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Research article

Effect of time-varying delays' dynamic characteristics on the stability of Hopfield neural networks

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Abstract: This manuscript investigated the stability of Hopfield neural networks with time-varying transmission delays and leakage delays, and specially discussed the impact of the time-varying delays' dynamic characteristics on network stability. First, to characterize the dynamic features of time-varying delays transitioning continuously between short and large delays, two key parameters were introduced: a critical threshold for distinguishing whether the time-varying delays are short or large delays, and the ratio of the measure of the union of time periods, in which the time-varying delays appear as short delays, to the measure of the whole time interval. Then, by utilizing the Lyapunov stability method, some sufficient conditions for the global exponential stability of the neural networks were derived. It is pointed out that when the measure of time periods in which the time-varying delays appear as short delays is large enough, the above two parameters related to the time-varying delays' dynamic characteristics will have an important impact on the system stability, and the upper bound that the time-varying delays can achieve in the whole time interval will no longer be the dominant factor influencing stability. Additionally, the relationship between the leakage delays and the stabilization ability of the negative feedback terms was explored. Two admissible upper bounds were presented, below which the leakage delays do not completely undermine the capacity of negative feedback terms to stabilize the neural networks. Finally, some simulation experiments were conducted to validate our theoretical findings.

Keywords: neural networks; stability; transmission delays; leakage delays; dynamic characteristics of time-varying delays

1. Introduction

Hopfield neural networks (NNs), as a kind of classical artificial NN, have been continuously attracting great attention of scientists since they were proposed by Hopfield in 1982 [1]. So far, this kind of network model has been widely applied in various realms including secure communications, pattern recognition, medical imaging, and so on [2–4], showing its great potential in practical applications. Stability [5] is a significant dynamical property of NNs and is also a key indicator for evaluating the performance of neural network models. Hence, as the theoretical foundation for the actual applications of NNs, the exploration on the stability theory of Hopfield NNs has continuously garnered extensive attention from scientists and remains a research focus to this day.

Time delays unavoidably exist in NNs and can significantly affect the dynamic performance of systems, potentially leading to instability, bifurcation, or oscillations [6, 7]. Meanwhile, different transmission circumstances can give rise to the occurrence of delays with different attributes. NNs often encounter various forms of time delays during the progress of information transmission, like transmission delays (or called discrete delays), proportional delays, distributed delays, leakage delays, and so forth [8, 9]. Among these, transmission delays and leakage delays are the most common types of delays. Transmission delays can be utilized to describe the phenomenon that information cannot be transmitted instantaneously between neurons on account of the intricate distribution pathways of various axons within NNs. Leakage delays arise in the negative feedback components of NNs and have been shown to possess the potential to destabilize the system stability within realistic nervous system architectures [10]. It holds considerable practical significance to investigate the stability of NNs with these two kinds of delays.

From the perspective of correlation with the time variable, time delays can be sorted into two categories: time-varying delays [11] and constant delays [12]. In comparison to constant delays, time-varying delays are more prevalent in real-life circumstances. Moreover, the models that incorporate time-varying delays provide a more accurate representation of the characteristics inherent in actual systems. Consequently, studying the stability issues associated with time-varying delays is particularly significant. During the previous two decades, the research on the stability of NNs subject to time-varying delays has garnered considerable interest from numerous investigators, resulting in a wealth of outstanding findings. Wang et al. conducted an analysis on the stability of mixed recurrent NNs characterized by leakage delays and impulsive perturbations [13]. Chen et al. explored the passivity properties of NNs incorporating not only leakage delays but time-varying transmission delays [14]. Zhang et al. integrated the concept of switched delays with an admissible edge-dependent average dwell-time method to address synchronization challenges for NNs experiencing not only time-varying leakage delays but transmission delays [15]. Wang et al. introduced a degree-dependent polynomial-based reciprocally convex matrix inequality and employed this inequality to establish several stability criteria for a specific sort of NN with time-varying delayed dynamics [16].

The Lyapunov-Krasovskii functional method [17] serves as a primary theoretical tool for researching the stability issues of time-varying delayed NNs. Researchers have integrated this method with various algebraic approaches, including Jensen's inequality technique, the free-weighting matrix method [18], and the reciprocally convex inequality approach [19], thereby constructing diverse stability criteria for delayed NNs [20–25]. With the utilization of the Lyapunov-Krasovskii functional method, Zhu and Cao investigated the exponential stability within a sort of Markovian jump impulsive stochastic Cohen-Grossberg NN characterized by mixed time delays and unknown parameters [26]. Zhang et al. explored the H_∞ synchronization problem associated with switching inertial NNs that suffer from mixed time-varying delays, presenting several related synchronization conditions [27]. Chen et al. examined the scenarios where delays in NNs are periodically varying bounded functions with constrained derivatives, proposing stability criteria through the application of both the Lyapunov-Krasovskii functional method and certain integral inequalities [28]. Qin et al. utilized this same methodological framework alongside an adaptive controller to address finite-time projective synchronization challenges in memristor-based NNs featuring multi-links and leakage delays [29]. However, it is important to note that in these aforementioned studies, upper bounds on time-varying delays and the bounds of their derivatives are universally regarded as critical factors influencing system stability. Moreover, most stability conditions derived using the Lyapunov-Krasovskii functional method within these references closely relate to these bounds. Unfortunately, these investigations often overlook potential impact stemming from other characteristics of time-varying delays on the overall system stability. The function value of a time-varying delay fluctuates continuously with the progression of the time variable. In certain intervals, it may present as a short delay characterized by a small value, while in other intervals, it transitions into a large delay with a larger value. Intuitively, the dynamic characteristics of time-varying delays that transition continuously between short and large delays may influence the stability of NNs. However, there exists a deficiency in comprehensive and in-depth research concerning the relationship between the time-varying delays' dynamic characteristics and neural network stability.

Leakage delays, as a common category of time delays, have been considered in numerous studies related to NNs [13–15, 23, 29]. Unlike other types of delays, leakage delays occur in the negative feedback terms and can influence the ability of negative feedback terms to stabilize NNs. Intuitively, smaller leakage delays exert less adverse effects on the negative feedback terms, whereas larger leakage delays tend to have more significant negative impacts on them. However, in many existing research articles [13–15,23,29], the existence of leakage delays is always regarded as a disruptive factor that can definitely render the negative feedback terms ineffective. Such an assumption is undoubtedly simplistic and conservative. Yet, to date, there remains an absence of thorough investigation into the relationship between leakage delays and the stabilization capability of negative feedback terms.

Motivated by the previous discussion, this manuscript explores the stability issues associated with a specific sort of Hopfield NN that incorporates time-varying transmission delays and leakage delays. Particular emphasis is placed on examining how the dynamic characteristics of these time-varying delays influence network stability. First, two parameters are introduced to characterize the dynamic features of time-varying delays transitioning continuously between short and large delays. These two parameters are, respectively, a critical threshold that can be used to distinguish whether the time-varying delays are short delays or large delays, and the ratio of the measure of the union of time periods, in which the time-varying delays appear as short delays, to the measure of the whole time interval. Then, by employing the Lyapunov stability method alongside algebraic inequality techniques, we establish several criteria for the stability of NNs. It is emphasized that when the duration of time intervals in which time-varying delays manifest as short delays is sufficiently large, the above two parameters significantly influence system stability, and the upper bound that time-varying delays can attain over the entire time interval will no longer be a predominant factor affecting stability. Furthermore, we investigate the relationship between leakage delays and the stabilization capacity of negative feedback terms. Two admissible upper bounds are presented, less than which the leakage delays can not completely undermine the capacity of negative feedback terms to stabilize the NNs.

The remaining portions of this manuscript are structured as follows. In Section 2, we provide a model description for a type of delayed Hopfield NN. Section 3 presents research findings related to neural network stability. Numerical simulations illustrated in Section 4 validate our primary theoretical results. In the end, Section 5 concludes this article.

Notations: Let $\mathbb{R} = (-\infty, +\infty)$. $\mathbb{R}^i$ is the Euclidean space with i dimensions. $\| \xi \|$ is the Euclidean norm of vector $\xi = (\xi_1, \cdots, \xi_n)^T$ and $\| \xi \| = \sqrt{\sum_{i=1}^n \xi_i^2}$. $| v |$ is the absolute value of scalar function v. $\Psi_1 \bigcup \Psi_2$ denotes the union of set Ψ_1 and set Ψ_2. $\Psi_1 \bigcap \Psi_2$ denotes the intersection of set Ψ_1 and set Ψ_2.

2. Problem formulation and preliminaries

Consider a Hopfield NN characterized by time-varying delays, which is constituted by n neurons. Its dynamical model is expressed by

$$\begin{cases} \dot{x}_i(t) = -b_i x_i(t - \delta_i(t)) + \sum_{j=1}^n \omega_{ij}\varphi_j(x_j(t - \tau_{ij}(t))) + I_i, \quad t \geq t_0, \\ x_i(s) = x_{i0}(s), \quad s \in [t_0 - h, t_0], \end{cases} \tag{2.1}$$

where $i = 1, 2, \ldots, n$, t stands for the time variable, $x_i(t) \in \mathbb{R}$ denotes the state of the ith neuron, $b_i > 0$ is the self-inhibition coefficient, and I_i is a constant input. ω_{ij} represents the weight of the connection from the jth neuron to the ith neuron. $\varphi_j : \mathbb{R} \to \mathbb{R}$ represents the nonlinear activation function associated with the jth neuron. The continuous function $\delta_i(t)$ is the leakage delay satisfying $0 \leq \delta_i(t) \leq \bar{\delta}_i$, where $\bar{\delta}_i$ is a positive constant. The continuous function $\tau_{ij}(t)$ stands for the transmission delay between the ith neuron and the jth neuron with $0 \leq \tau_{ij}(t) \leq \bar{\tau}_{ij}$, where $\bar{\tau}_{ij}$ is a positive constant. In addition, suppose that $\dot{\delta}_i(t) \leq 1$ and $\dot{\tau}_{ij}(t) \leq 1$ for $i, j \in \{1, 2, \ldots, n\}$ and $t \geq t_0$. The continuous function $x_{i0}(s) : [t_0 - h, t_0] \to \mathbb{R}$ is the initial value of system (2.1) where $h = \max_{i,j=1,2,\ldots,n} \{\bar{\delta}_i, \bar{\tau}_{ij}\}$.

