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Research article

Modification and the cohomology groups of compact solvmanifolds II

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Abstract: In this article, we refine the modification theorem for a compact solvmanifold given in 2006 and completely solve the problem of finding the cohomology ring on compact solvmanifolds.

Keywords: solvmanifolds; cohomology; invariant structure; homogeneous space; product; fiber bundles; symplectic manifolds; splittings; prealgebraic group; decompositions; modification; Lie group; compact manifolds; uniform discrete subgroups; locally flat parallelizable manifolds

1. Introduction

A compact real homogeneous manifold $M = G/H$ is called a solvmanifold if G is solvable and H is discrete. Here, we always assume that G is *simply connected.* If $\mathrm{Ad}(G)$ has the same real algebraic closure as that of $\mathrm{Ad}(H)$, we say that M has the Mostow condition with respect to G and H. When M is a solvmanifold and satisfies the Mostow condition, the cohomology of M can be calculated by the cohomology of the Lie algebra [1] (see also [2]). But, in general, it is very difficult to calculate the cohomology for a general compact solvmanifold. In 2006, we proved the following in [3]:

Proposition 1. *If $M = G/H$ is a compact real homogeneous solvmanifold, there is a finite covering space $M' = G/H'$, i.e., H/H' is a finite group, such that there is another simply connected solvable real Lie group G' which contains H' and is diffeomorphic to G such that (1) $M' = G'/H'$, (2) M' satisfies the Mostow condition with respect to G' and H'.*

In particular, we have:

Proposition 2.: *If G is solvable and H is discrete, $M = G/H$ is compact, and then we have $H^*(M', \mathbf{R}) = H^*(\mathcal{G}')$, where $\mathcal{G}'$ is the Lie algebra of the Lie group G' in Proposition 1.*

In [4], the last paragraph of the second section, we also gave a result on the cohomology group of the manifold M. In this paper, we will prove following Theorems 1, 2 and 3.

Instead of using an algebraic closure, we use the imaginary semisimple representation. We further use a simple twisted product on the original Lie group induced from the rotation part of the semisimple part of the inner automorphism to obtain the new modified Lie group. An application of this result is a simple way to calculate the cohomology groups of compact quotients of a real solvable Lie group over a cocompact discrete subgroup, which we stated in an earlier paper [4] on compact pseudo-kählerian complex solvmanifolds. The latter takes care of finite covering with a Lie algebra level method instead of the topological method. We also apply these results to the compact solvmanifolds with symplectic structure, including the equivalence to the almost symplectic property and extending the Benson-Gordon theorem on the 1-Lefschetz compact solvmanifolds by removing the completely real condition as we promised earlier, which also classifies these manifolds.

Before we state our new results, we give some construction first. Let G be a simply connected solvable Lie group. Then G can be realized as an upper triangular linear group over a linear space V such that its nilradical consists of upper triangle matrices with 1's on the diagonal (see [5]). There is a natural homomorphism of Lie groups which sends the upper triangle matrices to their diagonals $D: \ g \to D(g)$. The images $D(g)$ induce automorphisms on the Lie algebra, as a subspace of $V \otimes V^*$, and are also semisimple, which are called the semisimple parts $S(\mathrm{Ad}(g))$. We simply denote it by S. S is a product of two diagonal matrices, where one is positive and the other is a rotation. We denote the first by R and the latter by I. We shall see that $G, \ \Gamma$ satisfies the Mostow condition if and only if the algebraic closure $\mathbf{I} = \mathcal{A}(I(\Gamma))$ of $I(\Gamma)$ is the same as that of $I(G)$. Let $I_{\Gamma,0}$ be the identity component of $\mathbf{I}$. Then G is Mostow if and only if $I_{\Gamma,0}$ is the identity component of the algebraic closure $\mathbf{A}$ of $I(G)$. Therefore, there is a sub-torus T in $\mathbf{A}$ such that $\mathbf{A}$ is locally a product of T and $I_{\Gamma,0}$. There is a Lie algebra morphism from $\mathcal{G}$ to the Lie algebra of T induced by the morphism I and the projection $\mathbf{A}$ to T. We denote it by i_T. This morphism induces a Lie group morphism I_T from G to T.

There is a cofinite subgroup Γ_1 of Γ such that its image under I is in $I_{\Gamma,0}$. That is, let π be the morphism from $\mathbf{I}$ to $\mathbf{I}/I_{\Gamma,0}$. Then $(\pi \cdot I)(\Gamma_1) = 1$. We can simply take Γ_1 to be the kernel of $\pi \cdot I$. We also let Γ' be the kernel of I_T on Γ. Then Γ' is cofinite in Γ_1 and Γ. Let i be the endomorphism i_T of the Lie algebras. Then $I_T = \exp(i)$, and $[A, B]_i = [A, B] - i(A)B + i(B)A$ induces a new Lie structure on the Lie algebra and so on for the Lie group. We denote the new Lie group by G^i.

