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*Research article*

# Kronecker product bases and their applications in approximation theory

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**Abstract:** The Kronecker product is widely utilized to construct higher-dimensional spaces from lower-dimensional ones, making it an indispensable tool for efficiently analyzing multi-dimensional systems across various fields. This paper investigates the representation of analytic functions within hyper-elliptical regions through infinite series expansions involving sequences of Kronecker product bases of polynomials. Additionally, we examine the growth order and type and $T_\rho$-property of series composed of Kronecker product bases that represent entire functions. We also delve into the convergence properties of Kronecker product bases associated with special functions, including Bessel, Chebyshev, Bernoulli, Euler, and Gontcharoff polynomials. The obtained results extend and enhance the existing findings of such representations in hyper-spherical regions.

**Keywords:** bases of polynomials; Kronecker product bases; growth of bases; effectiveness; hyper-elliptical regions

## 1. Introduction

The elegant algebraic structure of the Kronecker product facilitates the development of efficient, elegant, and highly practical algorithms. Several contributions in scientific computing indicate that this significant matrix operation will be increasingly employed in different research trends and applications. Like any significant mathematical operation, the Kronecker product has been adapted and refined in a wide range of applications across numerous fields, including matrix equations [1, 2], matrix calculus [3–6], system theory [7–10], system identification [11, 12], signal processing [13], image processing [14], quantum mechanics [15], statistics and econometrics [16, 17], approximation theory [18] and other applications [19].

A rich area of research in approximation theory is the study of polynomial bases. The concept of polynomial bases was introduced by Cannon [20, 21] and Whittaker [22–24]. This theory has been generalized in three principal directions. The first direction involves studying the convergence properties of polynomial bases for several complex variables in hyper-spherical, hyper-elliptical, and polycylindrical regions. In the context of several complex variables, the effectiveness of basic sets of polynomials by holomorphic functions in hyperelliptical regions were explored in [25, 26]. The convergence properties of composite sets of polynomials were investigated in [29] while their order and type in complete Reinhardt domains were studied in [27]. Additionally, the authors of [28] introduced the representations of certain regular functions of several complex variables using exponential sets of polynomials in hyperelliptical regions. The second direction extends this study to the convergence properties of polynomial bases in Clifford analysis. Particularly, the theory of base of polynomials was adopted to the context of Clifford analysis in [30] and the order of functions represented by these bases were deduced in [31]. Further contributions in this direction, authors studied the approximation properties of monogenic functions by hypercomplex Ruscheweyh derivative bases [32], and Hasse derivative bases [33]. The third direction concerns the approximation of analytic functions by certain types of complex conformable fractional derivative bases of polynomials in Fréchet spaces [34–36].

It is of great interest to perform different mathematical structures on certain bases to produce new bases. Then the challenging task is examining the convergence properties of the new bases to determine whether they retain the same effectiveness properties of their constituents bases or not. For instance, the author of [37] demonstrated the convergence properties of Hadamard product bases of polynomials which defined using $n$ bases of polynomials of several complex variables in complete Reinhardt domains. The extension of the generalized Hadamard product set in several complex variables was introduced in [38] using hyperelliptical regions. Our emphasis in the current study is on exploring some important results related to the approximation theory in multi-dimensional expansions in hyper-elliptical regions. In this context, the Kronecker product plays a pivotal role in multi-dimensional polynomial approximation. Regarding to the approximation of multi-dimensional functions using polynomial series, the Kronecker product is employed to construct multi-variable polynomials from one-dimensional polynomial bases. This approach enables the efficient representation and computation of multi-dimensional polynomials, which are essential for approximating the functions of several complex variables.

The importance of the constructed Kronecker product bases lies in their ability to represent one-dimensional functions as multi-dimensional expansions. When dealing with multivariate functions, the Kronecker product enables the existence of a basic series expansion in terms of a one-dimensional basic series for each variable, resulting in a more efficient representation. The purpose of introducing these new bases is to represent analytic functions as infinite series within regions such as hyper-spheres, hyper-ellipses, or polycylinders.

Investigating the approximation properties of certain special functions, including Bessel, Chebyshev, Bernoulli, Euler, and Gontcharoff polynomials, has considerable significance in the theory of polynomial bases. These polynomials exhibit unique structural and analytical properties that elaborate their efficiency in various applications within mathematical analysis and applied mathematics. In the context of approximation theory, such polynomials serve as bases for the representation and approximation of more complex functions. Notably, the authors of [46] demonstrated that both the Bernoulli and Euler polynomials are of order 1 and are not effective anywhere. In [44], it was shown that both proper Bessel polynomials and general Bessel polynomials are effective everywhere. Moreover, the authors of [45] conducted a detailed analysis of the approximation properties of Chebyshev polynomials within the unit disk. In this study, we extend the corresponding approximation properties of certain bases constructed using the aforementioned polynomials.

We now provide a brief overview of the paper. Section 2 provides the key definitions, notations, and relevant results from previous work that will be utilized throughout the paper. We define and construct the Kronecker product of certain bases and prove it is indeed a base in Section 3. In Section 4, we investigate the effectiveness of Kronecker product bases for polynomials of several complex variables within closed hyper-elliptical regions. We establish the effectiveness criteria of Kronecker product bases in closed hyper-ellipses, whose constituents are Cannon bases of polynomials of one complex variable, which are effective in closed circles. Section 5 explores the order and type of the Kronecker product bases of polynomials. We examine the $T_\rho$-property in closed hyper-ellipse in Section 6. In Section 7, we discuss several applications of the Kronecker product bases of polynomials, which have links with the classical special functions such as Bessel, Chebyshev, Bernoulli, Euler, and Gontcharaff polynomials. Section 8 concludes the paper by summarizing the obtained results.

## 2. Preliminaries and notations

For simplicity purposes, the following notation is used throughout this study (see [26, 28]).

$$\begin{gathered}\mathbf{m} = m_1, m_2, \ldots, m_k; \quad \langle \mathbf{m} \rangle = m_1 + m_2 + \ldots m_k; \\ \mathbf{z} = z_1, z_2, \ldots z_k; \quad \mathbf{0} = 0, 0, \ldots, 0; \quad |\mathbf{z}|^2 = |z_1|^2 + |z_2|^2 + \ldots + |z_k|^2; \\ \mathbf{z}^{\mathbf{m}} = \prod_{s=1}^{k} z_s^{m_s}; \quad \mathbf{t}^{\mathbf{m}} = \prod_{s=1}^{k} t_s^{m_s}; \\ [\mathbf{r}] = [r_1, r_2, \ldots, r_k]; \quad [\mathbf{r}]^* = [r, r, \ldots, r]; \\ [\alpha \mathbf{r}] = [\alpha_1 r, \alpha_2 r, \ldots, \alpha_k r].\end{gathered} \tag{2.1}$$

In this notation, $m_s, s \in J = \{1, 2, \ldots, k\}$ are non-negative integers, while $t_s, s \in J$ are non-negative numbers, where $0 \leq t_s \leq 1, s \in J, |\mathbf{t}| = \left[\sum_{s=1}^{k} t_s^2\right]^{(1/2)} = 1$ and $r_s, s \in J$ are positive numbers. In the space $\mathbb{C}^k$ of the complex variables $z_s, s \in J$, the symbol $E_{[\mathbf{r}]}$ stands for an open hyper-ellipse of radius $r_s > 0, s \in J$, and its closure is denoted by $\bar{E}_{[\mathbf{r}]}$. The regions $E_{[\mathbf{r}]}$ and $\bar{E}_{[\mathbf{r}]}$ satisfy the following inequalities (see [26, 28]):

$$\begin{aligned} E_{[\mathbf{r}]} &= \{\mathbf{w} : |\mathbf{w}| < 1\}, \\ \bar{E}_{[\mathbf{r}]} &= \{\mathbf{w} : |\mathbf{w}| \leq 1\}, \end{aligned}$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_k)$, $w_s = \frac{z_s}{r_s}$, $s \in J$.

**Definition 2.1.** *[26, 28] Suppose that $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\} = \{\mathcal{P}_0[\mathbf{z}], \mathcal{P}_1[\mathbf{z}], \ldots, \mathcal{P}_{\mathbf{n}}[\mathbf{z}], \ldots\}$ is a set of polynomials. Then $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ is a base if every polynomial in the complex variables $z_s$; $s \in J$ can be uniquely expressed as a finite linear combination of the elements of the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$. Hence, the set $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ is a base if and only if there exists a unique row-finite matrix $\mathcal{P}$ that satisfies*

$$\mathcal{P}\mathcal{P}^{-1} = \mathcal{P}^{-1}\mathcal{P} = I, \tag{2.2}$$

*where $\mathcal{P} = \{\mathcal{P}_{\mathbf{m},\mathbf{h}}\}$ is the matrix of coefficients and $\mathcal{P}^{-1} = \left\{\mathcal{P}^{-1}_{\mathbf{m},\mathbf{h}}\right\}$ is the matrix of operators of the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$. Accordingly, for the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$, we have*

$$\mathcal{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \mathcal{P}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$
$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \mathcal{P}^{-1}_{\mathbf{m},\mathbf{h}} \mathcal{P}_{\mathbf{h}}[\mathbf{z}]. \tag{2.3}$$

Let $f(\mathbf{z}) = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ be any regular function at the origin $\mathbf{0}$. Using (2.3), we substitute for $\mathbf{z}^{\mathbf{m}}$ and rearrange the terms to obtain the following:

$$f(\mathbf{z}) = \sum_{\mathbf{m}} \Pi_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}[\mathbf{z}], \quad \Pi_{\mathbf{m}} = \sum_{\mathbf{h}} \mathcal{P}^{-1}_{\mathbf{m},\mathbf{h}} a_{\mathbf{h}},$$

which represents the associated basic series of the function $f(\mathbf{z})$.