Remark 1: Two typical types of time-varying delays, namely transmission delays and leakage delays, are considered in the neural network model (2.1). The transmission delays are caused by the fact that the signal transmission between neurons is restricted by the limited channel transmission speed or transmission capacity. This type of time delay is ubiquitous in NNs and can result in processing delays for information within the networks. The leakage delays primarily arise from measurement errors. This type of delay occurs within the negative feedback terms of NNs and can adversely affect the capacity of negative feedback terms to stabilize NNs. The presence of these two forms of time delays significantly impacts the stability of NNs and may even result in chaos, oscillations, or other complex dynamics. Hence, the research concerning these time delays holds substantial importance.

Assume that there exists a constant vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ that satisfies

$$-b_i x_i^* + \sum_{j=1}^n \omega_{ij}\varphi_j(x_j^*) + I_i = 0, \quad i = 1, 2, \ldots, n.$$

That is, the point x^* serves as an equilibrium point for the system described in (2.1). The primary objective of this article is to investigate the global exponential stability of the equilibrium point x^* and to establish corresponding criteria for its stability.

Definition 1 [30]: The equilibrium point x^* of system (2.1) is globally exponentially stable, if there exist two positive constants μ and ϑ such that, for every solution $x(t)$ of (2.1), the subsequent inequality is satisfied,

$$\| x(t) - x^* \| \leq \mu e^{-\vartheta(t-t_0)} \sup_{s\in[t_0-h,t_0]} \| x_0(s) - x^* \|,$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and $x_0(s) = (x_{10}(s), x_{20}(s), \ldots, x_{n0}(s))^T$. The parameter ϑ is the exponential convergence rate which describes how fast $x(t)$ converges to the equilibrium point x^*.

Remark 2: What is noteworthy is that our primary concern in this article is not to establish the conditions under which equilibrium points of system (2.1) exist. Consequently, rather than discussing the existence of equilibrium points, we will assume the existence of an equilibrium point x^* that is associated with the system (2.1). For research findings related to the presence of equilibrium points in delayed NNs, please refer to article [31].

Unlike constant delays, time-varying delays often exhibit some distinct and unique dynamic characteristics. The function values of time-varying delays exhibit continuous fluctuations as the time variable progresses. In some time intervals, they may manifest as short delays with small values, while at other intervals, they transition into large delays characterized by large values. A primary focus of this article is to examine the effect of the dynamic characteristics associated with time-varying delays that transition continuously between short and long delays on network stability. Furthermore, the impact of the leakage delays on the capacity of the negative feedback terms to stabilize neural networks will also be examined.

Let τ be defined as the critical threshold to distinguish whether the time-varying delays are short or large delays. It is defined that when the time-varying delays are less than or equal to τ at a given time t, they are called short delays at that moment. When they exceed τ at time t, they are called large delays at time t.

Next, two assumptions on time-varying delays in the system (2.1) are given.

Assumption 1: There is a time instant sequence $\{t_m\}_{m=1}^{\infty}$ satisfying $t_0 \leq t_1 < t_2 < \ldots$. Meanwhile, assume that a positive constant $\alpha > 1$ exists ,such that $t_{2k} - t_{2k-1} > \alpha\tau$, and the following inequalities hold for $t \in [t_{2k-1}, t_{2k}]$:

$$\delta_i(t) \leq \tau, \quad \tau_{ij}(t) \leq \tau, \quad \delta_i(t_{2k-1}) = 0, \quad \tau_{ij}(t_{2k-1}) = 0,$$

where $k = 1, 2, \ldots$ and $i, j = 1, 2, \ldots, n$.

Remark 3: The short delay and large delay are two opposing concepts. By introducing the critical threshold τ, the short delay and large delay are clearly divided. It should be pointed out that there are no specific requirements for the setting of this critical threshold, and usually this threshold can be set to different values depending on the system and the problem under study. In this paper, the setting of this critical threshold will be related to the exponential stability of NN (2.1).

Remark 4: When Assumption 1 holds, the interval $[t_0, \infty)$ can be split into an infinite number of small intervals $[t_{m-1}, t_m]$ $(m = 1, 2, \ldots)$ through the time instant sequence $\{t_m\}_{m=1}^{\infty}$. When $t \in [t_{2k-1}, t_{2k}]$ $(k = 1, 2, \ldots)$, the leakage delays $\delta_i(t)$ and transmission delays $\tau_{ij}(t)$ all appear as short delays. Furthermore, it should be noted that the small intervals $[t_{m-1}, t_m]$ $(m = 1, 2, \ldots)$ do not have to be equidistant. For instance, consider the delay function $\delta_i(t)$ satisfying the following form:

$$\delta_i(t) = 0.3(\cos(\sqrt{t}) + 1),$$

where $t \geq 1$. Clearly, the function takes the value of 0 at countless discrete moments, but it is not a periodic function.

Assumption 2: There are two positive constants N^* and γ, such that, for any given time instants t' and t'' $(t'' > t')$, the subsequent inequality is satisfied,

$$\mathcal{N}(t', t'') \geq \gamma(t'' - t') - N^*,$$

where $\mathcal{N}(t', t'')$ is the measure of the set $\{s \mid\ s \in [t', t''] \bigcap (\bigcup_{k=1}^{\infty} [t_{2k-1}, t_{2k}])\}$.

Remark 5: The set $\{s \mid\ s \in [t', t''] \bigcap (\bigcup_{k=1}^{\infty} [t_{2k-1}, t_{2k}])\}$ is the union of the time periods, in which the time-varying delays appear as short delays, within the interval $[t', t'']$. Due to the arbitrariness of t' and t'', the following relation can be derived from Assumption 2,

$$\lim_{t \to \infty} \frac{\mathcal{N}(t_0, t)}{t - t_0} \geq \gamma,$$

where $\mathcal{N}(t_0, t)$ is the measure of the set $\{s \mid\ s \in [t_0, t] \bigcap (\bigcup_{k=1}^{\infty} [t_{2k-1}, t_{2k}])\}$. This means that the parameter γ can be used to represent the ratio of the measure of the union of time periods, in which the time-varying delays appear as short delays, to the measure of the overall time interval $[t_0, \infty)$. The parameters τ and γ can be used, to a certain extent, to characterize the dynamic features of time-varying delays transitioning continuously between short and large delays. Therefore, the implication of the time-varying delays' dynamic characteristics on the neural network stability will be investigated via analyzing the bond between the stability of system (2.1) and these two parameters.

Then an assumption on activation functions in the system (2.1) is given as follows.

Assumption 3: There exists a positive constant ξ_j such that

$$\| \varphi_j(u) - \varphi_j(v) \| \leq \xi_j \| u - v \|, \quad for\ all\ u, v \in \mathbb{R},$$

where $j = 1, 2, \ldots, n$.

3. Main results

The global exponential stability of NN (2.1) is researched in this section, and several corresponding stability theorems are established. Particularly, the impact of time-varying delays' dynamic characteristics on neural network stability, as well as the influence of leakage delays on the stabilization ability of negative feedback terms, are discussed separately.

Let $e(t) = x(t) - x^*$ where $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T$, and then the system (2.1) can be converted into the following system,

$$\begin{cases} \dot{e}_i(t) = -b_i e_i(t - \delta_i(t)) + \sum_{j=1}^{n} \omega_{ij}[\varphi_j(e_j(t - \tau_{ij}(t)) + x_j^*) - \varphi_j(x_j^*)], \quad t \geq t_0, \\ e_i(s) = e_{i0}(s) = x_{i0}(s) - x_i^*, \quad s \in [t_0 - h, t_0], \end{cases} \tag{3.1}$$

where $i = 1, 2, \ldots, n$. Obviously, the origin is an equilibrium point of the system (3.1) and its stability is equivalent to the stability of equilibrium x^* for system (2.1). Therefore, the stability of NN (2.1) will be studied through an analysis of the stability of the zero equilibrium point in system (3.1).

First, a proposition is provided as follows, on the basis of which the main theorems of this paper will be established.

