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Research article

The asymptotic behavior of the reciprocal sum of generalized Fibonacci numbers

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Abstract: Let $(u_n)_{n\geq 0}$ be the special Lucas u-sequence defined by

$$u_{n+2} = Au_{n+1} - Bu_n, \quad u_0 = 0,\ u_1 = 1,$$

where $n \geq 0$, $B = \pm 1$, and A is an integer such that $A^2 - 4B > 0$. Let

$$a_k = \frac{1}{u_{mk}^s}, \frac{1}{u_{mk} + u_{mk+l}}, \frac{1}{\sum_{i=0}^{l} u_{mk+i}}, \frac{1}{u_{mk}u_{mk+2l}}, \frac{1}{u_{mk}u_{mk+2l-1}}, \frac{1}{u_{mk} + C},$$

where m, l are positive integers, $s = 1, 2, 3, 4$, and C is any constant. The aim of this paper is to find a form g_n such that

$$\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty} a_k\right)^{-1} - g_n\right) = 0.$$

For example, we show that

$$\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty} \frac{1}{u_{mk}}\right)^{-1} - (u_{mn} - u_{m(n-1)})\right) = 0.$$

Keywords: generalized Fibonacci number; reciprocal sum; asymptotic formulas

1. Introduction

In the past years, many mathematicians were interested in finding the formula for the integer part of the reciprocal tails of the convergent series. That is, find the explicit value of $\left\lfloor \left(\sum_{k=n}^{\infty} a_k\right)^{-1} \right\rfloor$ when $\sum_{k=1}^{\infty} a_k$

converges. The motivation of such research comes from the reciprocal sum of Fibonacci numbers. Let us recall that the Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0,\ F_1 = 1,$$

where $n \geq 0$. In [1], Ohtsuka and Nakamura proved

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \geq 2 \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ is even;} \\ F_{n-1}F_n, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. In [2], Wang proved

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^3} \right)^{-1} \right\rfloor = \begin{cases} F_nF_{n-1}^2 + F_{n-2}F_n^2 + \left\lfloor \frac{1}{11}(14F_{n-2} - 5F_n) \right\rfloor, & \text{if } n \geq 2 \text{ is even;} \\ F_nF_{n-1}^2 + F_{n-2}F_n^2 + \left\lfloor \frac{1}{11}(5F_n - 14F_{n-2}) \right\rfloor, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

where $F_{-1} = F_1 = 1$. In [3], Hwang et al. provided the relevant formula for the reciprocal sum of the fourth power of the Fibonacci numbers, that is,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^4} \right)^{-1} \right\rfloor = F_n^4 - F_{n-1}^4 + \frac{2(-1)^n}{5} F_{2n-1} - \left\{ \frac{n+2}{5} \right\},$$

where $\{x\} = x - \lfloor x \rfloor$. In addition, some mathematicians studied the reciprocal sums of Fibonacci, Lucas, and Pell sequences, such as [4–7], while some mathematicians studied the reciprocal sums of other types of sequences, such as [8–10].

Many mathematicians also considered the asymptotic behavior of these sequences. That is, find a suitable function g_n such that

$$\left(\sum_{k=n}^{\infty} a_k \right)^{-1} \sim g_n$$

when $\sum_{k=1}^{\infty} a_k$ converges. Here the notation $A_n \sim B_n$ means that

$$\lim_{n\to\infty} (A_n - B_n) = 0.$$

In [11], Lee et al. proved that

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_{mk-l}} \right)^{-1} \sim F_{mn-l} - F_{m(n-1)-l}$$

for any positive integer m and $0 \leq l \leq m-1$. In [12], Marques et al. proved that for any positive integer m there exists a positive constant C_m such that

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_{mk}^2} \right)^{-1} \sim F_{mn}^2 - F_{m(n-1)}^2 + (-1)^{mn} C_m.$$

Moreover, they gave an explicit form for C_m as follows:

$$C_m = \begin{cases} -\frac{2(L_{2m}-2)}{25F_{2m}}\sqrt{5}, & \text{if } m \text{ is even,} \\ \frac{2(L_{2m}+2)}{5L_{2m}}, & \text{if } m \text{ is odd,} \end{cases}$$

where L_n is the nth Lucas number. In [3], Hwang et al. studied the asymptotic behavior of the reciprocal sum of the fourth power of the Fibonacci numbers, and they proved that