**Definition 2.2.** *[26,28] The associated basic series $\sum_{\mathbf{m}} \Pi_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}[\mathbf{z}]$ represents $f(\mathbf{z})$ in $\bar{E}_{[\mathbf{r}]}$ if it converges uniformly to $f(z)$ in $\bar{E}_{[\mathbf{r}]}$.*

**Definition 2.3.** *[26,28] The base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\bar{E}_{[\mathbf{r}]}$ when the associated basic series represents every regular function in $\bar{E}_{[\mathbf{r}]}$.*

To examine the convergence properties of such bases, we use the following notations:

$$M\left(\mathcal{P}_{\mathbf{m}}, \bar{E}_{[\mathbf{r}]}\right) = \sup_{\bar{E}_{[\mathbf{r}]}} \left|\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\right|, \tag{2.4}$$

and

$$\Omega\left(\mathcal{P}_{\mathbf{m}}, \bar{E}_{[\mathbf{r}]}\right) = \sigma_{\mathbf{m}} \prod_{s=1}^{k} \{r_s\}^{\langle \mathbf{m} \rangle - m_s} \sum_{\mathbf{h}} \left|\mathcal{P}^{-1}_{\mathbf{m},\mathbf{h}}\right| M\left(\mathcal{P}_{\mathbf{h}}, \bar{E}_{[\mathbf{r}]}\right), \tag{2.5}$$

where

$$\sigma_{\mathbf{m}} = \inf_{|\mathbf{t}|=1} \frac{1}{\mathbf{t}^{\mathbf{m}}} = \frac{\{\langle \mathbf{m} \rangle\}^{(1/2)\langle \mathbf{m} \rangle}}{\prod_{s=1}^{k} m_s^{(1/2)m_s}}, \quad 1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{k})^{\langle \mathbf{m} \rangle}. \tag{2.6}$$

Moreover, the Cannon function is defined as

$$\Omega\left(\mathcal{P}, \bar{E}_{[\mathbf{r}]}\right) = \limsup_{\langle \mathbf{m} \rangle \to \infty} \left\{\Omega\left(\mathcal{P}_{\mathbf{m}}, \bar{E}_{[\mathbf{r}]}\right)\right\}^{\frac{1}{\langle \mathbf{m} \rangle}}. \tag{2.7}$$

Let $N_{\mathbf{m}} = N_{m_1, m_2, \ldots, m_k}$ denote the number of non-zero coefficients in the representation (2.3). As defined in [29], a base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials is called a Cannon base, if $N_{\mathbf{m}}$ satisfies

$$\lim_{\langle \mathbf{m} \rangle \to \infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle \mathbf{m} \rangle}} = 1. \tag{2.8}$$

If $\lim_{\langle \mathbf{m} \rangle \to \infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle \mathbf{m} \rangle}} = a$, where $a > 1$, then the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ is called a general base.

The following result characterizes the effectiveness property of the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$.

**Theorem 2.1.** *[26] A Cannon base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\bar{E}_{[\mathbf{r}]}$ if and only if*

$$\Omega\left(\mathcal{P}, \bar{E}_{[\mathbf{r}]}\right) = \prod_{s=1}^{k} r_s.$$

## 3. The Kronecker product bases of polynomials

The Kronecker product of two matrices of an arbitrary size was introduced over any ring (see [39–41]). It maps the matrices $\mathbf{A}_{i\times j}$ and $\mathbf{B}_{s\times t}$ into the $is \times jt$ matrix, which is denoted by $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$. Although the usual matrix product of two matrices $\mathbf{A}, \mathbf{B}$ requires that $j = s$ and that either $\mathbf{A}$ or $\mathbf{B}$ is a scalar, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is defined regardless the size of these matrices. The Kronecker product has the following properties:

(1) $\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C} = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$,
(2) $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ exist,
(3) If $\mathbf{A}$ and $\mathbf{B}$ are non-singular, then $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$,
(4) $I_m \otimes I_n = I_{mn}$.

Let $\{\mathcal{P}_{1,m_1}(z_1)\}, \{\mathcal{P}_{2,m_2}(z_2)\}, \ldots \{\mathcal{P}_{k,m_k}(z_k)\}$ (in short, $\{\mathcal{P}_{s,m_s}(z_s)\}$) where $s \in J$ be a finite numbers of bases of polynomials of one variable. Consider the product element

$$\mathcal{P}_{1,m_1}(z_1) \otimes \mathcal{P}_{2,m_2}(z_2) \otimes \ldots \otimes \mathcal{P}_{k,m_k}(z_k).$$

If, for any mode of arrangement, we put

$$\mathcal{K}_{m_1,m_2,\ldots,m_k}(z_1, z_2, \ldots, z_k) = \mathcal{P}_{1,m_1}(z_1) \otimes \mathcal{P}_{2,m_2}(z_2) \otimes \ldots \otimes \mathcal{P}_{k,m_k}(z_k), \tag{3.1}$$

the sequence $\{\mathcal{K}_{m_1,m_2,\ldots,m_k}(z_1, z_2, \ldots, z_k)\}$ represents a set consisting of multi-complex variable polynomials in $z_1, z_2, \ldots, z_k$ which indeed determined the Kronecker product base of the polynomials bases $\{P_{s,m_s}(z_s)\}$; $s \in J$.

In the current paper, we propose to investigate the convergence characterizations of the Kronecker product bases in multi-dimensional polynomials approximation in terms of their constituents bases in one-dimensional polynomial bases.

When $\{\mathcal{P}_{s,m_s}(z_s)\}$; $s \in J$ are Cannon bases of polynomials of complex variables $z_s$; $s \in J$, the product set

$$\mathcal{K}_{\mathbf{m}}[\mathbf{z}] = \bigotimes_{s=1}^{k} \mathcal{P}_{s,m_s}(z_s) \tag{3.2}$$

defines the Kronecker product bases of polynomials of several complex variables $z_s$; $s \in J$ where

$$\begin{aligned} \mathcal{K}_{\mathbf{m}}[\mathbf{z}] &= \sum_{\mathbf{h}} \mathcal{K}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \\ \mathbf{z}^{\mathbf{m}} &= \sum_{\mathbf{h}} \mathcal{K}^{-1}_{\mathbf{m},\mathbf{h}} \mathcal{K}_{\mathbf{h}}[\mathbf{z}]. \end{aligned} \tag{3.3}$$

Note that $\mathcal{K} = (\mathcal{K}_{\mathbf{m},\mathbf{h}})$ is the coefficients matrix and $\mathcal{K}^{-1} = \left(\mathcal{K}^{-1}_{\mathbf{m},\mathbf{h}}\right)$ is the operators matrix of the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ for which

$$\begin{aligned} \mathcal{K} &= \mathcal{P}_1 \otimes \mathcal{P}_2 \ldots \otimes \mathcal{P}_k, \\ \mathcal{K}^{-1} &= \mathcal{P}_1^{-1} \otimes \mathcal{P}_2^{-1} \ldots \otimes \mathcal{P}_k^{-1}, \end{aligned}$$

where $\mathcal{P}_s$ and $\mathcal{P}_s^{-1}$, $s \in J$ are, respectively, the matrix of coefficients and the matrix operators of the base $\{\mathcal{P}_{s,m_s}(z_s)\}$.

Now, we prove that the Kronecker product set $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ is indeed a base of polynomials. Observe that

$$\begin{aligned}\mathcal{K}\mathcal{K}^{-1} &= (\mathcal{P}_1 \otimes \mathcal{P}_2 \ldots \otimes \mathcal{P}_k)\left(\mathcal{P}_1^{-1} \otimes \mathcal{P}_2^{-1} \ldots \otimes \mathcal{P}_k^{-1}\right)\\ &= \mathcal{P}_1\mathcal{P}_1^{-1} \otimes \mathcal{P}_2\mathcal{P}_2^{-1}. \otimes \mathcal{P}_k\mathcal{P}_k{}^{-1}\\ &= I \otimes I \otimes \ldots \otimes I = I\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}^{-1}\mathcal{K} &= \left(\mathcal{P}_1^{-1} \otimes \mathcal{P}_2^{-1} \ldots \otimes \mathcal{P}_k^{-1}\right)(\mathcal{P}_1 \otimes \mathcal{P}_2 \ldots \otimes \mathcal{P}_k)\\ &= \mathcal{P}_1^{-1}\mathcal{P}_1 \otimes \mathcal{P}_2^{-1}\mathcal{P}_2. \otimes \mathcal{P}_k^{-1}\mathcal{P}_k\\ &= I \otimes I \otimes \ldots \otimes I = I.\end{aligned}$$

Owing to (2.2), we conclude that the Kronecker product set is a base.