Proposition 1: Let $V(t) = \frac{1}{2}\sum_{i=1}^{n} e_i^2(t)$. Suppose that Assumption 3 holds, and there are two time instants t' and t'' such that $t'' - t' > \tau$, $\delta_i(t') = 0$, $\tau_{ij}(t') = 0$, $\delta_i(t) \le \tau$ and $\tau_{ij}(t) \le \tau$, where $t \in [t', t'']$ and $i, j = 1, 2, \ldots, n$. Then

$$\dot{V}(t) \le -2\lambda V(t) \tag{3.2}$$

for $t \in [t', t'']$, if there are three positive constants ε, σ and θ such that

$$b_i - \varepsilon b_i - \frac{\sigma}{2}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j - \frac{\bar{b}\theta}{2\varepsilon} - \frac{4\theta(1+\theta)}{\varepsilon}\sum_{i=1}^{n} b_i - \frac{2(1+\theta)^2}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j > 0 \tag{3.3}$$

and

$$\tau = \frac{1}{1+\sqrt{\theta}}\sqrt{\frac{\theta}{2(1+\theta)}}\frac{1}{\sum_{i=1}^{n} b_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j}, \tag{3.4}$$

where $i = 1, 2, \ldots, n$, $\bar{b} = \max_{i=1,\ldots,n}\{b_i\}$, $\lambda = \min_{i=1,\ldots,n}\{\lambda_i\}$ and $\lambda_i = b_i - \varepsilon b_i - \frac{\sigma}{2}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j - \frac{\bar{b}\theta}{2\varepsilon} - \frac{4\theta(1+\theta)}{\varepsilon}\sum_{i=1}^{n} b_i - \frac{2(1+\theta)^2}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j$.

Furthermore, if $\delta_i(t) \le \tau$ and $\tau_{ij}(t) \le \tau$ for $t \in [t', \infty)$ while the other conditions still hold, then the inequality (3.2) holds for $t \in [t', \infty)$.

Proof: The time derivative of $V(t)$ along the trajectories of the system (3.1) is given by

$$\dot{V}(t) = \sum_{i=1}^{n} e_i(t)\{-b_i e_i(t - \delta_i(t)) + \sum_{j=1}^{n} \omega_{ij}[\varphi_j(e_j(t - \tau_{ij}(t)) + x_j^*) - \varphi_j(x_j^*)]\}, \tag{3.5}$$

where $t \ge t_0$. Then based on Assumption 3,

$$\dot{V}(t) \le -\sum_{i=1}^{n} b_i e_i(t) e_i(t - \delta_i(t)) + \sum_{i=1}^{n} e_i(t)\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j \parallel e_j(t - \tau_{ij}(t)) \parallel, \tag{3.6}$$

for $t \ge t_0$.

Let $\hat{m} = \lceil \frac{t''-t'}{\tau} \rceil - 1$, where $\lceil \frac{t''-t'}{\tau} \rceil$ is the smallest integer greater than or equal to $\frac{t''-t'}{\tau}$. Then let $t'_1 = t'$, $t'_2 = t' + \tau$, $t'_3 = t' + 2\tau$, $\ldots$, $t'_{\hat{m}+1} = t' + \hat{m}\tau$, and $t'_{\hat{m}+2} = t''$. Evidently, $[t', t''] = \bigcup_{k=1}^{\hat{m}+1} [t'_k, t'_{k+1}]$.

From (3.6), the following inequality holds for $t \in [t'_k, t'_{k+1}]$, $k = 1, 2, \ldots, \hat{m} + 1$,

$$\begin{aligned}
\dot{V}(t) \leq & -\sum_{i=1}^{n} b_i e_i^2(t) + \sum_{i=1}^{n} b_i e_i(t)[e_i(t) - e_i(t'_k)] \\
& + \sum_{i=1}^{n} b_i e_i(t)[e_i(t'_k) - e_i(t - \delta_i(t))] + \sum_{i=1}^{n} e_i(t) \sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j \parallel e_j(t - \tau_{ij}(t)) \parallel \\
& \leq -\sum_{i=1}^{n} b_i e_i^2(t) + \frac{\varepsilon}{2} \sum_{i=1}^{n} b_i e_i^2(t) + \frac{1}{2\varepsilon} \sum_{i=1}^{n} b_i [e_i(t) - e_i(t'_k)]^2 \\
& + \frac{\varepsilon}{2} \sum_{i=1}^{n} b_i e_i^2(t) + \frac{1}{2\varepsilon} \sum_{i=1}^{n} b_i [e_i(t'_k) - e_i(t - \delta_i(t))]^2 \\
& + \sum_{i=1}^{n} \sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j [\frac{\sigma}{2} e_i^2(t) + \frac{1}{2\sigma} e_j^2(t - \tau_{ij}(t))] \\
& \leq -a \parallel e(t) \parallel^2 + \frac{\bar{b}}{2\varepsilon} \parallel e(t) - e(t'_k) \parallel^2 + \frac{1}{2\varepsilon} \sum_{i=1}^{n} b_i \parallel e(t'_k) - e(t - \delta_i(t)) \parallel^2 \\
& + \frac{1}{2\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j \parallel e(t - \tau_{ij}(t)) \parallel^2,
\end{aligned} \tag{3.7}$$

where $a = \min\limits_{i=1,\ldots,n} \{b_i - \varepsilon b_i - \frac{\sigma}{2} \sum\limits_{j=1}^{n} \mid \omega_{ij} \mid \xi_j\}$ and $\bar{b} = \max\limits_{i=1,\ldots,n} \{b_i\}$.

Next, we can get a conclusion that

$$\parallel e(t) - e(t'_1) \parallel \ \leq \sqrt{\eta} \parallel e(t'_1) \parallel \ \ for\ t \in [t'_1, t'_2], \tag{3.8}$$

where $\eta = \frac{\theta}{2(1+\theta)}$. This conclusion is demonstrated as follows by contradiction. Suppose that there exists a time instant $t^* \in [t'_1, t'_2]$ such that $\parallel e(t^*) - e(t'_1) \parallel \ > \sqrt{\eta} \parallel e(t'_1) \parallel$ and

$$\parallel e(t) - e(t'_1) \parallel \ \leq \sqrt{\eta} \parallel e(t'_1) \parallel \ \ for\ t \in [t'_1, t^*). \tag{3.9}$$

Obviously, $t^* > t'_1$. This fact can be shown by contradiction. Suppose that $t^* = t'_1$. Then $0 = \parallel e(t^*) - e(t'_1) \parallel > \sqrt{\eta} \parallel e(t'_1) \parallel \ \geq 0$, which is a contradiction.

From (3.9),

$$\parallel e(t) - e(t'_1) \parallel^2 \ \leq \frac{\theta}{2(1+\theta)} \parallel e(t'_1) \parallel^2 \ \ for\ t \in [t'_1, t^*). \tag{3.10}$$

Then

$$\begin{aligned}
\parallel e(t) - e(t'_1) \parallel^2 & \leq \frac{\theta}{2} \parallel e(t) - (e(t) - e(t'_1)) \parallel^2 - \theta \parallel e(t) - e(t'_1) \parallel^2 \\
& \leq \frac{\theta}{2} \{\parallel e(t) \parallel^2 + 2 \parallel e(t) \parallel \parallel e(t) - e(t'_1) \parallel + \parallel e(t) - e(t'_1) \parallel^2\} - \theta \parallel e(t) - e(t'_1) \parallel^2 \\
& \leq \theta \parallel e(t) \parallel^2
\end{aligned} \tag{3.11}$$

for $t \in [t'_1, t^*)$. Next, the following inequality is deduced,

$$\parallel e(t'_1) \parallel \ \leq (1 + \sqrt{\theta}) \parallel e(t) \parallel \ \ for\ t \in [t'_1, t^*). \tag{3.12}$$

Since $\delta_i(t_1') = 0$, $\tau_{ij}(t_1') = 0$, $\dot{\delta}_i(t) \le 1$ and $\dot{\tau}_{ij}(t) \le 1$ where $i, j = 1, 2, \ldots, n$ and $t \ge t_0$, then $\delta_i(t) \le t - t_1'$ and $\tau_{ij}(t) \le t - t_1'$ for $t \in [t_1', t_2']$. This means that $t - \delta_i(t) \ge t_1'$ and $t - \tau_{ij}(t) \ge t_1'$ for $t \in [t_1', t_2']$. Then, from (3.9) and (3.12), the following two inequalities are acquired,

$$\| e(t - \delta_i(t)) - e(t_1') \|^2 \le \eta \| e(t_1') \|^2 \le \frac{\theta}{2(1+\theta)} 2(1+\theta) \| e(t) \|^2 = \theta \| e(t) \|^2 \tag{3.13}$$

and

$$\begin{aligned} \| e(t - \tau_{ij}(t)) \|^2 &\le \{\| e(t_1') \| + \| e(t - \tau_{ij}(t)) - e(t_1') \|\}^2 \\ &\le (1 + \sqrt{\eta})^2 \| e(t_1') \|^2 \\ &\le 2(1+\eta)2(1+\theta) \| e(t) \|^2 \\ &= (4 + 6\theta) \| e(t) \|^2, \end{aligned} \tag{3.14}$$

where $t \in [t_1', t^*)$ and $i, j = 1, 2, \ldots, n$.