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k^4}\right)^{-1} \sim F_n^4 - F_{n-1}^4 + \frac{2(-1)^n}{5}F_{2n-1} + \frac{2\sqrt{5}}{75}.$$

In [13], Lee and Park studied the asymptotic behavior of the reciprocal sum of F_kF_{k+m}, and they proved that

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_kF_{k+2l}}\right)^{-1} \sim F_{n+l-1}F_{n+l} - (F_l^2 + (-1)^l)\frac{(-1)^n}{3}$$

and

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_kF_{k+2l-1}}\right)^{-1} \sim F_{n+l-1}^2 - (F_{l-1}F_l + (-1)^l)\frac{(-1)^n}{3},$$

where l is a positive integer. In addition, some mathematicians studied other types of asymptotic behavior, such as [14, 15]. Inspired by the above results, we use the method of Yuan et al. [16] to study the asymptotic behavior of the sequences that are more general than the Fibonacci sequence. Let $(u_n)_{n\geq 0}$ be the special Lucas u-sequence defined by

$$u_{n+2} = Au_{n+1} - Bu_n, \quad u_0 = 0,\ u_1 = 1, \tag{1.1}$$

where $n \geq 0$, $B = \pm 1$, and A is an integer such that $A^2 - 4B > 0$. The relevant properties of the Lucas u-sequence can be found in Sun's book [17]. We know that the Binet formula is related to the sequence $(u_n)_{n\geq 0}$ has the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 0, \tag{1.2}$$

where

$$\alpha, \beta = \frac{A \pm \sqrt{A^2 - 4B}}{2}.$$

Let

$$a_k = \frac{1}{u_{mk}^s}, \frac{1}{u_{mk} + u_{mk+l}}, \frac{1}{\sum_{i=0}^{l} u_{mk+i}}, \frac{1}{u_{mk}u_{mk+2l}}, \frac{1}{u_{mk}u_{mk+2l-1}}, \frac{1}{u_{mk} + C},$$

where m and l are positive integers, $s = 1, 2, 3, 4$, and C is any constant. The aim of this paper is to find a form g_n such that

$$\left(\sum_{k=n}^{\infty} a_k\right)^{-1} \sim g_n.$$

The rest of this paper is organized as follows: in Section 2, we give our main results. In Section 3, we give the proof of our main results.

2. Main results

Let u_n be defined by the second-order linear recurrence sequence (1.1). In this paper, we shall prove the following eight theorems.

Theorem 2.1. *For any positive integer m, we have*

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}}\right)^{-1}\sim u_{mn}-u_{m(n-1)}.$$

Theorem 2.2. *For any positive integer m, we have*

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^2}\right)^{-1}\sim u_{mn}^2-u_{m(n-1)}^2+B^{mn}C_m,$$

where $C_m=\frac{2(1-B^m)}{(\alpha-\beta)^2}-\frac{2(\alpha^{2m}-1)^2}{(\alpha-\beta)^2\left(\alpha^{4m}-B^m\right)}$.

Theorem 2.3. *For any positive integer m, we have*

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^3}\right)^{-1}\sim u_{mn}^3-u_{m(n-1)}^3+3B^{mn}Q_m\left(u_{m(n+2)}-u_{m(n-3)}\right),$$

where $Q_m=\frac{u_m^2}{(1-(B\alpha)^{5m})(1-(B\beta)^{5m})}$.

Theorem 2.4. *For any positive integer m, we have*

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^4}\right)^{-1}\sim u_{mn}^4-u_{m(n-1)}^4+4B^{mn}U_m\left(u_{m(n+1)}^2-B^m u_{m(n-2)}^2\right)+V_m,$$

where $U_m=\frac{u_m^2}{(1-B^m\alpha^{6m})(1-B^m\beta^{6m})}$ *and* $V_m=\frac{(\alpha^{4m}-1)^2}{(\alpha-\beta)^4}\left(\frac{16(\alpha^{4m}-1)}{(\alpha^{6m}-B^m)^2}-\frac{10}{\alpha^{8m}-1}\right)$.

