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Research article

Variational approach to fractional Sturm-Liouville equations with mixed impulses

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Abstract: This research focused on solving fractional Sturm-Liouville equations subject to mixed boundary conditions and a combination of instantaneous and non-instantaneous impulses. Through the use of variational methods, the existence of both solutions and multiple solutions for the aforementioned problem was established under assumptions that were weaker than the classical superlinear Ambrosetti-Rabinowitz type growth requirement. Finally, the examples demonstrated the validity of the paper's results.

Keywords: Sturm-Liouville equation; mixed boundary conditions; non-instantaneous impulses; instantaneous impulses; variational methods

1. Introduction

Fractional differential equations (FDEs) overcome the limitations of traditional integer-order differential equations by introducing nonlocal fractional derivatives, enabling more precise characterization of dynamic behaviors in complex systems with memory, hereditary properties, or anomalous diffusion characteristics. Compared to integer-order models, they provide a superior framework for describing non-classical dynamic phenomena in practical fields such as physics, engineering, biology, and finance, offering a more effective mathematical tool for modeling and simulating complex phenomena. This research not only expands the theoretical framework of equations but also opens new avenues for addressing real-world scientific challenges.

Meanwhile, as a crucial subfield of differential equations, impulsive differential equations play a pivotal role in characterizing abrupt events and discontinuous dynamics inherent in systems like signal processing and multi-agent networks [1]. Due to this capability, they have been extensively studied by numerous researchers (see [2–4]). With the development of fractional-order impulsive differential equations (FIDEs) theory and the expansion of its interdisciplinary applications, recent studies have shifted their focus to the analysis of impulsive fractional-order boundary value problems [5–7]. In recent years, scholars have employed variational methods to achieve a series of results [8–10] for FIDEs subject to Dirichlet boundary conditions (BCs) and Sturm-Liouville BCs. The current body of research regarding FIDEs is predominantly based on the consideration of instantaneous impulses. This is well exemplified by the classical studies of Bonanno et al. [11] and Rodríguez-López and Tersian [12] in 2014, who examined the following FIDEs with Dirichlet BCs:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^CD_t^{\alpha}y(t)) + a(t)y(t) = \lambda g(t, y(t)),\ t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}y))(t_j) = \mu I_j(y(t_j)),\ j = 1, 2, \cdots, n, \\ y(0) = y(T) = 0, \end{cases}$$

where $\alpha \in (1/2, 1]$, $\lambda \in (0, +\infty)$, $\mu \in (0, +\infty)$ are two parameters. The operators ${}_0^CD_t^{\alpha}$ (left Caputo) and ${}_tD_T^{\alpha}$ (right Riemann-Liouville) are both of order α. The function is $g \in C([0, T] \times \mathbb{R}, \mathbb{R})$. $I_j \in C(\mathbb{R}, \mathbb{R})$, $j \in \{1, 2, \cdots, n\}$. The function is $a \in C([0, T])$, and there are two positive constants a_1 and a_2, such that $0 < a_1 \leq a(t) \leq a_2$. The authors employed variational methods and applied a three-critical-point theorem-which guarantees the existence of at least three critical points for the functional under specific geometric conditions-thereby proving that the aforementioned problem admits at least three solutions.

However, instantaneous impulses are often inadequate for describing numerous phenomena and models encountered in practice. This is evident in areas including, but not limited to, evolutionary dynamics and pharmacokinetics. In 2013, Hernádez and O'Regan [2] proposed the notion of non-instantaneous impulses for the first time, constructing the first non-instantaneous impulsive model based on pharmacokinetics. Unlike instantaneous impulses, non-instantaneous impulses describe a scenario where the state undergoes an abrupt change at a certain moment due to disturbance and maintains this altered state over a finite duration. This type of impulse offers distinct advantages in modeling processes such as population dynamics, and the absorption, diffusion, and metabolism of drugs in the human body [13–15]. Since then, significant progress has been made in both theoretical and applied research on non-instantaneous impulsive differential equations [16–18]. For example, in 2018, Khaliq and Rehman [19] employed the Lax-Milgram theorem to examine the solvability of FDEs with non-instantaneous impulses subject to BCs:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^CD_t^{\alpha}y(t)) = g_i(t), t \in (s_i, t_{i+1}], i = 0, 1, \cdots, m, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y(t)\right) = c_i, t \in (t_i, s_i], i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y\left(s_i^-\right)\right) = {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y\left(s_i^+\right)\right),\ i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y\left(s_i^-\right)\right) = c_0, y(0) = y(T) = 0, \end{cases}$$

where $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_m < t_{m+1} = T$, and ${}_0^CD_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are left Caputo fractional derivatives and right Riemann-Liouville fractional derivatives of the order $0 < \alpha \leq 1$, respectively. Keeping the derivative ${}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y(t)\right)$ constant on a finite interval $(t_i, s_i]$, the impulses begin suddenly at the points t_i. Here, $g_i : (s_i, t_{i+1}] \to \mathbb{R}$ are given functions, and c_i are given constants. In 2020, Zhang and Liu [20] pioneered the study of the FDEs with mixed instantaneous and non-instantaneous impulses under Dirichlet BCs:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^CD_t^{\alpha}y(t)) = g_i(t, y(t)), t \in (s_i, t_{i+1}], i = 0, 1, \cdots, m, \\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y\right)\right)(t_i) = I_i(y(t_i)), i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y(t)\right) = {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y\left(t_i^+\right)\right), t \in (t_i, s_i], i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y\left(s_i^-\right)\right) = {}_tD_T^{\alpha-1}\left({}_0^CD_t^{\alpha}y\left(s_i^+\right)\right),\ i = 1, 2, \cdots, m, \\ y(0) = y(T) = 0, \end{cases}$$

where $\alpha \in (1/2, 1]$, and ${}_0^C D_t^\alpha$ and ${}_t D_T^\alpha$ are left Caputo fractional derivatives and right Riemann-Liouville fractional derivatives of the order α, respectively. $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_m < t_{m+1} = T$, $I_i \in C(\mathbb{R}, \mathbb{R})$, and there exists $i \in \{1, 2, \cdots, m\}$ such that $I_i(u(t_i)) \neq 0$, $f_i \in C((s_i, t_{i+1}] \times \mathbb{R}, \mathbb{R})$. Using variational methods, the solvability of the aforementioned problem was proved under the condition that the nonlinear term satisfies sublinear growth. In 2024, Qiao et al. [21] employed critical point theory to FDEs featuring combined instantaneous and non-instantaneous impulses under Sturm-Liouville BCs:

$$\begin{cases} -\frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(y'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(y'(t))\right) + g(t)y(t) = \lambda f_i(t, y(t)), t \in (s_i, t_{i+1}], i = 0, 1, \cdots, m, \\ a\left(\frac{1}{2}{}_0D_t^{-\beta}(y'(0)) + \frac{1}{2}{}_tD_T^{-\beta}(y'(0))\right) - bh(y(0)) = A, \\ c\left(\frac{1}{2}{}_0D_t^{-\beta}(y'(T)) + \frac{1}{2}{}_tD_T^{-\beta}(y'(T))\right) + dh(y(T)) = B, \\ \Delta\left(\frac{1}{2}{}_0D_t^{-\beta}(y'(t_i)) + \frac{1}{2}{}_tD_T^{-\beta}(y'(t_i))\right) = \mu I_i(y(t_i)), i = 1, 2, \cdots, m, \\ {}_0D_t^{-\beta}(y'(t)) + {}_tD_T^{-\beta}(y'(t)) = {}_0D_t^{-\beta}\left(y'\left(t_i^+\right)\right) + {}_tD_T^{-\beta}\left(y'\left(t_i^+\right)\right), t \in (t_i, s_i], i = 1, 2, \cdots, m, \\ {}_0D_t^{-\beta}\left(y'\left(s_i^-\right)\right) + {}_tD_T^{-\beta}\left(y'\left(s_i^-\right)\right) = {}_0D_t^{-\beta}\left(y'\left(s_i^+\right)\right) + {}_tD_T^{-\beta}\left(y'\left(s_i^+\right)\right), \ i = 1, 2, \cdots, m, \end{cases}$$