From (3.7), (3.11), (3.13) and (3.14), it is inferred that

$$\begin{aligned} \dot{V}(t) &\le -a \| e(t) \|^2 + \frac{\bar{b}\theta}{2\varepsilon} \| e(t) \|^2 + \frac{\theta}{2\varepsilon} \sum_{i=1}^{n} b_i \| e(t) \|^2 + \frac{2+3\theta}{\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} | \omega_{ij} | \xi_j \| e(t) \|^2 \\ &\le -2\lambda V(t) \end{aligned} \tag{3.15}$$

for $t \in [t_1', t^*)$. Thus,

$$\| e(t) \| \le \| e(t_1') \| \quad for\ t \in [t_1', t^*). \tag{3.16}$$

From (3.1), the below-mentioned inequality is applicable for $t \ge t_0$,

$$\begin{aligned} \| \dot{e}(t) \| &\le \sum_{i=1}^{n} \| \dot{e}_i(t) \| \\ &\le \sum_{i=1}^{n} \{b_i \| e_i(t - \delta_i(t)) \| + \sum_{j=1}^{n} | \omega_{ij} | \xi_j \| e_j(t - \tau_{ij}(t)) \|\}. \end{aligned} \tag{3.17}$$

Then it can be inferred from (3.16) and (3.17) that

$$\| \dot{e}(t) \| \le [\sum_{i=1}^{n} b_i + \sum_{i=1}^{n} \sum_{j=1}^{n} | \omega_{ij} | \xi_j] \| e(t_1') \| \tag{3.18}$$

for $t \in [t_1', t^*)$. Thus,

$$\begin{aligned} \| e(t^*) - e(t_1') \| &= \| \int_{t_1'}^{t^*} \dot{e}(s) ds \| \\ &\le \int_{t_1'}^{t^*} \| \dot{e}(s) \| ds \\ &\le \tau [\sum_{i=1}^{n} b_i + \sum_{i=1}^{n} \sum_{j=1}^{n} | \omega_{ij} | \xi_j] \| e(t_1') \| \\ &\le \sqrt{\eta} \| e(t_1') \|, \end{aligned} \tag{3.19}$$

which is in contradiction to the assumption that $\| e(t^*) - e(t_1') \| > \sqrt{\eta} \| e(t_1') \|$. Therefore, the inequality (3.8) holds for $t \in [t_1', t_2']$. Then, similar to the derivation of (3.11) and (3.15), the following two inequalities are concluded,

$$\dot{V}(t) \leq -2\lambda V(t) \tag{3.20}$$

and

$$\| e(t) - e(t_1') \|^2 \leq \theta \| e(t) \|^2, \tag{3.21}$$

where $t \in [t_1', t_2']$.

Suppose that the following inequalities hold for $t \in [t_k', t_{k+1}']$,

$$\dot{V}(t) \leq -2\lambda V(t), \tag{3.22}$$

$$\| e(t) - e(t_k') \|^2 \leq \eta \| e(t_k') \|^2 \tag{3.23}$$

and

$$\| e(t) - e(t_k') \|^2 \leq \theta \| e(t) \|^2, \tag{3.24}$$

where $k \in \{1, 2, \ldots, \hat{m}\}$. Then consider the situation that $t \in [t_{k+1}', t_{k+2}']$.

Next, we can get a conclusion that

$$\| e(t) - e(t_{k+1}') \| \leq \sqrt{\eta} \| e(t_{k+1}') \| \quad for\ t \in [t_{k+1}', t_{k+2}']. \tag{3.25}$$

This conclusion is demonstrated as follows by contradiction. Suppose that there occurs a time instant $\hat{t}^* \in [t_{k+1}', t_{k+2}']$ such that $\| e(\hat{t}^*) - e(t_{k+1}') \| > \sqrt{\eta} \| e(t_{k+1}') \|$ and

$$\| e(t) - e(t_{k+1}') \| \leq \sqrt{\eta} \| e(t_{k+1}') \| \quad for\ t \in [t_{k+1}', \hat{t}^*). \tag{3.26}$$

It is evident that $\hat{t}^* > t_{k+1}'$. This fact can be shown by contradiction. Suppose that $\hat{t}^* = t_{k+1}'$. Then $0 = \| e(\hat{t}^*) - e(t_{k+1}') \| > \sqrt{\eta} \| e(t_{k+1}') \| \geq 0$, which is a contradiction.

Similar to the derivation of (3.11) and (3.12), the following inequalities are derived from (3.26),

$$\| e(t) - e(t_{k+1}') \|^2 \leq \theta \| e(t) \|^2 \tag{3.27}$$

and

$$\| e(t_{k+1}') \| \leq (1 + \sqrt{\theta}) \| e(t) \|, \tag{3.28}$$

where $t \in [t_{k+1}', \hat{t}^*)$.

Since $\delta_i(t) \leq \tau$ and $\tau_{ij}(t) \leq \tau$ where $t \in [t', t'']$ and $i, j = 1, 2, \ldots, n$, then $t - \delta_i(t) \geq t_{k+1}' - \tau = t_k'$ and $t - \tau_{ij}(t) \geq t_{k+1}' - \tau = t_k'$ for $t \in [t_{k+1}', t_{k+2}']$.

If $t - \delta_i(t) \geq t_{k+1}'$, then it can be inferred from (3.26) and (3.28) that

$$\begin{aligned} \| e(t - \delta_i(t)) - e(t_{k+1}') \|^2 &\leq \eta \| e(t_{k+1}') \|^2 \\ &\leq \theta \| e(t) \|^2, \end{aligned}$$

where $t \in [t_{k+1}', \hat{t}^*)$. If $t_k' \leq t - \delta_i(t) \leq t_{k+1}'$, then it can be inferred from (3.23), (3.24) and (3.28) that

$$\begin{aligned} \| e(t - \delta_i(t)) - e(t_{k+1}') \|^2 &\leq \{\| e(t_{k+1}') - e(t_k') \| + \| e(t - \delta_i(t)) - e(t_k') \|\}^2 \\ &\leq \{[\sqrt{\theta} + \sqrt{\eta}(1 + \sqrt{\theta})] \| e(t_{k+1}') \|\}^2 \\ &\leq \{[\sqrt{\theta} + \sqrt{\eta}(1 + \sqrt{\theta})](1 + \sqrt{\theta}) \| e(t) \|\}^2 \\ &\leq 8\theta(1 + \theta) \| e(t) \|^2, \end{aligned}$$

where $t \in [t'_{k+1}, \hat{t}^*)$. Thus,

$$\| e(t-\delta_i(t)) - e(t'_{k+1}) \|^2 \le 8\theta(1+\theta) \| e(t) \|^2 \tag{3.29}$$

for $t \in [t'_{k+1}, \hat{t}^*)$ and $i = 1, 2, \dots, n$.

If $t - \tau_{ij}(t) \ge t'_{k+1}$, then it is inferred from (3.26) and (3.28) that

$$\begin{aligned} \| e(t-\tau_{ij}(t)) \|^2 &\le \{\| e(t-\tau_{ij}(t)) - e(t'_{k+1}) \| + \| e(t'_{k+1}) \|\}^2 \\ &\le \{(1+\sqrt{\eta}) \| e(t'_{k+1}) \|\}^2 \\ &\le (4+6\theta) \| e(t) \|^2, \end{aligned}$$

where $t \in [t'_{k+1}, \hat{t}^*)$. If $t'_k \le t - \tau_{ij}(t) \le t'_{k+1}$, thereafter it can be inferred from (3.22), (3.24) and (3.28) that

$$\begin{aligned} \| e(t-\tau_{ij}(t)) \|^2 &\le \| e(t'_k) \|^2 \\ &\le 2(1+\theta) \| e(t'_{k+1}) \|^2 \\ &\le 4(1+\theta)^2 \| e(t) \|^2, \end{aligned}$$

where $t \in [t'_{k+1}, \hat{t}^*)$. Thus,

$$\| e(t-\tau_{ij}(t)) \|^2 \le 4(1+\theta)^2 \| e(t) \|^2 \tag{3.30}$$

for $t \in [t'_{k+1}, \hat{t}^*)$ and $i, j = 1, 2, \dots, n$.

From (3.7), (3.27), (3.29) and (3.30),

$$\begin{aligned} \dot{V}(t) &\le -\{a - \frac{\bar{b}\theta}{2\varepsilon} - \frac{4\theta(1+\theta)}{\varepsilon} \sum_{i=1}^{n} b_i - \frac{2(1+\theta)^2}{\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} | \omega_{ij} | \xi_j\} \| e(t) \|^2 \\ &= -2\lambda V(t), \end{aligned} \tag{3.31}$$

where $t \in [t'_{k+1}, \hat{t}^*)$.

Similar to the derivation of (3.18) and (3.19), the following relation is derived from (3.17), (3.22), (3.24) and (3.31) that

$$\begin{aligned} \| e(\hat{t}^*) - e(t'_{k+1}) \| &\le \tau[\sum_{i=1}^{n} b_i + \sum_{i=1}^{n} \sum_{j=1}^{n} | \omega_{ij} | \xi_j](1+\sqrt{\theta}) \| e(t'_{k+1}) \| \\ &\le \sqrt{\eta} \| e(t'_{k+1}) \|, \end{aligned} \tag{3.32}$$

which contradicts the supposition that $\| e(\hat{t}^*) - e(t'_{k+1}) \| > \sqrt{\eta} \| e(t'_{k+1}) \|$. Therefore, the inequality (3.25) applies to $t \in [t'_{k+1}, t'_{k+2}]$. Then, similar to the derivation of (3.27) and (3.31), the following inequalities are concluded,

$$\dot{V}(t) \le -2\lambda V(t) \tag{3.33}$$

and

$$\| e(t) - e(t'_{k+1}) \|^2 \le \theta \| e(t) \|^2, \tag{3.34}$$

where $t \in [t'_{k+1}, t'_{k+2}]$.

Therefore, on the basis of the mathematical induction method, the following relation is valid for any given $k \in \{1, 2, \dots, \hat{m}+1\}$,

$$\dot{V}(t) \le -2\lambda V(t) \; where \; t \in [t'_k, t'_{k+1}].$$

That is,

$$\dot{V}(t) \leq -2\lambda V(t) \tag{3.35}$$

for $t \in [t', t'']$.