Theorem 2.5. *For all positive integers m and l, we have*

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+u_{mk+l}}\right)^{-1}\sim u_{mn+l}-u_{m(n-1)+l}+u_{mn}-u_{m(n-1)}.$$

Theorem 2.6. *For all positive integers m and l, we have*

$$\left(\sum_{k=n}^{\infty}\frac{1}{\sum_{i=0}^{l}u_{mk+i}}\right)^{-1}\sim\frac{1}{\alpha-1}\left(u_{mn+l+1}-u_{m(n-1)+l+1}-u_{mn}+u_{m(n-1)}\right).$$

Remark 2.1. Note that when $l=1$, the two main terms of Theorems 2.5 and 2.6 are different. However, there is no contradiction since they are equivalent.

Theorem 2.7. *For all positive integers m and l, we have*

(i)
$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l}}\right)^{-1}\sim u_{mn+l}^2-u_{m(n-1)+l}^2-\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C_{m,l},$$
where $C_{m,l}=u_l^2+\frac{2B^l}{A^2-4B}\left(1-\frac{(1-B^m)(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)$.

(ii)
$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l-1}}\right)^{-1}\sim \alpha(1-\beta^{2m})u_{mn+l-1}^2-\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C'_{m,l},$$
where $C'_{m,l}=u_lu_{l-1}+\frac{B^{l-1}}{A^2-4B}\left(A-\frac{2(\alpha-\alpha\beta^{2m})(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)$.

Theorem 2.8. *For any positive integer m and constant C, we have*
$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+C}\right)^{-1}\sim u_{mn}-u_{m(n-1)}+C\frac{\alpha^m-1}{\alpha^m+1}.$$

Let $A=1$ and $B=-1$ in (1.1). Then u_n becomes the nth Fibonacci number. So we can obtain some known results when we take some special values for m, A, B.

If we take $A=1$ and $B=-1$ in Theorem 2.2, then
$$C_m=\frac{2(1-B^m)}{(\alpha-\beta)^2}-\frac{2(\alpha^{2m}-1)^2}{(\alpha-\beta)^2\left(\alpha^{4m}-B^m\right)}=\frac{2(1-(-1)^m)}{5}-\frac{2(\alpha^{2m}-1)^2}{5\left(\alpha^{4m}-(-1)^m\right)}. \tag{2.1}$$
If m is even, then it follows from (2.1) that
$$\begin{aligned}C_m&=-\frac{2(\alpha^{2m}-1)^2}{5\left(\alpha^{4m}-(-1)^m\right)}=\frac{-2(\alpha^{2m}-\alpha^m\beta^m)^2}{5(\alpha^{4m}-\alpha^{2m}\beta^{2m})}=\frac{-2(\alpha^m-\beta^m)^2}{5(\alpha^{2m}-\beta^{2m})}\\&=\frac{-2(\alpha^{2m}+\beta^{2m}-2)(\alpha-\beta)}{5(\alpha^{2m}-\beta^{2m})(\alpha-\beta)}=\frac{-2(L_{2m}-2)}{5\sqrt{5}u_{2m}}=\frac{-2(L_{2m}-2)}{25u_{2m}}\sqrt{5},\end{aligned}$$
where L_n is the nth Lucas number with $L_0=2$, $L_1=1$. If m is odd, then it follows from (2.1) that
$$\begin{aligned}C_m&=\frac{4}{5}-\frac{2(\alpha^{2m}-1)^2}{5\left(\alpha^{4m}+1\right)}=\frac{4}{5}-\frac{2(\alpha^{2m}+\alpha^m\beta^m)^2}{5(\alpha^{4m}+\alpha^{2m}\beta^{2m})}=\frac{4}{5}-\frac{2(\alpha^m+\beta^m)^2}{5(\alpha^{2m}+\beta^{2m})}\\&=\frac{4}{5}-\frac{2(\alpha^{2m}+\beta^{2m}-2)}{5(\alpha^{2m}+\beta^{2m})}=\frac{4L_{2m}}{5L_{2m}}-\frac{2(L_{2m}-2)}{5L_{2m}}=\frac{2(L_{2m}+2)}{5L_{2m}}.\end{aligned}$$
So we have the following corollary, which is given by [12].