where $\beta \in [0, 1)$, $\lambda > 0, a, b, c, d > 0$. A and B are constants. ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ denote the left and right Riemann-Liouville fractional integrals of order β, respectively, and $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_m < t_{m+1} = T$. The function is $g \in C^1([0, T], \mathbb{R})$, and there are g_1 and g_2 such that $0 < g_1 \leq g(t) \leq g_2$. $f_i \in C^1([s_i, t_{i+1}] \times \mathbb{R}, \mathbb{R})$ for $i = 0, 1, 2, \cdots, m$. The function is $h \in C^1(\mathbb{R}, \mathbb{R})$. The instantaneous impulses $I_i \in C^1(\mathbb{R}, \mathbb{R})$ start to change suddenly at the points t_i and the non-instantaneous impulses continue during the finite intervals $(t_i, s_i]$, for $i = 1, \cdots, m$. The aforementioned literature has discussed the existence of solutions for boundary value problems of FDEs with instantaneous or non-instantaneous impulses under various BCs. However, research on mixed BCs remains scarce. Therefore, this paper will further investigate the existence of solutions for FDEs involving both instantaneous and non-instantaneous impulses under mixed BCs.

Mixed BCs are crucial in differential equations, as they incorporate a linear combination of both the variable values and their derivatives, enabling more accurate modeling of practical phenomena such as heat conduction, fluid dynamics, and electrical circuits. Research on such problems contributes to constructing mathematical models that better align with real-world scenarios, providing more reliable theoretical foundations for engineering and scientific computations. Progress in fractional calculus has given rise to extensive research on mixed boundary value problems for FDEs [22–25]. In a 2025 study, Wang et al. [26] addressed a fractional Sturm-Liouville equation subject to mixed BCs and featuring both types of impulses (instantaneous and non-instantaneous). Application of the variational approach led to a set of solvability criteria for the problem over different parameter ranges. Through an in-depth review of relevant literature, we observed that current research on such problems primarily focuses on cases where the nonlinear term satisfies sublinear growth conditions or superlinear growth conditions. A typical example of the latter is the classical Ambrosetti-Rabinowitz condition, which requires the existence of constants $\theta > 2$ and $R > 0$ such that for all $|u| \geq R$ and a.e. $t \in [0, T]$, the nonlinearity $f(t, u)$ satisfies

$$0 < \theta F(t, u) \leq u f(t, u),$$

where $F(t, u) = \int_0^u f(t, s)\, ds$ is the primitive of f.

Based on this, this paper aims to investigate the existence and multiplicity of solutions for such problems under weaker growth conditions. Specifically, we will employ variational methods and critical point theory to further explore the following class of mixed boundary value problems for fractional Sturm-Liouville equations with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}_tD_T^{\alpha}\left(p(t)\,{}_0^CD_t^{\alpha}u(t)\right)+q(t)\,u(t)=\lambda f_i(t,u(t)),\ t\in(s_i,t_{i+1}], i=0,1,2,\cdots,n,\\ -\Delta\left({}_tD_T^{\alpha-1}\left(p(t_i)\,{}_0^CD_t^{\alpha}u(t_i)\right)\right)=\mu I_i(u(t_i)), i=1,2,\cdots,n,\\ {}_tD_T^{\alpha-1}\left(p(t)\,{}_0^CD_t^{\alpha}u(t)\right)={}_tD_T^{\alpha-1}\left(p(t_i^+)\,{}_0^CD_t^{\alpha}u(t_i^+)\right),\ t\in(t_i,s_i], i=1,2,\cdots,n,\\ {}_tD_T^{\alpha-1}\left(p(s_i^-)\,{}_0^CD_t^{\alpha}u(s_i^-)\right)={}_tD_T^{\alpha-1}\left(p(s_i^+)\,{}_0^CD_t^{\alpha}u(s_i^+)\right), i=1,2,\cdots,n,\\ u(0)=0,\ \beta u(T)+\dfrac{1}{p(T)}\,{}_tD_T^{\alpha-1}\left(p(T)\,{}_0^CD_t^{\alpha}u(T)\right)=b. \end{cases} \tag{1.1}$$

The instantaneous impulse phenomenon triggers abrupt jumps in the state quantity at specific moments t_i, while the non-instantaneous impulse effect evolves continuously over the time intervals $(t_i, s_i]$, that is,

$$\Delta\left({}_tD_T^{\alpha-1}\left(p(t_i)\,{}_0^CD_t^{\alpha}u(t_i)\right)\right)={}_tD_T^{\alpha-1}\left(p(t_i^+)\,{}_0^CD_t^{\alpha}u(t_i^+)\right)-{}_tD_T^{\alpha-1}\left(p(t_i^-)\,{}_0^CD_t^{\alpha}u(t_i^-)\right),$$
$${}_tD_T^{\alpha-1}\left(p(t_i^{\pm})\,{}_0^CD_t^{\alpha}u(t_i^{\pm})\right)=\lim_{t\to t_i^{\pm}}{}_tD_T^{\alpha-1}\left(p(t)\,{}_0^CD_t^{\alpha}u(t)\right),$$
$${}_tD_T^{\alpha-1}\left(p(s_i^{\pm})\,{}_0^CD_t^{\alpha}u(s_i^{\pm})\right)=\lim_{t\to s_i^{\pm}}{}_tD_T^{\alpha-1}\left(p(t)\,{}_0^CD_t^{\alpha}u(t)\right).$$

In (1.1), $\alpha\in(\frac{1}{2},1]$, $\lambda,\mu,\beta>0$. The operators ${}_0^CD_t^{\alpha}$ (left Caputo) and ${}_tD_T^{\alpha}$ (right Riemann-Liouville) are both of order α. $0=s_0<t_1<s_1<t_2<\cdots<s_n<t_{n+1}=T$, $I_i\in C^1(\mathbb{R},\mathbb{R})$ with at least one $i\in\{1,2,\cdots,n\}$ such that $I_i(u(t_i))\neq 0$, b is a constant, $f_i\in C^1((s_i,t_{i+1}]\times\mathbb{R},\mathbb{R})$, $p(t)\in C^1([0,T],\mathbb{R}^+)$, $0<p_0=\min_{[0,T]}p(t)$, $q(t)\in C^1([0,T],\mathbb{R}^+)$, and $0<q_0=\min_{[0,T]}q(t)\le q(t)\le q^0=\max_{[0,T]}q(t)$.

Existence results for solutions of problem (1.1) are obtained through variational approaches. By constructing a variational framework, the existence results for problem (1.1) are first proved under weakened super Ambrosetti-Rabinowitz type growth conditions using the mountain pass theorem. Subsequently, the nonlinearity is bifurcated into distinct regimes: a superlinear growth condition and a sublinear growth condition. Based on this decomposition, multiple existence of solutions for (1.1) is proved via genus theory. The innovations of this work are mainly reflected in the following aspects: On one hand, the assumptions on the nonlinearity are weaker than the classical Ambrosetti-Rabinowitz condition. On the other hand, by introducing nonlinear terms that combine both concave and convex features along with weakened superlinear growth conditions, the existing theoretical framework has been extended. In particular, compared with reference [26] which only deals with the Ambrosetti-Rabinowitz case, this paper establishes richer existence and multiplicity results under more general hypotheses, thereby substantially advancing the research on this problem.

The organizational framework of this work is arranged in the following manner: The second section assembles the requisite definitions and technical lemmas that underpin our later theoretical arguments. The variational approach developed in Section 3 yields existence results for problem (1.1). Finally, the fourth section summarizes the work of this paper and provides an outlook on future research directions.