If $\delta_i(t) \leq \tau$ and $\tau_{ij}(t) \leq \tau$ for $t \in [t', \infty)$ while the other conditions in Proposition 1 still hold, then, consistent with the preceding proof, it can be demonstrated by the mathematical induction method that the inequality (3.35) holds when $t \in [t', \infty)$.

This concludes the proof.

Next, a theorem is presented in which some stability criteria for NN (2.1) are constructed.

Theorem 1: Suppose that Assumptions 1–3 hold. If there exist three positive constants ε, σ and θ such that the conditions (3.3), (3.4) and the following inequality hold,

$$\gamma > \frac{\zeta}{2\lambda(1-\frac{1}{\alpha}) + \zeta}, \tag{3.36}$$

where $\zeta = \bar{a} + \frac{1}{\varepsilon}\sum_{i=1}^{n} b_i + \frac{1}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j$ and $\bar{a} = \max_{i=1,\ldots,n}\{\varepsilon b_i + \sigma \sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j\}$, then the equilibrium point x^* of NN (2.1) is globally exponentially stable and the state vector $x(t)$ satisfies the following relation for $t \geq t_0$,

$$\parallel x(t) - x^* \parallel \leq \sup_{s\in[t_0-h,t_0]} \{\parallel x_0(s) - x^* \parallel\}\mu e^{-\upsilon(t-t_0)}, \tag{3.37}$$

where $\mu = e^{\frac{1}{2}[2\lambda(1-\frac{1}{\alpha})+\zeta]N^*}$, $\upsilon = \frac{1}{2}[2\lambda(1-\frac{1}{\alpha})\gamma - \zeta(1-\gamma)]$ and $x_0(s) = (x_{10}(s), x_{20}(s), \ldots, x_{n0}(s))^T$.

Proof: According to Assumption 1 and Proposition 1,

$$\dot{V}(t) \leq -2\lambda V(t), \tag{3.38}$$

where $t \in [t_{2k-1}, t_{2k}]$, $k = 1, 2, \ldots$. Thus,

$$V(t) \leq V(t_{2k-1})e^{-2\lambda(t-t_{2k-1})}, \tag{3.39}$$

where $t \in [t_{2k-1}, t_{2k}]$, $k = 1, 2, \ldots$.

Furthermore, according to Assumption 1, $t_{2k} - t_{2k-1} > \alpha\tau > \tau$. Hence, from (3.38) and (3.39),

$$V(t) \leq V(t_{2k} - \tau) \leq V(t_{2k-1})e^{-2\lambda(t_{2k}-t_{2k-1}-\tau)}, \tag{3.40}$$

where $t \in [t_{2k} - \tau, t_{2k}]$, $k = 1, 2, \ldots$.

It is inferred from (3.6) that

$$\begin{aligned}\dot{V}(t) &\leq \sum_{i=1}^{n}\{\varepsilon b_i + \sigma\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j\}\frac{1}{2}e_i^2(t) + \frac{1}{\varepsilon}\sum_{i=1}^{n} b_i\frac{1}{2}e_i^2(t-\delta_i(t)) + \frac{1}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j\frac{1}{2}e_j^2(t-\tau_{ij}(t)) \\ &\leq \bar{a}V(t) + \frac{1}{\varepsilon}\sum_{i=1}^{n} b_i V(t-\delta_i(t)) + \frac{1}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j V(t-\tau_{ij}(t)),\end{aligned} \tag{3.41}$$

where $\bar{a} = \max_{i=1,\ldots,n}\{\varepsilon b_i + \sigma\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j\}$ and $t \geq t_0$.

Next, consider the situation that $t \in [t_{2k}, t_{2k+1}]$, $k = 1, 2, \ldots$. According to Assumption 1, $\delta_i(t_{2k}) \leq \tau$ and $\tau_{ij}(t_{2k}) \leq \tau$, where $i, j = 1, 2, \ldots, n$. Since $\dot{\delta}_i(t) \leq 1$ and $\dot{\tau}_{ij}(t) \leq 1$ for $t \geq t_0$, then $t - \delta_i(t) \geq t_{2k} - \tau$ and $t - \tau_{ij}(t) \geq t_{2k} - \tau$ for $t \in [t_{2k}, t_{2k+1}]$.

Then an auxiliary system is constructed as below,

$$\begin{cases} \dot{\bar{V}}(t) = \bar{a}\bar{V}(t) + \dfrac{1}{\varepsilon}\sum_{i=1}^{n} b_i \bar{V}(t - \delta_i(t)) + \dfrac{1}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j \bar{V}(t - \tau_{ij}(t)), \\ \bar{V}(s) = V(t_{2k-1})e^{-2\lambda(t_{2k}-t_{2k-1}-\tau)} \ \ for \ s \in [t_{2k} - \tau, t_{2k}], \end{cases} \tag{3.42}$$

where $t \in [t_{2k}, t_{2k+1}]$. It is easy to find that $V(t) \leq \bar{V}(t)$ for $t \in [t_{2k}, t_{2k+1}]$.

From Eq (3.42), $\dot{\bar{V}}(t) \geq 0$ for $t \in [t_{2k}, t_{2k+1}]$. Then, $\bar{V}(t - \delta_i(t)) \leq \bar{V}(t)$ and $\bar{V}(t - \tau_{ij}(t)) \leq \bar{V}(t)$ for $t \in [t_{2k}, t_{2k+1}]$. Therefore,

$$\dot{\bar{V}}(t) \leq \zeta \bar{V}(t), \tag{3.43}$$

where $\zeta = \bar{a} + \frac{1}{\varepsilon}\sum_{i=1}^{n} b_i + \frac{1}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \omega_{ij} \mid \xi_j$ and $t \in [t_{2k}, t_{2k+1}]$.

From (3.43), the subsequent inequality is derived,

$$V(t) \leq \bar{V}(t) \leq \bar{V}(t_{2k})e^{\zeta(t-t_{2k})} = V(t_{2k-1})e^{\zeta(t-t_{2k})}e^{-2\lambda(t_{2k}-t_{2k-1}-\tau)}, \tag{3.44}$$

where $t \in [t_{2k}, t_{2k+1}]$, $k = 1, 2, \ldots$.

Then consider the situation that $t \in [t_0, t_1]$. Similar to the previous discussion, it is inferred that

$$V(t) \leq V_{max}e^{\zeta(t-t_0)}, \tag{3.45}$$

where $V_{max} = \sup_{s \in [t_0 - h, t_0]} \{\frac{1}{2} \parallel x_0(s) - x^* \parallel^2\}$ and $t \in [t_0, t_1]$.

It can be inferred from (3.39), (3.44) and (3.45) that

$$\begin{aligned} V(t) &\leq V(t_{2m-1})e^{-2\lambda(t-t_{2m-1})} \\ &\leq V(t_{2m-3})e^{-2\lambda(t-t_{2m-1})}e^{\zeta(t_{2m-1}-t_{2m-2})}e^{-2\lambda(t_{2m-2}-t_{2m-3}-\tau)} \\ &\vdots \\ &\leq V(t_1)e^{-2\lambda(t-t_{2m-1})}e^{\zeta\sum_{k=2}^{m}(t_{2k-1}-t_{2k-2})}e^{-2\lambda\sum_{k=2}^{m}(t_{2k-2}-t_{2k-3}-\tau)} \\ &\leq V(t_1)e^{-2\lambda(t-t_{2m-1})}e^{\zeta\sum_{k=2}^{m}(t_{2k-1}-t_{2k-2})}e^{-2\lambda(1-\frac{1}{\alpha})\sum_{k=2}^{m}(t_{2k-2}-t_{2k-3})} \\ &\leq V_{max}e^{-2\lambda(1-\frac{1}{\alpha})[(t-t_{2m-1})+\sum_{k=2}^{m}(t_{2k-2}-t_{2k-3})]+\zeta\sum_{k=1}^{m}(t_{2k-1}-t_{2k-2})} \\ &\leq V_{max}e^{-2\lambda(1-\frac{1}{\alpha})[\gamma(t-t_0)-N^*]+\zeta(1-\gamma)(t-t_0)+\zeta N^*} \\ &\leq V_{max}e^{[2\lambda(1-\frac{1}{\alpha})+\zeta]N^*}e^{-[2\lambda(1-\frac{1}{\alpha})\gamma-\zeta(1-\gamma)](t-t_0)}, \end{aligned} \tag{3.46}$$

where $t \in [t_{2m-1}, t_{2m}]$ and $m = 2, 3, \ldots$. Assumptions 1 and 2 are used in the derivation of (3.46). Similarly, it can be concluded from (3.44) and (3.45) that the following relation holds when $t \in [t_{2m}, t_{2m+1}]$, $m = 2, 3, \ldots$,

$$\begin{aligned} V(t) &\leq V(t_{2m-1})e^{\zeta(t-t_{2m})}e^{-2\lambda(t_{2m}-t_{2m-1}-\tau)} \\ &\leq V_{max}e^{-2\lambda\sum_{k=2}^{m+1}(t_{2k-2}-t_{2k-3}-\tau)+\zeta[(t-t_{2m})+\sum_{k=1}^{m}(t_{2k-1}-t_{2k-2})]} \\ &\leq V_{max}e^{[2\lambda(1-\frac{1}{\alpha})+\zeta]N^*}e^{-[2\lambda(1-\frac{1}{\alpha})\gamma-\zeta(1-\gamma)](t-t_0)}. \end{aligned} \tag{3.47}$$

Then it is inferred from (3.46) and (3.47) that

$$V(t) \leq V_{max} e^{[2\lambda(1-\frac{1}{\alpha})+\zeta]N^*} e^{-[2\lambda(1-\frac{1}{\alpha})\gamma-\zeta(1-\gamma)](t-t_0)}, \tag{3.48}$$

where $t \geq t_3$. Further, it is not difficult to verify that the inequality (3.48) still holds for $t \in [t_0, t_3]$.

