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http://www.aimspress.com/journal/era

ERA, 33(2): 938–972.
DOI: 10.3934/era.2025043
Received: 26 October 2024
Revised: 30 December 2024
Accepted: 20 January 2025
Published: 21 February 2025

Research article

Large time behavior of strong solution to the magnetohydrodynamics system with temperature-dependent viscosity, heat-conductivity, and resistivity

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Abstract: In this paper, we investigate an initial boundary value problem of a planar magnetohydrodynamics system with temperature-dependent viscosity, heat conductivity, and resistivity. When all of the relative coefficients mentioned above are power functions of temperature, the existence and uniqueness of a global-in-time non-vacuum strong solutions are proved under some special assumptions. At the same time, we obtain the nonlinear exponential stability of the solution. In fact, the initial data could be large if the power of viscosity is small enough.

Keywords: magnetohydrodynamic equation; existence; uniqueness; large-time behavior

1. Introduction

The governing equations of a planar magnetohydrodynamics (MHD) compressible flow can be written in Lagrange variable form as:

$$v_t = u_x, \tag{1.1}$$

$$u_t + (P + \frac{1}{2}|\boldsymbol{b}|^2)_x = \left(\mu \frac{u_x}{v}\right)_x, \tag{1.2}$$

$$\boldsymbol{w}_t - \boldsymbol{b}_x = \left(\lambda \frac{\boldsymbol{w}_x}{v}\right)_x, \tag{1.3}$$

$$(v\boldsymbol{b})_t - \boldsymbol{w}_x = \left(\nu \frac{\boldsymbol{b}_x}{v}\right)_x, \tag{1.4}$$

$$\left(e+\frac{u^2+|\boldsymbol{w}|^2+v|\boldsymbol{b}|^2}{2}\right)_t+\left(u(P+\frac{1}{2}|\boldsymbol{b}|^2)-\boldsymbol{w}\cdot\boldsymbol{b}\right)_x=\left(\kappa\frac{\theta_x}{v}+\mu\frac{uu_x}{v}+\lambda\frac{\boldsymbol{w}\cdot\boldsymbol{w}_x}{v}+\nu\frac{\boldsymbol{b}\cdot\boldsymbol{b}_x}{v}\right)_x. \tag{1.5}$$

Here $t > 0$ represents the time, and $x \in \Omega = (0, 1)$ denotes the Lagrange mass coordinate. The unknown functions $v(x, t) > 0$, $u(x, t)$, $\boldsymbol{w} = (w_1(x, t), w_2(x, t))$, $\boldsymbol{b} = (b_1(x, t), b_2(x, t))$, $\theta(x, t) > 0$, e, and P are the specific volume of the gas, longitudinal velocity, transverse velocity, transverse magnetic field, absolute temperature, internal energy, and pressure, respectively. μ and λ are the viscosity of the flow, ν is the resistivity, and κ is the heat conductivity.

In this paper, we consider the MHD flow of a perfect gas. Thus, P and e satisfy:

$$P=\frac{R\theta}{v} \quad \text{and} \quad e=C_v\theta+Const. \tag{1.6}$$

Here $R > 0$ denotes the specific gas constant, and $C_v > 0$ stands for the heat capacity at constant volume. It is assumed that μ, λ, ν, and κ satisfy

$$\mu=\tilde{\mu}\theta^{\alpha}, \quad \lambda=\tilde{\lambda}\theta^{\alpha}, \quad \nu=\tilde{\nu}\theta^{\alpha}, \quad \text{and} \quad \kappa=\tilde{\kappa}\theta^{\beta}, \tag{1.7}$$

which contain the positive constants $\tilde{\mu} > 0$, $\tilde{\lambda} > 0$, $\tilde{\nu} > 0$, $\tilde{\kappa} > 0$, $\alpha > 0$, and $\beta > 0$.

The systems (1.1)–(1.7) are supplemented with initial conditions

$$(v,u,\boldsymbol{w},\boldsymbol{b},\theta)(x,0)=(v_0,u_0,\boldsymbol{w}_0,\boldsymbol{b}_0,\theta_0), \tag{1.8}$$

and boundary ones

$$(u,\boldsymbol{w},\boldsymbol{b},\theta_x)\big|_{\partial\Omega}=0. \tag{1.9}$$

Obviously, the initial data (1.8) should be compatible with the boundary conditions (1.9).

When $\boldsymbol{w} = \boldsymbol{b} = \boldsymbol{0}$, the Eqs (1.1)–(1.5) are converted into the compressible Navier–Stokes equation, which can be derived from Boltzmann's equation, assuming that the space and time scales are larger than all inherent scale–lengths, such as the Debye length or the gyro-radii of the charged particles [1–5]. Also, one can deduce from the Chapman–Enskog expansion for the first level of approximation in kinetic theory that the viscosity μ and λ may depend on the temperature or the density (see Chapman and Cowling [6]). Experimental results [7] show that the transport coefficients μ and κ vary according to gas temperature and density at very high temperatures and densities.

The central point of magnetohydrodynamics theory is that conductive fluids can support magnetic fields. Li and Shang [8] proved the existence and uniqueness of the global-in-time classical solution to the initial-boundary value problem when the viscosity, resistivity, and heat conductivity depend on the specific volume v and the temperature θ. In that paper, the coefficients are assumed to be proportional to $h(v)\theta^{\alpha}$, where $h(v)$ is a non-degenerate and smooth function satisfying some natural conditions, and the absolute value of the exponent α is sufficiently small. It's worth noting that Li and Shang considered the planar compressible magnetohydrodynamic system for the domain $[0, 1] \times \mathbb{R}^2$. Besides, Huang et al. [9] proved the large-time behavior of strong solutions to equations of compressible planar magnetohydrodynamic flow with the heat conductivity is the power function of temperature. Similar results can be observed in various other reports [10–16].

Recently, Sun et al. [17] verified the existence and uniqueness of a global-in-time non-vacuum strong solution to a one-dimensional compressible Navier–Stokes system for a viscous and heat-conducting ideal polytropic gas. It was assumed that the viscosity μ and heat conductivity κ depend on temperature θ with $\mu(\theta) = \theta^{\alpha}$ and $\kappa(\theta) = \theta^{\beta}$ for sufficiently small $\alpha > 0$ and arbitrary $\beta \geq 0$.

Before presenting our main results, we need to provide some explanations of the symbols first. Throughout this paper, the positive general constant C will be different in different lines. For $1 \leq p \leq \infty$, and integer $k \geq 0$, we adopt the simplified notations for the standard Sobolev space as follows:

$$L^p = L^p(\Omega),\ W^{k,p} = W^{k,p}(\Omega),\ H^k = W^{k,2}(\Omega).$$

Without loss of generality, we assume that $\tilde{\lambda} = \tilde{\mu} = \tilde{\nu} = \tilde{\kappa} = R = c_v = 1$, and

$$\int_0^1 v_0 \mathrm{d}x = 1, \quad \int_0^1 \left(\theta_0 + \frac{u_0^2 + |\boldsymbol{w}_0|^2 + v_0|\boldsymbol{b}_0|^2}{2}\right)\mathrm{d}x = 1. \tag{1.10}$$

Inspired above, we have the following main results.

Theorem 1.1. *For given positive constants $M_0 > 0$ and $V_0 > 0$. Assume that*

$$\|(v_0, u_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \theta_0)\|_{H^2} \leq M_0, \qquad \inf_{x\in[0,1]}\{v_0, \theta_0\} \geq V_0 > 0.$$

Then there exist $\epsilon_0 > 0$, $C_0 > 0$ and $C_1 > 0$ which depend only on β, M_0, V_0, such that the initial boundary value problem (1.1)–(1.9) *with* $0 \leq \alpha \leq \varepsilon_0$ *(see (3.5)) admits a unique global-in-time strong solution* $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)(x, t)$ *on* $[0, 1] \times [0, +\infty)$ *satisfying*

$$C_0^{-1} \leq v(x,t) \leq C_0, \qquad C_1 \leq \theta(x,t) \leq C_1^{-1}, \tag{1.11}$$

and

$$\begin{cases} (v, u, \boldsymbol{w}, \boldsymbol{b}, \theta) \in C([0, +\infty); H^2), \\ v_x \in L^2(0, +\infty; H^1), \\ (u_x, \boldsymbol{w}_x, \boldsymbol{b}_x, \theta_x) \in L^2(0, +\infty; H^2). \end{cases}$$

Furthermore, for any $t \geq 0$, it holds that

$$\|(v - 1, u, \boldsymbol{w}, \boldsymbol{b}, \theta - 1)\|_{H^1} \leq Ce^{-\eta_0 t},$$

where $C, \eta_0 > 0$ are some positive constants.

Remark 1.1. *From the view of physics, the resistance is a function of temperature (e.g., [18]). This implies that our results is physical. Dou et al.'s 2021 study [19], published in Scientific Reports, delves into a variety of issues, including Enhanced Oil Recovery, where the technology can potentially improve the extraction of residual oil from oil fields. The equations presented in our study, especially those related to magnetic force distribution and the relationship between magnetic force and displacement, are pivotal for understanding and optimizing these applications. They assist in predicting the behavior of magnetic foams under various conditions, which is crucial for designing effective systems in the aforementioned fields.*

In this paper, we try to use the framework of Li and Liang [20] to prove the global wellposedness of the solution. It should be emphasized that the key step is to derive the time–independent positive lower and upper bounds of specific volume and temperature. The foremost obstacles lie in the strong non-linearities caused by the temperature-dependent viscosity, resistivity, and heat-conductivity from Eq (1.7). Fortunately, these nonlinear terms are involved with μ_x, λ_x, ν_x, μ_t, λ_t, or ν_t which can be controlled by the smallness of α. With the help of upper and lower bounds of the specific volume, we can then estimate the higher-order derivatives of the solutions, and the upper and lower bounds of the temperature.

The rest of this paper is organized as follows. Section 2 is devoted to a discussion of a number of *a priori* estimates independent of time, which are required to extend the local solution to the time global. Based on the estimates given in Section 2, the main results of Theorem 1.1 are established in Section 3.

2. A priori estimates

For constants N, m_1, m_2, and T, define

$$\begin{aligned}
&X(0,T;m_1,m_2,N)\\
&:= \{(v,u,\theta,\boldsymbol{w},\boldsymbol{b}) : (v-1,u,\boldsymbol{w},\boldsymbol{b},\theta-1) \in C([0,T];H^2),\\
&v_x \in L^2(0,T;H^1), (u_x,\theta_x,\boldsymbol{w}_x,\boldsymbol{b}_x) \in L^2(0,T;H^2),\\
&v_t \in C([0,T];H^1), (u_t,\theta_t,\boldsymbol{w}_t,\boldsymbol{b}_t) \in L^2(0,T;H^1),\\
&v(x,t) \ge m_1, \theta(x,t) \ge m_2, \mathcal{E}(0,T) \le N^2, \forall (x,t) \in [0,1]\times[0,T]\},
\end{aligned}$$

where

$$\mathcal{E}(0,T) := \sup_{0\le t\le T} \|(u,\boldsymbol{w},\boldsymbol{b},v_x,\theta_x)\|_{H^1}^2 + \int_0^T \|\theta_t\|_{L^2}^2 \mathrm{d}t.$$

The main purpose of this section is to derive certain t-dependent *a priori* estimates for the variables $(v,u,\theta,\boldsymbol{w},\boldsymbol{b})$ in the function space $X(0,T;m_1,m_2,N)$, relevant to the initial boundary value problem (1.1)–(1.9) for $T>0$ and $0<m_i\le 1 (i=1,2)$, $2\le N<+\infty$. It follows from Sobolev's inequality that

$$m_1 \le v(x,t) \le 2N, \qquad m_2 \le \theta(x,t) \le 2N, \qquad \text{for} \qquad \forall (x,t) \in [0,1]\times[0,T].$$

Firstly, let us derive the time-independent lower and upper bounds of v.

Lemma 2.1. *Assume that the conditions listed in Theorem 1.1 hold; then there exists a constant* $0<\epsilon_1\le 1$ *depending only on* β, M_0, *and* V_0, *such that if*

$$m_2^{-\alpha} \le 2, \qquad N^\alpha \le 2, \qquad \alpha H(m_1,m_2,N) \le \epsilon_1, \tag{2.1}$$

where

$$H(m_1,m_2,N) := (m_1^{-1}+m_2^{-1}+N+1)^8.$$

Then for $(x,t)\in[0,1]\times[0,T]$,

$$C_0^{-1} \le v(x,t) \le C_0. \tag{2.2}$$

Here (and in what follow), $C_0, C_i (i=1,2,\cdots,10)$, *and* C *denote some generic positive constants depending only on* β, $\|(v_0,u_0,\boldsymbol{w}_0,\boldsymbol{b}_0,\theta_0)\|_{H^2}$, $\inf_{x\in(0,1)} v_0(x)$, *and* $\inf_{x\in(0,1)} \theta_0(x)$.

Proof. The proof is divided into four steps.

Step 1. (Basic energy estimate)

According to (1.1), (1.5), (1.9), and (1.10), for $t > 0$, one has

$$\int_0^1 v(x,t)\mathrm{d}x = 1, \int_0^1 \left(\theta + \frac{u^2 + |\boldsymbol{w}|^2 + v|\boldsymbol{b}|^2}{2}\right)(x,t)\mathrm{d}x = 1. \tag{2.3}$$

In light of (2.1), it is deduced that

$$\|\theta^{\alpha} + \theta^{-\alpha}\|_{L^\infty([0,1]\times[0,T])} \le m_2^{-\alpha} + (2N)^{\alpha} \le 2 + 4 = 6. \tag{2.4}$$

Simplifying (1.5) gives

$$\theta_t + \frac{\theta}{v}u_x = \left(\frac{\theta^{\beta}\theta_x}{v}\right)_x + \frac{\theta^{\alpha}(u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2)}{v}. \tag{2.5}$$

Then, multiplying (1.1)–(1.4), and (2.5) by $(1 - v^{-1})$, u, $\boldsymbol{w}$, $\boldsymbol{b}$, and $(1 - \theta^{-1})$, respectively, and integrating them over $[0, 1] \times [0, T]$, together with (2.4) gives

$$\sup_{0\le t\le T} \int_0^1 \left(\frac{u^2 + |\boldsymbol{w}|^2 + v|\boldsymbol{b}|^2}{2} + (v - \ln v) + (\theta - \ln\theta)\right)\mathrm{d}x + \int_0^T W(s)\mathrm{d}s \le E_0, \tag{2.6}$$

where

$$W(t) = \int_0^1 \left(\frac{\theta^{\alpha}\theta_x^2}{v\theta^2} + \frac{\theta^{\alpha}(u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2)}{v\theta}\right)(x,t)\mathrm{d}x, \tag{2.7}$$

and E_0 is the initial total entropy defined by

$$E_0 = \int_0^1 \left(\frac{u_0^2 + |\boldsymbol{w}_0|^2 + v|\boldsymbol{b}_0|^2}{2} + (v_0 - \ln v_0) + (\theta_0 - \ln\theta_0)\right)\mathrm{d}x.$$

Step 2. (Representation formula for v)