2. Preliminaries

Definition 2.1 ([27, Definition 2.2]). The left Caputo derivative ${}_0^CD_t^{\alpha}u(t)$ and the right Riemann-Liouville derivative ${}_tD_T^{\alpha}u(t)$ of order $\alpha>0$ ($n=\lceil\alpha\rceil$) for the function $u(t)$ are defined, respectively, as:

$$ {}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)ds, \ {}_t D_T^\alpha u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^T (s-t)^{n-\alpha-1} u(s)ds. $$

Definition 2.2 ([28, Definition 4.1]). Let $\alpha \in (\frac{1}{2}, 1]$. The fractional derivative space

$$ E^\alpha = \left\{ u : [0,T] \to \mathbb{R}^N : u \text{ is absolutely continous and } {}_0^C D_t^\alpha u \in L^2\left([0,T], \mathbb{R}^N\right) \right\} $$

is defined by the closure of $C_0^\infty\left([0,T], \mathbb{R}^N\right)$ with the norm

$$ \|u\|_{E^\alpha} = \left(\int_0^T \left| {}_0^C D_t^\alpha u(t) \right|^2 dt + \int_0^T |u(t)|^2 dt \right)^{\frac{1}{2}}. $$

E^α is a reflexive and separable Banach space.

Remark 2.1. For all $u \in E^\alpha$, we have $u \in L^2([0,T], \mathbb{R}^N)$ and ${}_0^C D_t^\alpha u \in L^2([0,T], \mathbb{R}^N)$.

Lemma 2.1 ([29, Remark 2.1]). For $u \in E^\alpha$, we prove that $\|u\|_{E^\alpha}$ is equivalent to

$$ \|u\|_\alpha = \left(\int_0^T p(t) \left| {}_0^C D_t^\alpha u(t) \right|^2 dt + \int_0^T q(t) |u(t)|^2 dt \right)^{\frac{1}{2}}, $$

where $p(t) \in L^\infty([0,T])$, $q(t) \in C([0,T])$, $\operatorname{essinf}_{[0,T]} p(t) > 0$, and $0 < q_0 \le q(t) \le q^0$.

Lemma 2.2 ([27, Property 2.2]). Let $q_1, q_2 \ge 1, \beta > 0$, and either $\frac{1}{q_1} + \frac{1}{q_2} \le 1 + \beta$ or $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \beta$, $q_1 \ne 1$, $q_2 \ne 1$. Then, for any $x \in L^{q_1}\left([0,T], \mathbb{R}^N\right)$, $y \in L^{q_2}\left([0,T], \mathbb{R}^N\right)$, the following equality holds:

$$ \int_0^T \left[{}_0 D_t^{-\beta} x(t) \right] y(t)\, dt = \int_0^T \left[{}_t D_T^{-\beta} y(t) \right] x(t)\, dt. $$

Definition 2.3 ([26, Definition 2.5]). Let $\frac{1}{2} < \alpha \le 1$. We define $E_0^\alpha = \{u \in E^\alpha : u(0) = 0\}$ to be the completion of $C_0^\infty\left([0,T], \mathbb{R}^N\right)$ with respect to $\|u\|_\alpha$.

Lemma 2.3 ([26, Lemma 2.5]). Let $\alpha \in (\frac{1}{2}, 1]$. E_0^α is a reflexive and separable Banach space.

Lemma 2.4 ([26, Lemma 2.6]). Let $\alpha \in (\frac{1}{2}, 1]$. If $u_k \rightharpoonup u$ in E_0^α, then $u_k \to u$ in $C([0,T], \mathbb{R}^N)$ with $\|u_k - u\|_\infty \to 0$ as $k \to \infty$.

In the space E_0^α, define a new norm

$$ \|u\| = \left(\int_0^T p(t) \left| {}_0^C D_t^\alpha u(t) \right|^2 dt + \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} q(t) |u(t)|^2 dt \right)^{\frac{1}{2}}, \tag{2.1} $$

where $p(t) \in C^1([0,T], \mathbb{R}^+)$, $0 < p_0 = \min_{[0,T]} p(t)$, $q(t) \in C^1([0,T], \mathbb{R}^+)$, and $0 < q_0 = \min_{[0,T]} q(t) \le q(t) \le q^0 = \max_{[0,T]} q(t)$.

Lemma 2.5 ([26, Lemma 3.1]). For $\forall u \in E_0^\alpha$, it follows that $\|u\|_\alpha$ and $\|u\|$ are equivalent, in the sense that there are $\varrho_1, \varrho_2 > 0$ for which

$$ \varrho_1 \|u\|_\alpha \le \|u\| \le \varrho_2 \|u\|_\alpha. $$

Lemma 2.6 ([26, Remark 3.1]). Assuming $\alpha \in (\frac{1}{2}, 1]$, the infinity norm of u satisfies the estimate $\|u\|_\infty \le M \|u\|$, in which the constant M is defined as

$$ M := \frac{T^{\alpha - \frac{1}{2}} p_0^{-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha - 1)^{\frac{1}{2}}}, $$

where $T, p_0 > 0$.

Lemma 2.7 ([30]). Suppose E is a reflexive Banach space. Additionally, let Θ be bounded and weakly order-closed, with φ being order-weakly lower semicontinuous over Θ. Then, for the function $\varphi : \Theta \subseteq E \to [-\infty, +\infty]$, and under the condition that $\Theta \neq \emptyset$, the minimization problem $\min_{u\in\Theta}\varphi(u) = \varepsilon$ admits a solution.

Definition 2.4 ([31]). (Palais-Smale Condition). Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$. The functional φ is said to satisfy the Palais-Smale condition (PS-condition), if any sequence $\{u_n\} \subset E$ such that $\{\varphi(u_n)\}$ is bounded, and $\varphi'(u_n) \to 0$ in the dual space E^* as $n \to \infty$, admits a convergent subsequence in E.

Lemma 2.8 ([31, Theorem 2.2]). (Mountain Pass Theorem). Assume $\varphi \in C^1(E, \mathbb{R})$ fulfills the PS-condition (see Definition 2.4) with $\varphi(0) = 0$. Suppose additionally that φ verifies the following two conditions:

(i) We can find $\rho, \sigma > 0$ with the following property: if any element $u_0 \in E$ has its norm equal to ρ, then the functional value $\varphi(u_0)$ is bounded below by σ;

(ii) The functional φ attains values below σ at some point $u_1 \in E$ beyond the norm threshold ρ.

Then, φ has a critical value $\varepsilon \geq \sigma$. Moreover, ε can be expressed as $\varepsilon = \inf_{k\in Y}\max_{s\in[0,1]}\varphi(k(s))$, where

$$Y = \left\{k \in C^1([0,1], E) : k(0) = u_0, k(1) = u_1\right\}.$$

Definition 2.5 ([31]). (Krasnoselskii Genus). Let E be a real Banach space and U be a closed symmetric subset of $E \setminus \{0\}$. The Krasnoselskii genus (or genus) of U is defined as the smallest integer n for which there exists an odd continuous mapping $\Psi : U \to \mathbb{R}^n \setminus \{0\}$. If no such integer exists, we define $\gamma(U) = +\infty$. By convention, $\gamma(\emptyset) = 0$.

Lemma 2.9 ([31]). Let φ be an even C^1 functional on E and satisfy the PS-condition (see Definition 2.4). For any $n \in \mathbb{N}$, $z \in \mathbb{R}$, set $\Sigma = \{U \subset E - \{0\} : U \subset E$ to be closed and 0-symmetric $\}$, $\Sigma_n = \{U \in \Sigma : \gamma(U) \geq n\}$, $K_z = \{u \in E : \varphi(u) = z, \varphi'(u) = 0\}$, $z_n = \inf_{U\in\Sigma_n}\sup_{u\in U}\varphi(u)$. Consequently,

(i) Given a nonempty class Σ_n and a real number z_n, the value z_n constitutes a critical value for the functional φ.

(ii) Suppose for some natural number l, the sequence $z_n = z_{n+1} = z_{n+2} = \cdots = z_{n+l} = z \in \mathbb{R}$ stabilizes at a real value $z \neq \varphi(0)$. Then, the Krasnoselskii genus of K_z satisfies $\gamma(K_z) \geq l + 1$.

Remark 2.2 [31, Remark 7.3]. Implies that when the critical set K_z belongs to Σ and has genus greater than 1, K_z must contain an infinite number of distinct elements.

