{1}{\xi_{i,j}}} \|b_i\|^3_{L^{\vec{\xi}_i}(C_l(R))} \to 0 \quad \text{as } R \to \infty, \end{aligned} \tag{3.12}$$

where

$$\frac{1}{\xi_{i,j}} + \frac{1}{\tau_{i,j}} = \frac{1}{3}, \quad \forall i, j = 1, 2, 3.$$

Now we estimate I_3. By the definition of ψ_R and χ_R, we have

$$\begin{aligned} I_3 = & - C_\alpha \sum_{l=1}^{3} \int_0^{2R} \int_{C_l(R)} y^\lambda u_i^* \partial_l u_i^* \cdot \frac{1}{R} \psi'\left(\frac{x_l}{R}\right) \psi\left(\frac{x_j}{R}\right) \psi\left(\frac{x_k}{R}\right) \chi_R(y) dx dy \\ & - C_\alpha \int_R^{2R} \int_{\mathbb{R}^3} y^\lambda \bar{\nabla} u^* \cdot u^* \psi_R \bar{\nabla} \chi_R(y) dx dy. \end{aligned}$$

It follows by using Hölder's inequality and (2.5) that

$$
\begin{aligned}
|I_3| \leq & \sum_{l=1}^{3} \frac{C}{R}\left(\int_0^{2R}\int_{C_l(R)} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy\right)^{\frac{1}{2}} \left(\int_0^{2R}\int_{C_l(R)} y^\lambda |u^*|^{\frac{2(5-2\alpha)}{3-2\alpha}} dxdy\right)^{\frac{3-2\alpha}{2(5-2\alpha)}} \\
& \times \left(\int_0^{2R}\int_{C_l(R)} y^\lambda dxdy\right)^{\frac{1}{5-2\alpha}} \\
& + \frac{C}{R}\left(\int_R^{2R}\int_{\mathbb{R}^3} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy\right)^{\frac{1}{2}} \left(\int_R^{2R}\int_{\mathbb{R}^3} y^\lambda |u^*|^{\frac{2(5-2\alpha)}{3-2\alpha}} dxdy\right)^{\frac{3-2\alpha}{2(5-2\alpha)}} \\
& \times \left(\int_R^{2R}\int_{\mathbb{R}^3} y^\lambda \psi_R^{5-2\alpha}(x) dxdy\right)^{\frac{1}{5-2\alpha}} \\
\leq & C\sum_{l=1}^{3} \left(\int_0^{2R}\int_{C_l(R)} y^\lambda |\bar{\nabla} u^*|^2 dxdy\right)^{\frac{1}{2}} \left(\int_0^{2R}\int_{C_l(R)} y^\lambda |u^*|^{\frac{2(5-2\alpha)}{3-2\alpha}} dxdy\right)^{\frac{3-2\alpha}{2(5-2\alpha)}} \\
& + C\left(\int_R^{2R}\int_{\mathbb{R}^3} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy\right)^{\frac{1}{2}} \left(\int_R^{2R}\int_{\mathbb{R}^3} y^\lambda |u^*|^{\frac{2(5-2\alpha)}{3-2\alpha}} dxdy\right)^{\frac{3-2\alpha}{2(5-2\alpha)}} \\
\leq & C\|u\|_{\dot{H}^\alpha(\mathbb{R}^3)} \sum_{l=1}^{3} \left(\int_0^{2R}\int_{C_l(R)} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy\right)^{\frac{1}{2}} \\
& + C\|u\|_{\dot{H}^\alpha(\mathbb{R}^3)} \left(\int_R^{2R}\int_{\mathbb{R}^3} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy\right)^{\frac{1}{2}}.
\end{aligned}
$$

Recall the fact that

$$
\int_0^{2R}\int_{C_l(R)} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy + \int_R^{2R}\int_{\mathbb{R}^3} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy \leq 2\|u\|^2_{\dot{H}^\alpha(\mathbb{R}^3)},
$$

we immediately get that $I_3 \to 0$ as $R \to \infty$. Similarly, $I_4 \to 0$ as $R \to \infty$.

It remains to estimate I_5. We need a separate treatment for $\gamma > 1$ and $\gamma = 1$.