First, (1.2) can be written as

$$u_t + (P + \frac{1}{2}|\boldsymbol{b}|^2)_x = \mu(\ln v)_{xt} + \mu_x\frac{u_x}{v},$$

that is,

$$\left(\frac{u}{\mu}\right)_t + g + \left(\mu^{-1}(P + \frac{1}{2}|\boldsymbol{b}|^2)\right)_x = (\ln v)_{xt}, \tag{2.8}$$

where

$$g = -(\mu^{-1})_t u - (\mu^{-1})_x(P + \frac{1}{2}|\boldsymbol{b}|^2) - \frac{\mu_x u_x}{\mu v}.$$

Integrating (2.8) over $[0, t] \times [x_1(t), x]$, it follows

$$\begin{aligned}\int_{x_1(t)}^x \left(\frac{u}{\mu} - \frac{u_0}{\mu_0}\right)\mathrm{d}\xi + \int_0^1 \int_{x_1(t)}^x g\mathrm{d}\xi\mathrm{d}s + \int_0^t \left(\frac{P + \frac{1}{2}|\boldsymbol{b}|^2}{\mu} - \frac{P + \frac{1}{2}|\boldsymbol{b}|^2}{\mu}(x_1)\right)\mathrm{d}s \\ = \ln v(x,t) - \ln v(x_1(t),t) - [\ln v_0(x) - \ln v(x_1(t),0)],\end{aligned}$$

where $x_1(t) \in [0,1]$ is determined by the following steps and $\mu_0 = \mu(\theta_0)$. Moreover, for ease of notation, define

$$F = \frac{u_x}{v} - \mu^{-1}(P + \frac{1}{2}|\boldsymbol{b}|^2) - \int_0^x g(\zeta)\mathrm{d}\zeta,$$
$$\varphi = \int_0^t F(x,s)\mathrm{d}s + \int_0^x \frac{u_0}{\mu_0}\mathrm{d}\zeta.$$

Based on the above definitions that

$$\varphi_x = \frac{u}{\mu}, \varphi_t = F. \tag{2.9}$$

It is easy to show that

$$\begin{aligned}
&\int_0^t \left[\mu^{-1}(P + \frac{1}{2}|\boldsymbol{b}|^2)(x_1(t),s) + \int_0^{x_1(t)} g(\xi,s)\mathrm{d}\xi\right]\mathrm{d}s \\
&= \int_0^t \left(\frac{u_x}{\mu} - F\right)(x_1(t),s)\mathrm{d}s = \int_0^t [(\ln v)_t - F](x_1(t),s)\mathrm{d}s \\
&= \ln v(x_1(t),t) - \ln v(x_1(t),0) - \int_0^t F(x_1(t),s)\mathrm{d}s.
\end{aligned} \tag{2.10}$$

With the help of (1.1) and (2.9) that

$$\begin{aligned}
(v\varphi)_t - (u\varphi)_x &= v\varphi_t - u\varphi_x = vF - \frac{u^2}{\mu} \\
&= u_x - \frac{v}{\mu}(P + \frac{1}{2}|\mathbf{b}|^2) - v\int_0^x g(\xi)\mathrm{d}\xi - \frac{u^2}{\mu}.
\end{aligned} \tag{2.11}$$

Integrating (2.11) over $[0,t]\times[0,1]$, it follows

$$\int_0^1 v\varphi\mathrm{d}x = \int_0^1 v_0 \int_0^x \frac{u_0}{\mu_0}\mathrm{d}\xi\mathrm{d}x - \int_0^t\int_0^1 \left[\frac{v}{\mu}(P + \frac{1}{2}|\boldsymbol{b}|^2) + v\int_0^x g(\xi)\mathrm{d}\xi + \frac{u^2}{\mu}\right]\mathrm{d}x\mathrm{d}s. \tag{2.12}$$

Hence, according to the mean value theorem, there exists $x_1(t) \in [0,1]$ such that $\varphi(x_1(t),t) = \int_0^1 v\varphi\mathrm{d}x$. By the definition of φ, (2.9) and (2.12), one obtains

$$\begin{aligned}
\int_0^t F(x_1(t),s)\mathrm{d}s &= \varphi(x_1(t),t) - \int_0^{x_1(t)} \frac{u_0}{\mu_0}\mathrm{d}\xi = \int_0^1 v_0\int_0^x \frac{u_0}{\mu_0}\mathrm{d}\xi\mathrm{d}x \\
&- \int_0^t\int_0^1 \left[\frac{v}{\mu}(P + \frac{1}{2}|\boldsymbol{b}|^2) + v\int_0^x g(\xi)\mathrm{d}\xi + \frac{u^2}{\mu}\right]\mathrm{d}x\mathrm{d}s - \int_0^{x_1(t)} \frac{u_0}{\mu_0}\mathrm{d}\xi.
\end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.10) gives

$$\begin{aligned}
&\int_0^t \left[\mu^{-1}(P + \frac{1}{2}|\boldsymbol{b}|^2)(x_1(t),s) + \int_0^{x_1(t)} g(\xi,s)\mathrm{d}\xi\right]\mathrm{d}s \\
&= \ln v(x_1(t),t) - \ln v(x_1(t),0) - \int_0^1 v_0\int_0^x \frac{u_0}{\mu_0}\mathrm{d}\xi\mathrm{d}x + \int_0^{x_1(t)} \frac{u_0}{\mu_0}\mathrm{d}\xi \\
&+ \int_0^t\int_0^1 \left[\frac{v}{\mu}(P + \frac{1}{2}|\boldsymbol{b}|^2) + v\int_0^x g(\xi)\mathrm{d}\xi + \frac{u^2}{\mu}\right]\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{2.14}$$

Moreover, substituting (2.14) into (2.9),

$$\begin{aligned}&\int_0^t \mu^{-1}(P+\frac{1}{2}|\boldsymbol{b}|^2)\mathrm{d}s+\int_0^t\int_0^x g\mathrm{d}\xi\mathrm{d}s-\int_0^t\int_0^1\left[\frac{v}{\mu}(P+\frac{1}{2}|\boldsymbol{b}|^2)+v\int_0^x g(\xi)\mathrm{d}\xi+\frac{u^2}{\mu}\right]\mathrm{d}x\mathrm{d}s\\&\quad+\int_{x_1(t)}^x\left(\frac{u}{\mu}-\frac{u_0}{\mu_0}\right)\mathrm{d}\xi+\int_0^1 v_0\int_0^x\frac{u_0}{\mu_0}\mathrm{d}\xi\mathrm{d}x-\int_0^{x_1(t)}\frac{u_0}{\mu_0}\mathrm{d}\xi=\ln v(x,t)-\ln v(x,0).\end{aligned}\tag{2.15}$$

It follows from (2.15) that

$$v=B^{-1}AD,\tag{2.16}$$

where

$$\begin{aligned}A&:=\exp\left\{\int_0^t\left[\mu^{-1}(P+\frac{1}{2}|\boldsymbol{b}|^2)+\int_0^x g\mathrm{d}\zeta\right]\mathrm{d}s\right\};\\B&:=\exp\left\{\int_0^t\int_0^1\left[\frac{v}{\mu}(P+\frac{1}{2}|\boldsymbol{b}|^2)+v\int_0^x g(\zeta)\mathrm{d}\zeta+\frac{u^2}{\mu}\right]\mathrm{d}x\mathrm{d}s\right\};\\D&:=v_0\exp\left\{\int_{x_1(t)}^x\left(\frac{u}{\mu}-\frac{u_0}{\mu_0}\right)\mathrm{d}\zeta+\int_0^1 v_0\int_0^x\frac{u_0}{\mu_0}\mathrm{d}\zeta\mathrm{d}x-\int_0^{x_1(t)}\frac{u_0}{\mu_0}\mathrm{d}\zeta\right\}.\end{aligned}$$

From (2.16), one has

$$vD^{-1}B=A.\tag{2.17}$$

Furthermore, we define

$$J:=\frac{1}{\mu}(P+\frac{1}{2}|\boldsymbol{b}|^2)+\int_0^x\mathrm{d}\zeta.$$

Then, multiplying (2.17) by J, we have

$$vD^{-1}BJ=\frac{\partial}{\partial t}A.$$

Since $A(0)=1$, integrating the above equality over $(0,t)$ with respect to time, one obtains

$$v=DB^{-1}+\int_0^t\frac{B(s)}{B(t)}\cdot\frac{D(t)}{D(s)}\cdot v\left[\frac{1}{\mu}(P+\frac{1}{2}|\boldsymbol{b}|^2)+\int_0^x g\mathrm{d}\xi\right]\mathrm{d}s.\tag{2.18}$$

Step 3. (Lower bound for v)

Applying Jensen's inequality to the convex function $\theta-\ln\theta$ leads to

$$\int_0^1\theta\mathrm{d}x-\ln\int_0^1\theta\mathrm{d}x\le\int_0^1(\theta-\ln\theta)\mathrm{d}x,$$

which, together with (2.3) and (2.6), leads to

$$\bar\theta(t)=\int_0^1\theta(x,t)\mathrm{d}x\in[\alpha_1,1],\tag{2.19}$$

where $0<\alpha_1<\alpha_2$ are two roots of

$$x-\ln x=e_0.$$

The expression of D and

$$\int_{x_1(t)}^{x} \frac{u}{\mu}\mathrm{d}\xi \leq \int_0^1 \theta^{-\alpha} u \mathrm{d}x \leq m_2^{-2\alpha} + \int_0^1 u^2 \mathrm{d}x \leq C,$$

imply that

$$C^{-1} \leq D \leq C. \tag{2.20}$$

Next, we will estimate B. It follows from (2.3) and (2.4) that

$$\begin{aligned}\int_0^1 \left(\frac{v}{\mu}(P + \frac{1}{2}|\boldsymbol{b}|^2) + \frac{u^2}{\mu}\right)\mathrm{d}x &= \int_0^1 \theta^{-\alpha}(\theta + \frac{1}{2}v|\boldsymbol{b}|^2 + u^2)\mathrm{d}x \\ &\leq m_2^{-\alpha}\int_0^1 (\theta + \frac{1}{2}v|\boldsymbol{b}|^2 + u^2)\mathrm{d}x \leq 4,\end{aligned} \tag{2.21}$$

and

$$\int_0^1 \left(\frac{v}{\mu}(P + \frac{1}{2}|\boldsymbol{b}|^2) + \frac{u^2}{\mu}\right)\mathrm{d}x \geq \int_0^1 \theta^{-\alpha}(\theta + \frac{1}{2}v|\boldsymbol{b}|^2 + u^2)\mathrm{d}x \geq (2N)^{-\alpha}\int_0^1 \theta \mathrm{d}x \geq \frac{\alpha_1}{4}. \tag{2.22}$$

On the other hand, by the expression of g, there exists a sufficiently small $\epsilon > 0$ such that

$$\begin{aligned}\int_0^1 \int_0^x g(\xi)\mathrm{d}\xi &= \int_0^1 v \int_0^x \alpha\theta^{-\alpha-1}(\theta_t u + \theta_x(P + \frac{1}{2}|\boldsymbol{b}|^2) - \frac{\theta^\alpha \theta_x u_x}{v})\mathrm{d}\xi\mathrm{d}x \\ &\leq \alpha m_2^{-\alpha-1}\int_0^1 v \int_0^1 (\theta_t u + m_1^{-1}\theta\theta_x + \frac{1}{2}\theta_x|\boldsymbol{b}|^2 - m_2^{-\alpha}m_1^{-1}\theta_x u_x)\mathrm{d}\xi\mathrm{d}x \\ &\leq \alpha m_2^{-\alpha-1}(\|\theta_t\|_{L^2}^2 + \|u\|_{L^2}^2 + m_1^{-1}\|\theta_x\|_{L^2}\|\theta\|_{L^2} + \|\theta_x\|_{L^2}\|u_x\|_{L^2}) \leq \epsilon t.\end{aligned} \tag{2.23}$$

Putting (2.21)–(2.23) into the expression of B, we will find that there exist C_2, C_3, such that

$$e^{C_2 t} \leq B(t) \leq e^{C_3 t}.$$

That means

$$e^{-C_2(t-s)} \leq \frac{B(s)}{B(t)} \leq e^{-C_3(t-s)}.$$

Thus, for $0 \leq t < t_0$, one has

$$v \geq DB^{-1} - C\varepsilon\int_0^t e^{-C_2(t-s)}\mathrm{d}s \geq Ce{-}Ct_0 - C\varepsilon(1 - e^{-C_2 t_0}).$$

For large enough $t > t_0$, it follows

$$\inf_{x\in\Omega} v(x,t) \geq C\int_0^t \frac{B(s)}{B(t)}\theta^{1-\alpha}\mathrm{d}s - C\varepsilon(1 - e^{-C_3 t}). \tag{2.24}$$

Therefore, one needs the estimates of θ and $\frac{B(s)}{B(t)}$. By the mean value theorem and (2.3), there exists $x_2(t) \in [0,1]$, such that $C^{-1} \leq \theta(x_2(t), t) \leq C$. Based on Cauchy-Schwarz's inequality, one has

$$\begin{aligned}&\left|[\ln(\theta + 1)]^{\frac{\beta}{2}+1} - [\ln(\theta(x_2(t), t) + 1)]^{\frac{\beta}{2}+1}\right| = \left|\int_{x_2}^x \frac{(\ln(\theta + 1))^\beta \cdot \theta_x}{\sqrt{v}(\theta + 1)} \cdot \sqrt{v}(\zeta)\mathrm{d}\zeta\right| \\ &\leq C\left(\int_0^1 \frac{(\ln(\theta + 1))^\beta\,\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}} \cdot \left(\int_0^1 v\mathrm{d}x\right)^{\frac{1}{2}} \leq C\left(\int_0^1 \frac{(\ln(\theta + 1))^\beta\,\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}},\end{aligned}$$

which implies

$$\theta \geq C - CW(t).$$

From (2.18), (2.20), (2.21), and (2.23), one has

$$\begin{aligned}
\int_0^t \frac{B(s)}{B(t)}\theta^{1-\alpha}\mathrm{d}s &\geq \int_0^t \frac{B(s)}{B(t)}\left(1-\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)\mathrm{d}s \\
&\geq C - e^{-Ct} - C\left\{\int_0^{\frac{t}{2}} \frac{B(s)}{B(t)}\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\mathrm{d}s + \int_{\frac{t}{2}}^t \frac{B(s)}{B(t)}\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\mathrm{d}s\right\} \\
&\geq C - e^{-Ct} - Ce^{-\frac{C}{2}t} - C\int_{\frac{t}{2}}^t\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\mathrm{d}s \geq C.
\end{aligned} \tag{2.25}$$

For the large enough time T_0, when $t > T_0$, plugging (2.25) into (2.24) gives

$$\inf_{x\in\Omega} v(x,t) \geq C.$$

Step 4. (Upper bound for v)

According to Holder's inequality, for $0 < \beta \leq 1$, one has

$$\begin{aligned}
\left|\theta^{\frac{1}{2}}(x,t) - \theta^{\frac{1}{2}}(x_2(t),t)\right| &\leq \int_0^1 \theta^{-\frac{1}{2}}\cdot\theta_x\mathrm{d}x \\
&\leq \|v\|_\infty^{\frac{1}{2}}\cdot\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_0^1 \theta^{1-\beta}\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \|v\|_\infty^{\frac{1}{2}}\cdot\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.26}$$

That means

$$\theta(x,t) \leq C + \|v\|_\infty\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x. \tag{2.27}$$

For $1 < \beta < \infty$, one has

$$\left|\theta^{\frac{\beta}{2}}(x,t) - \theta^{\frac{\beta}{2}}(x_2(t),t)\right| \leq \int_0^1 \frac{\theta^{\frac{\beta}{2}-1}\cdot\theta_x}{\theta}\mathrm{d}x \leq \left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_0^1 \theta^{1-\beta}\mathrm{d}x\right)^{\frac{1}{2}} \leq \left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}},$$

which means

$$\theta(x,t) \leq C + \int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x. \tag{2.28}$$

Then the standard calculations give

$$\begin{aligned}
\max_{x\in[0,1]} |\boldsymbol{b}|^2(x,t) &\leq C\int_0^1 |\boldsymbol{b}\cdot\boldsymbol{b}_x|\mathrm{d}x \leq C\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_x|^2}{v\theta}\mathrm{d}x + C\int_0^1 v\theta^{1-\alpha}|\boldsymbol{b}|^2\mathrm{d}x \\
&\leq C\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_x|^2}{v\theta}\mathrm{d}x + C.
\end{aligned} \tag{2.29}$$

It follows from the expression of v and (2.26)–(2.29) that

$$\|v\|_\infty \leq Ce^{-Ct} + C\int_0^t e^{-C(t-s)}\left((1+\|v\|_\infty)\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x + \int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_x|^2}{v\theta}\mathrm{d}x\right)\mathrm{d}s.$$

By using Grönwall's inequality, one has

$$\|v\|_\infty \leq C.$$

Up to now, the proof of Lemma 2.1 has been finished.