Definition 2.6. The function $u \in E_0^\alpha$ is a weak solution of (1.1) if u satisfies the following equation

$$\int_0^T p(t)\,{}_0^C D_t^\alpha u(t)\,{}_0^C D_t^\alpha v(t)\,dt + \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} q(t)\,u(t)\,v(t)\,dt - (b - \beta u(T))\,p(T)\,v(T)$$
$$= \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} f_i(t, u(t))\,v(t)\,dt + \mu\sum_{i=1}^{n} I_i(u(t_i))\,v(t_i),\ \forall v \in E_0^\alpha.$$

Consider the energy functional $\varphi_\lambda : E_0^\alpha \to \mathbb{R}$ given by

$$\varphi_\lambda(u) = \frac{1}{2}\|u\|^2 - \mu\sum_{i=1}^{n}\int_0^{u(t_i)} I_i(s)\,ds + \frac{p(T)}{2\beta}(b - \beta u(T))^2 - \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t, u(t))\,dt,\ u \in E_0^\alpha, \quad (2.2)$$

with $F_i(t,u(t)) = \int_0^u f_i(t,s)\,ds$. Due to the continuity of f_i and I_i, it follows that $\varphi_\lambda \in C^1\left(E_0^\alpha, \mathbb{R}\right)$, and

$$\begin{aligned}\langle \varphi_\lambda'(u), v\rangle = &\int_0^T p(t)\,{}_0^C D_t^\alpha u(t)\,{}_0^C D_t^\alpha v(t)\,dt + \sum_{i=0}^n \int_{s_i}^{t_{i+1}} q(t)\,u(t)\,v(t)\,dt \\ &- \lambda \sum_{i=0}^n \int_{s_i}^{t_{i+1}} f_i(t,u(t))\,v(t)\,dt - \mu \sum_{i=1}^n I_i(u(t_i))\,v(t_i) - (b - \beta u(T))\,p(T)\,v(T),\ \forall u, v \in E_0^\alpha.\end{aligned} \tag{2.3}$$

In this framework, weak solutions to (1.1) bijectively correspond to critical points of φ_λ.

3. Main results

As a foundation for the main theorems, we first establish the necessary assumptions.

(H_1) We can find growth exponents $0 \le l_i < 1, K_i > 0, L_i > 0, (i = 1, 2, \cdots, n)$ satisfying the subcritical growth condition:

$$|I_i(u)| \le K_i |u|^{l_i} + L_i,\ u \in \mathbb{R}.$$

(H_2) $\limsup\limits_{|u|\to 0} \frac{F_i(t,u)}{|u|^2} = 0$, $(i = 0, 1, \cdots, n)$, holds almost everywhere uniformly for $t \in (s_i, t_{i+1}]$.

(H_3) There exists a $\theta > 2$ such that the asymptotic inequality

$$\limsup_{|u|\to\infty} \frac{\theta F_i(t,u) - f_i(t,u)\,u}{|u|^2} \le 0,\ (i = 0, 1, \cdots, n),$$

holds uniformly for almost every $t \in (s_i, t_{i+1}]$.

(H_4) There are positivity subsets $\Omega_i \subset (s_i, t_{i+1}]$, $\mathrm{meas}(\Omega_i) > 0$, where meas(.) denotes the Lebesgue measure. Then, the nonlinear potential satisfies

$$\liminf_{|u|\to\infty} \frac{F_i(t,u)}{|u|^2} > 0,\ (i = 0, 1, \cdots, n),$$

uniformly for almost every $t \in \Omega_i$.

Lemma 3.1 . Given the validity of conditions (H_1) and (H_3), the energy functional φ_λ fulfills the PS-condition.

Proof. Consider a sequence $\{u_m\}_{m\in\mathbb{N}}$ in E_0^α with $\{\varphi_\lambda(u_m)\}_{m\in\mathbb{N}}$ bounded and $\lim\limits_{m\to\infty} \varphi'_\lambda(u_m) = 0$. Consequently, we can find $N > 0$ satisfying the uniform bounds:

$$|\varphi_\lambda(u_m)| \le N,\ \left\|\varphi'_\lambda(u_m)\right\|_* \le N,\ m \in \mathbb{N}, \tag{3.1}$$

with $\|\cdot\|_*$ being the dual space norm of E_0^α.

We proceed by contradiction to prove the boundedness of $\{u_m\}$. Suppose $\{u_m\}$ is unbounded, then $\|u_m\| \to \infty$. Let $v_m = \frac{u_m}{\|u_m\|}$, then $\|v_m\| = 1$. According to Lemma 2.4, assuming $v_m \rightharpoonup v_0$ in E_0^α, then $v_m \to v_0$ in $C^1([0,T], \mathbb{R})$, as $m \to \infty$. Combining Lemma 2.6 and condition (H_1), it follows that

$$I_i(u_m(t_i))\,u_m(t_i) \ge -K_i M^{l_i+1} \|u_m\|^{l_i+1} - L_i M\,\|u_m\|,$$

and

$$\left|\sum_{i=1}^{n}\int_{0}^{u_m(t_i)} I_i(s)\,ds\right| \le \sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}\|u_m\|^{l_i+1}}{l_i+1} + L_i M\,\|u_m\|\right).$$

From $\theta > 2$ and Lemma 2.6, it follows that

$$\begin{aligned}
&\frac{\theta p(T)}{2\beta}(b-\beta u_m(T))^2 + p(T)(b-\beta u_m(T))\,u_m(T) \\
&= \left(\frac{\theta}{2}-1\right)\frac{p(T)}{\beta}(b-\beta u_m(T))^2 + \frac{b}{\beta}p(T)(b-\beta u_m(T)) \\
&\ge -|b|\,\frac{p(T)}{\beta}\,|b-\beta u_m(T)| \ge -|b|\,\frac{p(T)}{\beta}\,(|b| + \beta M\,\|u_m\|)\,.
\end{aligned}$$

In summary, there is a fixed constant $N_1 > 0$ satisfying

$$\begin{aligned}
\left(\frac{\theta}{2}-1\right)\|u_m\|^2 &= \theta\varphi_\lambda(u_m) - \varphi'_\lambda(u_m)\,u_m + \theta\mu\sum_{i=1}^{n}\int_0^{u_m(t_i)} I_i(s)\,ds - \mu\sum_{i=1}^{n} I_i(u_m(t_i))\,u_m(t_i) \\
&\quad - \frac{\theta p(T)(b-\beta u_m(T))^2}{2\beta} - p(T)(b-\beta u_m(T))\,u_m(T) \\
&\quad + \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\left[\theta F_i(t,u_m(t)) - f_i(t,u_m(t))\,u_m(t)\right]dt \\
&\le N_1(1+\|u_m\|) + \theta\mu\sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}\|u_m\|^{l_i+1}}{l_i+1} + L_i M\|u_m\|\right) \\
&\quad + \mu\sum_{i=1}^{n}\left(K_i M^{l_i+1}\|u_m\|^{l_i+1} + L_i M\|u_m\|\right) + |b|\,\frac{p(T)}{\beta}\,(|b| + \beta M\,\|u_m\|) \\
&\quad + \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\left(\theta F_i(t,u_m(t)) - f_i(t,u_m(t))\,u_m(t)\right)dt.
\end{aligned}$$

From $\|u_m\| \to \infty$, it follows that

$$\begin{aligned}
\left(\frac{\theta}{2}-1\right)\|v_m\|^2 &\le \frac{N_1(1+\|u_m\|)}{\|u_m\|^2} + \theta\mu\sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}\|u_m\|^{l_i+1}}{(l_i+1)\|u_m\|^2} + \frac{L_i M\|u_m\|}{\|u_m\|^2}\right) \\
&\quad + \mu\sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}\|u_m\|^{l_i+1}}{\|u_m\|^2} + \frac{L_i M\|u_m\|}{\|u_m\|^2}\right) + |b|\,\frac{p(T)}{\beta}\left(\frac{|b|}{\|u_m\|^2} + \frac{\beta M\,\|u_m\|}{\|u_m\|^2}\right) \\
&\quad + \frac{\lambda}{\|u_m\|^2}\times\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\left[\theta F_i(t,u_m(t)) - f_i(t,u_m(t))\,u_m(t)\right]dt.
\end{aligned} \tag{3.2}$$