Corollary 2.1. *Let F_n be the nth Fibonacci number with $F_0=0$, $F_1=1$ and let L_n be the nth Lucas number with $L_0=2$, $L_1=1$. Then for any positive integer m, we have*
$$\left(\sum_{k=n}^{\infty}\frac{1}{F_{mk}^2}\right)^{-1}\sim F_{mn}^2-F_{m(n-1)}^2+(-1)^{mn}C_m,$$
where
$$C_m=\begin{cases}-\frac{2(L_{2m}-2)}{25F_{2m}}\sqrt{5}, & \text{if } m \text{ is even},\\ \frac{2(L_{2m}+2)}{5L_{2m}}, & \text{if } m \text{ is odd}.\end{cases}$$

If we take $m = A = 1$ and $B = -1$ in Theorem 2.4, then

$$U_m = \frac{u_m^2}{(1 - B^m\alpha^{6m})(1 - B^m\beta^{6m})} = \frac{1}{(1 + \alpha^6)(1 + \beta^6)} = \frac{1}{2 + \alpha^6 + \beta^6} = \frac{1}{20}$$

and

$$\begin{aligned} V_m &= \frac{(\alpha^{4m} - 1)^2}{(\alpha - \beta)^4}\left(\frac{16(\alpha^{4m} - 1)}{(\alpha^{6m} - B^m)^2} - \frac{10}{\alpha^{8m} - 1}\right) = \frac{(\alpha^4 - 1)^2}{(\alpha - \beta)^4}\left(\frac{16(\alpha^4 - 1)}{(\alpha^6 + 1)^2} - \frac{10}{\alpha^8 - 1}\right) \\ &= \frac{1}{25}\left(\frac{16(\alpha^4 - 1)^3}{(\alpha^6 + 1)^2} - \frac{10(\alpha^4 - 1)}{\alpha^4 + 1}\right) = \frac{1}{25}\left(4\sqrt{5} - \frac{10\sqrt{5}}{3}\right) = \frac{2\sqrt{5}}{75}, \end{aligned}$$

which imply that

$$\begin{aligned} &u_{mn}^4 - u_{m(n-1)}^4 + 4B^{mn}U_m\left(u_{m(n+1)}^2 - B^m u_{m(n-2)}^2\right) + V_m \\ &= u_n^4 - u_{n-1}^4 + 4(-1)^n \cdot \frac{1}{20} \cdot \left(u_{n+1}^2 + u_{n-2}^2\right) + \frac{2\sqrt{5}}{75} \\ &= u_n^4 - u_{n-1}^4 + \frac{(-1)^n}{5} \cdot \left(u_{n+1}^2 + u_{n-2}^2\right) + \frac{2\sqrt{5}}{75} \\ &= u_n^4 - u_{n-1}^4 + \frac{(-1)^n}{5} \cdot \left(u_{n+1}^2 - u_{n-1}^2 + u_{n-1}^2 + u_{n-2}^2\right) + \frac{2\sqrt{5}}{75} \\ &= u_n^4 - u_{n-1}^4 + \frac{(-1)^n}{5} \cdot (u_{2n} + u_{2n-3}) + \frac{2\sqrt{5}}{75} \\ &= u_n^4 - u_{n-1}^4 + \frac{2(-1)^n}{5} u_{2n-1} + \frac{2\sqrt{5}}{75}. \end{aligned}$$

So we have the following corollary, which is given by [3, Corollary 4.3].

Corollary 2.2. *Let F_n be the nth Fibonacci number with $F_0 = 0,\ F_1 = 1$. Then we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k^4}\right)^{-1} \sim F_n^4 - F_{n-1}^4 + \frac{2(-1)^n}{5} F_{2n-1} + \frac{2\sqrt{5}}{75}.$$

If we take $m = A = 1$ and $B = -1$ in Theorem 2.7, then

$$\begin{aligned} C_{1,l} &= u_l^2 + \frac{2B^l}{A^2 - 4B}\left(1 - \frac{(1 - B^m)(\alpha^{4m} - B^m)}{(\alpha^{2m} - 1)^2}\right) = u_l^2 + \frac{2(-1)^l}{5}\left(1 - \frac{2(\alpha^4 + 1)}{(\alpha^2 - 1)^2}\right) \\ &= u_l^2 + \frac{2(-1)^l}{5}\left(1 - \frac{2(\alpha^2 + \beta^2)}{(\alpha + \beta)^2}\right) = u_l^2 - 2(-1)^l \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} C'_{1,l} &= u_l u_{l-1} + \frac{B^{l-1}}{A^2 - 4B}\left(A - \frac{2(\alpha - \alpha\beta^{2m})(\alpha^{4m} - B^m)}{(\alpha^{2m} - 1)^2}\right) = u_l u_{l-1} + \frac{(-1)^{l-1}}{5}\left(1 - \frac{2(\alpha - \alpha\beta^2)(\alpha^4 + 1)}{(\alpha^2 - 1)^2}\right) \\ &= u_l u_{l-1} + \frac{(-1)^{l-1}}{5}\left(1 - \frac{2(\alpha^4 + 1)}{(\alpha^2 - 1)^2}\right) = u_l u_{l-1} + (-1)^l. \end{aligned} \tag{2.3}$$