In the forthcoming sections, we strive to provide justifications for the following questions:

1) If $k$ bases of polynomials $\{\mathcal{P}_{s,m_s}(z_s)\}$ are effective in the closed circles $\bar{C}_{r_s}$ for $0 < r_s \leq R_s$, is the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ as determined in (3.2) effective in the closed hyper-ellipse $\bar{E}_{[\mathbf{R}]}$ for $R_s \geq 0$?
2) If $k$ bases of polynomials $\{\mathcal{P}_{s,m_s}(z_s)\}$ are of orders $(\gamma_s)$ and types $(\tau_s)$, $s \in J$, what are the order and type of the Kronecker product $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ of these bases?
3) If $k$ bases of polynomials $\{\mathcal{P}_{s,m_s}(z_s)\}$ have the $\mathbb{T}_{\rho_s}$-property in $\bar{C}_{r_s}$, $s \in J$, what is the corresponding $\mathbb{T}_{\rho}$ -property of the Kronecker product $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ in $\bar{E}_{[\mathbf{r}]}$ ?
4) Can the obtained results be applied to some special functions such as Bessel, Chebyshev, Bernoulli, Euler, and Gontcharoff polynomials?

## 4. Effectiveness of the Kronecker product bases of polynomials in closed hyper-ellipses

Let $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ be the Kronecker product base of polynomials of several complex variables $z_s$, $s \in J$ whose constituents are considered to be Cannon bases $\{\mathcal{P}_{s,m_s}(z_s)\}$; $s \in J$. Furthermore, suppose that $z_s^{m_s}$; $s \in J$ admit the following finite representation

$$z_s^{m_s} = \sum_{h_s} \mathcal{P}^{-1}_{s,m_s,h_s}\mathcal{P}_{s,h_s}(z_s)\,;\, s \in J. \tag{4.1}$$

According to the definition of Kronecker product bases (3.1), $\mathbf{z}^{\mathbf{m}}$ admits the finite representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \mathcal{K}^{-1}_{\mathbf{m},\mathbf{h}}\mathcal{K}_{\mathbf{h}}[\mathbf{z}], \tag{4.2}$$

where

$$\mathcal{K}^{-1}_{\mathbf{m},\mathbf{h}} = \prod_{s=1}^{k} \mathcal{P}^{-1}_{s,m_s,h_s}. \tag{4.3}$$

The Cannon sum for the bases $\mathcal{P}_{s,m_s}(z_s)$ for the circles $\bar{C}_{r_s}$; $\in J$ is given by

$$\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{r_s}\right) = \sum_{h_s} \left|\mathcal{P}^{-1}_{s,m_s,h_s}\right| M\left(\mathcal{P}_{s,h_s}, \bar{C}_{r_s}\right) \tag{4.4}$$

where

$$M\left(\mathcal{P}_{s,h_s}, \bar{C}_{r_s}\right) = \sup_{\bar{C}_{r_s}} \left|\mathcal{P}_{s,h_s}(z_s)\right|,$$
$$M\left(\mathcal{P}_{s,h_s}, 0\right) = \left|\mathcal{P}_{s,h_s}(0)\right|. \tag{4.5}$$

The Cannon functions for the same bases for $\bar{C}_{r_s}$ are given by

$$\omega\left(\mathcal{P}_s, \bar{C}_{r_s}\right) = \limsup_{m_s\to\infty} \left\{\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{r_s}\right)\right\}^{\frac{1}{m_s}}. \tag{4.6}$$

Since $\{\mathcal{P}_{s,m_s}(z_s)\}, s \in J$ are Cannon bases, then the numbers of non-zero coefficients $N_{s,m_s}$ satisfy

$$N_{\mathbf{m}} = \prod_{s=1}^{k} N_{s,m_s}, \quad \lim_{\langle\mathbf{m}\rangle\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle\mathbf{m}\rangle}} = 1. \tag{4.7}$$

Thus, the Condition (2.8) holds and the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ will be also a Cannon base. A combination of (2.4), (3.1), and (4.5) leads to

$$M\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{r}]}\right) = \sup_{\bar{E}_{[\mathbf{r}]}} |\mathcal{K}_{\mathbf{m}}[\mathbf{z}]| = \sup_{|\mathbf{t}|=1} \prod_{s=1}^{k} M\left(\mathcal{P}_{s,m_s}, \bar{C}_{r_s t_s}\right). \tag{4.8}$$

Therefore, the Cannon sum for the base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ can be obtained by using (4.3), (4.4), and (4.8). In fact, the following double inequality can be easily verified:

$$\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{r_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1} \left\{\prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{r_s t_s}\right)\right\}\right] \leq \Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{r}]}\right)$$
$$\leq N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{r_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1} \left\{\prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{r_s t_s}\right)\right\}\right]. \tag{4.9}$$

Concerning the effectiveness of the Kronecker product bases in closed hyper-ellipses, the following result is established.

**Theorem 4.1.** *Suppose that $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ is the Kronecker product base of polynomials of several complex variables $z_s, s \in J$ whose constituents are the Cannon bases $\{\mathcal{P}_{s,m_s}(z_s)\}$. Then the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ is effective in the closed hyper-ellipse $\bar{E}_{[\mathbf{R}]}; R_s \geq 0, s \in J$, if and only if the bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ are effective in the closed circles $\bar{C}_{r_s}$ for $0 < r_s \leq R_s, s \in J$.*

*Proof.* First, suppose that each of the bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ is effective in $\bar{C}_{r_s}$ for $0 < r_s \leq R_s, s \in J$. Thus, the Cannon functions given in (4.6) satisfy

$$\omega\left(\mathcal{P}_s, \bar{C}_{r_s}\right) = r_s, (0 < r_s \leq R_s, s \in J).$$

Let $1 < \alpha < \infty$, and fix the positive number $\tau$ by

$$\tau^2 = \left(1 - \alpha^{-1}\right)/k. \tag{4.10}$$

Hence for any number $\alpha$, there exists a finite number $K > 1$ such that

$$\begin{cases} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s\tau}\right) < K\left\{\sqrt{\alpha}R_s\tau\right\}^{m_s} \\ \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s}\right) < K\left\{\sqrt{\alpha}R_s\right\}^{m_s} \quad (m_s \geq 0; s \in J) \end{cases} \tag{4.11}$$

Since the functions $\mathcal{P}_{s,m_s}(z_s)/z_s^{m_s}$ are regular in the region $R_s\tau \leq |z_s| \leq R_s$, $s \in J$, then in terms of (4.4) and (4.5), and by using (4.11), we obtain

$$\begin{aligned} \frac{\left|\mathcal{P}^{-1}_{s,m_s,h_s}\right| M\left(\mathcal{P}_{s,h_s}, \bar{C}_{R_st_s}\right)}{(R_st_s)^{m_s}} &\leq \sup\left\{\frac{\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s\tau}\right)}{(R_s\tau)^{m_s}}, \frac{\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s}\right)}{R_s^{m_s}}\right\} \\ &< K\alpha^{(m_s/2)} \end{aligned}$$

for $\tau \leq t_s \leq 1$. Therefore, it follows that

$$\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_st_s}\right) < KN_{s,m_s}\left(\sqrt{\alpha}R_st_s\right)^{m_s} (\tau \leq t_s \leq 1, m_s \geq 0, s \in J.) \tag{4.12}$$

Now, by considering the parameters $t_s$ where $s \in J$ in (4.9), we find that some of these parameters may lie in the interval $(0, \tau)$ and the others in $(\tau, 1)$. Thus, as in [29], for the numbers $t_s$ where $0 \leq t_s \leq 1$, $s \in J$, we consider three cases.

Case (i): $\tau < t_s \leq 1;\ (1 \leq s \leq l),\ 0 \leq t_u \leq \tau;\ (l+1 < u < k)$,
Case (ii): $0 \leq t_u \leq \tau;\ s \in J$,
Case (iii): $\tau \leq t_s \leq 1;\ s \in J$.

where $l$ is any integer not exceeding $K$. Now, since

$$\sup_{|\mathbf{t}|=1} \mathbf{t}^{\mathbf{m}} = \frac{1}{\sigma_{\mathbf{m}}}; \sum_{s=1}^{k} t_s^2 = 1, \tag{4.13}$$

it follows that

$$\sup_{|\mathbf{t}'|\leq 1} \mathbf{t}'^{\mathbf{m}'} \leq \frac{1}{\sigma_{\mathbf{m}'}}; \quad \mathbf{m}' = (m_1, m_2, \ldots, m_l) \tag{4.14}$$

and

$$\tau^{\langle\mathbf{m}\rangle - \langle\mathbf{m}'\rangle}\left\{\sup_{|\mathbf{t}'|=1-(k-1)\tau^2} \mathbf{t}'^{\mathbf{m}'}\right\} \leq \frac{1}{\sigma_{\mathbf{m}}}, \mathbf{t}' = (t_1, t_2, \ldots, t_l). \tag{4.15}$$

However, we have

$$\sup_{|\mathbf{t}'|=1-(k-1)\tau^2} \mathbf{t}'^{\mathbf{m}} = \frac{\left(1-k\tau^2\right)^{\langle\mathbf{m}'/2\rangle}}{\sigma_{\mathbf{m}'}}.$$

Thus,

$$\sigma_{\mathbf{m}}\tau^{\langle\mathbf{m}\rangle - \langle\mathbf{m}'\rangle} \leq \sigma_{\mathbf{m}'}\left(1-k\tau^2\right)^{-(\langle\mathbf{m}'\rangle/2)}. \tag{4.16}$$

