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*Expository*

# Number theoretic subsets of the real line of full or null measure

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**Abstract:** During a first or second course in number theory, students soon encounter several sets of "number theoretic interest". These include basic sets such as the rational numbers, algebraic numbers, transcendental numbers, and Liouville numbers, as well as more exotic sets such as the constructible numbers, normal numbers, computable numbers, badly approximable numbers, the Mahler sets S, T and U, and sets of irrationality exponent $m$, among others. Those exposed to some measure theory soon make a curious observation regarding a common property seemingly shared by all these sets: each of the sets has Lebesgue measure equal to zero, or its complement has Lebesgue measure equal to zero. In this expository note, we explain this phenomenon.

**Keywords:** full measure; null measure; Lebesgue measure; real line; transcendental numbers; Liouville numbers

## 1. Introduction

When you study number theory you meet many sets of numbers, such as

(i) the set of all rational numbers;
(ii) the set of all algebraic numbers [1];
(iii) the set of all transcendental numbers [1];
(iv) the set of all Liouville numbers [1];
(v) the set of all constructible numbers [2];
(vi) the set of all numbers normal to some base $b \in \mathbb{N}, b > 2$ [3,4];
(vii) the set of normal numbers [3,4];

(viii) the set of absolutely normal numbers [3,4];
(ix) the set of computable numbers [5];
(x) the set of badly approximable numbers [4];
(xi) the set of numbers in the Mahler set $S$ [4,6];
(xii) the set of numbers in the Mahler set $T$ [4,6];
(xiii) the set of numbers in the Mahler set $U$ [4,6];
(xiv) the set of numbers of irrationality exponent $m$, where $m$ is some positive real number [4,7];

and it may seem astonishing that each of these sets has Lebesgue measure equal to zero or its complement has Lebesgue measure equal to zero; that is, it is a set of *null measure* or a set of *full measure*.

While you may not be familiar with all of these sets, this observation should still surprise and interest you. The explanation is given in the next section.

## 2. The key observation

In this section, we state the key observation, Theorem 2.1, which is an immediate consequence of a result in ergodic theory.

We shall denote by $\mathbb{R}$ the set of all real numbers with its usual group structure and topology; by $\mathbb{Q}$ the topological subgroup of $\mathbb{R}$ of all rational numbers; by $\mathbb{Z}$ the set of all integers; by $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ the set of all natural numbers; by $\mathbb{Q}_t$ the subgroup of $\mathbb{Q}$ of all rational numbers with terminating decimal representations; by $\mathbb{R}^{>0}$ the multiplicative topological group of positive real numbers; and by $\mathbb{Q}^{>0}$ the subset of $\mathbb{R}$ of all positive rational numbers.

For an introduction to ergodic theory, see [8]. For our purposes, consider a group $G$ acting on the topological group $\mathbb{R}$ with its usual Lebesgue measure, such that the map $\phi : G \times \mathbb{R} \to \mathbb{R}$ is *measurable*, that is, the inverse image of each measurable set is measurable. The action is said to be *ergodic* if every set which is $G$-invariant is such that it or its complement in $\mathbb{R}$ has measure zero.

**Lemma 2.1.** *[9, Lemma 2.2.13] If $H$ is a subgroup of $\mathbb{R}$ with its Lebesgue measure, then the action of translation by members of $H$ is ergodic on $\mathbb{R}$ if and only if $H$ is a dense subgroup of $\mathbb{R}$.*

From Lemma 2.1 we immediately obtain our key observation, Theorem 2.1.