Note that

$$\frac{B^{mn}(\alpha^{2m} - 1)^2}{\alpha^{4m} - B^m} = \frac{(-1)^n(\alpha^2 - 1)^2}{\alpha^4 + 1} = (-1)^n\frac{(\alpha + \beta)^2}{\alpha^2 + \beta^2} = \frac{(-1)^n}{3}. \tag{2.4}$$

Then it follows from (2.2)–(2.4) that

$$\begin{aligned} u_{mn+l}^2 - u_{m(n-1)+l}^2 - \frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C_{m,l} =& u_{n+l}^2 - u_{n-1+l}^2 - \frac{(-1)^n}{3}C_{1,l} \\ =& u_{n+l}u_{n+l-1} + (-1)^{n+l-1} - \frac{(-1)^n}{3}(u_l^2 - 2(-1)^l) \\ =& u_{n+l}u_{n+l-1} - \frac{(-1)^n}{3}(u_l^2 - 2(-1)^l + 3(-1)^l) \\ =& u_{n+l}u_{n+l-1} - \frac{(-1)^n}{3}(u_l^2 + (-1)^l) \end{aligned}$$

and

$$\begin{aligned} \alpha(1-\beta^{2m})u_{mn+l-1}^2 - \frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C'_{m,l} &= \alpha(1-\beta^2)u_{n+l-1}^2 - \frac{(-1)^n}{3}C'_{1,l} \\ &= u_{n+l-1}^2 - \frac{(-1)^n}{3}(u_l u_{l-1} + (-1)^l). \end{aligned}$$

So we have the following corollary, which is given by [13, Sections 3 and 4].

Corollary 2.3. *Let F_n be the nth Fibonacci number with $F_0 = 0$, $F_1 = 1$. Then for any positive integer l, we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+2l}}\right)^{-1} \sim F_{n+l-1}F_{n+l} - (F_l^2 + (-1)^l)\frac{(-1)^n}{3}$$

and

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+2l-1}}\right)^{-1} \sim F_{n+l-1}^2 - (F_{l-1}F_l + (-1)^l)\frac{(-1)^n}{3}.$$

3. Proof of the theorem

In this section, we give the proofs of the above eight theorems. We first give some identities that will be used in the proofs of our main results. Let m and k be positive integers. Then it follows from (1.2) that

$$\frac{1}{u_{mk}} = \frac{\alpha-\beta}{\alpha^{mk}-\beta^{mk}} = \frac{\alpha-\beta}{\alpha^{mk}}\left(1 - \frac{B^{mk}}{\alpha^{2mk}}\right)^{-1}. \tag{3.1}$$

Moreover, we have the following identities:

$$(1+x)^{-1} = 1 - x + x^2 - \frac{x^3}{1+x}, \tag{3.2a}$$

$$(1-x)^{-1} = 1 + x + x^2 + \frac{x^3}{1-x}, \tag{3.2b}$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \frac{x^3(4-3x)}{(x-1)^2}, \tag{3.2c}$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + \frac{x^3(10-15x+6x^2)}{(1-x)^3}, \tag{3.2d}$$

$$(1-x)^{-4} = 1 + 4x + 10x^2 + \frac{x^3(20-45x+36x^2-10x^3)}{(x-1)^4}. \tag{3.2e}$$

To prove the above eight theorems, we may split the proofs into eight subsections as follows:

3.1. The proof of Theorem 2.1

In this subsection, we will provide a proof of Theorem 2.1.