Lemma 2.2. *Assume that the conditions listed in Lemma 2.1 hold; then for any $p > 0$, there exists some positive constant $C(p)$ such that*

$$\sup_{0\leq t\leq T}\int_0^1 \theta^{1-p}\mathrm{d}x + \int_0^T\int_0^1\left(\frac{\theta^\beta\theta_x^2}{\theta^{p+1}} + \frac{u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2}{\theta^p}\right)\mathrm{d}x\mathrm{d}t \leq C(p). \tag{2.30}$$

Proof. From (2.6), we see that (2.30) holds for $p = 1$. Then we assume $p > 0$ and $p \neq 1$. Multiplying (2.5) by θ^{-p} and integrating by parts, one can arrive at

$$\begin{aligned}
&\frac{1}{p-1}\left(\int_0^1 \theta^{1-p}\mathrm{d}x\right)_t + p\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^{p+1}}\mathrm{d}x + \int_0^1 \frac{\theta^\alpha(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)}{v\theta^p}\mathrm{d}x\\
&= \int_0^1 \frac{(\theta^{1-p}-1)u_x}{v}\mathrm{d}x + \int_0^1 \frac{u_x}{v}\mathrm{d}x \leq C(p)\int_0^1 |\theta^{\frac12}-1|(\theta^{\frac12-p}+1)|u_x|\mathrm{d}x + \int_0^1 \frac{v_t}{v}\mathrm{d}x\\
&\leq C(p)\max_{x\in[0,1]}|\theta^{\frac12}-1|\int_0^1(\theta^{\frac12-p}+1)|u_x|\mathrm{d}x + \int_0^1 (\ln v)_t\mathrm{d}x\\
&\leq C(p)\max_{x\in[0,1]}|\theta^{\frac12}-1|\left(\int_0^1(\theta^{\frac12-p})|u_x|\mathrm{d}x + \int_0^1|u_x|\mathrm{d}x\right) + \left(\int_0^1 \ln v\mathrm{d}x\right)_t\\
&\leq C(p)\max_{x\in[0,1]}|\theta^{\frac12}-1|\left(\int_0^1 \frac{v\theta^{1-p}}{\theta^\alpha}\mathrm{d}x\right)^{\frac12}\left(\int_0^1 \frac{\theta^\alpha u_x^2}{v\theta^p}\mathrm{d}x\right)^{\frac12}\\
&+ C(p)\max_{x\in[0,1]}|\theta^{\frac12}-1|\left(\int_0^1 \frac{u_x^2}{v\theta}\mathrm{d}x\right)^{\frac12}\left(\int_0^1 v\theta\mathrm{d}x\right)^{\frac12} + \left(\int_0^1 \ln v\mathrm{d}x\right)_t\\
&\leq \frac12\int_0^1 \frac{\theta^\alpha u_x^2}{v\theta^p}\mathrm{d}x + C(p)\max_{x\in[0,1]}|\theta^{\frac12}-1|\left(\int_0^1 \frac{v\theta^{1-p}}{\theta^\alpha}\mathrm{d}x + 1\right) + \left(\int_0^1 \ln v\mathrm{d}x\right)_t.
\end{aligned} \tag{2.31}$$

Moreover, it follows from (2.3) and (2.19) that

$$\alpha_1 \leq \int_0^1 \theta\mathrm{d}x \leq \int_0^1\left(\theta + \eta\frac{\theta^\alpha\left(u^2+|\boldsymbol{w}|^2+v|\boldsymbol{b}|^2\right)}{2}\right)\mathrm{d}x \leq 1.$$

For any real number q, it follows from (2.2) and (2.4) that

$$
\begin{aligned}
|1-\bar{\theta}^q| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\int_0^1 \left(\theta+\eta\frac{\theta^\alpha\left(u^2+|\boldsymbol{w}|^2+v|\boldsymbol{b}|^2\right)}{2}\right)\mathrm{d}x\right)^q \mathrm{d}\eta\right| \\
&= \left|\int_0^1 q\left(\int_0^1 \theta+\eta\frac{\theta^\alpha\left(u^2+|\boldsymbol{w}|^2+v|\boldsymbol{b}|^2\right)}{2}\mathrm{d}x\right)^{q-1}\mathrm{d}\eta\cdot\int_0^1 \frac{\theta^\alpha\left(u^2+|\boldsymbol{w}|^2+v|\boldsymbol{b}|^2\right)}{2}\mathrm{d}x\right| \\
&\le C(q)\max_{x\in[0,1]}(|u|+|\boldsymbol{w}|+|\boldsymbol{b}|)\left(\int_0^1 \left(u^2+|\boldsymbol{w}|^2+v|\boldsymbol{b}|^2\right)\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C(q)\int_0^1 (|u_x|+|\boldsymbol{w}_x|+|\boldsymbol{b}_x|)\mathrm{d}x \\
&\le C(q)\left(\int_0^1 \frac{\theta^\alpha\left(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2\right)}{v\theta}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 v\theta^{1-\alpha}\mathrm{d}x\right) \\
&\le C(q)W^{\frac{1}{2}}(t).
\end{aligned}
\tag{2.32}
$$

After that, for $\beta\in(0,1)$, it follows from (2.27), (2.28), and (2.19) that

$$
\begin{aligned}
\max_{x\in[0,1]}|\theta^{\frac{1}{2}}-1| &\le \max_{x\in[0,1]}|\theta^{\frac{1}{2}}-\bar{\theta}^{\frac{1}{2}}|+\max_{x\in[0,1]}|\bar{\theta}^{\frac{1}{2}}-1|\le C\int_0^1 \theta^{-\frac{1}{2}}|\theta_x|\mathrm{d}x+CW^{\frac{1}{2}}(t) \\
&\le C\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}\cdot\left(\int_0^1 v\theta^{1-\beta}\mathrm{d}x\right)^{\frac{1}{2}}+CW^{\frac{1}{2}}(t)\le CW^{\frac{1}{2}}(t),
\end{aligned}
\tag{2.33}
$$

when $\beta\ge 1$, one has

$$
\begin{aligned}
&\max_{x\in[0,1]}|\theta^{\frac{1}{2}}-1|\le \max_{x\in[0,1]}|\theta^{\frac{1}{2}}-\bar{\theta}^{\frac{1}{2}}|+\max_{x\in[0,1]}|\bar{\theta}^{\frac{1}{2}}-1|\le \max_{x\in[0,1]}|\theta^{\frac{\beta}{2}}-\bar{\theta}^{\frac{\beta}{2}}|+CW^{\frac{1}{2}}(t) \\
&\le C\int_0^1 \theta^{\frac{\beta}{2}-1}|\theta_x|\mathrm{d}x+CW^{\frac{1}{2}}(t)\le C\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 v\theta^{1-\beta}\mathrm{d}x\right)^{\frac{1}{2}}+CW^{\frac{1}{2}}(t) \\
&\le CW^{\frac{1}{2}}(t).
\end{aligned}
\tag{2.34}
$$

Therefore, for $\beta>0$, it follows from (2.33), (2.34), and (2.6) that

$$
\int_0^T \max_{x\in[0,1]}(\theta^{\frac{1}{2}}-1)^2\mathrm{d}t\le C. \tag{2.35}
$$

Finally, we see that for $p\in[0,1]$, one has

$$
\int_0^1 \theta^{1-p}\mathrm{d}x\le\int_0^1 \theta\mathrm{d}x+1\le C.
$$

And for $\beta\ge 0$, it follows from (2.33), (2.34), and (2.6) that

$$
\sup_{0\le t<\infty}\int_0^1 |\ln v|\mathrm{d}x\le C. \tag{2.36}
$$

As a result, according to (2.33), (2.34), (2.7), (2.35), and Grönwall's inequality, we derive (2.30) from (2.31), which finishes the proof of Lemma 2.2.

Lemma 2.3. *Assume that the conditions listed in Lemma 2.1 hold; then for all $T > 0$,*

$$\sup_{0\le t\le T}\int_0^1 v_x^2\mathrm{d}x + \int_0^T\int_0^1 (v_x^2(1+\theta) + u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2)\mathrm{d}x\mathrm{d}t \le C_4. \tag{2.37}$$

Proof. First, integrating (2.5) over $[0,1]\times[0,t]$, by (2.36), one has

$$\begin{aligned}
&\int_0^T\int_0^1 \frac{\theta^\alpha\left(u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2\right)}{v}\mathrm{d}x\mathrm{d}t \\
&= \int_0^1 \theta\mathrm{d}x - \int_0^1 \theta_0\mathrm{d}x + \int_0^T\int_0^1 \frac{\theta-1}{v}u_x\mathrm{d}x\mathrm{d}t + \int_0^1 \ln v\mathrm{d}x - \int_0^1 \ln v_0\mathrm{d}x \\
&\le \frac{1}{2}\int_0^T\int_0^1 \frac{\theta^\alpha u_x^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 \frac{(\theta-1)^2}{v\theta^\alpha}\mathrm{d}x\mathrm{d}t + C \\
&\le \frac{1}{2}\int_0^T\int_0^1 \frac{\theta^\alpha u_x^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\theta^{\frac{1}{2}} - 1|^2\int_0^1 (\theta^{\frac{1}{2}}+1)^2\mathrm{d}x\mathrm{d}t + C \\
&\le \frac{1}{2}\int_0^T\int_0^1 \frac{\theta^\alpha u_x^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\theta^{\frac{1}{2}} - 1|^2\mathrm{d}t + C \\
&\le \frac{1}{2}\int_0^T\int_0^1 \frac{\theta^\alpha u_x^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T W(t)\mathrm{d}t + C \le C,
\end{aligned}$$

thus it follows from (2.2) and (2.4) that

$$\int_0^T\int_0^1 \left(u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2\right)\mathrm{d}x\mathrm{d}t \le C. \tag{2.38}$$

Next, since

$$\begin{aligned}
\left(\frac{\theta^\alpha v_x}{v}\right)_t &= \theta^\alpha\left(\frac{v_x}{v}\right)_t + \alpha\theta^{\alpha-1}\theta_t\frac{v_x}{v} = \theta^\alpha\left(\frac{v_t}{v}\right)_x + \alpha\theta^{\alpha-1}\theta_t\frac{v_x}{v} \\
&= \left(\theta^\alpha\frac{v_t}{v}\right)_x - \alpha\theta^{\alpha-1}\theta_x\frac{v_t}{v} + \alpha\theta^{\alpha-1}\theta_t\frac{v_x}{v} = \left(\theta^\alpha\frac{v_t}{v}\right)_x + \frac{\alpha\theta^{\alpha-1}}{v}(v_x\theta_t - \theta_x v_t),
\end{aligned}$$

the momentum equation (1.2) can be rewritten as

$$\left(u - \theta^\alpha\frac{v_x}{v}\right)_t = -\left(\frac{\theta}{v} + \frac{1}{2}|\boldsymbol{b}|^2\right)_x - \frac{\alpha\theta^{\alpha-1}}{v}(v_x\theta_t - \theta_x v_t). \tag{2.39}$$

Multiplying (2.39) by $\left(u-\frac{\theta^\alpha v_x}{v}\right)$ and integrating it over $[0,1]\times[0,t]$ yields that for any $t\in[0,T]$,

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left(u-\frac{\theta^\alpha v_x}{v}\right)^2\mathrm{d}x-\frac{1}{2}\int_0^1\left(u-\frac{\theta^\alpha v_x}{v}\right)^2(x,0)\mathrm{d}x=\int_0^T\int_0^1\left(\frac{\theta v_x}{v^2}-\frac{\theta_x}{v}-\mathbf{b}\cdot\mathbf{b}_x\right)\\
&\quad\cdot\left(u-\frac{\theta^\alpha v_x}{v}\right)\mathrm{d}x\mathrm{d}t-\int_0^T\int_0^1\frac{\alpha\theta^{\alpha-1}}{v}(v_x\theta_t-\theta_x v_t)\left(u-\frac{\theta^\alpha v_x}{v}\right)\mathrm{d}x\mathrm{d}t\\
&=-\int_0^T\int_0^1\left(\frac{\theta^{\alpha+1}v_x^2}{v^3}\right)\mathrm{d}x\mathrm{d}t+\int_0^T\int_0^1\frac{\theta u v_x}{v^2}\mathrm{d}x\mathrm{d}t-\int_0^T\int_0^1\frac{\theta_x}{v}\left(u-\frac{\theta^\alpha v_x}{v}\right)\mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^T\int_0^1\boldsymbol{b}\cdot\boldsymbol{b}_x\left(u-\frac{\theta^\alpha v_x}{v}\right)\mathrm{d}x\mathrm{d}t-\int_0^T\int_0^1\frac{\alpha\theta^{\alpha-1}}{v}(v_x\theta_t-\theta_x v_t)\left(u-\frac{\theta^\alpha v_x}{v}\right)\mathrm{d}x\mathrm{d}t\\
&=-\int_0^T\int_0^1\left(\frac{\theta^{\alpha+1}v_x^2}{v^3}\right)\mathrm{d}x\mathrm{d}t+\sum_{i=1}^4 I_i.
\end{aligned}\tag{2.40}$$

Each $I_i (i=1,2,3,4)$ can be estimated as follows. First, based on Cauchy's inequality, we have

$$\begin{aligned}
|I_1|&=\left|\int_0^T\int_0^1\frac{\theta u v_x}{v^2}\mathrm{d}x\mathrm{d}t\right|\\
&\le\frac{1}{8}\int_0^T\int_0^1\frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t+C\int_0^T\int_0^1\frac{\theta^{1-\alpha}u^2}{v}\mathrm{d}x\mathrm{d}t\\
&\le\frac{1}{8}\int_0^T\int_0^1\frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t+C,
\end{aligned}\tag{2.41}$$

where it has been used

$$\begin{aligned}
\int_0^T\int_0^1\frac{\theta^{1-\alpha}u^2}{v}\mathrm{d}x\mathrm{d}t&\le C\int_0^T\int_0^1\theta u^2\mathrm{d}x\mathrm{d}t\le C\int_0^T\max_{x\in[0,1]}|u|^2\int_0^1\theta\mathrm{d}x\mathrm{d}t\\
&\le C\int_0^T\max_{x\in[0,1]}|u|^2\mathrm{d}t\le C\int_0^T\int_0^1 u_x^2\mathrm{d}x\mathrm{d}t\le C.
\end{aligned}$$