**Theorem 2.1.** *Let $X$ be a Lebesgue measurable subset of $\mathbb{R}$. If for each $q \in \mathbb{Q}$ and each $x \in X$, $q + x \in X$, then either $X$ or $\mathbb{R} \setminus X$ has Lebesgue measure zero.*

**Corollary 2.1.** *Let $I$ be any index set, and $X_i$, $i \in I$, Lebesgue measurable subsets of $\mathbb{R}$ such that $\mathbb{R} = \bigcup_{i \in I} X_i$ and $\mathbb{Q} + X_i = X_i$, for each $i \in I$. Then, precisely one of the $X_i$ has full measure and each of the others has measure zero.*

**Remark 2.1.** *When applying Lemma 2.1 to obtain Theorem 2.1, we choose $H$ to be the dense subgroup $\mathbb{Q}$ of $\mathbb{R}$. We could instead have chosen $H$ to be the set $\mathbb{Q}_t$ which is also a dense subgroup of $\mathbb{R}$. When we apply Theorem 2.1 later, we are seeking subsets $X$ of $\mathbb{R}$ such that $\mathbb{Q} + X = X$. However, the weaker condition $\mathbb{Q}_t + X = X$ would suffice.*

**Corollary 2.2.** *Let $X$ be a subset of $\mathbb{R}$ which is closed with respect to algebraic dependence; that is, $x \in X$ implies every element $y$ in $\mathbb{R}$ with the property that $x$ and $y$ are algebraically dependent is also in $X$. Then, $X$ satisfies the conditions $\mathbb{Q} + X = X$ and $\mathbb{Q}^{>0} \cdot X = X$. If $X$ is also Lebesgue measurable, then either $X$ has full measure or null measure.*

**Corollary 2.3.** *For $X$, a subset of $\mathbb{R}$, $\mathbb{Q} + X = X$ if and only if $\mathbb{Q} + (\mathbb{R} \setminus X) = \mathbb{R} \setminus X$. Also, $\mathbb{Q}^{>0} \cdot X = X$ if and only if $\mathbb{Q}^{>0} \cdot (\mathbb{R} \setminus X) = (\mathbb{R} \setminus X)$. Indeed, if $F$ is any subfield of $\mathbb{R}$, then (i) $F + X = X$ if and only if $F + (\mathbb{R} \setminus X) = \mathbb{R} \setminus X$, and (ii) $(F \setminus \{0\}) \cdot X = X$ if and only if $(F \setminus \{0\}) \cdot (\mathbb{R} \setminus X) = \mathbb{R} \setminus X$.*

## 3. Applications of the key observation

### 3.1. Irrationality exponent

Let $x$ be a real number. Then, $x$ is said to have *irrationality exponent* $\mu(x)$ if $\mu(x)$ is the infimum of all $\mu$ such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has at most finitely many solutions $\frac{p}{q}$, for $p, q \in \mathbb{Z}$, $q > 0$.

For a discussion of the irrationality exponent (previously referred to as the *irrationality measure*), see [7, 10–16].

Note that if

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has at most finitely many solutions, then for any $r, s \in \mathbb{Z}$, with $s > 0$,

$$\left|x - \frac{p}{q}\right| = \left|\left(x + \frac{r}{s}\right) - \left(\frac{p}{q} + \frac{r}{s}\right)\right| = \left|\left(x + \frac{r}{s}\right) - \frac{ps + qr}{qs}\right| < \frac{1}{(qs)^\mu}$$

has at most finitely many solutions, as $\frac{1}{(qs)^\mu} \leq \frac{1}{q^\mu}$.

From this, we can easily see that if $S_m$ consists of all real numbers of irrationality exponent $m$, for $m \in [2, \infty) \cup \{\infty\} \cup \{1\}$, then $\mathbb{Q} + S_m = S_m$.

We also note that if

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has at most finitely many solutions, then for each $n \in \mathbb{N}$,

$$\left|nx - \frac{np}{q}\right| < \frac{n}{q^\mu}$$

has at most finitely many solutions, which in turn implies that

$$\left|nx - \frac{np}{q}\right| < \frac{1}{q^\mu}$$

has at most finitely many solutions. So, $nS_m = S_m$, for every $n \in \mathbb{N}$.