For Case (i), we substitute $t_u = \tau$ where $l+1 \le u \le k$), for the elements of the product on the right-hand side of (4.9). Using (4.7), (4.10), (4.11), (4.12), (4.14), (4.15), and (4.16) then leads to the following relations:

$$\begin{aligned}
\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{R}]}\right) &\le N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1} \left\{ \prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s t_s}\right)\right\}\right] \\
&= N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1:0\le t_u\le\tau<t_s\le 1} \left\{ \prod_{s=1}^{l} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s t_s}\right) \prod_{u=l+1}^{k} \omega\left(\mathcal{P}_{u,m_u}, \bar{C}_{R_u t_u}\right)\right\}\right] \\
&\le N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|t'|\le 1} \left\{ \prod_{s=1}^{l} K N_{s,m_s} \left(\sqrt{\alpha} R_s t_s\right)^{m_s} \prod_{u=1+1}^{l} K\left(\sqrt{\alpha} R_u \tau\right)^{m_u}\right\}\right] \\
&\le K^N N_{\mathbf{m}}^2 \sigma_{\mathbf{m}} \left(\sqrt{\alpha} \prod_{s=1}^{k} R_s\right)^{\langle\mathbf{m}\rangle} \tau^{\langle\mathbf{m}\rangle - \langle\mathbf{m}'\rangle} \left[\sup_{|\mathbf{t}'|\le 1} \mathbf{t}'^{\mathbf{m}'}\right] \\
&\le K^N N_{\mathbf{m}}^2 \alpha^{(\langle\mathbf{m}\rangle/2)+(\langle\mathbf{m}'\rangle/2)} \left(\prod_{s=1}^{k} R_s\right)^{\langle\mathbf{m}\rangle} \\
&\le K^N N_{\mathbf{m}}^2 \left(\alpha \prod_{s=1}^{k} R_s\right)^{\langle\mathbf{m}\rangle}.
\end{aligned}$$

For Case (ii), we use (2.5) and (4.11) to get

$$\begin{aligned}
\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{R}]}\right) &\le N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1} \left\{ \prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s t_s}\right)\right\}\right] \\
&\le N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1} \left\{ \prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s}\tau\right)\right\}\right] \\
&\le N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \prod_{s=1}^{k} K\left(\sqrt{\alpha} R_s \tau\right)^{m_s} \\
&\le K^N N_{\mathbf{m}}\sigma_{\mathbf{m}} \tau^{\langle\mathbf{m}\rangle} \left(\sqrt{\alpha} \prod_{s=1}^{k} R_s\right)^{\langle\mathbf{m}\rangle} \\
&\le K^N N_{\mathbf{m}} \left(\alpha \prod_{s=1}^{k} R_s\right)^{\langle\mathbf{m}\rangle}.
\end{aligned}$$

For Case (iii), we appeal to the relations (2.5), (4.7), and (4.12) to obtain

$$\begin{aligned}
\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{R}]}\right) &\le N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1} \left\{ \prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s t_s}\right)\right\}\right] \\
&\le N_{\mathbf{m}}\sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle\mathbf{m}\rangle - m_s} \left[\sup_{|\mathbf{t}|=1} \left\{ \prod_{s=1}^{k} K N_{s,m_s} \left(\sqrt{\alpha} R_s t_s\right)^{m_s}\right\}\right]
\end{aligned}$$

$$\leq K^N N_{\mathbf{m}}^2 \sigma_{\mathbf{m}} \left( \sqrt{\alpha} \prod_{s=1}^{k} R_s \right)^{\langle \mathbf{m} \rangle} \left[ \sup_{|\mathbf{t}| \leq 1} \mathbf{t}^{\mathbf{m}} \right]$$
$$\leq K^N N_{\mathbf{m}}^2 \left( \alpha \prod_{s=1}^{k} R_s \right)^{\langle \mathbf{m} \rangle}.$$

Therefore, for (i), (ii), and (iii), we have

$$\Omega\left(\mathcal{K}, \bar{E}_{[\mathbf{R}]}\right) = \limsup_{\langle \mathbf{m} \rangle \to \infty} \left\{\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{R}]}\right)\right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \alpha \prod_{s=1}^{k} R_s. \tag{4.17}$$

Moreover, using (4.2), it follows that

$$\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{R}]}\right) \geq \sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m} \rangle - m_s} \sup_{\bar{E}_{[\mathbf{R}]}} |\mathbf{z}^{\mathbf{m}}| = \left( \prod_{s=1}^{k} R_s \right)^{\langle \mathbf{m} \rangle},$$

which means that

$$\Omega\left(\mathcal{K}, \bar{E}_{[\mathbf{R}]}\right) \geq \prod_{s=1}^{k} R_s. \tag{4.18}$$

Since $\alpha$ can be chosen arbitrarily near to one, then (4.17) and (4.18), imply that $\Omega\left(\mathcal{K}, \bar{E}_{[\mathbf{R}]}\right) = \prod_{s=1}^{k} R_s$, and the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ is effective on $\bar{E}_{[\mathbf{R}]}$.

For the converse, suppose that the base $\{\mathcal{P}_{1,m_1}(z_1)\}$, for example, is not effective in $\bar{C}_{R_1}$. Taking the values $m_1 = m, t_1 = 1, m_s = t_s = 0; 2 \leq s \leq k$ on the left-hand side of (4.9), then using (2.5), (2.6), and (4.6), we have

$$\begin{aligned}
\Omega\left(\mathcal{K}, \bar{E}_{[\mathbf{R}]}\right) &= \limsup_{\langle \mathbf{m} \rangle \to \infty} \left\{\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{R}]}\right)\right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \\
&\geq \limsup_{\langle \mathbf{m} \rangle \to \infty} \left\{ \sigma_{\mathbf{m}} \prod_{s=1}^{k} \{R_s\}^{\langle \mathbf{m} \rangle - m_s} \left[ \sup_{|\mathbf{t}|=1} \left\{ \prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{R_s t_s}\right) \right\} \right] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \\
&\geq \limsup_{m \to \infty} \left\{ \sigma_{m,0,0\ldots,0} \prod_{s=1}^{k} \{R_s\}^{m} \omega\left(\mathcal{P}_{1,m}, \bar{C}_{R_1}\right) \prod_{s=2}^{k} \omega\left(\mathcal{P}_{s,0}, 0\right) \right\}^{\frac{1}{m}} \\
&= \prod_{s=2}^{k} R_s \omega\left(\mathcal{P}_1, \bar{C}_{R_1}\right) > \prod_{s=1}^{k} R_s.
\end{aligned} \tag{4.19}$$

Therefore, the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ is not effective in $\bar{E}_{[\mathbf{R}]}$. Moreover, let $\mu$ be any positive integer and suppose that the base $\{\mathcal{P}_{1,m_1}(z_1)\}$ is not effective in $\bar{C}_{\frac{R_1}{\sqrt{1+\mu}}}$. Then there is a number $\beta > 1$ and a sequence $\left(n_j\right)$ of positive integers such that

$$\omega\left(\mathcal{P}_{1,n_j}, \bar{C}_{\frac{R_1}{\sqrt{1+\mu}}}\right) > \left\{ \frac{\beta R_1}{\sqrt{1+\mu}} \right\}^{n_j}; j \geq 1. \tag{4.20}$$

Observe that we take $m_1 = m, m_2 = \mu m, t_1 = (1+\mu)^{\frac{-1}{2}}, t_2 = \left(\frac{\mu}{1+\mu}\right)^{\frac{1}{2}}, m_s = t_s = 0; 3 \le s \le k$ in (4.9). This is always possible since $K \ge 2$. In this case, as in [29], we deduce that

$$\sigma_{\mathbf{m}} = \sigma_{m,\mu m,0,\ldots,0} = \frac{(1+\mu)^{\frac{1}{2}(1+\mu)m}}{\mu^{\frac{1}{2}\mu m}}. \tag{4.21}$$

A combination of (2.4), (4.9), (4.18), and (4.20) yields

$$\begin{aligned}
\Omega\left(\mathcal{K}, \bar{E}_{[\mathbf{R}]}\right) &= \limsup_{\langle \mathbf{m}\rangle \to \infty} \left\{\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\mathbf{R}]}\right)\right\}^{\frac{1}{\langle \mathbf{m}\rangle}} \\
&\ge \limsup_{m\to\infty} \left\{\Omega\left(\mathcal{K}_{m,\mu m,0,\ldots,0}, \bar{E}_{[\mathbf{R}]}\right)\right\}^{\frac{1}{(1+\mu)m}} \\
&\ge \limsup_{m\to\infty} \left\{\sigma_{m,\mu m,0,\ldots,0} R_1^m R_2^{\mu m} \prod_{s=3}^{k} (R_s)^{(1+\mu)m}\right. \\
&\quad \left.\omega\left(\mathcal{P}_{1,m}, \bar{C}_{\frac{R_1}{\sqrt{1+\mu}}}\right) \omega\left(\mathcal{P}_{2,\mu m}, \bar{C}_{R_2}\sqrt{\frac{\mu}{1+\mu}}\right) \prod_{s=3}^{k} \omega\left(\mathcal{P}_{s,0}, 0\right)\right\}^{\frac{1}{(1+\mu)m}} \\
&\ge R_1^{\frac{\mu}{1+\mu}} \prod_{s=2}^{k} (R_s)(1+\mu)^{\frac{1}{2(1+\mu)}} \limsup_{n_j\to\infty} \left\{\omega\left(\mathcal{P}_{1,n_j}, \bar{C}_{\frac{R_1}{\sqrt{1+\mu}}}\right)\right\}^{\frac{1}{(1+\mu)n_j}} \\
&\ge \beta^{\frac{1}{1+\mu}} \prod_{s=1}^{k} R_s \\
&> \prod_{s=1}^{k} R_s.
\end{aligned}$$