Therefore,

$$V(t) \leq V_{max} e^{[2\lambda(1-\frac{1}{\alpha})+\zeta]N^*} e^{-[2\lambda(1-\frac{1}{\alpha})\gamma-\zeta(1-\gamma)](t-t_0)} \; for \; t \geq t_0. \tag{3.49}$$

That is,

$$\| x(t) - x^* \| \leq \sup_{s\in[t_0-h,t_0]} \{\| x_0(s) - x^* \|\} \mu e^{-\frac{1}{2}[2\lambda(1-\frac{1}{\alpha})\gamma-\zeta(1-\gamma)](t-t_0)}, \tag{3.50}$$

where $t \geq t_0$ and $\mu = e^{\frac{1}{2}[2\lambda(1-\frac{1}{\alpha})+\zeta]N^*}$.

This ends the proof.

Remark 6: In Theorem 1, some stability conditions pertaining to the parameters τ and γ are established. These two parameters can be used to describe the dynamic characteristics of time-varying delays continuously varying between short and large delays. By observing the condition (3.36) and other stability conditions in Theorem 1, the following peroration can be drawn: when the measure of the union of time periods in which the time-varying delays appear as short delays is large enough, the parameters τ and γ will have an important impact on the stability of NN (2.1) and the upper bound that the time-varying delays can achieve in the whole time interval $[t_0, \infty)$ will no longer be the dominant factor influencing stability. This is different from most existing research results on the stability of delayed NNs, like the references [20–29]. In these references, the derivatives and upper bounds of time-varying delays are usually regarded as key factors affecting stability, and limited attention has been given to the impact of time-varying delays' dynamic characteristics on the neural network stability.

Remark 7: By observing the inequality (3.37), the value of the exponential convergence rate υ hinges on the parameters λ and γ, and the value of υ is greater when these two parameters are larger. Additionally, by observing conditions (3.3) and (3.4), it can be found that when the value of parameter τ is smaller, the value of the parameter θ can be smaller to make the condition (3.4) true, thereby making the value of the parameter λ larger. Hence, it can be deduced that a smaller value of τ and a larger value of γ will give rise to a greater value of υ. This means that the convergence speed of the network state vector $x(t)$ to the equilibrium point x^* will be faster, if the critical threshold of short delays is smaller and the ratio of the measure of time periods, in which the time-varying delays appear as short delays, to the measure of the whole time interval is larger. In addition, since the conditions established in Theorem 1 can guarantee the equilibrium point x^* is globally exponentially stable, then it is concluded that the equilibrium point x^* is unique for system (2.1) when the conditions hold.

Two corollaries related to Theorem 1 are presented as follows.

Corollary 1: Suppose that Assumption 3 holds. If there are three positive constants ε, σ and θ such that the conditions (3.3) and (3.4) hold, and there exits a time instant $\hat{t}$ such that $\delta_i(\hat{t}) = 0$, $\tau_{ij}(\hat{t}) = 0$, $\delta_i(t) \leq \tau$ and $\tau_{ij}(t) \leq \tau$ for $i, j =, 1, 2, \ldots, n$ and $t \geq \hat{t}$, then the equilibrium point x^* of NN (2.1) is globally exponentially stable and the state vector $x(t)$ satisfies the following relation for $t \geq \hat{t}$,

$$\| x(t) - x^* \| \leq \sup_{s\in[t_0-h,t_0]} \{\| x_0(s) - x^* \|\} e^{\frac{1}{2}\zeta(\hat{t}-t_0)} e^{-\lambda(t-\hat{t})}.$$

Proof: According to Proposition 1,

$$\dot{V}(t) \leq -2\lambda V(t) \ \ for\, t \geq \hat{t}.$$

Thus,

$$V(t) \leq V(\hat{t})e^{-2\lambda(t-\hat{t})} \tag{3.51}$$

for $t \geq \hat{t}$.

Similar to the derivation of the inequality (3.45), an inequality is obtained as below,

$$V(t) \leq V_{max}e^{\zeta(\hat{t}-t_0)}, \tag{3.52}$$

where $t \in [t_0, \hat{t}]$.

From (3.51) and (3.52),

$$V(t) \leq V_{max}e^{\zeta(\hat{t}-t_0)}e^{-2\lambda(t-\hat{t})},$$

where $t \geq \hat{t}$. That is, $\| x(t) - x^* \| \leq \sup_{s\in[t_0-h,t_0]} \{\| x_0(s) - x^* \|\}e^{\frac{1}{2}\zeta(\hat{t}-t_0)}e^{-\lambda(t-\hat{t})}$ for $t \geq \hat{t}$.

This concludes the proof.

Corollary 2: Suppose that Assumptions 1–3 are true. In addition, suppose that $\tau_{ij}(t) = 0$ for $t \geq t_0$ and $i, j = 1, 2, \ldots, n$, that is, there is no transmission delay in the network (2.1). If there exist three positive constants ε, σ and θ such that

$$b_i - \varepsilon b_i - \frac{\sigma}{2}\sum_{j=1}^{n} | \omega_{ij} | \xi_j - \frac{1}{2\sigma}\sum_{j=1}^{n} | \omega_{ji} | \xi_i - \frac{\bar{b}\theta}{2\varepsilon} - \frac{4\theta(1+\theta)}{\varepsilon}\sum_{i=1}^{n} b_i > 0, \tag{3.53}$$

$$\tau = \sqrt{\frac{\theta}{2(1+\theta)}}\frac{1}{(1+\sqrt{\theta})\sum_{i=1}^{n} b_i + \sum_{i=1}^{n}\sum_{j=1}^{n} | \omega_{ij} | \xi_j}, \tag{3.54}$$

and

$$\gamma > \frac{\hat{\zeta}}{2\hat{\lambda}(1-\frac{1}{\alpha}) + \hat{\zeta}}, \tag{3.55}$$

where $i = 1, 2, \ldots, n$, $\hat{\lambda} = \min_{i=1,\ldots,n}\{b_i - \varepsilon b_i - \frac{\sigma}{2}\sum_{j=1}^{n} | \omega_{ij} | \xi_j - \frac{1}{2\sigma}\sum_{j=1}^{n} | \omega_{ji} | \xi_i - \frac{\bar{b}\theta}{2\varepsilon} - \frac{4\theta(1+\theta)}{\varepsilon}\sum_{i=1}^{n} b_i\}$ and $\hat{\zeta} = \max_{i=1,\ldots,n}\{\varepsilon b_i + \sigma\sum_{j=1}^{n} | \omega_{ij} | \xi_j + \frac{1}{\sigma}\sum_{j=1}^{n} | \omega_{ji} | \xi_i + \sum_{i=1}^{n}\frac{b_i}{\varepsilon}\}$, then the equilibrium point x^* of NN (2.1) is globally exponentially stable and the state vector $x(t)$ satisfies the following relation for $t \geq t_0$,

$$\| x(t) - x^* \| \leq \sup_{s\in[t_0-h,t_0]}\{\| x_0(s) - x^* \|\}\hat{\mu}e^{-\hat{\upsilon}(t-t_0)}, \tag{3.56}$$

where $\hat{\mu} = e^{\frac{1}{2}[2\hat{\lambda}(1-\frac{1}{\alpha})+\hat{\zeta}]N^*}$ and $\hat{\upsilon} = \frac{1}{2}[2\hat{\lambda}(1-\frac{1}{\alpha})\gamma - \hat{\zeta}(1-\gamma)]$.

In Theorem 1, Corollaries 1 and 2, an assumption is requisite that the time-varying delays $\delta_i(t)$ and $\tau_{ij}(t)$ have to be equal to zero at some discrete time instants. This constraint will no longer be needed in the theorem to be presented next.

Theorem 2: Suppose that $x_i(s) = \rho_i$ where $i = 1, 2 \cdots, n$, $s \in [t_0 - h, t_0]$ and ρ_i is any given constant, that is, the initial values of system (2.1) are constant initial values. Suppose that Assumption 3 is true. If there exist three positive constants ε, σ and θ such that conditions (3.3) and (3.4) hold, and

$$\delta_i(t) \leq \tau, \quad \tau_{ij}(t) \leq \tau \tag{3.57}$$

for $t \geq t_0$ and $i, j = 1, 2, \ldots, n$, then the equilibrium point x^* of NN (2.1) is globally exponentially stable and the state vector $x(t)$ fulfills the following relation for $t \geq t_0$,

$$\| x(t) - x^* \| \leq \| \rho - x^* \| e^{-\lambda(t-t_0)}, \tag{3.58}$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_n)^T$.