Also, if

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^\mu}$$

has at most finitely many solutions, then for each $n \in \mathbb{N}$,

$$\left|\frac{1}{n}x - \frac{p}{nq}\right| < \frac{1}{nq^\mu}$$

has at most finitely many solutions, which in turn implies that

$$\left|\frac{1}{n}x - \frac{p}{nq}\right| < \frac{1}{(nq)^{\mu}}$$

has at most finitely many solutions, as $\frac{1}{(nq)^{\mu}} < \frac{1}{nq^{\mu}}$. From this statement and the previous one, we see that $\mathbb{Q}^{>0} \cdot S_m = S_m$. Then, by Corollary 2.3, $\mathbb{Q}^{>0} \cdot (\mathbb{R} \setminus S_m) = (\mathbb{R} \setminus S_m)$.

It is well known (for example, see [4, Theorem E.3, p. 246] or [12]) that $S_2$ is a Lebesgue measurable set of full measure. Thus, the complement of $S_2$ has measure zero. As the Lebesgue measure is complete, each subset of the complement of $S_2$ is measurable and has measure zero. In particular, each $S_m$, $m \neq 2$, is measurable and has Lebesgue measure zero.

It is not easy to determine the irrationality exponent of specific numbers. However, all rational numbers have irrationality exponent 1, the Liouville numbers have infinite irrationality exponent, and the number $e$ has irrationality exponent 2. In 1955, Klaus Roth published that that every irrational algebraic number has irrationality exponent 2. For this result, which is usually referred to as the Thue-Siegel-Roth Theorem, he was awarded a Fields Medal in 1958.

**Remark 3.1.** *Amongst the sets $S_m$ of irrationality exponent equal to m, defined above, precisely one has full measure, and each of the others has measure zero. Of course, $S_2$ is known to be the one with full measure.*

### *3.2. Normal numbers*

As our next application, we turn to *normal numbers*. For a discussion of normal numbers, see [3, 17–22]. Only a few specific numbers are known to be normal. For example it is not known if $\sqrt{2}$, $\pi$, or $e$ are normal. However, the set of normal numbers has full measure in $\mathbb{R}$.

Let $b$ be an integer greater than or equal to 2, and $N_b$ the set of real numbers which are normal to base $b$, and $A_b$ the set of real numbers which are not normal to base $b$. Note that a rational number written to any base $b$, terminates or repeats. So, it is easy to verify that $\mathbb{Q} + N_b = N_b$. Further, $N_b$ is known to be Lebesgue measurable. In addition, Wall in [22] proved that if $x$ is normal in base $b$, then $qx$ is a normal number to base $b$, for $q \in \mathbb{Q}$, $q \neq 0$. So, $\mathbb{Q} \cdot N_b = N_b$.

**Remark 3.2.** *Let $N_b$ and $A_b$ be as defined above. Then, precisely one of $N_b$ and $A_b$ has full measure, and the other has zero measure. Of course, it is $N_b$ which has full measure. In fact, over 100 years ago, in 1909 to be precise, Émil Borel (1871–1956) [23, 24] proved that there is a subset Z of the unit interval [0,1] which has Lebesgue measure zero and is such that every number in [0,1] which is not in Z is normal to base 2. If N is the set of all normal numbers (also called absolutely normal numbers) and AB is the set of (absolutely) non-normal numbers, N has full measure and AB has zero measure.*

### *3.3. Mahler sets*

We now turn to an important example in the study of *transcendental numbers*, namely *Mahler classes*, named after the German mathematician Kurt Mahler (1903–1988) who introduced them.

In 1844, Joseph Liouville (1809–1882) proved the existence of transcendental numbers [1, 6]. He introduced the set $\mathcal{L}$ of real numbers, now known as Liouville numbers, and showed that they are all transcendental. In 1873, Charles Hermite (1822–1901) proved that the number $e$ is transcendental. In

1882, Carl Louis Ferdinand von Lindemann (1852–1939) extended Hermite's argument to show that $\pi$ is also a transcendental number. This provided a negative answer to the 2000 year old problem of squaring the circle. For a full discussion, see [2, 3].

Recall that a real number $\xi$ is called a *Liouville number* if for every positive integer $n$, there exists a pair of integers $(p, q)$ with $q > 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^n}.$$

It was then reasonable to ask if the numbers $e$ and $\pi$ are Liouville numbers. This was answered in the negative by Kurt Mahler using Mahler sets. He showed that $e$ and $\pi$ do not belong to the Mahler set $U$, whereas every Liouville number lies in $U$. (The Mahler sets are described below.)