Hence, the base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ is not effective in $\bar{E}_{[\mathbf{R}]}$. Therefore, for the base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ to be effective in the hyper-ellipse $\bar{E}_{[\mathbf{R}]}$, the base $\{\mathcal{P}_{1,m_1}(z_1)\}$ should be effective in the circles $\bar{C}_{R_1}$ and $\bar{C}_{\frac{R_1}{\sqrt{1+\mu}}}$, i.e., the base $\{\mathcal{P}_{1,m_1}(z_1)\}$ should be effective in $\bar{C}_{r_1}; \bar{C}_{\frac{R_1}{\sqrt{1+\mu}}} \le r_1 \le R_1$. Since $\mu$ can be chosen arbitrarily to be large, we infer that the base $\{\mathcal{P}_{1,m_1}(z_1)\}$ should be effective in $\bar{C}_{r_1}$ for $0 < r_1 \le R_1$. In the same way, it can be proved that to ensure the effectiveness of the base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ in the hyper-ellipse $\bar{E}_{[\mathbf{R}]}$, each of the constituent bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ should be effective in $\bar{C}_{r_s}$ for $0 < r_s \le R_s$. Theorem 3.1 is therefore established.

□

As an immediate consequence of Theorem 4.1, the following result is yielded.

**Corollary 4.1.** *The Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in the hyper-ellipse $\bar{E}_{[\mathbf{r}]}, 0 < r_s \le R_s$ if and only if each of the constituents bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ is effective in $\bar{C}_{r_s}$ for $0 < r_s \le R_s$, $s \in J$.*

Now, we provide the following examples to illustrate the effectiveness property of the Kronecker product bases of polynomials in closed hyper-ellipses in terms of certain bases of polynomials in closed circles.

**Example 4.1.** *For $s = 1, 2$, the two bases of polynomials $\{\mathcal{P}_{s,m_s}(z_s)\}$ are defined by*

$$\mathcal{P}_{s,m_s}(z_s) = \begin{cases} 1, & m_s = 0 \\ 1 + z_s^{m_s}, & m_s \geq 1. \end{cases}$$

*Then $z_s^{m_s}$ can be written as follows:*

$$z_s^{m_s} = \mathcal{P}_{s,m_s}(z_s) - \mathcal{P}_{s,0}(z_s).$$

*Note that $\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{r_s}\right) = 2 + r_s^{m_s}$ and $\omega\left(\mathcal{P}_s, \bar{C}_{r_s}\right) = r_s \; \forall r_s \geq 1, \; s = 1, 2$. Therefore, the bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ are effective in $\bar{C}_{r_s}$. By using Theorem 4.1, it follows that the Kronecker product base $\{\mathcal{K}_{m_1,m_2}(z_1, z_2)\} = \{\mathcal{P}_{1,m_1}(z_1)\} \otimes \{\mathcal{P}_{2,m_2}(z_2)\}$ is effective in the closed hyper-ellipse $\bar{E}_{r_1,r_2}$ $\forall\, r_1, r_2 \geq 1$.*

**Example 4.2.** *For $s = 1, 2$, the two bases of polynomials $\{\mathcal{P}_{s,m_s}(z_s)\}$ are defined by*

$$\mathcal{P}_{1,m_1}(z_1) = \begin{cases} 1, & m_1 = 0 \\ 1 + (\frac{z_1}{2})^{m_1}, & m_1 \geq 1, \end{cases}$$

*and*

$$\mathcal{P}_{2,m_2}(z_2) = \begin{cases} 1, & m_2 = 0 \\ 1 + (\frac{z_2}{3})^{m_2}, & m_2 \geq 1. \end{cases}$$

*Similar to the previous Example 4.1, we find that the resulting Kronecker product base $\{\mathcal{K}_{m_1,m_2}(z_1, z_2)\}$ is effective in the closed hyper-ellipse $\bar{E}_{r_1,r_2}$ for all $r_1 \geq 2$ and $r_2 \geq 3$.*

## 5. Mode of increase of the Kronecker product bases of polynomials

We determine the order and type of the Kronecker product bases of polynomials $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ in relation to the constituent bases $\{\mathcal{P}_{s,m_s}(z_s)\}$. Suppose that the Cannon bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ are of increasing orders $(\gamma_s)$ and types $(\tau_s)$. Suppose that the $\{\mathcal{P}_{1,m_1}(z_1)\}$ has the greater rate of increase. In other words, we consider either $\gamma_1 > \gamma_2, \gamma_3, \ldots, \gamma_k$ or $\gamma_1 = \gamma_2 = \gamma_3 = \ldots = \gamma_k$ and $\tau_1 > \tau_2, \tau_3, \ldots, \tau_k$, or $\tau_1 = \tau_2 = \tau_3 = \ldots = \tau_k$. We now evaluate the order $\Omega$ and type $\Gamma$ of the Kronecker product bases in terms of the increase mode of the constituents.

**Definition 5.1.** *[28] The order $\Omega$ of the base of polynomials $\mathcal{P}_{\mathbf{m}}[\mathbf{z}]$ in the closed hyper-ellipse $\bar{E}_{[\alpha\mathbf{R}]}$ is given by*

$$\Omega = \lim_{R\to\infty} \limsup_{\langle\mathbf{m}\rangle\to\infty} \frac{\log \Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\alpha\mathbf{R}]}\right)}{\langle\mathbf{m}\rangle \log\langle\mathbf{m}\rangle}.$$

If $0 < \Omega < \infty$, the type $\Gamma$ is given by

$$\Gamma = \lim_{R\to\infty} \frac{e}{\Omega} \limsup_{\langle\mathbf{m}\rangle\to\infty} \frac{\{\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\alpha\mathbf{R}]}\right)\}^{\frac{1}{\Omega\langle\mathbf{m}\rangle}}}{\langle\mathbf{m}\rangle}.$$

**Remark 5.1.** *Note that when the base of polynomials $\mathcal{P}_{\mathbf{m}}[\mathbf{z}]$ is of the finite order $\Omega$ and finite type $\Gamma$, then it represents every entire function of an order less than $\frac{1}{\Omega}$ and a type less than $\frac{1}{\Gamma}$ in any finite hyper-ellipse.*

**Theorem 5.1.** *Let $\{\mathcal{P}_{s,m_s}(z_s)\}$, $s \in J$ be Cannon bases of polynomials of increasing orders $(\gamma_s)$ and types $(\tau_s)$, where $\{\mathcal{P}_{1,m_1}(z_1)\}$ has the greater rate of increase. Then the order $\Omega$ of the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ equals $\gamma_1$, and the type $\Gamma$ is determined as follows:*

*(i) If $\gamma_1 \geq \frac{1}{2}$, then $\Gamma = \tau_1$.*
*(ii) If $\gamma_1 < \frac{1}{2}$, then $\tau_1 \leq \Gamma \leq \tau_1 2^{\frac{1}{2\gamma_1}-1}$.*

*Proof.* From the left-hand side of inequality (4.9), we have

$$\Omega\left(\mathcal{K}_{m,0,\ldots,0}, \bar{E}_{[a\mathbf{R}]}\right) \geq \sigma_{m,0,0\ldots,0} R^{(k-1)m} \prod_{s=1}^{k} \{\alpha_s\}^m \,\omega\left(\mathcal{P}_{1,m}, \bar{C}_{\alpha_1 R}\right) \prod_{s=2}^{k} \omega\left(\mathcal{P}_{s,0}, 0\right). \tag{5.1}$$

Hence,

$$\begin{aligned}\Omega &= \lim_{R\to\infty} \limsup_{\langle\mathbf{m}\rangle\to\infty} \frac{\log \Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\alpha\mathbf{R}]}\right)}{\langle\mathbf{m}\rangle \log\langle\mathbf{m}\rangle} \\ &\geq \lim_{R\to\infty} \limsup_{m\to\infty} \frac{\log \Omega\left(\mathcal{K}_{m,0,\ldots,0}, \bar{E}_{[\alpha\mathbf{R}]}\right)}{m \log m} \\ &\geq \lim_{R\to\infty} \limsup_{m\to\infty} \frac{\log \omega\left(\mathcal{P}_{1,m}, \bar{C}_{\alpha_1 R}\right)}{m \log m} = \gamma_1.\end{aligned} \tag{5.2}$$

From the definition of the number $\sigma_{\mathbf{m}}$, we observe that

$$\sup_{|\mathbf{t}|=1} \mathbf{t}^{\mathbf{m}} = \frac{1}{\sigma_{\mathbf{m}}}.$$

By taking $t_i = k^{-\frac{1}{2}}$, it readily follows that

$$\sigma_{\mathbf{m}} \leq k^{-\frac{1}{2}\langle\mathbf{m}\rangle}. \tag{5.3}$$

Now, if $\gamma_1 = \infty$, there is nothing to prove. So, for $\gamma_1 < \infty$, let $\gamma$ be any finite number greater than $\gamma_1$; we then have

$$\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{\alpha_s R}\right) < K m_s^{\gamma m_s}, \quad (m_s \geq 1, s \in J). \tag{5.4}$$