Thereom 1.: *G^i is a Lie group such that its image under I is in $I_{\Gamma,0}$. Moreover, Γ' is a subgroup of G^i and $G/\Gamma' = G^i/\Gamma'$.*

In particular, if we denote the new Lie algebra by $\mathcal{G}_i$, using Mostow's result on the cohomology group on a real compact solvmanifold in [1], we have the following theorem.

Theorem 2.: *Let $M' = G/\Gamma'$, then $H^*(M', \mathbf{R}) = H^*(\mathcal{G}_i)$.*

In an earlier version of this paper, we consider the special case in which $I_{\Gamma,0}$ is an identity. In that case, we could just apply [6] instead. But, in general, $I_{\Gamma,0}$ is not necessary the identity as we shall see later on.

Now we consider the linear action of $I_T(\Gamma/\Gamma')$ on the Lie algebra. We have the following theorem.

Theorem 3.: $H^*(M, \mathbf{R}) = H^*_{I_T(\Gamma/\Gamma')}(\mathcal{G}_i)$.

We are also interested in the structure of compact homogeneous manifolds with symplectic structure (which might not be invariant under the group action). With Propositions 1 and 2 we successfully treated the compact complex pseudo-kähler solvmanifolds in [4]. See [7] for the difficulties of classifying such manifolds. With Proposition 2, one can reduce the classification of compact solvmanifolds with real symplectic structures to a finite covering. With Theorem 3, we have further results in [8]. Here, we just give some of direct applications of our results, which we did not state earlier since they are quite clear from our results and we were concerned with other more difficult questions.

Corollary 1.: *We denote ρ_{I_T} as the induced representation of the Lie group on the Lie algebra by $g \to I_T(g^{-1})$. Then,*

$$H^k(M, \mathbf{R}) \subset H^k(\mathcal{G}, \otimes^k \rho_{I_T}).$$

Notice that ρ_{I_T} is split into line bundles of characters generated by their eigenvectors.

Corollary 2.: *Let $M = G/\Gamma$ be a compact solvmanifold. Then, it has a symplectic structure if and only if it is almost symplectic, i.e., if there is a 2 class $\alpha \in H^2(M, \mathbf{R})$ such that $\alpha^n \neq 0$ in $H^{2n}(M, \mathbf{R})$.*

We write g_a as the set of eigenvectors in the Lie algebra, and ρ_a as the inverse of the related characters. Then, $\mathcal{G}_i = \sum_a g_a \otimes \rho_a$. We have the following corollary.

Corollary 3.: *There is a finite dimensional complex*

$$\sum_{k=0}^{n} (\wedge^k (\sum_a g_a \otimes \rho_a)) \cap \Omega^k(M)$$

which induces the cohomology ring on M.

Furthermore, we can generalize the Benson-Gordon theorem in [9].

Corollary 4.: *If G is a solvable Lie group and G/Γ is a compact solvmanifold with a symplectic structure such that the 1-Lefschetz condition is satisfied, then:*

(i) There is an abelian complement a in the Lie algebra $\mathcal{G}$ of $n = [\mathcal{G}, \mathcal{G}]$.

(ii) a and n are even dimensional.

(iii) The center of $\mathcal{G}$ intersects n trivially.

(iv) The symplectic form is cohomologous to a G^i left invariant symplectic form on G^i/Γ' of the form $\omega = \omega_0 + \omega_1$, where $n = \ker(\omega_0)$ and $a = \ker(\omega_1)$.

(v) Both ω_0 and ω_1 are closed but not exact on $\mathcal{G}_i$.

(vi) The G^i adjoint action of a on n are symplectic automorphisms on (n, ω_1).

Moreover, if a compact solvmanifold with a symplectic structure has property (iv) with a complement a of n, then it is 1-Lefschetz.

Therefore, one can classify all 1-Lefschetz compact solvmanifolds by finding all the even dimensional abelian rational subgroup A of symplectic automorphisms over a compact symplectic nilmanifold such that N is the commutator of the semi-product $A \propto N$.

We just adopt their proof to the $I_T(\Gamma/\Gamma')$ invariant case. Actually, [9] proved the following theorem.