Case a: $\gamma \in (1, \infty)$. Rewrite

$$
\nabla P = a\nabla\rho^\gamma = \left(\frac{a\gamma}{\gamma-1}\right)\rho\nabla\rho^{\gamma-1}.
$$

This, along with $\operatorname{div}(\rho u) = 0$, derives

$$
\begin{aligned}
I_5 &= \frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3} \psi_R \rho u \cdot \nabla\rho^{\gamma-1} dx \\
&= -\frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3} \psi_R \operatorname{div}(\rho u)\rho^{\gamma-1} dx + \frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3} \rho^\gamma u \cdot \nabla\psi_R dx \\
&= \frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3} \rho^\gamma u \cdot \nabla\psi_R dx.
\end{aligned}
$$

Then it follows from $u_i \in L^{\vec{p}_i}(\mathbb{R}^3)$ and $\|\rho\|_{L^\infty(\mathbb{R}^3)} < \infty$ that

$$|I_5| \leq \frac{C}{R}\sum_{l=1}^{3}\int_{C_l(R)}\rho^\gamma|u|dx \leq \frac{C}{R}\sum_{l,i=1}^{3}\|\rho\|^\gamma_{L^\infty(\mathbb{R}^3)}\|u_i\|_{L^{\vec{p}_i}(C_l(R))}\|1\|_{L^{\vec{t}_i}(C_l(R))}$$
$$\leq C\sum_{l,i=1}^{3}\|\rho\|^\gamma_{L^\infty(\mathbb{R}^3)}R^{2-\Sigma_{j=1}^{3}\frac{1}{p_{i,j}}}\|u_i\|_{L^{\vec{p}_i}(C_l(R))},$$

where

$$1 = \frac{1}{p_{i,j}} + \frac{1}{t_{i,j}}, \quad \forall i, j = 1, 2, 3. \tag{3.13}$$

Hence, by (1.6), we have $I_5 \to 0$ as $R \to \infty$.

Case b: $\gamma = 1$. Under this circumstance we have

$$\nabla P = a\nabla\rho = a\rho\nabla\ln\rho.$$

By using $\mathrm{div}(\rho u) = 0$ again, we obtain

$$I_{13} = a\int_{\mathbb{R}^3}\psi_R\rho u\cdot\nabla\ln\rho dx$$
$$= -a\int_{\mathbb{R}^3}\psi_R\,\mathrm{div}(\rho u)\ln\rho dx + a\int_{\mathbb{R}^3}(\rho\ln\rho)u\cdot\nabla\psi_R dx$$
$$= a\int_{\mathbb{R}^3}\rho\ln\rho u\cdot\nabla\psi_R dx.$$

Note that

$$|t\ln t| \leq \begin{cases} Ct^2 & \text{as} \quad t\in(1,\infty); \\ Ct^{\frac{1}{2}} & \text{as} \quad t\in(0,1]. \end{cases}$$

So

$$\|\rho\ln\rho\|_{L^\infty(\mathbb{R}^3)} \leq C\|\rho\|^2_{L^\infty(\mathbb{R}^3)} + C\|\rho\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^3)}.$$

Accordingly, $u_i \in L^{\vec{p}_i}(\mathbb{R}^3)$ is used to deduce that

$$|I_{13}| \leq C\sum_{l=1}^{3}\sum_{i=1}^{3}\int_{C_l(R)}|\rho\ln\rho||u_i||\nabla\psi_R|dx$$
$$\leq \frac{C}{R}\sum_{l,i=1}^{3}\|\rho\ln\rho\|_{L^\infty(\mathbb{R}^3)}\|u_i\|_{L^{\vec{p}_i}(C_l(R))}\|1\|_{L^{\vec{t}_i}(C_l(R))}$$
$$\leq C\sum_{l,i=1}^{3}R^{2-\Sigma_{j=1}^{3}\frac{1}{p_{i,j}}}\|\rho\ln\rho\|_{L^\infty(\mathbb{R}^3)}\|u_i\|_{L^{\vec{p}_i}(C_l(R))}$$
$$\leq C\sum_{l,i=1}^{3}R^{2-\Sigma_{j=1}^{3}\frac{1}{p_{i,j}}}\left(\|\rho\|^2_{L^\infty(\mathbb{R}^3)} + \|\rho\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^3)}\right)\|u_i\|_{L^{\vec{p}_i}(C_l(R))},$$

where $\vec{t}_i$ is determined by (3.13). Hence, by (1.6), we have $I_5 \to 0$ as $R \to \infty$. Concluding the above two cases, we obtain

$$I_5 \to 0 \text{ as } R \to \infty.$$

Concluding the above estimates for I_1, I_2, I_3, I_4, and I_5 and letting $R \to \infty$ in (3.7), we obtain

$$C_\alpha \iint_{\mathbb{R}^4_+} y^\lambda \left|\bar{\nabla} u^*\right|^2 dxdy + C_\beta \iint_{\mathbb{R}^4_+} y^\mu \left|\bar{\nabla} b^*\right|^2 dxdy = 0,$$

which implies that $u^* = b^* =$ constant. Hence, $u = u^*(x, 0)$ and $b = b^*(x, 0)$ are both constant vector fields. Since $\left(u_j, b_j\right) \in L^{\vec{p}_j}\left(\mathbb{R}^3\right) \times L^{\vec{q}_j}\left(\mathbb{R}^3\right)$, we conclude that $u = b = 0$. This completes the proof of Theorem 1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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