Introducing (5.3) and (5.4) on the right-hand side of (4.9) , we obtain

$$\begin{aligned}\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\alpha\mathbf{R}]}\right) &\leq N_{\mathbf{m}} \sigma_{\mathbf{m}} R^{(k-1)\langle\mathbf{m}\rangle} \prod_{s=1}^{k} \{\alpha_s\}^{\langle\mathbf{m}\rangle - m_s} \prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{\alpha_s R}\right) \\ &\leq N_{\mathbf{m}} \sigma_{\mathbf{m}} (\alpha R)^{(k-1)\langle\mathbf{m}\rangle} \prod_{s=1}^{k} \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{\alpha_s R}\right) \\ &< K N_{\mathbf{m}} k^{-\frac{1}{2}\langle\mathbf{m}\rangle} (\alpha R)^{(k-1)\langle\mathbf{m}\rangle} \{\langle\mathbf{m}\rangle\}^{\gamma\langle\mathbf{m}\rangle}\end{aligned} \tag{5.5}$$

where $\alpha = \max_{1\leq s\leq k} \alpha_s$. Hence, as $\langle\mathbf{m}\rangle$ tends to infinity, (5.5) yields

$$\lim_{R\to\infty}\limsup_{\langle\mathbf{m}\rangle\to\infty}\frac{\log\Omega\left(\mathcal{K}_{\mathbf{m}},\bar{E}_{[\alpha\mathbf{R}]}\right)}{\langle\mathbf{m}\rangle\log\langle\mathbf{m}\rangle}\leq\gamma.$$

Consequently, we deduce that $\Omega \leq \gamma$, and as $\gamma$ can be arbitrarily chosen near to $\gamma_1$, it follows that $\Omega \leq \gamma_1$. The inequality (5.2) thus implies that $\Omega = \gamma_1$, as required.

The proof is now extended to evaluate the type $\Gamma$ of the Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$. As in the case of the order, we first obtain a lower bound by appealing to the relation (5.1). In fact,by virtue of this relation, it can be easily verified that

$$\begin{aligned}\Gamma &= \lim_{R\to\infty}\frac{e}{\Omega}\limsup_{\langle\mathbf{m}\rangle\to\infty}\frac{\{\Omega\left(\mathcal{K}_{\mathbf{m}},\bar{E}_{[\alpha\mathbf{R}]}\right)\}^{\frac{1}{\langle\mathbf{m}\rangle\Omega}}}{\langle\mathbf{m}\rangle}\\ &\geq \lim_{R\to\infty}\frac{e}{\gamma_1}\limsup_{m\to\infty}\frac{\{\omega\left(\mathcal{P}_{1,m},\bar{C}_{\alpha_1 R}\right)\}^{\frac{1}{\Omega m}}}{m}=\tau_1.\end{aligned}\tag{5.6}$$

Now, since the order $\gamma_1$ and type $\tau_1$ are the greater increase, unless $\tau_1$ is infinite (in which case, the proof is terminated), given any number $\tau > \tau_1$, we obtain

$$\omega\left(\mathcal{P}_{s,m_s},\bar{C}_{\alpha_s R}\right) < K(\frac{\tau\gamma_1 m_s}{e})^{\gamma_1 m_s},\quad (m_s\geq 1, s\in J)\,.\tag{5.7}$$

The right-hand inequality of (4.9) and the definition of $\sigma_{\mathbf{m}}$ as given in (2.6) are applied to (5.7) to get

$$\Omega\left(\mathcal{K}_{\mathbf{m}},\bar{E}_{[\alpha\mathbf{R}]}\right) < KN_{\mathbf{m}}(\alpha R)^{(k-1)\langle\mathbf{m}\rangle}(\frac{\tau\gamma_1\langle\mathbf{m}\rangle}{e})^{\gamma_1\langle\mathbf{m}\rangle}\prod_{s=1}^{k}\{\frac{m_s}{\langle\mathbf{m}\rangle}\}^{(\gamma_1-\frac{1}{2})m_s}.\tag{5.8}$$

Suppose now that $\gamma_1 \geq \frac{1}{2}$. In this case, (5.8) implies that

$$\begin{aligned}\Gamma &= \lim_{R\to\infty}\frac{e}{\Omega}\limsup_{\langle\mathbf{m}\rangle\to\infty}\frac{\{\Omega\left(\mathcal{K}_{\mathbf{m}},\bar{E}_{[\alpha\mathbf{R}]}\right)\}^{\frac{1}{\Omega\langle\mathbf{m}\rangle}}}{\langle\mathbf{m}\rangle}\\ &\leq \lim_{R\to\infty}\frac{e}{\gamma_1}[KN_{\mathbf{m}}(\alpha R)^{(k-1)\langle\mathbf{m}\rangle}(\frac{\tau\gamma_1\langle\mathbf{m}\rangle}{e})^{\gamma_1\langle\mathbf{m}\rangle}]^{\frac{1}{\gamma_1\langle\mathbf{m}\rangle}}\\ &= \tau,\end{aligned}$$

and as $\tau$ is chosen as near to $\tau_1$ as we please, we infer that $\Gamma \leq \tau_1$. Hence, by (5.6), we conclude that $\Gamma = \tau_1$, and the first assertion of the theorem concerning the type is established.

If, however, $\gamma_1 < \frac{1}{2}$, the extreme terms in (5.8), which were dispensed with in the case where $\gamma_1 \geq \frac{1}{2}$, will now be taken into account. In fact, we have

$$\prod_{s=1}^{k}\{\frac{\langle\mathbf{m}\rangle}{m_s}\}^{(\frac{1}{2}-\gamma_1)m_s}\leq 2^{(\frac{1}{2}-\gamma_1)\langle\mathbf{m}\rangle}$$

and hence (5.8) yields that

$$\Gamma = \lim_{R\to\infty} \frac{e}{\Omega} \limsup_{\langle \mathbf{m}\rangle\to\infty} \frac{\{\Omega\left(\mathcal{K}_{\mathbf{m}}, \bar{E}_{[\alpha\mathbf{R}]}\right)\}^{\frac{1}{\Omega\langle \mathbf{m}\rangle}}}{\langle \mathbf{m}\rangle}$$
$$\leq \lim_{R\to\infty} \frac{e}{\gamma_1}[KN_{\mathbf{m}}(\alpha R)^{(k-1)\langle \mathbf{m}\rangle}(\frac{\tau\gamma_1\langle \mathbf{m}\rangle}{e})^{\gamma_1\langle \mathbf{m}\rangle}\ 2^{(\frac{1}{2}-\gamma_1)\langle \mathbf{m}\rangle}]^{\frac{1}{\gamma_1\langle \mathbf{m}\rangle}}$$
$$= \tau 2^{\frac{1}{2\gamma_1}-1}.$$

The procedure followed above leads to the inequality

$$\Gamma \leq \tau_1 2^{\frac{1}{2\gamma_1}-1}.$$

By virtue of (5.6), the double inequality

$$\tau_1 \leq \Gamma \leq \tau_1 2^{\frac{1}{2\gamma_1}-1} \tag{5.9}$$

is proved, and the proof of Theorem 5.1 is completed. □

The following example shows that the upper bound of (5.9) can be attained.

**Example 5.1.** *Consider the base of polynomials* $\{\mathcal{P}_{s,m_s}(z_s)\}$, $s = 1, 2$ *given by*

$$\begin{cases} \mathcal{P}_{s,0}(z_s) & = 1 \\ \mathcal{P}_{s,m_s}(z_s) & = (m_s!)^{\frac{1}{4}} + z_s^{m_s}, (m_s \geq 1,\ s = 1, 2). \end{cases}$$

*Note that the simple base* $\{\mathcal{P}_{s,m_s}(z_s)\}$, $s = 1, 2$ *is of the order* $\gamma_1 = \frac{1}{4}$ *and the type* $\tau_1 = 4$. *By applying the double suffix notation, the Kronecker product base* $\{\mathcal{K}_{m_1,m_2}(z_1, z_2)\} = \{\mathcal{P}_{1,m_1}(z_1)\} \otimes \{\mathcal{P}_{2,m_2}(z_2)\}$ *is constructed in the following manner:*

$$\mathcal{K}_{0,0}(z_1, z_2) = 1$$
$$\mathcal{K}_{0,m_2}(z_1, z_2) = \mathcal{P}_{1,0}(z_1) \otimes \mathcal{P}_{2,m_2}(z_2) = (m_2!)^{\frac{1}{4}} + z_2^{m_2}, (m_2 \geq 1)$$
$$\mathcal{K}_{m_1,0}(z_1, z_2) = \mathcal{P}_{1,m_1}(z_1) \otimes \mathcal{P}_{2,0}(z_2) = (m_1!)^{\frac{1}{4}} + z_1^{m_1}, (m_1 \geq 1)$$

*Therefore,*

$$\{\mathcal{K}_{m_1,m_2}(z_1, z_2)\} = \{\mathcal{P}_{m_1}(z_1)\} \otimes \{\mathcal{P}_{m_2}(z_2)\} = \{(m_1!)^{\frac{1}{4}} + z_1^{m_1}\} \otimes \{(m_2!)^{\frac{1}{4}} + z_2^{m_2}\}.$$

*Easy calculations show that* $\Omega = \frac{1}{4}$ *and the type* $\Gamma = 8$.