From (H_3), it follows that there exist $\Omega_{i0} \subset (s_i, t_{i+1}]$, $\operatorname{meas}(\Omega_{i0}) = 0$, such that the asymptotic inequality

$$\limsup_{|u|\to\infty}\frac{\theta F_i(t,u) - f_i(t,u)\,u}{|u|^2} \le 0$$

holds uniformly for $t \in (s_i, t_{i+1}] \setminus \Omega_{i0}$. We assert that

$$\limsup_{m\to\infty} \frac{\theta F_i(t, u_m(t)) - f_i(t, u_m(t))\, u_m(t)}{\|u_m\|^2} \le 0, \quad t \in (s_i, t_{i+1}] \setminus \Omega_{i0}. \tag{3.3}$$

If it does not exist, then there exist $t_0 \in (s_i, t_{i+1}] \setminus \Omega_{i0}$ and a subsequence (relabeled as $\{u_m\}$) satisfying

$$\limsup_{m\to\infty} \frac{\theta F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{\|u_m\|^2} > 0. \tag{3.4}$$

Assuming the boundedness of $\{u_m(t_0)\}$, we can find $N_2 > 0$ satisfying $|u_m(t_0)| \le N_2$, $\forall m \in \mathbb{N}$. Since f_i is continuous on Ω_{i0}, it follows that

$$\frac{\theta F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{\|u_m\|^2} \le \frac{(\theta+1)\, N_2}{\|u_m\|^2} \to 0, \ m \to \infty,$$

which contradicts (3.4). Therefore, $\{u_m(t_0)\}$ admits a subsequence exhibiting divergence: $|u_m(t_0)| \to \infty, \ m \to \infty$. Hence,

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{\theta F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{\|u_m\|^2} \\
&= \limsup_{m\to\infty} \frac{\theta F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{|u_m(t_0)|^2} \cdot |v_m(t_0)|^2 \\
&= \limsup_{m\to\infty} \frac{\theta F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{|u_m(t_0)|^2} \cdot \lim_{m\to\infty} |v_m(t_0)|^2 \\
&\le 0.
\end{aligned}$$

This contradicts (3.4). Hence, (3.3) holds.

Through Eqs (3.2) and (3.3), it follows that $\limsup\limits_{m\to\infty}\left(\frac{\theta}{2}-1\right)\|v_m\|^2 \le 0$. The condition $\theta > 2$ implies the convergence $\|v_m\|^2 \to 0$ as $m \to \infty$, which contradicts $\|v_m\| = 1$. As a result, $\{u_m\}$ is energy-bounded in E_0^α.

The following proves that $u_m \to u$ in E_0^α. Indeed, by reflexivity of E_0^α, we may extract a subsequence (still denoted u_m) converging weakly: $u_m \rightharpoonup u$ in E_0^α. Consequently, $u_m \to u$ in $C^1([0,T], \mathbb{R})$. It then follows that

$$u_m(T) \to u(T), \ m \to \infty,$$

$$\langle \varphi'_\lambda(u_m) - \varphi'_\lambda(u), u_m - u \rangle \to 0, \ m \to \infty,$$

$$(I_i(u_m(t)) - I_i(u(t)))(u_m(t) - u(t)) \to 0, \ m \to \infty,$$

$$\int_{s_i}^{t_{i+1}} (f_i(t, u_m(t)) - f_i(t, u(t)))(u_m(t) - u(t))\, dt \to 0, \ m \to \infty.$$

Thus,

$$
\begin{aligned}
&\|u_m - u\|^2 = \langle \varphi'_\lambda(u_m) - \varphi'_\lambda(u), u_m - u\rangle \\
&+ \mu \sum_{i=1}^{n} \left(I_i(u_m(t)) - I_i(u(t))\right)(u_m(t) - u(t)) \\
&+ bp(T)(u_m(T) - u(T)) - p(T)(u_m(T) - u(T))^2\beta \\
&+ \lambda \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} \left(f_i(t, u_m(t)) - f_i(t, u(t))\right)(u_m(t) - u(t))\,dt \\
&\to 0, \quad m \to \infty.
\end{aligned}
$$

This means that $u_m \to u$ in E_0^α.□ □

Theorem 3.1 . If (H_1)–(H_4) hold and the constant $\Lambda = \frac{\delta^2}{2M^2} - \frac{p(T)\beta\delta^2}{2} - \mu\sum_{i=1}^{n}\left(\frac{K_i}{l_i+1}\delta^{l_i+1} + L_i\delta\right) - \frac{p(T)b^2}{2\beta} > 0$, where $\delta \in (0,\kappa)$, $\kappa > 0$, then for each $\lambda \in \left(0, \frac{\Lambda}{\kappa T\delta^2}\right)$, problem (1.1) admits at least two distinct weak solutions.

Proof. Denote by B_r the open ball of radius r centered at the origin in the space E_0^α. Here, ∂B_r stands for the boundary of B_r, and $\overline{B}_r$ for its closure. Standard arguments demonstrate the weak closedness and boundedness of $\overline{B}_{\frac{\delta}{M}}$. Through meticulous verification, we demonstrate that $\varphi_\lambda(u)$ is weakly lower semicontinuous (w.l.s.c.) in E_0^α. According to Lemma 2.7, $\varphi_\lambda(u)$ attains a local minimizer u_0 within $\overline{B}_{\frac{\delta}{M}}$, that is, $\varphi_\lambda(u_0) \le \varphi_\lambda(0) = 0$.

By hypothesis (H_2), when $\kappa > 0$, $i = 0, 1, \cdots, n$, one can find $\delta \in (0,\kappa)$ ensuring that for a.e. $t \in (s_i, t_{i+1}]$, $u \in \mathbb{R}$ satisfying $|u| \le \delta$, and the following holds:

$$
|F_i(t,u)| \le \kappa|u|^2. \tag{3.5}
$$

Assume $\|u\| \le \frac{\delta}{M}$. By Lemma 2.6, we have $\|u\|_\infty \le \delta$. Then, for $u \in \partial B_r, (r \le \frac{\delta}{M})$, combining Lemma 2.6, (H_1), and (3.5), we obtain

$$
\begin{aligned}
\varphi_\lambda(u) &\ge \frac{1}{2}\|u\|^2 - \lambda\kappa TM^2\|u\|^2 - \frac{p(T)}{2\beta}\left(b^2 + \beta^2M^2\|u\|^2\right) - \mu\sum_{i=1}^{n}\left(\frac{K_iM^{l_i+1}}{l_i+1}\|u\|^{l_i+1} + L_iM\|u\|\right) \\
&= \left(\frac{1}{2} - \frac{p(T)\beta M^2}{2} - \lambda\kappa TM^2\right)r^2 - \mu\sum_{i=1}^{n}\left(\frac{K_iM^{l_i+1}}{l_i+1}r^{l_i+1} + L_iMr\right) - \frac{p(T)b^2}{2\beta}.
\end{aligned}
$$

Then, for $u \in \partial B_{\frac{\delta}{M}}$, we get

$$
\varphi_\lambda(u) \ge \left(\frac{1}{2} - \frac{p(T)\beta M^2}{2} - \lambda\kappa TM^2\right)\frac{\delta^2}{M^2} - \mu\sum_{i=1}^{n}\left(\frac{K_i}{l_i+1}\delta^{l_i+1} + L_i\delta\right) - \frac{p(T)b^2}{2\beta} = N_\lambda.
$$

Given that $\lambda \in \left(0, \frac{\Lambda}{\kappa T\delta^2}\right)$, it follows that $\varphi_\lambda(u) = N_\lambda > 0 \ge \varphi_\lambda(u_0)$ holds for all $u \in \partial B_{\frac{\delta}{M}}$. Therefore, $\inf_{u\in\partial B_{\frac{\delta}{M}}}\varphi_\lambda(u) > \varphi_\lambda(u_0)$.