Theorem 4.: *If $M = G/\Gamma$ is a solvmanifold, then $H^j(M, \mathbf{R}) = H^j(\mathcal{G})$, $j = 1, n-1$. Moreover, let $n = [\mathcal{G}, \mathcal{G}]$, a be a complement of n, $k = \dim a$, and $l = \dim n$. Then, $d(\wedge^{n-2}\mathcal{G}) = (\wedge^k a^*) \wedge (\wedge^{l-1} n^*)$, $H^1(M, \mathbf{R}) = H^1(\mathcal{G}) = a^*$, and $H^{n-1}(M) = H^{n-1}(\mathcal{G}) = (\wedge^{k-1} a^*) \wedge (\wedge^l n^*)$.*

The first part comes from the relation between the first homology and the fundamental group. The second part was proven in [9]. See (2.7) there.

We see that the only matter is that ω is no more G-left invariant in our case but G^i-left invariant. Since $a \subset a^i$, $H^1_{I_T(\Gamma/\Gamma')}(\mathcal{G}_i) = (a^i)^*_{I_T(\Gamma/\Gamma')} = a^*$, and similarly for $H^{n-1}_{I_T(\Gamma/\Gamma')}$. Therefore, their arguments still hold for our case. But, we shall use their argument with the G^i left-invariant differential forms instead. See Section 3.

We obtained Propositions 1 and 2 in 2006 to deal with the pseudo-kählerian case. About the same time when we were trying to publish [4], Professor Fino showed us their paper [10] and told us the work of Professor Witte [11]. It turned out that the modification result in Proposition 1 was known to Witte in 1994 already, and moreover, only need to modify it by the imaginary part. However, there were some errors in Witte's paper, see [12] Section 6 for some corrections. This made things more complicated.

There are also some efforts from [13, 14]. In summer 2013, Professor Dorfmeister told us about Kasuya's work. That eventually lead us to the book [15] and then this version of results.

Here, I would like to thank Professor Fino for showing us about [11] and Professor Dorfmeister for telling us about [13, 14].

2. Proof of our main statements: Theorems 1, 2 and 3

In this section, we shall prove Theorems 1, 2, and 3 as our main statements.

As in [3], we shall use modifications of solvable Lie groups. The modifications we used in [3] first appeared in Azencott and Wilson's paper [16] in 1976. Later on, a similar construction was used in [17] in 1989. Our method in [3] was formally taken from [17], applied to a completely different setting following yet another different setting in an earlier paper. That was the reason that we did not concentrate on the solvable case and made things more complicated. Around 1985, Professor Dorfmeister had a different form of modification in [18], which also originally came from [19]. The modification which we shall use in this paper is similar to Dorfmeister's version, but more general and simpler. We take it from [15].

Let $i : \mathcal{G} \to \mathrm{Der}(\mathcal{G})$ be the homomorphism of the projection of the imaginary part of the adjoint action on to the Lie algebra of T, $A \to i_T(A)$. We define a new product by

$$[A, B]_i = [A, B] - i(A)B + i(B)A.$$

Then, it defines a new Lie algebraic structure. We only need to prove that the Jacobi identity is true. Let

$$s(\Psi(A, B, C)) = \Psi(A, B, C) + \Psi(B, C, A) + \Psi(C, A, B)$$

for any operation Ψ of three elements A, B, and C. Then,

$$s(\Psi(A, B, C)) = s(\Psi(B, C, A)) = s(\Psi(C, A, B)).$$

We notice that $i(i(A)B) = 0$, and have

$$\begin{aligned}
s([[A, B]_i, C]_i) &= s([[A, B] - i(A)B + i(B)A, C]_i) \\
&= s([i(B)A - i(A)B, C]) + s(i(C)([A, B]) - i([A, B])C) \\
&+ s(i(i(A)B - i(B)A)C - i(C)(i(A)B - i(B)A)) \\
&= s([i(B)A, C]) - s([i(A)B, C]) + s([i(C)A, B]) + s([A, i(C)B]) \\
&- s([i(A), i(B)])C - s(i(C)i(A)B) + s(i(C)i(B)A) \\
&= s([i(A)C, B]) - s([i(A)B, C]) + s([i(A)B, C]) - s([i(C)B, A])
\end{aligned}$$

$$
\begin{aligned}
&- \quad s([i(A), i(B)])C - s(i(A)i(B)C) + s(i(B)i(A)C) \\
&= \quad 0
\end{aligned}
$$

From [15] page 94 to 97, the Lie group product is $I_T(g,h) = (I_T(h)g)h$ and $I_T(I_T(g)h) = I_T(h)$. Especially, if Γ' is the kernel of I_T, then $I_T(g,\gamma) = g\gamma$ for any $g \in G,\ \gamma \in \Gamma'$. Therefore, $(\Gamma')^i = \Gamma'$ and $G/\Gamma' = G^i/\Gamma'$.