In Example 5.1, we took the two coincident constituent bases and this yields the upper bound in (5.9). In the following, we find the order and type of the Kronecker product bases of given bases.

**Example 5.2.** *Suppose that the base of polynomials* $\{\mathcal{P}_{s,m_s}(z_s)\}$ *where* $s = 1, 2$ *is defined by*

$$\mathcal{P}_{s,m_s}(z_s) = z_s^{m_s} + m_s^{m_s},\ \mathcal{P}_{s,0}(z_s) = 1.$$

*Note that*

$$\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_{r_s}\right) = m_s^{m_s}\left[2 + \left(\frac{R_s}{m_s}\right)^{m_s}\right].$$

*Direct calculations implies that the base* $\{\mathcal{P}_{s,m_s}(z_s)\}$ *is of the order* $\gamma_s = 1$ *and the type* $\tau_s = e$, $s = 1, 2$. *Consequently, the Kronecker product base* $\{\mathcal{K}_{m_1,m_2}(z_1, z_2)\} = \{\mathcal{P}_{1,m_1}(z_1)\} \otimes \{\mathcal{P}_{2,m_2}(z_2)\}$ *is of the order* $\Omega = 1$ *and the type* $\Gamma = e$.

## 6. The $\mathbb{T}_\rho$-property of the Kronecker product bases of polynomials

In this section, we discuss the $\mathbb{T}_\rho$-property of Kronecker product base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ in the closed hyperellipse $\bar{E}_{[\mathbf{r}]}$.

Now, we define the $\mathbb{T}_\rho$-property of the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ in $\bar{E}_{[\mathbf{r}]}$ as follows:

**Definition 6.1.** *If the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ represents all entire functions of order less than $\rho$ in $\bar{E}_{[\mathbf{r}]}$, then it is said to have the property $\mathbb{T}_\rho$ in $\bar{E}_{[\mathbf{r}]}$.*

Let

$$\Omega(\mathcal{P}, \bar{E}_{[\mathbf{r}]}) = \limsup_{\langle \mathbf{m}\rangle\to\infty} \frac{\log\ \Omega(\mathcal{P}_{\mathbf{m}}, \bar{E}_{[\mathbf{r}]})}{\langle \mathbf{m}\rangle \log\langle \mathbf{m}\rangle}.$$

The following theorem concerns the property $\mathbb{T}_\rho$ of the base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$.

**Theorem 6.1.** *A base $\{\mathcal{P}_{\mathbf{m}}[\mathbf{z}]\}$ to have the property $\mathbb{T}_\rho$ for all entire functions of an order less than $\rho$ in $\bar{E}_{[\mathbf{r}]}$ if and only if $\Omega(\mathcal{P}, \bar{E}_{[\mathbf{r}]}) \leq \frac{1}{\rho}$.*

*Proof.* The proof can be carried out in a very similar way to the case of complete Reinhardt domains (polycylinderical regions) (see [27]); therefore, it is omitted. □

Next, we construct the $\mathbb{T}_\rho$-property of the Kronecker product bases in $\bar{E}_{[\mathbf{r}]}$ in terms of the $\mathbb{T}_\rho$-property of their constituents in $\bar{C}_{r_s}$.

**Theorem 6.2.** *Let $\{\mathcal{P}_{s,m_s}(z_s)\}$, $s \in J$ be the base of polynomials and suppose that $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ is their Kronecker product bases. Then the base $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ has the $T_\rho$-property in $\bar{E}_{[\mathbf{r}]}$ if and only if the bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ have the property $\mathbb{T}_{\rho_s}$ in $\bar{C}_{r_s}$, where $s \in J$ and $\rho = \min\{\rho_s,\ s \in J\}$.*

*Proof.* Suppose that the bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ has the property $\mathbb{T}_{\rho_s}$, $s \in J$ in $\bar{C}_{r_s}$. According to Theorem 6.1, we have

$$\omega(\mathcal{P}_s, \bar{C}_{r_s}) \leq \frac{1}{\rho_s}, \tag{6.1}$$

where

$$\omega(\mathcal{P}_s, \bar{C}_{r_s}) = \limsup_{m_s\to\infty} \frac{\log\ \omega(\mathcal{P}_{s,m_s}, \bar{C}_{r_s})}{m_s \log m_s}. \tag{6.2}$$

If $\rho^{'} < \min\{\rho_s\}$, then from (6.1) and (6.2), we get

$$\omega(\mathcal{P}_{s,m_s}, \bar{C}_{r_s}) < k_s\{m_s\}^{\frac{m_s}{\rho^{'}}}, \tag{6.3}$$

where $k_s > 1$ are constants. Introducing (6.3) in (4.9), it follows that

$$\Omega(\mathcal{P}_{\mathbf{m}}, \bar{E}_{[\mathbf{r}]}) < \prod_{s=1}^{k} k_s\{m_s\}^{\frac{m_s}{\rho^{'}}} < K\{\langle \mathbf{m}\rangle\}^{\frac{\langle \mathbf{m}\rangle}{\rho^{'}}},$$

where $K = \max\{k_s\}$, $s \in J$. Hence, as $\langle \mathbf{m}\rangle \to \infty$, we obtain

$$\Omega(\mathcal{P},\bar{E}_{[\mathbf{r}]})=\limsup_{\langle\mathbf{m}\rangle\to\infty}\frac{\log\ \Omega(\mathcal{P}_{\mathbf{m}},\bar{E}_{[\mathbf{r}]})}{\langle\mathbf{m}\rangle\log\langle\mathbf{m}\rangle}\leq\frac{1}{\rho'}.$$

Since, $\rho'$ can be chosen arbitrarily near to $\rho$, then

$$\Omega(\mathcal{P},\bar{E}_{[\mathbf{r}]})\leq\frac{1}{\rho}.$$

In view of Theorem 6.1, the Kronecker product bases $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ has the $T_\rho$-property in $\bar{E}_{[\mathbf{r}]}$. To complete the proof, suppose that the base $\{\mathcal{P}_{1,m_1}(z_1)\}$ for example, does not have the property $\mathbb{T}_{\rho_1}$, in $\bar{C}_{r_1}$. In that case

$$\omega(\mathcal{P}_1,\bar{C}_{r_1})>\frac{1}{\rho_1}.$$

Hence,

$$\begin{aligned}\Omega(\mathcal{P},\bar{E}_{[\mathbf{r}]})&=\limsup_{\langle\mathbf{m}\rangle\to\infty}\frac{\log\ \Omega(\mathcal{P}_{\mathbf{m}},\bar{E}_{[\mathbf{r}]})}{\langle\mathbf{m}\rangle\log\langle\mathbf{m}\rangle}\\&\geq\limsup_{m_1\to\infty}\frac{\log\ \omega(\mathcal{P}_1,\bar{C}_{r_1})}{m_1\log m_1}\\&=\omega\left(\mathcal{P}_1,\bar{C}_{R_1}\right)>\frac{1}{\rho_1}.\end{aligned}$$

Therefore, according to Theorem 6.1, the Kronecker product bases $\{\mathcal{K}_{\mathbf{m}}[\mathbf{z}]\}$ cannot have the $T_{\rho_1}$-property in $\bar{E}_{[\mathbf{r}]}$, for any $\rho\leq\rho_1$ Hence in the case where $\rho_s\geq\rho_1$, the Kronecker product bases cannot have the property $T_\rho$, where $\rho=\min\{\rho_s,\ s\in J\}$. In the case where $\rho_s>\rho_1$, we have $\rho=\min\{\rho_s,\ s\in J\}=\rho_2$, and hence the Kronecker product bases cannot have the $T_\rho$-property in $\bar{E}_{[\mathbf{r}]}$. Thus Theorem 6.2 is completely established. □

The following example calculates the $\mathbb{T}_\rho$-property of the Kronecker product base of polynomials in $\bar{E}_{[\mathbf{r}]}$.

**Example 6.1.** *Suppose that* $\{\mathcal{P}_{s,m_s}(z_s)\}$ ; $s=1,2$ *are defined by:*

$$\mathcal{P}_{s,m_s}(z_s)=\begin{cases}z_s^{m_s}, & m_s\text{ is even,}\\ z_s^{m_s}+\frac{z_s^{t(m_s)}}{2^{m_s^{m_s}}}, & m_s\text{ is odd.}\end{cases}$$

*where* $t(m_s)$ *is the nearest even integer to* $m_s\log m_s+m_s^{m_s},\ s=1,2.$

When $m_s$ is odd, we obtain:

$$z_s^{m_s}=\mathcal{P}_{s,m_s}(z_s)-\frac{\mathcal{P}_{s,t(m_s)}(z_s)}{2^{m_s^{m_s}}}.$$

Hence,

$$\omega\left(\mathcal{P}_{s,m_s},\bar{C}_{r_s}\right)=r_s^{m_s}+2\frac{r_s^{t(m_s)}}{2^{m_s^{m_s}}}.$$

Putting $r_s = 2$, it follows that

$$\omega\left(\mathcal{P}_{s,m_s}, \bar{C}_2\right) = 2^{m_s} + 2^{m_s \log m_s + 1},$$

thus

$$\omega\left(\mathcal{P}_s, \bar{C}_2\right) = \limsup_{m_s \to \infty} \frac{log\ \omega\left(\mathcal{P}_{s,m_s}, \bar{C}_2\right)}{m_s\ log\ m_s} \leq \log 2.$$

Therefore, the base $\{\mathcal{P}_{s,m_s}(z_s)\}$ has the $\mathbb{T}_{\frac{1}{\log 2}}$-property in $\overline{C}_2$. Applying Theorem 6.2, the Kronecker product base $\{\mathcal{K}_{m_1,m_2}(z_1, z_2)\}$ has the $\mathbb{T}_{\frac{1}{\log 2}}$-property in the closed equi-hyper-ellipse $\bar{E}_{2,2}$.