Next, by using (2.4), (2.6), and (2.30) with $p=\beta$, it follows

$$\begin{aligned}
|I_2|&=\left|\int_0^T\int_0^1\frac{\theta_x}{v}\left(u-\frac{\theta^\alpha v_x}{v}\right)\mathrm{d}x\mathrm{d}t\right|\\
&\le\frac{1}{8}\int_0^T\int_0^1\frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t+C\int_0^T\int_0^1\frac{\theta^{1-\alpha}u^2}{v}\mathrm{d}x\mathrm{d}t+C\int_0^T\int_0^1\frac{\theta^\alpha\theta_x^2}{v\theta}\mathrm{d}x\mathrm{d}t\\
&\le\frac{1}{8}\int_0^T\int_0^1\frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t+C.
\end{aligned}\tag{2.42}$$

Combining (2.42) with Cauchy's inequality leads to

$$\begin{aligned}
|I_3|&=\left|\int_0^T\int_0^1\boldsymbol{b}\cdot\boldsymbol{b}_x\left(u-\frac{\theta^\alpha v_x}{v}\right)\mathrm{d}x\mathrm{d}t\right|\\
&=\int_0^T\int_0^1\left(|\boldsymbol{b}_x|^2+|\boldsymbol{b}|^2\left(u-\frac{\theta^\alpha v_x}{v}\right)^2\right)\mathrm{d}x\mathrm{d}t\\
&\le C\int_0^T W(t)\int_0^1\left(u-\frac{\theta^\alpha v_x}{v}\right)^2\mathrm{d}x\mathrm{d}t+C.
\end{aligned}\tag{2.43}$$

Rewriting (2.5) as

$$\theta_t = -\frac{\theta^\beta \theta_x v_x}{v^2} + \frac{\beta\theta^{\beta-1}\theta_x^2}{v} + \frac{\theta^\beta \theta_{xx}}{v} + \frac{\theta^\alpha \left(u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2\right)}{v} - \frac{\theta}{v}u_x.$$

We set

$$\begin{aligned} Y &= -(uv - \theta^\alpha v_x)(v_x\theta_t - \theta_x v_t) = \theta^\alpha v_x^2\theta_t - uvv_x\theta_t - \theta^\alpha\theta_x u_x v_x + uv\theta_x u_x \\ &= uv\theta_x u_x + \left(\theta^\alpha\theta_t + \frac{u\theta^\beta\theta_x}{v}\right)v_x^2 - \Big(\beta\theta^{\beta-1}u\theta_x^2 + \theta^\beta u\theta_{xx} \\ &\quad + \theta^\alpha u(u_x^2 + |\boldsymbol{w}_x| + |\boldsymbol{b}_x|^2) - \theta u u_x + \theta^\alpha\theta_x u_x\Big)v_x \\ &= uv\theta_x u_x + \mathcal{R}_1 v_x^2 + \mathcal{R}_2 v_x, \end{aligned} \tag{2.44}$$

where

$$\mathcal{R}_1 := \theta^\alpha\theta_t + \frac{u\theta^\beta\theta_x}{v},$$

and

$$\mathcal{R}_2 := -(\beta\theta^{\beta-1}u\theta_x^2 + \theta^\beta u\theta_{xx} + \theta^\alpha u(u_x^2 + |\boldsymbol{w}_x| + |\boldsymbol{b}_x|^2) - \theta u u_x + \theta^\alpha\theta_x u_x).$$

Then from (2.44), one has

$$\begin{aligned} |I_4| &= \left|\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}}{v^2}\left(uv\theta_x u_x + \mathcal{R}_1 v_x^2 + \mathcal{R}_2 v_x\right)\mathrm{d}x\mathrm{d}t\right| \\ &\le \left|\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}u\theta_x u_x}{v}\mathrm{d}x\mathrm{d}t\right| + \left|\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}}{v^2}\mathcal{R}_1 v_x^2\mathrm{d}x\mathrm{d}t\right| \\ &\quad + \left|\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}}{v^2}\mathcal{R}_2 v_x\mathrm{d}x\mathrm{d}t\right| := \sum_{i=1}^3 J_i. \end{aligned} \tag{2.45}$$

Each $J_i (i = 1, 2, 3)$ can be estimated as follows. First, by means of Cauchy's inequality

$$J_1 \le \int_0^T \|\alpha\theta^{\frac{1+\alpha-\beta}{2}}\|_\infty\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x + \int_0^1 \frac{\theta^\alpha u_x^2}{v\theta}\mathrm{d}x\right)\mathrm{d}t \le C. \tag{2.46}$$

According to the definition of $\mathcal{R}_1$, one has

$$\begin{aligned} J_2 &= \left|\int_0^T\int_0^1 \left(\frac{\alpha\theta^{\alpha-1}v_x^2}{v^2}\cdot\frac{u\theta^\beta\theta_x}{v} + \frac{\alpha\theta^{\alpha-1}v_x^2}{v^2}\cdot\theta^\alpha\theta_t\right)\mathrm{d}x\mathrm{d}t\right| \\ &= \left|\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha+\beta-1}uv_x^2\theta_x}{v^3}\mathrm{d}x\mathrm{d}t\right| + \left|\int_0^T\int_0^1 \frac{\alpha\theta^{2\alpha-1}v_x^2\theta_t}{v^2}\mathrm{d}x\mathrm{d}t\right| \\ &\le \int_0^T \|\alpha\theta^{\frac{\alpha+\beta-1}{2}}\frac{v_x u}{v}\|_\infty\left(\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x + \int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\right)\mathrm{d}t \\ &\quad + \int_0^T \|\alpha\theta^{\frac{3}{2}(\alpha-1)}\frac{v_x}{\sqrt{v}}\|_\infty\left(\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x + \int_0^1 \theta_t^2\mathrm{d}x\right)\mathrm{d}t \\ &\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\mathrm{d}t + C\|\alpha\|_\infty\int_0^T\int_0^1 \theta_t^2\mathrm{d}x\mathrm{d}t \\ &\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t + C. \end{aligned} \tag{2.47}$$

It follows from the definition of $\mathcal{R}_2$ that

$$\mathcal{R}_2^2 \le C\left(\theta^{2\alpha}\theta_x^2u_x^2 + u^2\beta^2\theta^{2\beta-2}\theta_x^4 + \theta^{2\beta}u^2\theta_{xx}^2 + u^2\theta^{2\alpha}u_x^4 + u^2\theta^{2\alpha}|\boldsymbol{w}_x|^4 + u^2\theta^{2\alpha}|\boldsymbol{b}_x|^4 + u_x^2\theta^{2\alpha+2}\right),$$

then, one has

$$\begin{aligned} J_3 &= \left|\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}}{v^2}\mathcal{R}_2 v_x \mathrm{d}x\mathrm{d}t\right| \\ &\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 \frac{\alpha^2\theta^{\alpha-3}\mathcal{R}_2^2}{v}\mathrm{d}x\mathrm{d}t \\ &\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t + C. \end{aligned} \tag{2.48}$$

Inserting (2.46)–(2.48) into (2.45) gives

$$|I_4| \le \frac{1}{4}\int_0^T\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t + C. \tag{2.49}$$

Putting (2.41)–(2.43), and (2.49) into (2.40), combining Grönwall's inequality gives

$$\int_0^1 \left(u - \frac{v_x}{v}\right)^2 \mathrm{d}x + \int_0^T\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t \le C.$$

Note that

$$\begin{aligned} \int_0^1 \left(u - \frac{v_x}{v}\right)^2 \mathrm{d}x &= \int_0^1 \left(u^2 - 2u\frac{v_x}{v} + \frac{v_x^2}{v^2}\right)^2 \mathrm{d}x \\ &= \int_0^1 u^2\mathrm{d}x + \int_0^1 \frac{v_x^2}{v^2}\mathrm{d}x - 2\int_0^1 u\frac{v_x}{v}\mathrm{d}x, \end{aligned}$$

that means

$$\begin{aligned} &\int_0^1 u^2\mathrm{d}x + \int_0^1 \frac{v_x^2}{v^2}\mathrm{d}x + \int_0^T\int_0^1 \frac{\theta^{\alpha+1}v_x^2}{v^3}\mathrm{d}x\mathrm{d}t \\ &\le C + 2\int_0^1 u\frac{v_x}{v}\mathrm{d}x \le \frac{1}{2}\int_0^1 \frac{v_x^2}{v^2}\mathrm{d}x + C\int_0^1 u^2\mathrm{d}x + C \le C. \end{aligned}$$

On the other hand, it follows from (2.33) and (2.34) that

$$\begin{aligned} \int_0^1 v_x^2\mathrm{d}x &= \int_0^1 v_x^2(1-\theta)\mathrm{d}x + \int_0^1 \theta v_x^2\mathrm{d}x \\ &\le C\max_{x\in[0,1]}|\theta^{\frac{1}{2}} - 1|^2\int_0^1 v_x^2\mathrm{d}x + \int_0^1 \theta v_x^2\mathrm{d}x \\ &\le CW(t)\int_0^1 v_x^2\mathrm{d}x + \int_0^1 \theta v_x^2\mathrm{d}x. \end{aligned}$$

Together with (2.38), the proof of Lemma 2.3 has been completed.

Lemma 2.4. *Assume that the conditions listed in Lemma 2.1 hold; then for all $T > 0$, one has*

$$\sup_{0\le t\le T}\int_0^1 \left(|\boldsymbol{b}_x|^2 + |\boldsymbol{w}_x|^2\right)\mathrm{d}x + \int_0^T\int_0^1 \left(|\boldsymbol{b}_t|^2 + |\boldsymbol{w}_t|^2 + |\boldsymbol{b}_{xx}|^2 + |\boldsymbol{w}_{xx}|^2\right)\mathrm{d}x\mathrm{d}t \le C_5. \tag{2.50}$$

Proof. First, rewrite (1.3) as

$$\boldsymbol{w}_t = \frac{\theta^\alpha \boldsymbol{w}_{xx}}{v} - \frac{\theta^\alpha \boldsymbol{w}_x v_x}{v^2} + \frac{\alpha\theta^{\alpha-1}\theta_x \boldsymbol{w}_x}{v} + \boldsymbol{b}_x. \tag{2.51}$$

Multiplying (2.51) by w_{xx} and integrating over $[0,1]\times[0,T]$, one obtains:

$$\begin{aligned}
&\frac{1}{2}\int_0^1 |\boldsymbol{w}_x|^2 \mathrm{d}x + \int_0^T\int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v}\mathrm{d}x\mathrm{d}t = \int_0^T\int_0^1 \frac{\theta^\alpha v_x \boldsymbol{w}_x\cdot \boldsymbol{w}_{xx}}{v^2}\mathrm{d}x\mathrm{d}t \\
&+\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}\boldsymbol{w}_x\cdot\boldsymbol{w}_{xx}}{v}\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1 \boldsymbol{b}_x\cdot\boldsymbol{w}_{xx}\mathrm{d}x\mathrm{d}t = \sum_{i=1}^3 I_i.
\end{aligned} \tag{2.52}$$

Each $I_i (i = 1, 2, 3)$ is estimated as follows. From Cauchy's inequality and (2.4), it shows

$$\begin{aligned}
I_1 &= \int_0^T\int_0^1 \frac{\theta^\alpha v_x \boldsymbol{w}_x\cdot\boldsymbol{w}_{xx}}{v^2}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{w}_{xx}|^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{w}_x|^2 v_x^2}{v^3}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{w}_{xx}|^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{w}_x|^2 \int_0^1 \frac{\theta^\alpha v_x^2}{v^3}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{w}_{xx}|^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{w}_x|^2\mathrm{d}t \\
&\le \frac{1}{4}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{w}_{xx}|^2}{v}\mathrm{d}x\mathrm{d}t + C,
\end{aligned} \tag{2.53}$$

where it has been used

$$\begin{aligned}
&\int_0^T \max_{x\in[0,1]}|\boldsymbol{w}_x|^2\mathrm{d}t \le \int_0^T\int_0^1 |\boldsymbol{w}_x|\cdot|\boldsymbol{w}_{xx}|\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 |\boldsymbol{w}_x|^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{w}_{xx}|^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 |\boldsymbol{w}_x|^2\mathrm{d}x\mathrm{d}t \le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{w}_{xx}|^2}{v}\mathrm{d}x\mathrm{d}t + C.
\end{aligned} \tag{2.54}$$

Next, from the *a priori* assumption, one has

$$
\begin{aligned}
I_2 &= -\int_0^T \int_0^1 \frac{\alpha\theta^{\alpha-1}\theta_x \boldsymbol{w}_x \cdot \boldsymbol{w}_{xx}}{v} \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C\int_0^T \int_0^1 \frac{\alpha^2\theta^{\alpha-2}\theta_x^2 |\boldsymbol{w}_x|^2}{v} \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]} |\boldsymbol{w}_x|^2 \cdot |\alpha|^2 \int_0^1 \frac{\theta^{\alpha-2}\theta_x^2}{v} \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]} |\boldsymbol{w}_x|^2 \cdot |\alpha|^2 \cdot m_2^{\alpha-2} \cdot \|\theta_x^2\|_{L^2}^2 \mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]} |\boldsymbol{w}_x|^2 \cdot |\alpha|^2 \cdot H(m_1, m_2, N) \mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]} |\boldsymbol{w}_x|^2 \mathrm{d}t \\
&\le \frac{1}{4}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C.
\end{aligned} \tag{2.55}
$$

Furthermore, according to (2.37), one has

$$
\begin{aligned}
I_3 &= \int_0^T \int_0^1 \boldsymbol{b}_x \cdot \boldsymbol{w}_{xx} \mathrm{d}x\mathrm{d}t \le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C\int_0^T \int_0^1 \frac{|\boldsymbol{b}_x|^2\theta^{-\alpha}}{v} \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{w}_{xx}|^2}{v} \mathrm{d}x\mathrm{d}t + C.
\end{aligned} \tag{2.56}
$$

Substituting (2.53), (2.55), and (2.56) into (2.52), one has

$$
\int_0^1 |\boldsymbol{w}_x|^2 \mathrm{d}x + \int_0^T \int_0^1 |\boldsymbol{w}_{xx}|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.57}
$$

Combining (2.51) with (2.54) gives

$$
\begin{aligned}
&\int_0^T \int_0^1 |\boldsymbol{w}_t|^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_0^T \int_0^1 \left(\frac{\theta^{2\alpha}\boldsymbol{w}_{xx}^2}{v^2} + \frac{\theta^{2\alpha}|\boldsymbol{w}_x|^2 v_x^2}{v^4} + \frac{\alpha^2\theta^{2\alpha-2}\theta_x^2|\boldsymbol{w}_x|^2}{v^2} + |\boldsymbol{b}_x|^2 \right) \mathrm{d}x\mathrm{d}t \\
&\le C\int_0^T \max_{x\in[0,1]} |\boldsymbol{w}_x|^2 \mathrm{d}t + C \le C.
\end{aligned} \tag{2.58}
$$