For implementation purposes, each segment $i = 0, 1, \cdots, n$ under constraints (H_3) and (H_4) with continuous f_i generates design parameters $N_3 > 0$, $\rho > \frac{(\theta-2)M^2\left[p(T)\beta M^2 + \frac{1}{2}\right]}{2\lambda}$ and operational domain Ω_i

(positive measure) satisfying

$$|F_i(t,u)| \geq \frac{2\rho}{\theta-2}|u|^2 - N_3,\ t \in \Omega_i, u \in \mathbb{R}. \tag{3.6}$$

Select $u_0(t) \in E_0^\alpha$ with $\|u_0\| \leq M$, $\int_{\Omega_i} |u_0(t)|^2 dt = 1$. For $\xi > 0$, it follows from (3.6) and (H_1) that,

$$\begin{aligned}
\varphi_\lambda(\xi u_0) \leq & \frac{\xi^2}{2}\|u_0\|^2 - \lambda(n+1)\frac{2\rho\xi^2}{\theta-2} + \lambda\sum_{i=0}^{n} N_3 \mathrm{meas}(\Omega_i) \\
& + \frac{p(T)}{\beta}\left(b^2 + \beta^2 M^2 \xi^2 \|u_0\|^2\right) \\
& + \mu\sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}\xi^{l_i+1}}{l_i+1}\|u_0\|^{l_i+1} + L_i \xi M \|u_0\|\right) \\
\leq & \left(\frac{M^2}{2} + p(T)\beta M^4 - \frac{2\rho\lambda(n+1)}{\theta-2}\right)\xi^2 + \lambda\sum_{i=0}^{n} N_3 \mathrm{meas}(\Omega_i) \\
& + \mu\sum_{i=1}^{n}\left(\frac{K_i M^{2(l_i+1)}\xi^{l_i+1}}{l_i+1} + L_i \xi M^2\right) + \frac{p(T)b^2}{\beta}.
\end{aligned}$$

Noting that $\rho > \frac{(\theta-2)M^2\left[p(T)\beta M^2 + \frac{1}{2}\right]}{2\lambda(n+1)}$, and it follows that $\varphi_\lambda(\xi u_0) \to -\infty$, $\xi \to \infty$. A rigorous analysis demonstrates the existence of $u_1 > 0$ with $\|u_1\| > \frac{\delta}{M}$ that satisfies $\inf_{u \in \partial B_{\frac{\delta}{M}}} \varphi_\lambda(u) > \varphi_\lambda(u_1)$. According to Lemmas 2.8 and 3.1, it is straightforward to see that there exists $u_2 \in E_0^\alpha$ satisfying $\varphi'_\lambda(u_2) = 0$, $\varphi_\lambda(u_2) > \max\{\varphi_\lambda(u_0), \varphi_\lambda(u_1)\}$. In conclusion, two distinct weak solutions (u_0 and u_2) are identified for (1.1). □ □

Remark 3.1. The assumptions in Theorem 3.1, while specific, define a broad and natural class of problems. Conditions (H_1)–(H_4) are standard in the variational framework for impulsive and superlinear equations, ensuring the necessary geometry and compactness for the mountain pass theorem. The constant condition $\Lambda > 0$ provides a verifiable criterion that links the system's parameters.

In what follows, we examine the scenario in problem (1.1) with the decomposition $f_i = f_{i1} + f_{i2}$, in which f_{i1} exhibits superlinear growth as $|u| \to \infty$, whereas f_{i2} shows sublinear behavior at infinity. We formally define the integral functions $F_{i1}(t,u) = \int_0^u f_{i1}(t,s)\,ds$ and $F_{i2}(t,u) = \int_0^u f_{i2}(t,s)\,ds$ for subsequent analysis. The rigorous mathematical assumptions are enumerated below:

(H_2') Given $i = 0, 1, \cdots, n$, one can find $\theta > 2$ for which $\lim\limits_{|u|\to\infty} \frac{F_{i1}(t,u)}{|u|^\theta} = \infty$, uniformly in $t \in (s_i, t_{i+1}]$;

(H_3') For each $i = 0, 1, \cdots, n$, one can find positive numbers $k_0 > 0$, $D > 0$ for which $f_{i1}(t,u)u - \theta F_{i1}(t,u) \geq -k_0 u^2$ is satisfied whenever $t \in (s_i, t_{i+1}]$, $|u| \geq D$;

(H_4') Given $i = 0, 1, \cdots, n$, one can find $1 < \tau < 2$ and $A_1 \in L^1((s_i, t_{i+1}], \mathbb{R}^+)$ for which $|f_{i2}(t,u)| \leq A_1(t)|u|^{\tau-1}$ is satisfied whenever $t \in (s_i, t_{i+1}]$, $u \in \mathbb{R}$;

(H_5) Given $i = 0, 1, \cdots, n$, one can find $A_2 \in C^1((s_i, t_{i+1}], \mathbb{R}^+)$ for which $F_{i2}(t,u) \geq A_2(t)|u|^\tau$ is satisfied whenever $t \in (s_i, t_{i+1}]$, $u \in \mathbb{R}$.

Lemma 3.2. The validity of conditions (H_1), (H_2')–(H_4') implies that φ_λ fulfills the PS-condition.

Proof. Our first step establishes the boundedness of $\{u_m\}_{m\in\mathbb{N}} \subset E_0^\alpha$. We verify this by contradiction. Assume that $\|u_m\| \to \infty, m \to \infty$. Let $v_m = \frac{u_m}{\|u_m\|}$, then $\|v_m\| = 1$. According to Lemma 2.4, if $v_m \rightharpoonup v_0$ in E_0^α, then $v_m \to v_0$ in $C^1([0,T],\mathbb{R})$ as $m \to \infty$. By virtue of hypothesis $(H_4{}')$, we obtain the growth conditions:

$$|f_{i2}(t,u)u| \le A_1(t)|u|^\tau,\ |F_{i2}(t,u)| \le \frac{1}{\tau}A_1(t)|u|^\tau. \tag{3.7}$$

The subsequent analysis divides into two mutually exclusive cases: the trivial case $v_0 \equiv 0$ and the nontrivial case $v_0 \neq 0$.

Case 1: $v_0 \equiv 0$. In view of assumption $(H_3{}')$ and the continuity of f_i, we can find a constant $k_1 > 0$ for which the following inequality holds:

$$f_{i1}(t,u)u - \theta F_{i1}(t,u) \ge -k_0u^2 - k_1, \quad t \in (s_i,t_{i+1}],\ u \in \mathbb{R}. \tag{3.8}$$

Therefore, from (3.1), (3.7), and Eq (3.8), we obtain,

$$\begin{aligned}
o(1) &= \frac{\theta N + N\|u_m\|}{\|u_m\|^2} \ge \frac{\theta\varphi_\lambda(u_m) - \varphi'(u_m)u_m}{\|u_m\|^2}\\
&= \left(\frac{\theta}{2}-1\right) + \frac{\mu}{\|u_m\|^2}\sum_{i=1}^{n}\left(I_i(u_m(t_i))u_m(t_i) - \theta\int_0^{u_m(t_i)} I_i(s)\,ds\right)\\
&+ \frac{\theta p(T)}{2\beta\|u_m\|^2}(b-\beta u_m(T))^2 + \frac{p(T)}{\|u_m\|^2}(b-\beta u_m(T))u_m(T)\\
&+ \frac{\lambda}{\|u_m\|^2}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}(f_i(t,u_m(t))u_m(t) - \theta F_i(t,u_m(t)))\,dt\\
&\ge \left(\frac{\theta}{2}-1\right) - |b|\frac{p(T)}{\beta\|u_m\|^2}(|b| + \beta M\|u_m\|)\\
&- \frac{\mu}{\|u_m\|^2}\sum_{i=1}^{n}\left(\left(\frac{K_i\theta}{l_i+1}+K_i\right)\|u_m\|_\infty^{l_i+1} + (\theta+1)L_i\|u_m\|_\infty\right)\\
&- \frac{\lambda}{\|u_m\|^2}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\left(k_0|u_m|^2 + k_1 + \left(1+\frac{\theta}{\tau}\right)A_1(t)|u_m|^\tau\right)dt\\
&\ge \left(\frac{\theta}{2}-1\right) - \frac{p(T)|b|}{\beta\|u_m\|^2}(|b| + \beta M\|u_m\|)\\
&- \frac{\mu}{\|u_m\|^2}\sum_{i=1}^{n}\left(\left(\frac{K_i\theta}{l_i+1}+K_i\right)M^{l_i+1}\|u_m\|^{l_i+1} + (\theta+1)L_iM\|u_m\|\right)\\
&- \frac{\lambda T}{\|u_m\|^2}\left(k_0M^2\|u_m\|^2 + k_1\right) - \frac{\lambda}{\|u_m\|^2}\left(1+\frac{\theta}{\tau}\right)\|A_1\|_{L^1}M^\tau\|u_m\|^\tau\\
&\ge \left(\frac{\theta}{2}-1\right) - \lambda T k_0 M^2,\ m \to \infty.
\end{aligned}$$

This indicates that $\{u_m\}$ is bounded in E_0^α.