Proof: According to the assumptions in this theorem, the initial values of the system (3.1) become constant initial values which are given as below,

$$e_i(s) = \rho_i - x_i^* \;\; for\ s \in [t_0 - h, t_0]\ and\ i = 1, \ldots, n.$$

Let $t_m = t_0 + m\tau$, $m = 1, 2, \ldots$. Similar to the derivation of (3.7), the following relation is obtained,

$$\begin{aligned} \dot{V}(t) \leq & -a \| e(t) \|^2 + \frac{\bar{b}}{2\varepsilon} \| e(t) - e(t_m) \|^2 + \frac{1}{2\varepsilon} \sum_{i=1}^{n} b_i \| e(t_m) - e(t - \delta_i(t)) \|^2 \\ & + \frac{1}{2\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} | \omega_{ij} | \xi_j \| e(t - \tau_{ij}(t)) \|^2, \end{aligned} \tag{3.59}$$

where $t \in [t_m, t_{m+1}]$, $m = 0, 1, 2, \ldots$, $a = \min_{i=1,\ldots,n} \{b_i - \varepsilon b_i - \frac{\sigma}{2} \sum_{j=1}^{n} | \omega_{ij} | \xi_j\}$ and $\bar{b} = \max_{i=1,\ldots,n} \{b_i\}$.

Next, we can get a conclusion that

$$\| e(t) - e(t_0) \| \leq \sqrt{\eta} \| e(t_0) \| \quad for\ t \in [t_0, t_1], \tag{3.60}$$

where $\eta = \frac{\theta}{2(1+\theta)}$. This conclusion is demonstrated as follows by contradiction. Suppose that there is a time instant $\tilde{t} \in [t_0, t_1]$ such that $\| e(\tilde{t}) - e(t_0) \| > \sqrt{\eta} \| e(t_0) \|$ and

$$\| e(t) - e(t_0) \| \leq \sqrt{\eta} \| e(t_0) \| \quad for\ t \in [t_0, \tilde{t}). \tag{3.61}$$

Similar to the derivation of (3.11), the following relation is derived from (3.61),

$$\| e(t) - e(t_0) \|^2 \leq \theta \| e(t) \|^2 \quad for\ t \in [t_0, \tilde{t}). \tag{3.62}$$

If $t - \delta_i(t) \geq t_0$ and $t \in [t_0, \tilde{t})$, then, similar to the derivation of (3.13), the subsequent inequality can be gained,

$$\| e(t - \delta_i(t)) - e(t_0) \|^2 \leq \theta \| e(t) \|^2 .$$

If $t - \delta_i(t) < t_0$ and $t \in [t_0, \tilde{t})$, then $\| e(t - \delta_i(t)) - e(t_0) \|^2 = 0 \leq \theta \| e(t) \|^2$. Hence,

$$\| e(t - \delta_i(t)) - e(t_0) \|^2 \leq \theta \| e(t) \|^2 \tag{3.63}$$

for $i = 1, 2, \ldots, n$ and $t \in [t_0, \tilde{t})$.

If $t - \tau_{ij}(t) \geq t_0$ and $t \in [t_0, \tilde{t})$, then, similar to the derivation of (3.14), the subsequent inequality can be obtained,

$$\| e(t - \tau_{ij}(t)) \|^2 \leq (4 + 6\theta) \| e(t) \|^2 .$$

If $t - \tau_{ij}(t) < t_0$ and $t \in [t_0, \tilde{t})$, then it follows from (3.62) that $\| e(t - \tau_{ij}(t)) \|^2 = \| e(t_0) \|^2 \leq 2(1 + \theta) \| e(t) \|^2 \leq (4 + 6\theta) \| e(t) \|^2$. Thus,

$$\| e(t - \tau_{ij}(t)) \|^2 \leq (4 + 6\theta) \| e(t) \|^2 \tag{3.64}$$

for $i, j = 1, 2, \ldots, n$ and $t \in [t_0, \tilde{t})$.

Then, resembling the derivation of inequalities (3.15)–(3.19), the following relation can be deduced,

$$\| e(\tilde{t}) - e(t_0) \| \leq \sqrt{\eta} \| e(t_0) \|,$$

which contradicts the assumption that $\| e(\tilde{t}) - e(t_0) \| > \sqrt{\eta} \| e(t_0) \|$. Therefore, the inequality (3.60) holds for $t \in [t_0, t_1]$. Then, similar to the derivation of (3.20) and (3.21), the following two inequalities can be concluded,

$$\dot{V}(t) \leq -2\lambda V(t)$$

and

$$\| e(t) - e(t_0) \|^2 \leq \theta \| e(t) \|^2,$$

where $t \in [t_0, t_1]$.

The subsequent proof is the same as the proof of Proposition 1. By using the mathematical induction method, the following inequality can be proved to be true,

$$\dot{V}(t) \leq -2\lambda V(t)$$

for $t \geq t_0$. Consequently, $\| x(t) - x^* \| \leq \| \rho - x^* \| e^{-\lambda(t-t_0)}$ for $t \geq t_0$.

This concludes the proof.

Remark 8: In fact, the parameter τ given in conditions (3.4) and (3.57) can be regarded as an admissible upper bound on the time-varying delays, below which the leakage delays and transmission delays will not destroy the neural network stability. Furthermore, by observing the relationship between the parameters in condition (3.3), it can be found that the presence of negative feedback terms in NN (2.1) is one of the main reasons why the network is stable. This implies that the occurrence of leakage delays will not completely undermine the capacity of negative feedback terms to stabilize the network when these delays are smaller than the admissible upper bound τ. At present, many studies on NNs have considered and researched leakage delays, like [13–15, 23, 29]. However, in these studies, the existence of leakage delays is always regarded as a disruptive factor that can definitely render the negative feedback terms ineffective. Obviously, such a setting adopted for leakage delays may make the theoretical results obtained in these references somewhat conservative. Meanwhile, it should be pointed out that there are few studies on the admissible upper bounds of leakage delays related to the stabilization ability of the negative feedback terms.

Next, a corollary related to Theorem 2 is presented.

Corollary 3: Suppose that Assumption 3 holds, and $x_i(s) = \rho_i$ where $i = 1, 2 \cdots, n$, $s \in [t_0 - h, t_0]$ and ρ_i is any given constant. Furthermore, suppose that $\tau_{ij}(t) = 0$ for $t \geq t_0$ and $i, j = 1, 2, \ldots, n$. If there are three positive constants ε, σ and θ such that the conditions (3.53) and (3.54) hold, and

$$\delta_i(t) \leq \tau \tag{3.65}$$

for $t \geq t_0$ and $i = 1, 2, \ldots, n$, then the equilibrium point x^* of NN (2.1) is globally exponentially stable and the state vector $x(t)$ satisfies the following relation for $t \geq t_0$,

$$\| x(t) - x^* \| \leq \| \rho - x^* \| e^{-\hat{\lambda}(t-t_0)}. \tag{3.66}$$

Remark 9: The parameter $\tau = \sqrt{\frac{\theta}{2(1+\theta)}} \frac{1}{(1+\sqrt{\theta})\sum_{i=1}^{n} b_i + \sum_{i=1}^{n}\sum_{j=1}^{n} |\omega_{ij}| \xi_j}$ given in conditions (3.54) and (3.65) can be regarded as an admissible upper bound of the time-varying leakage delays when there is no transmission delay in the network (2.1).

4. Numerical simulations

Example 1: A neural network consisting of three neurons is considered as below,

$$\begin{cases} \dot{x}_1(t) = -2x_1(t - \delta_1(t)) + 0.02tanh(x_1(t - \tau_{11}(t))) - 0.012tanh(x_2(t - \tau_{12}(t))), \\ \dot{x}_2(t) = -2x_2(t - \delta_2(t)) + 0.018tanh(x_1(t - \tau_{21}(t))) + 0.0171tanh(x_2(t - \tau_{22}(t))) \\ \qquad + 0.0115tanh(x_3(t - \tau_{23}(t))), \\ \dot{x}_3(t) = -2x_3(t - \delta_3(t)) - 0.0475tanh(x_1(t - \tau_{31}(t))) + 0.011tanh(x_3(t - \tau_{33}(t))), \end{cases} \tag{4.1}$$

where $t_0 = 0$. The time-varying delays in this system are supposed to be that $\delta_i(t) = \tau_{ij}(t) = \psi(t)$ and

$$\psi(t) = \begin{cases} 0.01[cos(t + \pi) + 1], & t \in [100(k-1)\pi,\ 100(k-1)\pi + 96\pi), \\ cos(t + \pi) + 1, & t \in [100(k-1)\pi + 96\pi,\ 100(k-1)\pi + 97\pi), \\ cos(t) + 3, & t \in [100(k-1)\pi + 97\pi,\ 100(k-1)\pi + 99\pi), \\ cos(t + \pi) + 1, & t \in [100(k-1)\pi + 99\pi,\ 100k\pi), \end{cases} \tag{4.2}$$

where $i, j = 1, 2, 3$, $k = 1, 2, \ldots$, and π stands for PI. Obviously, the function $\psi(t)$ is continuous and $\dot{\psi}(t) \leq 1$ for $t \geq 0$. Furthermore, it can be found that there exists a zero equilibrium point for this system.