Proof. By (3.1) and (3.2b), we have

$$\begin{aligned}\frac{1}{u_{mk}} &= \frac{\alpha-\beta}{\alpha^{mk}}\left(1-\frac{B^{mk}}{\alpha^{2mk}}\right)^{-1} = \frac{\alpha-\beta}{\alpha^{mk}}\left(1+\frac{B^{mk}}{\alpha^{2mk}}+\frac{1}{\alpha^{4mk}}+\frac{B^{mk}}{\alpha^{4mk}(\alpha^{2mk}-B^{mk})}\right)\\ &= (\alpha-\beta)\left(\frac{1}{\alpha^{mk}}+\frac{B^{mk}}{\alpha^{3mk}}+\frac{1}{\alpha^{5mk}}+\frac{B^{mk}}{\alpha^{5mk}(\alpha^{2mk}-B^{mk})}\right).\end{aligned} \tag{3.3}$$

Let n be a positive integer. Then it follows from (3.3) that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_{mk}} =&(\alpha-\beta)\left(\sum_{k=n}^{\infty}\frac{1}{\alpha^{mk}}+\sum_{k=n}^{\infty}\frac{B^{mk}}{\alpha^{3mk}}+\sum_{k=n}^{\infty}\frac{1}{\alpha^{5mk}}+\sum_{k=n}^{\infty}\frac{B^{mk}}{\alpha^{5mk}(\alpha^{2mk}-B^{mk})}\right)\\ =&(\alpha-\beta)\left(\frac{\alpha^m}{\alpha^{mn}(\alpha^m-1)}+\frac{B^{mn}\alpha^{3m}}{\alpha^{3mn}(\alpha^{3m}-B^m)}+\frac{\alpha^{5m}}{\alpha^{5mn}(\alpha^{5m}-1)}\right)+(\alpha-\beta)\sum_{k=n}^{\infty}\frac{B^{mk}}{\alpha^{5mk}(\alpha^{2mk}-B^{mk})}\\ =&\frac{\alpha^m(\alpha-\beta)}{\alpha^{mn}(\alpha^m-1)}\left(1+\frac{B^{mn}\alpha^{2m}(\alpha^m-1)}{\alpha^{2mn}(\alpha^{3m}-B^m)}+\frac{\alpha^{4m}(\alpha^m-1)}{\alpha^{4mn}(\alpha^{5m}-1)}+O\left(\frac{1}{\alpha^{6mn}}\right)\right).\end{aligned} \tag{3.4}$$

Here the notation $f(x) = O(g(x))$ means that there is a constant C such that $|f(x)| \le Cg(x)$ for all large enough real numbers x. By (3.2a) and (3.4), we have

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}}\right)^{-1} &= \frac{\alpha^{mn}(\alpha^m-1)}{\alpha^m(\alpha-\beta)}\left(1+\frac{B^{mn}\alpha^{2m}(\alpha^m-1)}{\alpha^{2mn}(\alpha^{3m}-B^m)}+\frac{\alpha^{4m}(\alpha^m-1)}{\alpha^{4mn}(\alpha^{5m}-1)}+O\left(\frac{1}{\alpha^{6mn}}\right)\right)^{-1}\\ &= \frac{\alpha^{mn}(\alpha^m-1)}{\alpha^m(\alpha-\beta)}\left(1-\frac{B^{mn}\alpha^{2m}(\alpha^m-1)}{\alpha^{2mn}(\alpha^{3m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\ &= \frac{\alpha^{mn}(\alpha^m-1)}{\alpha^m(\alpha-\beta)}-\frac{B^{mn}\alpha^m(\alpha^m-1)^2}{\alpha^{mn}(\alpha-\beta)(\alpha^{3m}-B^m)}+O\left(\frac{1}{\alpha^{3mn}}\right)\\ &= \frac{\alpha^{mn}}{\alpha-\beta}-\frac{\alpha^{m(n-1)}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right)\\ &= \frac{\alpha^{mn}-\beta^{mn}+\beta^{mn}}{\alpha-\beta}-\frac{\alpha^{m(n-1)}-\beta^{m(n-1)}+\beta^{m(n-1)}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right)\\ &= u_{mn}-u_{m(n-1)}+\frac{\beta^{mn}-\beta^{m(n-1)}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right).\end{aligned}$$

Then

$$\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}}\right)^{-1}-(u_{mn}-u_{m(n-1)})\right)=\lim_{n\to\infty}\left(\frac{\beta^{mn}-\beta^{m(n-1)}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right)\right)=0.$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}}\right)^{-1}\sim u_{mn}-u_{m(n-1)}.$$

□

3.2. *The proof of Theorem 2.2*

In this subsection, we will provide a proof of Theorem 2.2.