## 7. Applications involving special polynomials

Orthogonal polynomials such as the Legendre, Chebyshev, Hermite, Bernoulli, Euler, and Bessel polynomials play a significant role in the approximation theory, particularly when it comes to expressing and approximating functions using polynomial series.

### *7.1. Effectiveness of the Kronecker product bases involving certain special polynomials*

To show the validity of our main results concerning effectiveness, we begin by considering simple bases of the proper and general Bessel polynomials $\{\mathcal{P}_{s,m_s}(z_s)\}$ and $\{Q_{s,m_s}(z_s)\}$, given as follows:

$$\begin{cases} \mathcal{P}_{s,0}(z_s) & = 1, \\ \mathcal{P}_{s,m_s}(z_s) & = \sum_{h_s=0}^{m_s} \frac{(m_s+h_s)!}{h_s!(m_s+h_s)!}\left(\frac{z_s}{2}\right)^{h_s}, \quad m_s \geq 1,\ \forall s \in J \end{cases}$$

and

$$\begin{cases} Q_{s,0}(z_s) & = 1, \\ Q_{s,m_n}(z_s) & = 1 + \sum_{h_s=1}^{m_s} \frac{m_s!\prod_{j_s=1}^{h_s}(m_s+j_s+a_s-2)}{h_s!(m_s+h_s)!}\left(\frac{z_s}{b_s}\right)^{h_s}, m_s \geq 1,\ \forall s \in J, \end{cases}$$

where $a_s$ and $b_s$ are given numbers. The authors of [43, 44] proved that both the bases $\{\mathcal{P}_{s,m_s}(z_s)\}$ and $\{Q_{s,m_s}(z_s)\}$ are effective in $\bar{C}_{r_s}$ for all $r_s > 0$. By applying Theorem 4.1, the resulting Kronecker product bases

$$\left\{\mathcal{K}_{\mathbf{m}}^{(\mathcal{P})}[\mathbf{z}]\right\} = \bigotimes_{s=1}^{k} \{\mathcal{P}_{s,m_s}(z_s)\}$$

and

$$\left\{\mathcal{K}_{\mathbf{m}}^{(Q)}[\mathbf{z}]\right\} = \bigotimes_{s=1}^{k} \{Q_{s,m_s}(z_s)\}$$

are effective in the closed hyper-ellipse $\bar{E}_{[\mathbf{r}]}$ for all $r_s > 0, s \in J$.

Recently, in [45], the Chebyshev polynomials

$$\begin{cases} T_{s,0}(z_s) & = 1, \quad m_s = 0\ \forall s \in J \\ T_{s,m_s}(z_s) & = \sum_{h_s=0}^{\left[\frac{m_s}{2}\right]} \frac{(m_s!}{2h_s!(m_s-2h_s)!} z_s^{m_s-2h_s}\left(z_s^2 - 1\right)^{h_s}, \quad m_s \geq 1,\ \forall s \in J \end{cases}$$

was proved to be effective in $\bar{C}_1$. As an immediate consequence of Theorem 4.1, the Kronecker product bases $\left\{\mathcal{K}_{\mathbf{m}}^{(T)}[\mathbf{z}]\right\}$ given by

$$\left\{\mathcal{K}_{\mathbf{m}}^{(T)}[\mathbf{z}]\right\} = \bigotimes_{s=1}^{k} \{T_{s,m_s}(z_s)\}$$

is effective in the closed equi-hyper-ellipse $\bar{E}_{[\mathbf{1}]}$.

Now, the base of Gontcharaff polynomials $\{\mathcal{G}_{s,m_s}(z_s)\}$, associated with a given set $(a_s t_s^{m_s})$ of points as introduced in [42], is defined in the form

$$\begin{cases} \mathcal{G}_{s,0}(z_s) = 1, \\ \mathcal{G}_{s,m_s}(z_s; a_s, a_s t_s, a_s t_s^2, \ldots, a_s t_s^{m_s-1}) = \int_{a_s}^{z_s} dl_1^s \int_{a_s t_s}^{l_1^s} dl_2^s \int_{a_s t_s^2}^{l_2^s} dl_3^s \ldots \int_{a_s t_s^{m_s-1}}^{l_{m_s-1}^s} dl_{m_s}^s, \ m_s \geq 1 \end{cases}$$

where $a_s$ and $t_s$ are given complex numbers. For the case when $|t_s| < 1$, the authors of [42] proved that the Gontcharaff polynomials $\{\mathcal{G}_{s,m_s}(z_s)\}$ are effective in $\bar{C}_{r_s}$ for all $r_s \geq |a_s|$. Applying Theorem 4.1, we conclude that the Kronecker product bases

$$\left\{\mathcal{K}_{\mathbf{m}}^{(\mathcal{G})}[\mathbf{z}]\right\} = \bigotimes_{s=1}^{k} \{\mathcal{G}_{s,m_s}(z_s)\}$$

is effective in the closed hyper-ellipse $\bar{E}_{[\mathbf{r}]}$ for all $r_s \geq |a_s|$.

### *7.2. Mode of increase and the $\mathbb{T}_\rho$-property of the Kronecker product bases involving certain special polynomials*

Regarding the mode of increase of the constructed Kronecker product bases, we consider the simple bases of the Bernoulli polynomials $\{B_{s,m_s}(z_s)\}$ and the Euler polynomials $\{E_{s,m_s}(z_s)\}$ given by

$$B_{s,m_s}(z_s) = \sum_{h_s=0}^{m_s} \binom{m_s}{h_s} B_{m_s-h_s}\, z_s^{h_s}$$

and

$$E_{s,m_s}(z_s) = \sum_{h_s=0}^{m_s} \binom{m_s}{h_s} E_{m_s-h_s} \left(z_s - \frac{1}{2}\right)^{h_s}.$$

Recently, the authors of [46] proved that the Bernoulli polynomials $\{B_{s,m_s}(z)\}$ are of the order 1 and type $\frac{1}{2\pi}$, and the Euler polynomials $\{E_{s,m_s}(z)\}$ are of order 1 and type $\frac{1}{\pi}$. Applying Theorem 5.1, we deduce that the Kronecker product bases of Bernoulli polynomials

$$\left\{\mathcal{K}_{\mathbf{m}}^{(B)}[\mathbf{z}]\right\} = \bigotimes_{s=1}^{k} \{B_{s,m_s}(z_s)\}$$

are of order 1 and type $\frac{1}{2\pi}$, and the Kronecker product bases of the Euler polynomials

$$\left\{\mathcal{K}_{\mathbf{m}}^{(E)}[\mathbf{z}]\right\} = \bigotimes_{s=1}^{k} \{E_{s,m_s}(z_s)\}$$

are of order 1 and type $\frac{1}{\pi}$.

Furthermore, the author of [47] concluded that when $|t_s| = 1$, the previously mentioned Gontcharaff polynomials $\{\mathcal{G}_{s,m_s}(z_s)\}$ have order 1 and type $\frac{|a|}{\mu}$, where $\mu$ is the modulus of a zero of the function

$$f(z_s) = \sum_{m_s=0}^{\infty} t_s^{\frac{m_s(m_s-1)}{2}} \frac{z_s^{m_s}}{m_s!}$$

of the least modulus. Consequently, by applying Theorem 5.1, the Kronecker product bases of Gontcharaff polynomials are of order 1 and type $\frac{|a|}{\mu}$.

Additionally, as an application of Theorem 6.2, the recent results in [46] indicated that the Bernoulli and Euler polynomials have the property $\mathbb{T}_1$. Moreover, the author of [47] proved that the Gontcharaff polynomials have the property $\mathbb{T}_1$. Applying Theorem 6.2, we conclude that the order of the Kronecker product bases of Bernoulli, Euler, and Gontcharaff polynomials have also the property $\mathbb{T}_1$.

**Remark 7.1.**

1) *Similar results for the Kronecker product bases $\{\mathcal{K}_{\boldsymbol{m}}[z]\}$ in hyper-elliptical regions can be obtained when the constituents bases are taken to be general bases.*
2) *To get the analogous results concerning the convergence properties in hyper-spherical regions [29, 48], we put $R_1 = R_1 = \ldots = R_k = R$.*

## 8. Conclusions

The present paper is a study of convergence properties (effectiveness, order, type, and $T_\rho$-property) of a new kind of bases of polynomials in multi-dimensional polynomial approximation, namely Kronecker product bases. The region of representation are hyper-elliptical regions. Some applications on these bases of the problem of classical special functions such as Bessel, Chebyshev, Bernoulli, Euler, and Gontcharaff polynomials have been studied. The results obtained are natural generalizations of the results obtained in hyper-spherical regions.

## Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

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