According to the parameter settings in the system (4.1), it can be inferred that there exist three positive constants $\varepsilon = 0.5$, $\sigma = 4$ and $\theta = 0.01$ such that the conditions (3.3), (3.4) and (3.36) in Theorem 1 hold. Then it is concluded from Theorem 1 that the zero equilibrium point of system (4.1) is globally exponentially stable. Next, eight sets of initial values for this system are randomly picked as: $x^{(1)}(s) = (3, -4, 9)$, $x^{(2)}(s) = (0.3, -1, 1)$, $x^{(3)}(s) = (5, 8, -0.1)$, $x^{(4)}(s) = (-6, -11, -0.9)$, $x^{(5)}(s) = (-7.5, 13, -6)$, $x^{(6)}(s) = (8.8, -14.5, -13.6)$, $x^{(7)}(s) = (1.9, 2.6, 0)$ and $x^{(8)}(s) = (3.8, 6.3, -2.9)$, where $s \in [-4, 0]$. Figure 1 shows the evolution of the norm of the state vectors of system (4.1) satisfying these different initial conditions. It is observed in this figure that the state vectors satisfying these different initial conditions converge to the origin as the time variable goes to infinity. This verifies the correctness of Theorem 1.

To further elucidate the novelty of the theoretical findings presented in this paper, consider another neural network as follows,

$$\begin{cases} \dot{x}_1(t) = -2x_1(t-\delta_1(t)) + 0.02tanh(x_1(t-\tau_{11}(t))) - 0.012tanh(x_2(t-\tau_{12}(t))), \\ \dot{x}_2(t) = -2x_2(t-\delta_2(t)) + 0.018tanh(x_1(t-\tau_{21}(t))) + 0.0171tanh(x_2(t-\tau_{22}(t))) \\ \qquad + 0.0115tanh(x_3(t-\tau_{23}(t))), \\ \dot{x}_3(t) = -2x_3(t-\delta_3(t)) - 0.0475tanh(x_1(t-\tau_{31}(t))) + 0.011tanh(x_3(t-\tau_{33}(t))), \end{cases} \tag{4.3}$$

where $\delta_i(t) = \tau_{ij}(t) = \varpi(t)$,

$$\varpi(t) = \begin{cases} cos(t+\pi)+1, & t \in [20(k-1)\pi,\ 20(k-1)\pi+\pi), \\ cos(t)+3, & t \in [20(k-1)\pi+\pi,\ 20(k-1)\pi+19\pi), \\ cos(t+\pi)+1, & t \in [20(k-1)\pi+19\pi,\ 20k\pi), \end{cases} \tag{4.4}$$

$i, j = 1, 2, 3$ and $k = 1, 2, \ldots$. Next, eight sets of initial values for this system are chosen that are the same as the initial conditions of the system (4.1) above. Figure 2 shows the evolution of the norm of the state vectors of system (4.3) fulfilling these initial conditions. It can be observed in Figure 2 that the state vectors of system (4.3) do not converge to the zero equilibrium point over time. Hence, from this numerical experiment result, it can be inferred that the zero equilibrium point of system (4.3) is unstable.

By observing the functions $\varpi(t)$ and $\psi(t)$, it can be found that the derivatives of both functions are less than or equal to 1, and these two functions have the same supremum 4 and infimum 0. Furthermore, the parameter settings in system (4.1) and system (4.3) are exactly the same. But, according to the experimental results in Figures 1 and 2, it is clear that these two systems have completely different stability properties. The reason why this happens is that the time-varying delays in these systems have different dynamic characteristics. It can be found from (4.2) and (4.4) that, compared with the function $\varpi(t)$, the fraction of time periods in which the time-varying delay $\psi(t)$ appears as a short delay over the whole time interval $[0, \infty)$ is larger. This indicates that, in addition to system parameters, derivatives of time-varying delays and their upper bounds, the time-varying delays' dynamic characteristics also have a significant effect on the stability of NNs. Thus, it is of significance to study the relationship between the time-varying delays' dynamic characteristics and the stability of NNs in this paper.

Furthermore, in most existing research papers on the stability of delayed NNs, only the stability criteria related to the derivatives and upper bounds of time-varying delays are established. It is not easy to employ these existing stability criteria to effectively analyze the stability of some systems such as the systems (4.1) and (4.3), which differ only in the dynamic characteristics of time-varying delays. In contrast, the stability of system (4.1) can be validly analyzed by making use of the stability theorems presented in this paper. Thus, the comparison can show that the theoretical results exhibited in this paper are somewhat innovative.

Remark 10: Compared with the classic Lyapunov-Krasovskii functional method, the stability analysis method proposed in this paper can effectively establish the correlation between the stability of delayed neural networks and the time-varying delays' dynamic characteristics. However, this does not mean that the stability analysis method in this paper is superior to the Lyapunov-Krasovskii functional method. In fact, the stability criteria established in this paper and those based on the Lyapunov-Krasovskii functional method are suitable for different types of dynamical systems in terms of stability analysis. In practical applications, one should choose to use these two types of methods according to the characteristics of the systems. The stability conditions based on the Lyapunov-Krasovskii functional method are usually related to the upper bound of time-varying delays. If the upper bound of time-varying delays of a delayed system is particularly large, then it may not be suitable to use the Lyapunov-Krasovskii functional method to analyze the stability of the system. Besides, if the parameters τ and γ that characterize the time-varying delays' dynamic characteristics of this system satisfy the conditions in the stability theorems of this paper, then the stability of the system can be analyzed using the stability criteria of this paper. Although the stability criteria proposed in this paper are independent of the upper bound of time-varying delays, they have strong constraints on the parameters τ and γ. If for a delayed system, the parameters τ and γ fail to satisfy the conditions in the theorems of this paper, then the analysis method presented in this paper will no longer be applicable. But if the upper bound of time-varying delays is not particularly large, then it is more suitable to use the Lyapunov-Krasovskii functional method to analyze the stability.

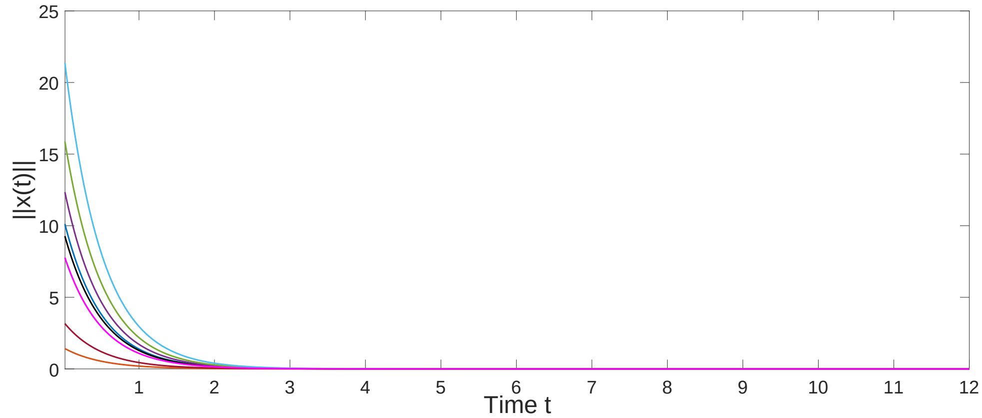

Figure 1. Evolution of the state vectors of system (4.1).

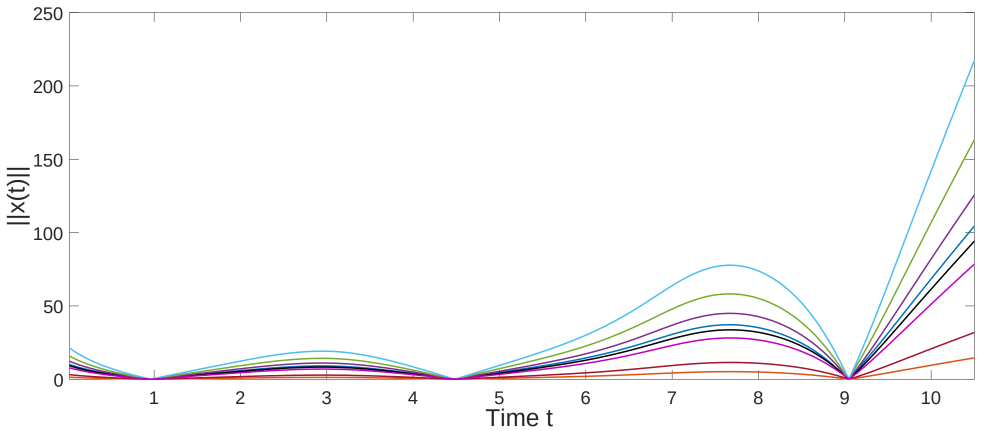

Figure 2. Evolution of the state vectors of system (4.3).

5. Conclusions

In this article, the stability of Hopfield NNs with time-varying transmission delays and leakage delays is investigated, and a special focus is placed on the impact of the time-varying delays' dynamic characteristics on the neural network stability. To characterize the dynamic features of time-varying delays continuously varying between short and large delays, two parameters are introduced: a critical threshold used to distinguish whether the time-varying delays are short or large delays, and the ratio of the measure of the time periods, in which the time-varying delays appear as short delays, to the measure of the whole time interval. Subsequently, some stability conditions for the NNs are proposed. Notably, it is revealed that, when the measure of the time periods in which the time-varying delays appear as short delays is large enough, these two parameters related to the time-varying delays' dynamic characteristics can exert a significant influence on system stability, and the upper bound of time-varying delays across the whole time interval ceases to be the primary determinant of stability. Finally, the relationship between leakage delays and the stabilization ability of negative feedback terms is explored, and two admissible upper bounds are constructed, less than which leakage delays do not completely compromise the capacity of negative feedback terms to stabilize the NNs.

In the future, the investigation on stability of delayed coupled neural networks with time-varying topologies and a limited communication data rate will be carried out.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Qingyun Wang is an editorial board member for the Electronic Research Archive and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no conflicts of interest.

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