Moreover,

$$
g\gamma = (I_T(\gamma)I_T(\gamma^{-1})g)\gamma = I_T(I_T(\gamma^{-1})g, \gamma)
$$

for any $g \in G^i,\ \gamma \in \Gamma$. Therefore, the action of Γ on the right invariant Lie algebra $\mathcal{G}_i$ is the same as the inverse action of $I_T(\gamma)$. We have all our Theorems 1, 2, and 3.

A remark from the author is that the algebraic closures of $\mathrm{Ad}(G)$, $\mathrm{Ad}(G^i)$, and $\mathrm{Ad}(\Gamma')$ have the same image of R, but only the last two have the same image from I with $I_{\Gamma,0}$ as the identity components. The first part also comes from that G/Γ' is compact.

3. The generalized Benson-Gordon Theorem: Corollary 4

Here, we shall prove our Corollary 4 following [9] page 974 with the G^i left-invariant differential forms. Since $\mathcal{G}^i$ is a modification of $\mathcal{G}$, as a vector space, we could identify $\mathcal{G}_i$ with $\mathcal{G}$. Then, $\mathcal{G}_i = a \oplus n$.

Lemma 1. *Any* $(n-1)$*-form is closed.*

Proof: Indeed, we can write $A = i(X)vol$. Then, $dA = -\mathrm{tr}(\mathrm{ad}(X))vol = 0$.

Lemma 2. *We write* $\Lambda_i^{s,t} = (a^*)^s \wedge (n^*)^t$ *as a vector subspace of the* G^i *left-invariant vector fields. Then,*

$$
\begin{aligned}
d :\ \Lambda_i^{s,t} &\to \Lambda_i^{s+2,t-1} + \Lambda_i^{s+1,t} + \Lambda_i^{s,t+1}; \\
d :\ \Lambda_i^{k,l-2} &\to \Lambda_i^{k,l-1}; \\
d :\ \Lambda_i^{k-1,l-1} &\to \Lambda_i^{k,l-1}; \\
d :\ \Lambda_i^{k-2,l} &\to \Lambda_i^{k,l-1}.
\end{aligned}
$$

Therefore, $d(\Lambda^{n-2}(\mathcal{G}^i)^* \subset \Lambda_i^{k,l-1}$.

Proof: Since $d(a^*) = 0$, we have $d(\Lambda_i^{s,t}) = \Lambda_i^{s,0} \wedge d(\Lambda_i^{0,t})$ and follow exactly the same argument as in [9].

Lemma 3. *Every element in* $(\Lambda_i^{k,l-1})_{I_T(\Gamma/\Gamma')}$ *is exact and* $H^{n-1}(\mathcal{G}) = (\Lambda_i^{k-1,l})_{I_T(\Gamma/\Gamma')}$.

Proof: Same as in [9].

Next, for any $X \in n$, we have $B = \omega(X,\ ) \wedge \omega^{n-1} \in \Lambda_i^{k,l-1}$. Therefore, B is exact by Lemma 3. That is, $C = \omega(X,\ )$ is not in a^*. Therefore, ω is non-degenerate on n. We let $a = \{A|_{\omega(A,n)\neq 0}\}$. We have $\omega \in \Lambda^{2,0} + \Lambda_i^{0,2}$ with components $\omega^{2,0}$ and $\omega^{0,2}$.

Since $\omega^{2,0}$ is closed, so is $\omega^{0,2}$. $\omega^{2,0}$ is G invariant.

Lemma 4. *a is abelian.*

Proof: $\omega([A,B],X) = \omega(A,[B,X]) + \omega(B,[X,A]) = 0$ for all $X \in \mathcal{G}_i,\ A,B \in a$, and therefore, $[A,B] = 0$.

Similarly, we can get (vi) in Corollary 4 with a as a $\mathcal{G}_i$ subalgebra acting on n.

Following the argument in [9], we have the following lemma.

Lemma 5. *If $Z \in n$ and is in the center of $\mathcal{G}$, then $Z = 0$.*

Proof: $C = \omega(Z, \ )$ is closed. By Lemma 3, $C \wedge \omega^{n-1}$ is exact, and therefore, $C = 0$ and $Z = 0$ by the 1-Lefschetz condition.

This proves (iii). Once we have (iv) in Corollary 4, one can easily see that the 1-Lefschetz condition holds. Therefore, we have proved Corollary 4.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declare there is no conflicts of interest.

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