Case 2: $v_0 \neq 0$. Let $\Omega'_i = \{t \in (s_i,t_{i+1}] : |v_0(t)| > 0\}$, $i = 0,1,\cdots,n$, then $\mathrm{meas}(\Omega'_i) > 0$. Observe that $\|u_m\| \to \infty,\ (m \to \infty)$ together with the identity $|u_m(t)| = |v_m(t)|\cdot\|u_m\|$ necessarily leads to $|u_m(t)| \to$

$\infty, (m \to \infty)$ when $t \in \Omega'_i$. We therefore deduce from the combination of (3.1) and (3.7) that

$$\begin{aligned}
&\sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} F_{i1}\left(t, u_m(t)\right) dt \\
&= -\frac{1}{\lambda}\varphi_\lambda\left(u_m\right) + \frac{1}{2\lambda}\|u_m\|^2 - \frac{\mu}{\lambda}\sum_{i=1}^{n}\int_0^{u_m(t_i)} I_i(s)\, ds \\
&+ \frac{p(T)}{2\beta\lambda}(b - \beta u_m(T))^2 - \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_{i2}\left(t, u_m(t)\right) dt \\
&\le \frac{N}{\lambda} + \frac{1}{2\lambda}\|u_m\|^2 + \frac{\mu}{\lambda}\sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}}{l_i+1}\|u_m\|^{l_i+1} + L_i M\|u_m\|\right) \\
&+ \frac{p(T)}{\beta\lambda}\left(b^2 + \beta^2 M^2\|u_m\|^2\right) + \frac{M^\tau}{\tau}\|A_1\|_{L^1}\|u_m\|^\tau .
\end{aligned}$$

Observing the parameter constraints $\theta > 2$, where $1 \le l_i + 1 < 2$ and simultaneously $1 < \tau < 2$, we consequently derive

$$\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} \frac{F_{i1}\left(t, u_m(t)\right)}{\|u_m\|^\theta} dt \le o(1), \quad m \to \infty. \tag{3.9}$$

Nevertheless, through an application of Fatou's lemma combined with hypothesis ($H_2{}'$), one obtains

$$\begin{aligned}
&\lim_{m\to\infty}\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} \frac{F_{i1}\left(t, u_m(t)\right)}{\|u_m\|^\theta} dt \ge \lim_{m\to\infty}\sum_{i=0}^{n}\int_{\Omega'_i} \frac{F_{i1}\left(t, u_m(t)\right)}{\|u_m\|^\theta} dt \\
&= \lim_{m\to\infty}\sum_{i=0}^{n}\int_{\Omega'_i} \frac{F_{i1}\left(t, u_m(t)\right)|v_m(t)|^\theta}{|u_m(t)|^\theta} dt = \infty,
\end{aligned}$$

which contradicts (3.9). In summary, the sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in E_0^α. Subsequently, employing an analogous approach to that used in proving Lemma 3.1, we establish the convergence $\|u_m - u\| \to 0$ in the space E_0^α when $m \to \infty$. □ □

Theorem 3.2 . Under hypotheses (H_1), ($H_2{}'$)–($H_4{}'$), and (H_5), if the functions $f_i(t,u)(i = 0, 1, \cdots, n)$ and $I_i(u)(i = 1, 2, \cdots, n)$ are odd with respect to u, then (1.1) possesses an infinite number of weak solutions.

Proof. Clearly, φ_λ is an even function and $\varphi_\lambda(0) = 0$. Let $\{e_m\}_{m=1}^\infty$ be an orthonormal basis of E_0^α, i.e., $\left\|e_q\right\| = 1$, $\left\langle e_q, e_{q'}\right\rangle = 0$, $1 \le q \ne q'$. For each natural number m, let E_m be the linear span of the vectors $e_1, e_2, \cdots, e_m$, and let S_m denote the set of all unit vectors in E_m. So for every vector $u \in E_m$, we can find real numbers $\chi_1, \chi_2, \cdots, \chi_m$ satisfying

$$u(t) = \sum_{j=1}^{m}\chi_j e_j(t),\ t \in [0, T]. \tag{3.10}$$

In other words,

$$\begin{aligned}\|u\|^2 &= \int_0^T p(t)\left|{}_0^C D_t^\alpha u(t)\right|^2 dt + \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} q(t)\,|u(t)|^2 dt \\ &= \sum_{j=0}^{m}\chi_j^2\left(\int_0^T p(t)\left|{}_0^C D_t^\alpha e_j(t)\right|^2 dt + \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} q(t)\left|e_j(t)\right|^2 dt\right) \\ &= \sum_{j=0}^{m}\chi_j^2\left\|e_j\right\|^2 = \sum_{j=0}^{m}\chi_j^2.\end{aligned} \tag{3.11}$$

Conversely, hypothesis (H_5) guarantees that for each open bounded interval Π_i contained in $(s_i, t_{i+1}]$ (where i ranges over 0 to n), we may select a uniform lower bound $A_3 > 0$ ensuring

$$F_{i2}(t, u(t)) \geq A_2(t)\,|u(t)|^\tau \geq A_3|u(t)|^\tau \tag{3.12}$$

holds throughout $\Pi_i \times \mathbb{R}$. Regarding the open sets Π_i previously introduced, the growth condition $(H_2{}')$ requires positive coefficients A_4, A_5 with the property that

$$F_{i1}(t, u(t)) \geq A_4|u|^\theta - A_5 \tag{3.13}$$

holds uniformly for all $t \in \Pi_i$ and $u \in \mathbb{R}$. Therefore, for $\forall u \in S_m$, it follows from (3.11)–(3.13) that

$$\begin{aligned}\varphi_\lambda(\eta u) &= \frac{1}{2}\|\eta u\|^2 + \frac{p(T)}{2\beta}(b - \beta\eta u(T))^2 - \mu\sum_{i=1}^{n}\int_0^{\eta u(t_i)} I_i(s)\,ds - \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t, \eta u(t))\,dt \\ &\leq \frac{1}{2}\|\eta u\|^2 + \mu\sum_{i=1}^{n}\int_0^{\eta u(t_i)} I_i(s)\,ds + \frac{p(T)}{2\beta}(b - \beta\eta u(T))^2 - \lambda\sum_{i=0}^{n}\int_{\Pi_i} F_i(t, \eta u(t))\,dt \\ &\leq \frac{\eta^2}{2}\|u\|^2 - \lambda\eta^\theta A_4\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\chi_j e_j(t)\right|^\theta dt + \lambda A_5 T \\ &\quad + \frac{p(T)}{\beta}\left(b^2 + \beta^2\eta^2 M^2\|u\|^2\right) - \lambda\eta^\tau A_3\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\chi_j e_j(t)\right|^\tau dt \\ &\quad + \mu\sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}\eta^{l_i+1}}{l_i+1}\|u\|^{l_i+1} + L_i M\eta\|u\|\right) \\ &= \frac{\eta^2}{2} - \lambda\eta^\theta A_4\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\chi_j e_j(t)\right|^\theta dt + \lambda A_5 T + \frac{p(T)}{\beta}\left(b^2 + \beta^2\eta^2 M^2\right) \\ &\quad - \lambda\eta^\tau A_3\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\chi_j e_j(t)\right|^\tau dt + \mu\sum_{i=1}^{n}\left(\frac{K_i M^{l_i+1}\eta^{l_i+1}}{l_i+1} + L_i M\eta\right).\end{aligned}$$

Furthermore, one may readily establish the positivity: $\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\chi_j e_j(t)\right|^\theta dt > 0$. Noting the parameter constraints $\theta > 2$, $1 < \tau < 2$, and $1 \leq l_i + 1 < 2$ (for all $i = 1, 2, \cdots, n$), we can find positive numbers

ξ, w guaranteeing

$$\varphi_\lambda(wu) < -\xi,\ u \in S_m. \tag{3.14}$$

Let $S_m^w = \{wu : u \in S_m\}$, $\Theta = \left\{(\omega_1, \omega_2, \cdots, \omega_m) \in \mathbb{R}^m : \sum_{j=1}^{m} \chi_j^2 < w^2\right\}$. From (3.14), it follows that $\varphi_\lambda(u) < -\xi,\ u \in S_m^w$. Combined with the even function $\varphi_\lambda \in C^1\left(E_0^\alpha, \mathbb{R}\right)$, we obtain $S_m^w \subset \varphi_\lambda^{-\xi} \in \Sigma$.