**Remark 3.3.** *In 1962, Paul Erdős (1913–1996) [25] proved that* every real number is a sum of two Liouville numbers.

Alan Baker (1939–2018) in his classic work [6] on transcendental number theory said: A classification of the set of all transcendental numbers into three disjoint aggregates, termed $S$-, $T$-, and $U$-numbers, was introduced by Kurt Mahler [26] in 1932, and it has proved to be of considerable value in the general development of the subject.

The classification of Mahler partitions the set of real numbers into four sets, characterized by the rate with which a nonzero polynomial with integer coefficients approaches zero when evaluated at a particular number.

Given a polynomial $P(X)$, recall that the height of $P$, denoted by $H(P)$, is the maximum of the absolute values of the coefficients of $P$. Given a complex number $\xi$, a positive integer $n$, and a real number $H \geq 1$, we define the quantity

$$w_n(\xi, H) = \min\{|P(\xi)| : P(X) \in \mathbb{Z}[X],\ H(P) \leq H,\ \deg(P) \leq n,\ P(\xi) \neq 0\}.$$

Furthermore, we set

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(\xi, H)}{\log H}$$

and

$$w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

With the above notation in mind, Mahler partitioned the set of all real numbers into the following: Let $\xi$ be a real number. We say $\xi$ is an

- $A$-number if $w(\xi) = 0$,
- $S$-number if $0 < w(\xi) < \infty$,
- $T$-number if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any $n \geq 1$,
- $U$-number if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for all $n \geq n_0$, for some positive integer $n_0$.

Note that the $A$-numbers are the algebraic numbers.

The following theorem of Mahler, see [4, Theorem 3.2], records a fundamental property of the Mahler classes.

**Theorem 3.1.** *If $\xi, \eta \in \mathbb{R}$ are algebraically dependent, then they belong to the same Mahler class.*

It follows immediately from Theorem 3.1 and Corollary 2.2 that:

**Corollary 3.1.** *Each of the Mahler sets $S$, $T$, $U$ and $A$ have full measure or null measure.*

In [4], we see that $S$ has full measure and the other Mahler sets have null measure.

Kurt Mahler proved in 1953 [27,28] that neither $\pi$ nor $e$ is a Liouville number, and therefore neither lies in the Mahler set $U$. Indeed, $e$ is in the Mahler set $S$ and $\pi$ lies in $S \cup T$, see [4].

### 3.4. *Countable sets*

For a discussion of computable numbers, which were introduced by Alan Turing (1912–1954), see the entry on computable numbers in Wikipedia. Every algebraic number is computable. So, every non-computable number is transcendental. While it is easily seen that the set of computable numbers has zero Lebesgue measure since it is countable, we nevertheless note that this too satisfies the pattern we have described, namely a set closed under the addition of rational numbers. Constructible numbers are described in [2].

### 3.5. *Badly approximable numbers*

A real number $x$ is said to be a *badly approximable number* if it has the the property that there exists a constant $c$ such that, for all rational numbers $\frac{p}{q}$, we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}.$$

For a discussion of badly approximable numbers, see [29]. It is easily verified that if $X$ is the set of all badly approximable numbers, then $\mathbb{Q} + X = X$ and $\mathbb{Q}^{>0} \cdot X = X$.

**Corollary 3.2.** *If $X$ is the set of numbers which are not badly approximable, then $X$ or $\mathbb{R} \setminus X$ has full Lebesgue measure.*

In fact, from [29], the set of badly approximable numbers has Lebesgue measure zero.

## 4. Conclusions

We could provide a myriad of further examples of well-known sets of real numbers which have either full measure or null measure. But, our point is now clear. If the subset $X$ of $\mathbb{R}$ has the property that $X + \mathbb{Q} = X$, then its Lebesgue measure is full or zero. In practice, if you can show that the Lebesgue measure of $X$ is non-zero, then you know it has full measure.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare there are no conflicts of interest.

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