Next, rewrite (1.4) as

$$
\mathbf{b}_t = \frac{\theta^\alpha \boldsymbol{b}_{xx}}{v^2} - \frac{\theta^\alpha \boldsymbol{b}_x v_x}{v^3} + \frac{\alpha\theta^{\alpha-1}\theta_x \boldsymbol{b}_x}{v^2} - \frac{u_x \boldsymbol{b}}{v} + \frac{\boldsymbol{w}_x}{v}. \tag{2.59}
$$

Multiplying (2.59) by b_{xx} and integrating the result over $[0,1]\times[0,T]$ yields

$$\begin{aligned}
&\frac{1}{2}\int_0^1 |\boldsymbol{b}_x|^2 \mathrm{d}x + \int_0^T\int_0^1 \frac{\theta^\alpha |\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t \\
&= -\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}\theta_x \boldsymbol{b}_x\cdot\boldsymbol{b}_{xx}}{v^2}\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1 \frac{\theta^\alpha v_x \boldsymbol{b}_x\cdot\boldsymbol{b}_{xx}}{v^3}\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\int_0^1 \frac{u_x \boldsymbol{b}\cdot\boldsymbol{b}_{xx}}{v}\mathrm{d}x\mathrm{d}t - \int_0^T\int_0^1 \frac{\boldsymbol{w}_x\cdot\boldsymbol{b}_{xx}}{v}\mathrm{d}x\mathrm{d}t := \sum_{i=1}^4 J_i.
\end{aligned} \tag{2.60}$$

Each $J_i (i = 1, 2, 3)$ is estimated as follows. From Cauchy's inequality and (2.4), one has

$$\begin{aligned}
J_1 &= -\int_0^T\int_0^1 \frac{\alpha\theta^{\alpha-1}\theta_x \boldsymbol{b}_x\cdot\boldsymbol{b}_{xx}}{v^2}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 \frac{\alpha^2\theta^{\alpha-2}\theta_x^2|\boldsymbol{b}_x|^2}{v^2}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{b}_x|^2\cdot|\alpha|^2\int_0^1 \frac{\theta^{\alpha-2}\theta_x^2}{v^2}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{b}_x|^2\cdot|\alpha|^2\cdot m_2^{\alpha-2}\cdot\|\theta_x^2\|_{L^2}^2\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\mathbf{w}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{b}_x|^2\cdot|\alpha|^2\cdot H(m_1,m_2,N)\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{b}_x|^2\mathrm{d}t \le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C,
\end{aligned} \tag{2.61}$$

where it has been used

$$\begin{aligned}
\int_0^T \max_{x\in[0,1]}|\boldsymbol{b}_x|^2\mathrm{d}t &\le \int_0^T\int_0^1 |\boldsymbol{b}_x|\cdot|\boldsymbol{b}_{xx}|\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 |\boldsymbol{b}_x|^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 |\boldsymbol{b}_x|^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C.
\end{aligned} \tag{2.62}$$

It follows from (2.2), (2.4), (2.37), and (2.62) that

$$\begin{aligned}
J_2 &= \int_0^T\int_0^1 \frac{\theta^\alpha v_x \boldsymbol{b}_x\cdot\boldsymbol{b}_{xx}}{v^3}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{b}_x|^2\int_0^1 \frac{\theta^\alpha v_x^2}{v^4}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|\boldsymbol{b}_x|^2\mathrm{d}t \\
&\le \frac{1}{4}\int_0^T\int_0^1 \frac{\theta^\alpha|\boldsymbol{b}_{xx}|^2}{v^2}\mathrm{d}x\mathrm{d}t + C.
\end{aligned} \tag{2.63}$$

From (2.2)–(2.4), one has

$$
\begin{aligned}
J_3 &= \int_0^T \int_0^1 \frac{u_x \boldsymbol{b} \cdot \boldsymbol{b}_{xx}}{v} \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8} \int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{b}_{xx}|^2}{v^2} \mathrm{d}x\mathrm{d}t + \max_{(x,t)\in[0,1]\times[0,T]} |\boldsymbol{b}|^2 \cdot C \int_0^T \int_0^1 u_x^2 \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8} \int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{b}_{xx}|^2}{v^2} \mathrm{d}x\mathrm{d}t + \max_{(x,t)\in[0,1]\times[0,T]} |\boldsymbol{b}|^2 \\
&\le \frac{1}{8} \int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{b}_{xx}|^2}{v^2} \mathrm{d}x\mathrm{d}t + C \sup_{0<t<T} \int_0^1 |\boldsymbol{b}| \cdot |\boldsymbol{b}_x| \mathrm{d}x \\
&\le \frac{1}{8} \int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{b}_{xx}|^2}{v^2} \mathrm{d}x\mathrm{d}t + \frac{1}{8} \sup_{0<t<T} |\boldsymbol{b}_x|^2 + C.
\end{aligned}
\tag{2.64}
$$

According to (2.37), it follows

$$
\begin{aligned}
J_4 &= \int_0^T \int_0^1 \frac{\boldsymbol{w}_x \cdot \boldsymbol{b}_{xx}}{v} \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8} \int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{b}_{xx}|^2}{v^2} \mathrm{d}x\mathrm{d}t + C \int_0^T \int_0^1 |\boldsymbol{w}_x|^2 \mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8} \int_0^T \int_0^1 \frac{\theta^\alpha |\boldsymbol{b}_{xx}|^2}{v^2} \mathrm{d}x\mathrm{d}t + C.
\end{aligned}
\tag{2.65}
$$

Inserting (2.61), (2.63)–(2.65) into (2.60), which implies

$$
\sup_{0<t<T} \int_0^1 |\boldsymbol{b}_x|^2 \mathrm{d}x + \int_0^T \int_0^1 |\boldsymbol{b}_{xx}|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.66}
$$

From (2.59), one obtains

$$
\begin{aligned}
\int_0^T \int_0^1 |\boldsymbol{b}_t|^2 \mathrm{d}x\mathrm{d}t \le C \int_0^T \int_0^1 \bigg(& \frac{\theta^{2\alpha} |\boldsymbol{b}_{xx}|^2}{v^4} + \frac{\alpha^2 \theta^{2\alpha-2} \theta_x^2 |\boldsymbol{b}_x|^2}{v^4} \\
&+ \frac{\theta^{2\alpha} |\boldsymbol{b}_x|^2 v_x^2}{v^6} + \frac{u_x^2 |\boldsymbol{b}|^2}{v^2} + \frac{|\boldsymbol{w}_x|^2}{v^2} \bigg) \mathrm{d}x\mathrm{d}t \le C.
\end{aligned}
\tag{2.67}
$$

Therefore, it follows from (2.57), (2.58), (2.66), and (2.67) that (2.50) is correct. Then the Lemma 2.4 has been proved.

Lemma 2.5. *Assume that the conditions listed in Lemma 2.1 hold; then for all $T > 0$,*

$$
\int_0^T \int_0^1 \theta_x^2 \mathrm{d}x\mathrm{d}t \le C_6.
$$

Proof. For the case of $\beta > 1$, setting $p = \beta - 1$ in (2.30) will give

$$
\int_0^T \int_0^1 \theta_x^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.68}
$$

For $0 < \beta \leq 1$, multiplying (2.5) by $\theta^{1-\frac{\beta}{2}}$ and integrating by parts, it gives

$$\begin{aligned}
&\frac{2}{4-\beta}\left(\int_0^1 \theta^{2-\frac{\beta}{2}}\mathrm{d}x\right)_t + \frac{2-\beta}{2}\int_0^1 \frac{\theta^{\frac{\beta}{2}}\theta_x^2}{v}\mathrm{d}x \\
&= -\int_0^1 \frac{\theta^{2-\frac{\beta}{2}}u_x}{v}\mathrm{d}x + \int_0^1 \frac{\theta^{\alpha+1-\frac{\beta}{2}}\left(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2\right)}{v}\mathrm{d}x \\
&= \int_0^1 \frac{(\bar{\theta}^{2-\frac{\beta}{2}}-\theta^{2-\frac{\beta}{2}})u_x}{v}\mathrm{d}x + \int_0^1 \frac{(1-\bar{\theta}^{2-\frac{\beta}{2}})u_x}{v}\mathrm{d}x - \int_0^1 \frac{u_x}{v}\mathrm{d}x \\
&\quad + \int_0^1 \frac{\theta^{\alpha+1-\frac{\beta}{2}}(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)}{v}\mathrm{d}x := \sum_{i=1}^4 I_i.
\end{aligned} \tag{2.69}$$

Each $I_i (i = 1, 2, 3, 4)$ can be estimated as follows. First, by (2.7), one has

$$\begin{aligned}
I_1 &= \int_0^1 \frac{(\bar{\theta}^{2-\frac{\beta}{2}}-\theta^{2-\frac{\beta}{2}})u_x}{v}\mathrm{d}x \\
&= \int_0^1 \left|\bar{\theta}^{1-\frac{\beta}{4}}-\theta^{1-\frac{\beta}{4}}\right|(\bar{\theta}^{1-\frac{\beta}{4}}+\theta^{1-\frac{\beta}{4}})|u_x|\mathrm{d}x \\
&\leq C\max_{x\in[0,1]}\left|\bar{\theta}^{1-\frac{\beta}{4}}-\theta^{1-\frac{\beta}{4}}\right|\left(\int_0^1(\theta^{2-\frac{\beta}{2}}+1)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 u_x^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \frac{1}{8}\max_{x\in[0,1]}\left|\bar{\theta}^{1-\frac{\beta}{4}}-\theta^{1-\frac{\beta}{4}}\right| + C\int_0^1(\theta^{2-\frac{\beta}{2}}+1)\mathrm{d}x\int_0^1 u_x^2\mathrm{d}x \\
&\leq \frac{1}{8}\left(\int_0^1 \theta^{-\frac{\beta}{4}}|\theta_x|\mathrm{d}x\right)^2 + C\int_0^1(\theta^{2-\frac{\beta}{2}}+1)\mathrm{d}x\int_0^1 u_x^2\mathrm{d}x \\
&\leq \frac{1}{8}\int_0^1 \frac{\theta^{\frac{\beta}{2}}\theta_x^2}{v}\mathrm{d}x + C\int_0^1 \frac{\theta^{\beta}\theta_x^2}{v\theta^2}\mathrm{d}x + C\int_0^1(\theta^{2-\frac{\beta}{2}}+1)\mathrm{d}x\int_0^1 u_x^2\mathrm{d}x \\
&\leq \frac{1}{8}\int_0^1 \frac{\theta^{\frac{\beta}{2}}\theta_x^2}{v}\mathrm{d}x + CW(t) + C\int_0^1 \theta^{2-\frac{\beta}{2}}\mathrm{d}x\int_0^1 u_x^2\mathrm{d}x + C\int_0^1 u_x^2\mathrm{d}x.
\end{aligned} \tag{2.70}$$

According to (2.32), one obtains

$$\begin{aligned}
I_2 &= \int_0^1 \frac{(1-\bar{\theta}^{2-\frac{\beta}{2}})u_x}{v}\mathrm{d}x \leq \max_{x\in[0,1]}\left|1-\bar{\theta}^{2-\frac{\beta}{2}}\right|\int_0^1 u_x\mathrm{d}x \\
&\leq \max_{x\in[0,1]}\left|1-\bar{\theta}^{2-\frac{\beta}{2}}\right|^2 + C\int_0^1 u_x^2\mathrm{d}x \leq CW(t) + C\int_0^1 u_x^2\mathrm{d}x.
\end{aligned} \tag{2.71}$$

It follows

$$I_3 = -\int_0^1 \frac{u_x}{v}\mathrm{d}x \leq C\int_0^1 u_x^2\mathrm{d}x \leq C. \tag{2.72}$$

Next, it follows from (2.4) that

$$\begin{aligned} I_4 &= \int_0^1 \frac{\theta^{\alpha+1-\frac{\beta}{2}}\left(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2\right)}{v}\mathrm{d}x \\ &\le C\max_{x\in[0,1]}\left(\left|\theta^{1-\frac{\beta}{2}}-\bar{\theta}^{1-\frac{\beta}{2}}\right|+1\right)\int_0^1(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)\mathrm{d}x \\ &\le \frac{1}{8}\int_0^1\theta^{-\frac{\beta}{2}}|\theta_x|\mathrm{d}x\int_0^1(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)\mathrm{d}x + C\int_0^1(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)\mathrm{d}x \\ &\le \frac{1}{8}\int_0^1\frac{\theta^{\frac{\beta}{2}}\theta_x^2}{v}\mathrm{d}x + C\int_0^1\frac{\theta^{\beta}\theta_x^2}{v\theta^2}\mathrm{d}x + C\int_0^1(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)\mathrm{d}x \\ &\quad + C\left(\int_0^1 u_x^2\mathrm{d}x\right)^2 + C\left(\int_0^1|\boldsymbol{w}_x|^2\mathrm{d}x\right)^2 + C\left(\int_0^1|\boldsymbol{b}_x|^2\mathrm{d}x\right)^2 \\ &\le \frac{1}{8}\int_0^1\frac{\theta^{\frac{\beta}{2}}\theta_x^2}{v}\mathrm{d}x + CW(t) + C\int_0^1(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)\mathrm{d}x \\ &\quad + C\left(\int_0^1 u_x^2\mathrm{d}x\right)^2 + C\left(\int_0^1|\boldsymbol{w}_x|^2\mathrm{d}x\right)^2 + C\left(\int_0^1|\boldsymbol{b}_x|^2\mathrm{d}x\right)^2. \end{aligned} \tag{2.73}$$

Substituting (2.70)–(2.73) into (2.69), integrating on $[0,T]$, and combining (2.37) and Grönwall's inequality, one has when $0<\beta\le 1$,

$$\int_0^T\int_0^1\theta_x^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.74}$$

Then from (2.68) and (2.74) the proof of Lemma 2.5 has ended.