As a direct consequence of (3.10) and (3.11), we deduce an odd C^1-smooth boundary mapping $\Psi : \partial\Theta \to S_m^w$ with homomorphic properties. From the genus theory, the energy level set satisfies:

$$\gamma\left(\varphi_\lambda^{-\xi}\right) \geq \gamma\left(S_m^w\right) = m. \tag{3.15}$$

Therefore, $\varphi_\lambda^{-\xi} \in \Sigma_m$, which implies $\Sigma_m \neq \phi$. Let $z_m = \inf_{U\in\Sigma_m}\sup_{u\in U}\varphi_\lambda(u)$. The minimax sequence satisfies: $-\infty \leq z_m < -\xi < 0$, as guaranteed by (3.15) and the lower semicontinuity of φ_λ in E_0^α. In other words, $\forall m \in \mathbb{N}$, $z_m \in \mathbb{R}^-$. Consequently, an application of Lemma 2.9 combined with Remark 2.2 shows that φ_λ admits an infinite sequence of nontrivial critical points. In other words, (1.1) possesses an infinite number of nontrivial solutions in the weak sense. □ □

Example 3.1. We now study the following problem:

$$\begin{cases} {}_tD_1^{0.75}((16+2t)_0^CD_t^{0.75}u(t)) + (1+2t^2)u(t) = \lambda f_i(t,u(t)), t \in (s_i, t_{i+1}], i = 0, 1, \\ -\Delta\left({}_tD_1^{-0.25}((16+2t_1)_0^CD_t^{0.75}u(t_1))\right) = I_1(u(t_1)), \\ {}_tD_1^{-0.25}((16+2t)_0^CD_t^{0.75}u(t)) = {}_tD_1^{-0.25}((16+2t_1^+)_0^CD_t^{0.75}u(t_1^+)), t \in (t_1, s_1], \\ {}_tD_1^{-0.25}((16+2s_1^-)_0^CD_t^{0.75}u(s_1^-)) = {}_tD_1^{-0.25}((16+2s_1^+)_0^CD_t^{0.75}u(s_1^+)), \\ u(0) = 0, u(1) + \dfrac{1}{18}\,{}_tD_1^{-0.25}\left(18_0^CD_t^{0.75}u(1)\right) = b, \end{cases} \tag{3.16}$$

where $\alpha = 0.75$, $T = n = \beta = \mu = 1$, $b = \frac{1}{3}$, $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$. Let us select $p(t) = 16+2t$, $q(t) = 1+2t^2$, where $t \in [0,1]$. Then, we have $p_0 = 16$, $M \doteq 0.377$. Choose $I_1(u) = \frac{1}{4}u^{\frac{1}{2}}$, and there exist $K_1 = L_1 = \frac{1}{4}, l_1 = \frac{1}{2}$, such that condition (H_1) holds. Let $f_i(t,u(t)) = \frac{1}{2}u^3$, then $F_i = \frac{1}{8}u^4$. It follows that $\Lambda \doteq 3.756 > 0$, and the conditions (H_2)–(H_4) are satisfied with $\theta = 4$. Therefore, according to Theorem 3.1, for each $\lambda \in (0, 24)$, problem (3.16) has at least two distinct solutions.

To demonstrate the validity of Theorem 3.2, we present below a concrete example satisfying assumptions (H_1), (H_2')–(H_4'), (H_5).

Example 3.2. We now study a problem:

$$\begin{cases} {}_tD_1^{0.75}((16+2t)_0^CD_t^{0.75}u(t)) + (\ln(1+t^2))u(t) \\ = \dfrac{u^5(1+\sin t)}{4} + \dfrac{u^{\frac{1}{3}}(2+\cos t)}{4}, t \in (s_i, t_{i+1}], i = 0, 1, \\ -\Delta_t D_1^{-0.25}(p(t_1)_0^CD_t^{0.75}u)(t_1) = 2I_1(u(t_1)), \\ {}_tD_1^{-0.25}(p(t)_0^CD_t^{0.75}u)(t) = {}_tD_1^{-0.25}(p(t_1^+)_0^CD_t^{0.75}u)(t_1^+), t \in (t_1, s_1], \\ {}_tD_1^{-0.25}(p(s_1^-)_0^CD_t^{0.75}u)(s_1^-) = {}_tD_1^{-0.25}((p(s_1^+)_0^CD_t^{0.75}u)(s_1^+), \\ u(0) = 0, u(1) + \dfrac{1}{18}\,{}_tD_1^{-0.25}\left(18_0^CD_t^{0.75}u(1)\right) = b, \end{cases} \tag{3.17}$$

where $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$. Let us select $p(t) = 16 + 2t$, $q(t) = \ln(1 + t^2)$, where $t \in [0, 1]$. Then, we have $p_0 = 16$, $M \doteq 0.39$. Choose $I_1(u) = \sin u$, and there exist $K_1 = L_1 = 2, l_1 = \frac{1}{2}$, such that condition (H_1) holds. Integration gives the potential terms: $F_{i1} = \frac{u^6(1+\sin t)}{6}, F_{i2} = \frac{3u^{\frac{4}{3}}(2+\cos t)}{4}$. Through calculation, when we set $k_0 = 2, D = 3, \theta = 4$, it can be verified that both conditions $(H_2{}')$ and $(H_3{}')$ are fulfilled. With the choices $\tau = \frac{4}{3}, A_1(t) = \frac{1}{3}t + 5, A_2(t) = \frac{1}{4}(1 + t)$, hypotheses $(H_4{}')$ and (H_5) hold. Trivially, both $f_i(t, u)$ and $I_1(u)$ possess odd parity in the u-variable. Whence, the full satisfaction of Theorem 3.2's assumptions yields the existence of an unbounded sequence of nontrivial weak solutions to (3.17).

4. Conclusions

This paper has investigated a class of mixed boundary value problems for fractional Sturm-Liouville equations incorporating both instantaneous and non-instantaneous impulses. First, using the mountain pass theorem, we establish the existence of at least two weak solutions for problem (1.1) under assumptions that are weaker than the standard Ambrosetti-Rabinowitz condition (see Theorem 3.1). Second, when the nonlinearity satisfies a combined condition of superlinear and sublinear growth, we proved a multiplicity result for weak solutions by applying the genus theory in critical point theory (see Theorem 3.2). Notably, the superlinear growth condition required here is less restrictive than the classical Ambrosetti-Rabinowitz condition. Our results significantly extend the restrictions on the growth of the nonlinear term found in the existing literature [26]. Building upon this work, several promising directions warrant further investigation. Future research will focus on extending the present framework to encompass more generalized impulsive patterns, such as state-dependent or random impulses, and exploring the problem under different boundary conditions like periodic or Robin types. Additionally, investigating the case of variable-order fractional derivatives represents a meaningful direction for generalization. The development of reliable numerical schemes to simulate these complex systems would also be valuable for verifying theoretical results and providing deeper dynamical insights.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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