Lemma 2.6. *Assume that the conditions listed in Lemma 2.1 hold; then for all $T>0$, one has*

$$\sup_{0\le t\le T}\int_0^1 u_x^2\mathrm{d}x + \int_0^T\int_0^1 u_{xx}^2\mathrm{d}x\mathrm{d}t \le C_7. \tag{2.75}$$

Proof. Rewrite (1.2) as

$$u_t = \frac{\theta^\alpha u_{xx}}{v} - \frac{\theta^\alpha u_x v_x}{v^2} + \frac{\alpha\theta^{\alpha-1}\theta_x u_x}{v} - \frac{\theta_x}{v} + \frac{\theta v_x}{v^2} - \boldsymbol{b}\cdot\boldsymbol{b}_x. \tag{2.76}$$

Multiplying (2.76) by u_{xx} and integrating the result over $[0,1]\times[0,T]$, it shows

$$\begin{aligned} &\frac{1}{2}\int_0^1 u_x^2\mathrm{d}x + \int_0^T\int_0^1\frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t \\ &\le \int_0^T\int_0^1\frac{\theta^\alpha v_x u_x u_{xx}}{v^2}\mathrm{d}x\mathrm{d}t - \int_0^T\int_0^1\frac{\alpha\theta^{\alpha-1}\theta_x u_x u_{xx}}{v}\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1\frac{\theta_x u_{xx}}{v}\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^T\int_0^1\frac{\theta v_x u_{xx}}{v^2}\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1\boldsymbol{b}\cdot\boldsymbol{b}_x u_{xx}\mathrm{d}x\mathrm{d}t = \sum_{i=1}^5 J_i. \end{aligned} \tag{2.77}$$

Each J_i can be estimated as follows. First, according to (2.37), one has

$$\begin{aligned}
J_1 &= \int_0^T \int_0^1 \frac{\theta^\alpha v_x u_x u_{xx}}{v^2}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|u_x|^2 \int_0^1 v_x^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|u_x|^2\mathrm{d}t \\
&\le \frac{1}{4}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C,
\end{aligned} \tag{2.78}$$

where it has been used

$$\begin{aligned}
&\int_0^T \max_{x\in[0,1]}|u_x|^2\mathrm{d}t \le \int_0^T \int_0^1 |u_x|\cdot|u_{xx}|\mathrm{d}x\mathrm{d}t + C\int_0^T \int_0^1 u_x^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \int_0^1 u_x^2\mathrm{d}x\mathrm{d}t \le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C.
\end{aligned} \tag{2.79}$$

Secondly, combining (2.30) with $p = \beta + 1 - \alpha$, one obtains

$$\begin{aligned}
J_2 &= -\int_0^T \int_0^1 \frac{\alpha\theta^{\alpha-1}\theta_x u_x u_{xx}}{v}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|u_x|^2\cdot|\alpha|^2 m_2^{\alpha-2}\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}|u_x|^2\mathrm{d}t \\
&\le \frac{1}{4}\int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x + C.
\end{aligned} \tag{2.80}$$

Next, from (2.2) and (2.4), one has

$$\begin{aligned}
J_3 &= \int_0^T \int_0^1 \frac{\theta_x u_{xx}}{v}\mathrm{d}x\mathrm{d}t \le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C.
\end{aligned} \tag{2.81}$$

Furthermore, from (2.37), one has

$$\begin{aligned}
J_4 &= -\int_0^T \int_0^1 \frac{\theta v_x u_{xx}}{v^2}\mathrm{d}x\mathrm{d}t \le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \int_0^1 \theta^2 v_x^2\mathrm{d}x \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \left(\max_{x\in[0,1]}(\theta-\bar{\theta})^2 + 1\right)\int_0^1 v_x^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}(\theta-\bar{\theta})^2\mathrm{d}t + C\int_0^T \int_0^1 v_x^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x + C\int_0^T \int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{8}\int_0^T \int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x + C,
\end{aligned} \tag{2.82}$$

where we have used

$$\int_0^T \max_{x\in[0,1]} |\theta-\bar{\theta}|^2 \mathrm{d}t \le C.$$

In fact, for $0<\beta<2$,

$$\int_0^T \max_{x\in[0,1]} |\theta-\bar{\theta}|^2 \mathrm{d}t \le C\int_0^T\int_0^1 |\theta_x|^2 \mathrm{d}x\mathrm{d}t \le C,$$

and for $\beta \ge 2$,

$$\begin{aligned}\int_0^T \max_{x\in[0,1]} |\theta-\bar{\theta}|^2 \mathrm{d}t &\le C\int_0^T \max_{x\in[0,1]} |\theta^{\frac{\beta}{2}}-\bar{\theta}^{\frac{\beta}{2}}|^2 \mathrm{d}t \le C\int_0^T\int_0^1 \theta^{\beta-2}|\theta_x|^2 \mathrm{d}x\mathrm{d}t\\ &\le C\left(\int_0^T\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T\int_0^1 v\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \le C.\end{aligned}$$

Finally, it follows from (2.62) that

$$\begin{aligned}J_5 = \int_0^T\int_0^1 \boldsymbol{b}\cdot\boldsymbol{b}_x u_{xx}\mathrm{d}x\mathrm{d}t &\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]} |\boldsymbol{b}_x|^2 \int_0^1 v|\boldsymbol{b}|^2\mathrm{d}x\mathrm{d}t\\ &\le \frac{1}{8}\int_0^T\int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x\mathrm{d}t + C\int_0^T \max_{x\in[0,1]} |\boldsymbol{b}_x|^2\mathrm{d}t \le \frac{1}{8}\int_0^1 \frac{\theta^\alpha u_{xx}^2}{v}\mathrm{d}x + C.\end{aligned} \tag{2.83}$$

Substituting (2.78) and (2.80)–(2.83) into (2.77) gives

$$\int_0^1 u_x^2\mathrm{d}x + \int_0^T\int_0^1 u_{xx}^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.84}$$

On the other hand, from (1.2), one has

$$|u_t|^2 \le C(u_{xx}^2 + v_x^2u_x^2 + \alpha^2\theta^{2\alpha-2}\theta_x^2u_x^2 + \theta_x^2 + \theta^2v_x^2 + |\boldsymbol{b}|^2|\boldsymbol{b}_x|^2),$$

and

$$\int_0^T\int_0^1 u_t^2\mathrm{d}x\mathrm{d}t \le C.$$

Combining this with (2.84), the proof of Lemma 2.6 has been finished.

Lemma 2.7. *Assume that the conditions listed in Lemma 2.1 hold; then for all $T>0$,*

$$C_1 \le \theta(x,t) \le C_1^{-1}, \tag{2.85}$$

$$\sup_{0\le t\le T}\int_0^1 \theta_x^2\mathrm{d}x + \int_0^T\int_0^1 (\theta_t^2+\theta_{xx}^2)\mathrm{d}x\mathrm{d}t \le C_8. \tag{2.86}$$

Proof. Multiplying (2.5) by θ gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta^2\mathrm{d}x + \int_0^1 \frac{\theta^\beta\theta_x^2}{v}\mathrm{d}x\\ &= \int_0^1 \frac{(1-\theta^2)u_x}{v}\mathrm{d}x - \left(\int_0^1 \ln v\mathrm{d}x\right)_t + \int_0^1 \frac{\theta^{\alpha+1}(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)}{v}\mathrm{d}x\\ &\le C\max_{x\in[0,1]}(|1-\theta|^2+u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2) - \left(\int_0^1 \ln v\mathrm{d}x\right)_t.\end{aligned}$$

It follows from (2.32), (2.74), and (2.6) that

$$\begin{aligned}
&\int_0^T \max_{x\in[0,1]}(\theta-1)^2\mathrm{d}t \\
&\le C\int_0^T \max_{x\in[0,1]}(\theta-\bar\theta)^2\mathrm{d}t + C\int_0^T \max_{x\in[0,1]}(\bar\theta-1)^2\mathrm{d}t \\
&\le C\int_0^T\int_0^1 \theta|\theta_x|\mathrm{d}x\mathrm{d}t + C\int_0^T V(t)\mathrm{d}t \\
&\le \int_0^T \theta_x^2\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 \theta^2\mathrm{d}x\mathrm{d}t + C\int_0^T V(t)\mathrm{d}t \\
&\le C\int_0^T\int_0^1 \theta^2\mathrm{d}x\mathrm{d}t + C.
\end{aligned}$$

By combining this with (2.3), (2.37), and Grönwall's inequality, one has

$$\int_0^T\int_0^1 \theta^\beta\theta_x^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.87}$$

Next, multiplying (2.5) by $\theta^\beta\theta_t$ and integrating it over $(0,1)$, by (2.4), one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{(\theta^\beta\theta_x)^2}{v}\mathrm{d}x + \int_0^1 \theta^\beta\theta_t^2\mathrm{d}x \\
&= -\frac{1}{2}\int_0^1 \frac{(\theta^\beta\theta_x)^2}{v^2}u_x\mathrm{d}x - \int_0^1 \frac{\theta^{\beta+1}\theta_t u_x}{v}\mathrm{d}x + \int_0^1 \frac{\theta^{\alpha+\beta}\theta_t(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)}{v}\mathrm{d}x \\
&\le \frac{1}{2}\int_0^1 \theta^\beta\theta_t^2\mathrm{d}x + C\max_{x\in[0,1]}|u_x|\theta^{\frac{\beta}{2}}\int_0^1 \frac{\theta^{\frac{3\beta}{2}}\theta_x^2}{v^2}\mathrm{d}x + C\int_0^1 \theta^{\beta+2}u_x^2\mathrm{d}x \\
&\quad + \int_0^1 \theta^\beta(u_x^4+|\boldsymbol{w}_x|^4+|\boldsymbol{b}_x|^4)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \theta^\beta\theta_t^2\mathrm{d}x + \sum_{i=1}^3 I_i.
\end{aligned} \tag{2.88}$$

Moreover, each $I_i (i = 1, 2, 3)$ can be estimated as follows. First,

$$\begin{aligned}
I_1 &= C\max_{x\in[0,1]}|u_x|\theta^{\frac{\beta}{2}}\int_0^1 \frac{\theta^{\frac{3\beta}{2}}\theta_x^2}{v^2}\mathrm{d}x \le C\max_{x\in[0,1]}u_x^2\theta^\beta + C\left(\int_0^1 \theta^{\frac{3}{2}\beta}\theta_x^2\mathrm{d}x\right)^2 \\
&\le C\max_{x\in[0,1]}u_x^2\max_{x\in[0,1]}(1+\theta^{2\beta+2}) + C\int_0^1 \theta^\beta\theta_x^2\mathrm{d}x\cdot\int_0^1 \theta^{2\beta}\theta_x^2\mathrm{d}x.
\end{aligned} \tag{2.89}$$

Second,

$$I_2 \le C\int_0^1 (1+\theta^{2\beta+2})u_x^2\mathrm{d}x \le C\max_{x\in[0,1]}u_x^2\max_{x\in[0,1]}(1+\theta^{2\beta+2}). \tag{2.90}$$

Finally,

$$I_3 \le C\max_{x\in[0,1]}(u_x^4+|\boldsymbol{w}_x|^4+|\boldsymbol{b}_x|^4)\max_{x\in[0,1]}(1+\theta^{2\beta+2}).$$

Substituting (2.88)–(2.90) into (2.87) gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{(\theta^\beta\theta_x)^2}{v}\mathrm{d}x + \int_0^1 \theta^\beta\theta_t^2\mathrm{d}x \\
&\le C\max_{x\in[0,1]}(u_x^2+u_x^4+|\boldsymbol{w}_x|^4+|\boldsymbol{b}_x|^4)\max_{x\in[0,1]}(1+\theta^{2\beta+2}) \\
&\quad + C\int_0^1 \theta^\beta\theta_x^2\mathrm{d}x\int_0^1 \theta^{2\beta}\theta_x^2\mathrm{d}x.
\end{aligned}$$

Direct calculations yield

$$\max_{x\in[0,1]}|\theta^{\beta+1}-\bar{\theta}^{\beta+1}| \le C + C\int_0^1 \theta^{2\beta}\theta_x^2\mathrm{d}x, \tag{2.91}$$

and

$$\max_{x\in[0,1]}(1+\theta^{2\beta+2}) \le \max_{x\in[0,1]}(1+\theta^{\beta+1})^2 \le C\left(\int_0^1 \theta^\beta|\theta_x|\mathrm{d}x\right)^2 \le C\int_0^1 \theta^{2\beta}\theta_x^2\mathrm{d}x.$$

From this and (2.91), and integrating over $[0,T]$, together with Grönwall's inequality, one has

$$\sup_{0\le t\le T}\int_0^1 \theta^{2\beta}\theta_x^2\mathrm{d}x + \int_0^T\int_0^1 \theta^\beta\theta_t^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.92}$$

Combining with (2.91), one has

$$\max_{(x,t)\in[0,1]\times[0,T]}\theta(x,t) \le C. \tag{2.93}$$

On the one hand, (2.93) gives

$$\int_0^T\int_0^1 (\theta^{\beta+1}-\bar{\theta}^{\beta+1})^2\mathrm{d}x\mathrm{d}t \le C\int_0^T\int_0^1 \theta^{2\beta}\theta_x^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.94}$$

Together with (2.4), (2.37), (2.91), and (2.92), one has

$$\begin{aligned}
&\int_0^T\left|\frac{d}{dt}\int_0^1 (\theta^{\beta+1}-\bar{\theta}^{\beta+1})^2\mathrm{d}x\right|\mathrm{d}t \\
&\le C\int_0^T\int_0^1 (\theta^{\beta+1}-\bar{\theta}^{\beta+1})^2\mathrm{d}x\mathrm{d}t + C\int_0^T\int_0^1 (\theta^{2\beta}\theta_t^2+\bar{\theta}_t^2)\mathrm{d}x\mathrm{d}t \\
&\le C\int_0^T\int_0^1 u_x^2\mathrm{d}x\mathrm{d}t + C \le C.
\end{aligned} \tag{2.95}$$

Combining (2.37), (2.93), and (2.94) leads to

$$\lim_{t\to+\infty}\int_0^1 (\theta^{\beta+1}-\bar{\theta}^{\beta+1})^2\mathrm{d}x = 0.$$

Then combining (2.91) gives

$$\max_{x\in[0,1]}(\theta^{\beta+1}-\bar{\theta}^{\beta+1})^4 \le \int_0^1 (\theta^{\beta+1}-\bar{\theta}^{\beta+1})^2\mathrm{d}x\int_0^1 \theta^{2\beta}\theta_x^2\mathrm{d}x \to 0, \qquad as \qquad t\to+\infty. \tag{2.96}$$

Therefore, it follows from (2.19) and (2.95) that there exists some T_0, such that

$$\theta(x,t) \geq \frac{\gamma_1}{2},$$

for all $(x,t) \in [0,1] \times [T_0, +\infty)$.

On the other hand, for $p > 2$, multiplying (2.5) by $\frac{1}{\theta^p}$, one has

$$\begin{aligned}\frac{1}{p-1}\left(\int_0^1 \left(\frac{1}{\theta}\right)^{p-1} \mathrm{d}x\right)_t + \int_0^1 \frac{\mu u_x^2}{v\theta^p}\mathrm{d}x &\leq \int_0^1 \frac{u_x}{v\theta^{p-1}}\mathrm{d}x \\ &\leq \frac{1}{2}\int_0^1 \frac{\mu u_x^2}{v\theta^p}\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{1}{\mu v\theta^{p-2}}\mathrm{d}x.\end{aligned}$$

That means

$$\left\|\frac{1}{\theta}\right\|_{L^{p-1}}^{p-2} \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{\theta}\right\|_{L^{p-1}} \leq C\left\|\frac{1}{\theta}\right\|_{L^{p-2}}^{p-2} \leq C\left\|\frac{1}{\theta}\right\|_{L^{p-1}}^{p-2}, \tag{2.97}$$

where the positive constant C independent of p and T. (2.97) gives

$$\sup_{0<t<T} \|\theta^{-1}\|_{L^{p-1}} \leq C(T+1).$$

Letting $p \to +\infty$, there exists a positive constant $C_1 \leq \frac{\gamma_1}{2}$ such that

$$\theta(x,t) \geq C_1,$$

for all $(x,t) \in [0,1] \times [0,T_0]$. Combining this, (2.96), and (2.92) yields that for all $(x,t) \in [0,1] \times [0,+\infty)$,

$$C_1 \leq \theta \leq C_1^{-1}. \tag{2.98}$$

Together with (2.91), one has

$$\sup_{0\leq t\leq T} \int_0^1 \theta_x^2 \mathrm{d}x + \int_0^T \int_0^1 \theta_t^2 \mathrm{d}x\mathrm{d}t \leq C. \tag{2.99}$$

Finally, it follows from (2.5) that

$$\frac{\theta^\beta \theta_{xx}}{v} = \theta_t + \frac{\theta}{v}u_x - \frac{\beta\theta^{\beta-1}\theta_x^2}{v} + \frac{\theta^\beta v_x\theta_x}{v^2} - \frac{\theta^\alpha(u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2)}{v},$$

from this and (2.37), (2.97), (2.98) yields

$$\int_0^T \int_0^1 \theta_{xx}^2 \mathrm{d}x\mathrm{d}t \leq C\int_0^T \int_0^1 (\theta_t^2 + u_x^2 + \theta_x^4 + v_x^2\theta_x^2 + u_x^4 + |\boldsymbol{b}_x|^4 + |\boldsymbol{w}_x|^4)\mathrm{d}x\mathrm{d}t \leq C. \tag{2.100}$$

Combining with (2.98)–(2.100), the proof of Lemma 2.7 has been finished.

Lemma 2.8. *Assume that the conditions listed in Lemma 2.1 hold; then for all $T > 0$, one has*

$$\begin{aligned}&\sup_{0<t<T} \int_0^1 (u_t^2 + |\boldsymbol{w}_t|^2 + |\boldsymbol{b}_t|^2 + \theta_t^2 + u_{xx}^2 + \theta_{xx}^2 + |\boldsymbol{w}_{xx}|^2 + |\boldsymbol{b}_{xx}|^2)\mathrm{d}x \\ &\quad + \int_0^T \int_0^1 (u_{xt}^2 + |\boldsymbol{w}_{xt}|^2 + |\boldsymbol{b}_{xt}|^2 + \theta_{xt}^2)\mathrm{d}x\mathrm{d}t \leq C_9.\end{aligned} \tag{2.101}$$

Proof. First, differentiating (1.2) with respect to t shows

$$u_{tt}+\left(\frac{v\theta_t-\theta u_x}{v^2}+\boldsymbol{b}\cdot\boldsymbol{b}_t\right)_x=\left((\frac{\mu}{v})_t u_x+\frac{\mu}{v}u_{xt}\right)_x.$$

Multiplying the above equation by u_t, one obtains after integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\mathrm{d}x+\int_0^1\frac{\mu}{v}u_{xt}^2\mathrm{d}x\\&=\int_0^1\frac{v\theta_t-\theta u_x}{v^2}u_{xt}\mathrm{d}x+\int_0^1\boldsymbol{b}\cdot\boldsymbol{b}_t u_{xt}\mathrm{d}x\mathrm{d}t-\int_0^1\left(\frac{\mu}{v}\right)_t u_x u_{xt}\mathrm{d}x\\&\le\frac{1}{2}\int_0^1\frac{\mu}{v}u_{xt}^2\mathrm{d}x+C\int_0^1(\theta_t^2+u_x^2+|\boldsymbol{b}|^2|\boldsymbol{b}_x|^2+\theta_t^2u_x^2+u_x^4)\mathrm{d}x\\&\le\frac{1}{2}\int_0^1\frac{\mu}{v}u_{xt}^2\mathrm{d}x+C\int_0^1(\theta_t^2+u_x^2+u_x^4)\mathrm{d}x+C\max_{x\in[0,1]}|\boldsymbol{b}|^2\int_0^1|\boldsymbol{b}_t|^2\mathrm{d}x\\&\quad+C\max_{x\in[0,1]}u_x^2\int_0^1\theta_t^2\mathrm{d}x,\end{aligned}\tag{2.102}$$

where in the last inequality it has been used

$$\int_0^1\left(\frac{\mu}{v}\right)_t u_x u_{xt}\mathrm{d}x=\int_0^1\frac{\mu_t v-\mu v_t}{v^2}u_x u_{xt}\mathrm{d}x\le\frac{1}{4}\int_0^1\frac{\mu}{v}u_{xt}^2\mathrm{d}x+C\int_0^1(\theta_t^2u_x^2+u_x^4)\mathrm{d}x.$$

Next, differentiating (1.3) with respect to t shows

$$\boldsymbol{w}_{tt}-\boldsymbol{b}_{xt}=\left(\left(\frac{\lambda}{v}\right)_t\boldsymbol{w}_x+\frac{\lambda}{v}\boldsymbol{w}_{xt}\right)_x.$$

Multiplying the above equation by w_t, one also gets after integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1|\boldsymbol{w}_t|^2\mathrm{d}x+\int_0^1\frac{\lambda}{v}|\boldsymbol{w}_{xt}|^2\mathrm{d}x\\&=-\int_0^1\boldsymbol{b}_t\cdot\boldsymbol{w}_{xt}\mathrm{d}x-\int_0^1\left(\frac{\lambda}{v}\right)_t\boldsymbol{w}_x\cdot\boldsymbol{w}_{xt}\mathrm{d}x\\&\le\frac{1}{2}\int_0^1\frac{\lambda}{v}|\boldsymbol{w}_{xt}|^2\mathrm{d}x+C\int_0^1(|\boldsymbol{b}_t|^2+\theta_t^2|\boldsymbol{w}_x|^2+u_x^2|\boldsymbol{w}_x|^2)\mathrm{d}x.\end{aligned}\tag{2.103}$$

Similarly, differentiating (1.4) with respect to t shows

$$\mathbf{b}_{tt}=\frac{\boldsymbol{w}_{xt}}{v}-\frac{u_{xt}\boldsymbol{b}}{v}-\frac{2u_x\boldsymbol{b}_t}{v}+\left(\left(\frac{\nu}{v}\right)_t\boldsymbol{b}_x+\frac{\nu}{v}\boldsymbol{b}_{xt}\right)_x.$$

Multiplying the above by $\boldsymbol{b}_t$, one gets after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1|\boldsymbol{b}_t|^2\mathrm{d}x+\int_0^1\frac{\nu}{v}|\boldsymbol{b}_{xt}|^2\mathrm{d}x\\&=-\int_0^1\frac{\boldsymbol{w}_t\cdot\boldsymbol{b}_{xt}}{v}\mathrm{d}x-\int_0^1\frac{u_{xt}\boldsymbol{b}\cdot\boldsymbol{b}_t}{v}\mathrm{d}x-2\int_0^1\frac{u_x|\boldsymbol{b}_t|^2}{v}\mathrm{d}x-\int_0^1\left(\frac{\nu}{v}\right)_t\boldsymbol{b}_x\cdot\boldsymbol{b}_{xt}\mathrm{d}x\\&\le\frac{1}{2}\int_0^1\frac{\nu}{v}|\boldsymbol{b}_{xt}|^2\mathrm{d}x+\frac{1}{8}\int_0^1\frac{\mu}{v}u_{xt}^2\mathrm{d}x+C\int_0^1(|\boldsymbol{w}_t|^2+|\boldsymbol{b}|^2|\boldsymbol{b}_t|^2+\theta_t^2|\boldsymbol{b}_x|^2+u_x^2|\boldsymbol{b}_x|^2)\mathrm{d}x\\&\quad+C\max_{x\in[0,1]}|u_x|\int_0^1|\boldsymbol{b}_t|^2\mathrm{d}x.\end{aligned}\tag{2.104}$$

At the end, differentiating (2.5) with respect to t shows

$$\begin{aligned}
&\theta_{tt}+\frac{\theta_t}{v}u_x+\frac{\theta u_{xt}}{v}-\frac{\theta u_x^2}{v^2}\\
&=\left(\left(\frac{\theta^\beta}{v}\right)_t\theta_x+\frac{\theta^\beta}{v}\theta_{xt}\right)_x+\left(\frac{\theta^\alpha}{v}\right)_t(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)\\
&\quad+\frac{2\theta^\alpha}{v}(u_xu_{xt}+\boldsymbol{w}_x\cdot\boldsymbol{w}_{xt}+\boldsymbol{b}_x\cdot\boldsymbol{b}_{xt}).
\end{aligned}$$

Multiplying the above by θ_t and integrating by parts, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\theta_t^2\mathrm{d}x+\int_0^1\frac{\theta^\beta}{v}\theta_{xt}^2\mathrm{d}x\\
&=-\int_0^1\left(\frac{\theta^\beta}{v}\right)_t\theta_x\theta_{xt}\mathrm{d}x-\int_0^1\left(\frac{\theta_t}{v}u_x+\frac{\theta u_{xt}}{v}-\frac{\theta u_x^2}{v^2}\right)\theta_t\mathrm{d}x\\
&\quad+\int_0^1\left(\frac{\theta}{v}\right)_t(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)\theta_t\mathrm{d}x+\int_0^1\frac{2\mu}{v}(u_xu_{xt}+\boldsymbol{w}_x\cdot\boldsymbol{w}_{xt}+\boldsymbol{b}_x\cdot\boldsymbol{b}_{xt})\theta_t\mathrm{d}x\\
&\le\frac{1}{2}\int_0^1\frac{\theta^\beta}{v}\theta_{xt}^2\mathrm{d}x+\frac{1}{8}\int_0^1\frac{\mu}{v}u_{xt}^2\mathrm{d}x+\frac{1}{8}\int_0^1\frac{\lambda}{v}|\boldsymbol{w}_{xt}|^2\mathrm{d}x\\
&\quad+\frac{1}{8}\int_0^1\frac{\nu}{v}|\boldsymbol{b}_{xt}|^2\mathrm{d}x+C\max_{x\in[0,1]}(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2+1)\int_0^1\theta_t^2\mathrm{d}x\\
&\quad+C\int_0^1(\theta_x^2\theta_t^2+u_x^2\theta_x^2+u_x^4+|\boldsymbol{w}_x|^4+|\boldsymbol{b}_x|^4)\mathrm{d}x.
\end{aligned}\tag{2.105}$$

According to (2.86), one has

$$\begin{aligned}
\int_0^1\theta_x^2\theta_t^2\mathrm{d}x&\le\max_{x\in[0,1]}\theta_t^2\int_0^1\theta_x^2\mathrm{d}x\le C\int_0^1|\theta_t||\theta_{xt}|\mathrm{d}x\\
&\le\frac{1}{8}\int_0^1\frac{\theta^\beta}{v}\theta_{xt}^2\mathrm{d}x+C\int_0^1\theta_t^2\mathrm{d}x.
\end{aligned}$$

Combining (2.102)–(2.105) and Grönwall's inequality, we deduce

$$\sup_{0<t<T}\int_0^1(u_t^2+|\boldsymbol{w}_t|^2+|\boldsymbol{b}_t|^2+\theta_t^2)\mathrm{d}x+\int_0^T\int_0^1(u_{xt}^2+|\boldsymbol{w}_{xt}|^2+|\boldsymbol{b}_{xt}|^2+\theta_{xt}^2)\mathrm{d}x\mathrm{d}t\le C.\tag{2.106}$$

Finally, we rewrite (1.2) as

$$\frac{\mu}{v}u_{xx}=u_t+\frac{\theta_xv-\theta v_x}{v^2}+\boldsymbol{b}\cdot\boldsymbol{b}_x-\frac{\mu_xv-\mu v_x}{v^2}u_x.$$

It follows from (2.106), (2.85), (2.86) and (2.37) that

$$\begin{aligned}
\int_0^1u_{xx}^2\mathrm{d}x&\le C\int_0^1(u_t^2+\theta_x^2+v_x^2+|\boldsymbol{b}|^2|\boldsymbol{b}_x|^2)\mathrm{d}x+C\max_{x\in[0,1]}u_x^2\int_0^1(\theta_x^2+v_x^2)\mathrm{d}x\\
&\le\frac{1}{2}\int_0^1u_{xx}^2\mathrm{d}x+C.
\end{aligned}\tag{2.107}$$

Similarly, rewriting (2.5) as

$$\frac{\theta^\beta}{v}\theta_{xx} = \theta_t + \frac{\theta}{v}u_x - \left(\frac{\theta^\beta}{v}\right)_x \theta_x - \frac{\theta^\alpha(u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2)}{v}.$$

Using (2.5), (2.106), (2.85), (2.86), (2.37), and (2.107), one has

$$\int_0^1 \theta_{xx}^2 \mathrm{d}x \le C\int_0^1 (u_x^2 + \theta_t^2 + \theta_x^4 + v_x^2\theta_x^2 + u_x^4 + |\boldsymbol{w}_x|^4 + |\boldsymbol{b}_x|^4)\mathrm{d}x \le C + \frac{1}{2}\int_0^1 \theta_{xx}^2 \mathrm{d}x. \tag{2.108}$$

Next, rewriting (1.3) as

$$\frac{\lambda}{v}\boldsymbol{w}_{xx} = \boldsymbol{w}_t - \boldsymbol{b}_x - \left(\frac{\lambda}{v}\right)_x \boldsymbol{w}_x.$$

Then it follows that

$$\int_0^1 |\boldsymbol{w}_{xx}|^2 \mathrm{d}x \le C\int_0^1 (|\boldsymbol{b}_x|^2 + |\boldsymbol{w}_t|^2 + \theta_x^2|\boldsymbol{w}_x|^2 + v_x^2|\boldsymbol{w}_x|^2)\mathrm{d}x \le C + \frac{1}{2}\int_0^1 |\boldsymbol{w}_{xx}|^2 \mathrm{d}x. \tag{2.109}$$

At the end, rewriting (1.4) as

$$\frac{\nu}{v}\boldsymbol{b}_{xx} = (v\boldsymbol{b})_t - \boldsymbol{w}_x - \left(\frac{\nu}{v}\right)_x \boldsymbol{b}_x,$$

thus, one has

$$\begin{aligned}\int_0^1 |\boldsymbol{b}_{xx}|^2 \mathrm{d}x &\le C\int_0^1 (v_t^2 + |\boldsymbol{b}_t|^2 + |\boldsymbol{w}_x|^2 + \theta_x^2|\boldsymbol{b}_x|^2 + v_x^2|\boldsymbol{b}_x|^2)\mathrm{d}x \\ &\le \frac{1}{2}\int_0^1 |\boldsymbol{b}_{xx}|^2 \mathrm{d}x + C.\end{aligned} \tag{2.110}$$

Combining (2.106)–(2.110), the proof of Lemma 2.8 has been proved.

Lemma 2.9. *Assume that the conditions listed in Lemma* (2.1) *hold; then for all* $T > 0$, *one has*

$$\sup_{0<t<T}\int_0^1 v_{xx}^2 \mathrm{d}x + \int_0^T\int_0^1 (v_{xx}^2 + v_{xxt}^2 + u_{xxx}^2 + |\boldsymbol{w}_{xxx}|^2 + |\boldsymbol{b}_{xxx}|^2 + \theta_{xxx}^2)\mathrm{d}x\mathrm{d}t \le C_{10}. \tag{2.111}$$

Proof. First, differentiating (1.2) with respect to x gives

$$u_{xt} - \mu\left(\frac{v_x}{v}\right)_{xt} = -\left(\frac{\theta_x v - \theta v_x}{v^2} + \boldsymbol{b}\cdot\boldsymbol{b}_x\right)_x + \mu_x\left(\frac{u_{xx}v - v_x u_x}{v^2}\right) + \left(\mu_x\frac{u_x}{v}\right)_x. \tag{2.112}$$

Multiplying (2.112) by $\left(\frac{v_x}{v}\right)_x$, and integrating it over [0, 1], we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mu\left(\frac{v_x}{v}\right)_x^2 \mathrm{d}x + \int_0^1 \frac{\theta}{v}\left(\frac{v_x}{v}\right)_x^2 \mathrm{d}x \\ &= \int_0^1 \left(u_{xt} + \left(\frac{\theta_x}{v}\right)_x + \left(\frac{\theta}{v}\right)_x \frac{v_x}{v}\right)\left(\frac{v_x}{v}\right)_x \mathrm{d}x \\ &\quad + \int_0^1 \left((\boldsymbol{b}\cdot\boldsymbol{b}_x)_x - \mu_x\left(\frac{u_{xx} - v_x u_x}{v^2}\right) - \left(\mu_x\frac{u_x}{v}\right)_x\right)\left(\frac{v_x}{v}\right)_x \mathrm{d}x \\ &\le \frac{1}{2}\int_0^1 \left(\frac{v_x}{v}\right)_x^2 \mathrm{d}x + C\int_0^1 (u_{xt}^2 + \theta_{xx}^2 + \theta_x^2 v_x^2 + v_x^4 + u_{xx}^2 + u_x^2 v_x^2 + |\boldsymbol{b}_x|^2)\mathrm{d}x \\ &\quad + C\max_{x\in[0,1]}(\theta_x^2 + u_x^2 + |\boldsymbol{b}|^2)\int_0^1 \left(\frac{v_x}{v}\right)_x^2 \mathrm{d}x.\end{aligned}$$

Together with (2.37), (2.75), (2.85), (2.86), (2.101), and Grönwall's inequality, one has

$$\sup_{0<t<T}\int_0^1 \mu\left(\frac{v_x}{v}\right)_x^2 \mathrm{d}x+\int_0^T\int_0^1 \frac{\theta}{v}\left(\frac{v_x}{v}\right)_x^2 \mathrm{d}x\mathrm{d}t \le C.$$

That means

$$\sup_{0<t<T}\int_0^1 v_{xx}^2 \mathrm{d}x+\int_0^T\int_0^1 v_{xx}^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.113}$$

Furthermore, (2.112) can be written as

$$\frac{\mu}{v}u_{xxx}=u_{xt}+\left(\frac{\theta_x v-\theta v_x}{v^2}\right)_x+(\boldsymbol{b}\cdot\boldsymbol{b}_x)_x+\frac{\mu v_x-\mu_x v}{v^2}u_{xx}+\left(\frac{\mu u_x v_x}{v^2}\right)_x-\left(\mu_x\frac{u_x}{v}\right)_x.$$

Together with (2.37), (2.75), (2.85), (2.86), (2.101), and (2.113), one has

$$\int_0^T\int_0^1 u_{xxx}^2 \mathrm{d}x\mathrm{d}t \le C\int_0^T \max_{x\in[0,1]}(\theta_x^2+|\boldsymbol{b}|^2+v_x^2+u_x^2+\theta_x^2 v_x^2)\mathrm{d}t+C\le C. \tag{2.114}$$

Next, differentiating (2.5) with respect to x, it shows

$$\frac{\theta^\beta\theta_{xxx}}{v}=\theta_{xt}-\left(\frac{\theta^\beta}{v}\right)_x\theta_{xx}+\left(\frac{\beta\theta^{\beta-1}\theta_x}{v}-\frac{\theta^\beta\theta_x v_x}{v^2}\right)_x+\left(\frac{\theta^\alpha(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2)}{v}\right)_x.$$

Thus, one has

$$\int_0^T\int_0^1 u_{xxx}^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.115}$$

Similarly, differentiating (1.4) with respect to x, one has

$$\frac{\nu}{v}\mathbf{b}_{xxx}=(u_x\boldsymbol{b}+v\boldsymbol{b}_t)_x-\boldsymbol{w}_{xx}-\left(\frac{\nu_x v-\nu v_x}{v^2}\right)_x\boldsymbol{b}_x-2\left(\frac{\nu}{v}\right)_x\boldsymbol{b}_{xx}.$$

It implies that

$$\int_0^T\int_0^1 |\boldsymbol{b}_{xxx}|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.116}$$

Finally, differentiating (1.3) with respect to x gives

$$\frac{\lambda}{v}\boldsymbol{w}_{xxx}=\boldsymbol{w}_{xt}-\boldsymbol{b}_{xx}-\left(\frac{\lambda_x v-\lambda v_x}{v^2}\right)_x\boldsymbol{w}_x-2\left(\frac{\lambda}{v}\right)_x\boldsymbol{w}_{xx}.$$

Then, one has

$$\int_0^T\int_0^1 |\boldsymbol{w}_{xxx}|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.117}$$

Combining with (2.113)–(2.117), we obtain (2.111). The Lemma 2.9 has been proved.

3. The proof of Theorem 1.1

With all the *a priori* estimates in Section 2 at hand, we will complete the proof of Theorem 1.1. For this purpose, it will be shown that the existence and uniqueness of local solutions to the initial-boundary value problem (1.1)–(1.9), which can be obtained by using the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval.

Lemma 3.1. *Letting the* (1.10) *holds, then there exists* $T_0 = T_0(V_0, V_0, M_0) > 0$*, depending only on* β*,* V_0 *and* M_0*, such that the initial boundary value problem* (1.1)–(1.9) *has a unique solution* $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta) \in X(0, T_0; \frac{1}{2}V_0, \frac{1}{2}V_0, 2M_0)$.

Proof of Theorem 1.1 First, using Lemma 3.1, the problem (1.1)–(1.9) has a unique solution $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta) \in X(0, T_1; \frac{1}{2}V_0, \frac{1}{2}V_0, 2M_0)$, where $T_1 = T_0(V_0, V_0, M_0)$.

For the positive constants $\alpha \le \alpha_1$ with α_1 being small enough such that

$$\left(\frac{1}{2}V_0\right)^{-\alpha_1} \le 2, \qquad (2M_0)^{\alpha_1} \le 2, \qquad \alpha_1 H(\frac{1}{2}V_0, \frac{1}{2}V_0, 2M_0) \le \epsilon_1, \tag{3.1}$$

where ϵ_1 is chosen in Lemma 2.1, one deduces from Lemmas 2.1–2.9 with $T = T_1$ that the solution $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ satisfies

$$C_0 \le v(x,t) \le C_0^{-1}, C_1 \le \theta(x,t) \le C_1^{-1}, \qquad x \in [0,1] \times [0, T_1], \tag{3.2}$$

and

$$\sup_{0\le t\le T_1} \|(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)\|_{H^2}^2 + \int_0^{T_1} \|\theta_t\|_{L^2}^2 \mathrm{d}t \le C_{11}^2, \tag{3.3}$$

where $C_i (i = 2, \cdots, 10)$ is chosen in Section 2, and $C_{11} := \sum_{i=2}^{10} C_i$. It follows from Lemmas 2.8 and 2.9 that $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta) \in C([0, T_1); H^2)$. If one takes $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)(\cdot, T_1)$ as the initial data and applies Lemma 3.1 again, the local solution $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ can be extended to the time interval $[T_1, T_1 + T_2]$ with $T_2(C_0, C_1, C_{11})$. Moreover, one obtains

$$v(x,t) \ge \frac{1}{2}C_0, \qquad \theta(x,t) \ge \frac{1}{2}C_1, \qquad (x,t) \in [0,1] \times [T_1, T_1 + T_2],$$

and

$$\sup_{0\le t\le T_1+T_2} \|(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)\|_{H^2}^2 + \int_0^{T_1+T_2} \|\theta_t\|_{L^2}^2 \mathrm{d}t \le 4C_{11}^2.$$

Combining with (3.2), (3.3) implies

$$v(x,t) \ge \frac{1}{2}C_0, \qquad \theta(x,t) \ge \frac{1}{2}C_1, \qquad (x,t) \in [0,1] \times [T_1, T_1 + T_2],$$

and

$$\sup_{0\le t\le T_1+T_2} \|(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)\|_{H^2}^2 + \int_0^{T_1+T_2} \|\theta_t\|_{L^2}^2 \mathrm{d}t \le 5C_{11}^2. \tag{3.4}$$

Taking $\alpha \le \min\{\alpha_1, \alpha_2\}$, where α_1 is chosen in (3.1) and α_2 is chosen to be such that

$$\left(\frac{1}{2}V_0\right)^{-\alpha_2} \le 2, \qquad \left(\sqrt{5}C_{11}\right)^{\alpha_2} \le 2, \qquad \alpha_2 H(\frac{1}{2}C_0, \frac{1}{2}C_1, \sqrt{5}C_{11}) \le \epsilon_1,$$

where ϵ_1 is chosen in Lemma 2.1. Then one can employ Lemmas 2.1–2.9 with $T = T_1 + T_2$ to infer the local solution $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ satisfies (3.2) and (3.3).

Thus, choosing

$$\epsilon_0 = \min\{\alpha_1, \alpha_2\}, \tag{3.5}$$

and repeating the above procedure, one can then extend the solution $(v, u, \mathbf{w}, \boldsymbol{b}, \theta)$ step by step to a global one provided that $0 \leq \alpha \leq \epsilon_0$. Furthermore, one derives the initial boundary value problem (1.1)–(1.9) has a unique global solution $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ satisfying (3.2) and (3.3). Moreover, $(v, u, \boldsymbol{w}, \boldsymbol{b}, \theta) \in X(0, +\infty; C_0, C_1, C_{11})$.

The large-time behavior (1.11) follows from Lemmas 2.3–2.9 by using a standard argument (see Reference [21]).

First, similar to (2.6), multiplying (1.1) by $(1-v^{-1})$, (1.2) by u, (1.3) by $\boldsymbol{w}$, (1.4) by $\boldsymbol{b}$, (2.5) by $1-\theta^{-1}$ and adding them altogether, integrating the resultant equality over $(0, 1)$, one has after using (2.2) and (2.85) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 &\left(\frac{u^2 + |\boldsymbol{w}|^2 + v|\boldsymbol{b}|^2}{2} + (v - \ln v - 1) + (\theta - \ln\theta - 1)\right)\mathrm{d}x \\ &+ C_{10}\int_0^1 (\theta_x^2 + u_x^2 + |\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2)\mathrm{d}x \leq 0,\end{aligned}$$

where (and in what follows) $C_i, i = 12, \cdots, 18$ and C, C^* denote some generic positive constants depending only on β and M_0, V_0.

By means of (2.87), (2.2), (2.85), (2.86), (2.101) and Sobolev's inequality, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{(\theta^\beta\theta_x)^2}{v}\mathrm{d}x + C_{13}\int_0^1 \theta_t^2\mathrm{d}x \leq C_{14}\int_0^1 \theta_x^2\mathrm{d}x + \epsilon\|u_{xx}\|_{L^2}^2 + C_\epsilon\|u_x\|_{L^2}^2. \tag{3.6}$$

Next, multiplying (2.39) by $(u - \theta^\alpha\frac{v_x}{v})$ and integrating the resultant equality over $(0, 1)$, using (2.2), (2.85), (2.50), (2.75), and Poincare's inequality yields that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(u - \theta^\alpha\frac{v_x}{v}\right)^2\mathrm{d}x + C_{15}\int_0^1 v_x^2\mathrm{d}x \\ &\leq C\int_0^1 \left(|uv_x| + |u\theta_x| + |\boldsymbol{b}\cdot\boldsymbol{b}_x| + |v_x\theta_t| + |u_x\theta_x| + |\theta_x v_X| + |\boldsymbol{b}_x v_x|\right)\mathrm{d}x \\ &\leq C_{16}\|(u_x, \theta_x, \theta_t, \boldsymbol{b}_x)\|_{L^2}^2 + \frac{C_{15}}{2}\|v_x\|_{L^2}^2.\end{aligned} \tag{3.7}$$

By the virtue of (2.50), (2.75), and (2.54), one obtains that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_x^2\mathrm{d}x + C_{18}\int_0^1 u_{xx}^2\mathrm{d}x \leq C_{17}\|(v_x, \theta_x)\|_{L^2}^2. \tag{3.8}$$

Furthermore, adding (3.5) multiplied by $\frac{2(C_{17}+1)}{C_{15}}$, (2.7) multiplied by $\frac{1}{C_{13}}\left(\frac{2C_{16}(C_{17}+1)}{C_{15}} + 1\right)$, and (3.4) multiplied by C^*, which satisfy

$$C^* := \frac{1}{C_{12}}\left[\frac{C_{14} + C_\epsilon}{C_{13}}\left(\frac{2C_{16}(C_{17} + 1)}{C_{15}} + 1\right) + \frac{2C_{16}(C_{17} + 1)}{C_{15}} + C_{17} + 1 + \frac{4C_{12}(C_{17} + 1)}{C_{15}}\right].$$

From (3.6) and choosing ϵ suitably small, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}Q + \frac{1}{2}\|(u_x, \theta_x, v_x)\|_{L^2}^2 \le 0,$$

where

$$Q := \int_0^1 C^* \left(\frac{u^2 + |\boldsymbol{w}|^2 + v|\boldsymbol{b}|^2}{2} + (v - \ln v - 1) + (\theta - \ln\theta - 1) \right) + \frac{2C_{16}(C_{17}+1) + C_{15}}{C_{13}C_{15}} \frac{(\theta^\beta \theta_x)^2}{v} + \frac{2(C_{17}+1)}{C_{15}} \left(u - \theta^\alpha \frac{v_x}{v} \right)^2 + u_x^2 \mathrm{d}x.$$

By using Cauchy–Schwarz's inequality, one obtains

$$\left| \frac{\mu u v_x}{v} \right| \le \left(\frac{\mu v_x}{v} \right)^2 + u^2,$$

which along with Poincare's inequality, yields that

$$C^{-1}\|v_x\|_{L^2}^2 - \|u\|_{L^2}^2 \le \int_0^1 \left(u - \theta^\alpha \frac{v_x}{v} \right)^2 \mathrm{d}x \le C\|(u_x, v_x)\|_{L^2}^2.$$

Finally, using Poincare's inequality and (3.7) implies that

$$C^{-1}\|(v-1, u, \theta-1)\|_{H^1}^2 \le Q \le C\|(u_x, v_x, \theta_x)\|_{L^2}^2,$$

where it has been used the conservation of energy implied by (1.5), (2.30), and the following fact:

$$\|\theta - 1\|_{L^2} \le \int_0^1 |\theta - \bar{\theta}|^2 \mathrm{d}x + C\|u\|_{L^2}^2 \le C\|(\theta_x, u_x)\|_{L^2}^2.$$

By means of (3.6) and (3.8), one obtains that

$$\|(v-1, u, \theta-1)(t)\|_{H^1}^2 \le Ce^{-\eta_0 t}.$$

Thus, the proof of Theorem 1.1 has been completed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are grateful to the anonymous referees for their helpful comments and valuable suggestions, which have improved the presentation of the manuscript. This work was supported by the Jiangxi Provincial Natural Science Foundation (Grant Nos. 20232BAB211001, 20242BAB25006).

Conflict of interest

The authors declare there are no conflicts of interest.

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