Proof. By (3.1) and (3.2c), we have

$$\begin{aligned}\frac{1}{u_{mk}^2} &= \frac{(\alpha-\beta)^2}{\alpha^{2mk}}\left(1-\frac{B^{mk}}{\alpha^{2mk}}\right)^{-2} = \frac{(\alpha-\beta)^2}{\alpha^{2mk}}\left(1+\frac{2B^{mk}}{\alpha^{2mk}}+\frac{3}{\alpha^{4mk}}+\frac{4B^{mk}\alpha^{2mk}-3}{\alpha^{4mk}(B^{mk}\alpha^{2mk}-1)^2}\right)\\ &= (\alpha-\beta)^2\left(\frac{1}{\alpha^{2mk}}+\frac{2B^{mk}}{\alpha^{4mk}}+\frac{3}{\alpha^{6mk}}+\frac{4B^{mk}\alpha^{2mk}-3}{\alpha^{6mk}(B^{mk}\alpha^{2mk}-1)^2}\right)\\ &= (\alpha-\beta)^2\left(\frac{1}{\alpha^{2mk}}+\frac{2B^{mk}}{\alpha^{4mk}}+\frac{3}{\alpha^{6mk}}+R_k\right),\end{aligned} \tag{3.5}$$

where

$$R_k = \frac{4B^{mk}\alpha^{2mk}-3}{\alpha^{6mk}(B^{mk}\alpha^{2mk}-1)^2}.$$

Let n be a positive integer. Then it follows from (3.5) that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_{mk}^2} =&(\alpha-\beta)^2\left(\sum_{k=n}^{\infty}\frac{1}{\alpha^{2mk}}+\sum_{k=n}^{\infty}\frac{2B^{mk}}{\alpha^{4mk}}+\sum_{k=n}^{\infty}\frac{3}{\alpha^{6mk}}+\sum_{k=n}^{\infty}R_k\right)\\ =&(\alpha-\beta)^2\left(\frac{\alpha^{2m}}{\alpha^{2mn}(\alpha^{2m}-1)}+\frac{2B^{mn}\alpha^{4m}}{\alpha^{4mn}(\alpha^{4m}-B^m)}+\frac{3\alpha^{6m}}{\alpha^{6mn}(\alpha^{6m}-1)}+\sum_{k=n}^{\infty}R_k\right)\\ =&\frac{(\alpha-\beta)^2\alpha^{2m}}{\alpha^{2mn}(\alpha^{2m}-1)}\left(1+\frac{2B^{mn}\alpha^{2m}(\alpha^{2m}-1)}{\alpha^{2mn}(\alpha^{4m}-B^m)}+\frac{3\alpha^{4m}(\alpha^{2m}-1)}{\alpha^{4mn}(\alpha^{6m}-1)}\right)\\ &+\frac{(\alpha-\beta)^2\alpha^{2m}}{\alpha^{2mn}(\alpha^{2m}-1)}\cdot\frac{\alpha^{2mn}(\alpha^{2m}-1)}{\alpha^{2m}}\sum_{k=n}^{\infty}R_k\\ =&\frac{(\alpha-\beta)^2\alpha^{2m}}{\alpha^{2mn}(\alpha^{2m}-1)}(1+\omega),\end{aligned} \tag{3.6}$$

where

$$\omega = \frac{2B^{mn}}{\alpha^{2mn}}\cdot\frac{\alpha^{2m}(\alpha^{2m}-1)}{(\alpha^{4m}-B^m)}+\frac{3}{\alpha^{4mn}}\cdot\frac{\alpha^{4m}}{(\alpha^{4m}+\alpha^{2m}+1)}+\frac{\alpha^{2mn}(\alpha^{2m}-1)}{\alpha^{2m}}\sum_{k=n}^{\infty}R_k.$$

Note that

$$\omega = \frac{2B^{mn}}{\alpha^{2mn}}\cdot\frac{\alpha^{2m}(\alpha^{2m}-1)}{(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right).$$

Then we have

$$\omega^2-\frac{\omega^3}{1+\omega} = O\left(\frac{1}{\alpha^{4mn}}\right). \tag{3.7}$$

From (3.2a), (3.6), and (3.7), it follows that

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^2}\right)^{-1} =\frac{\alpha^{2mn}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}(1+\omega)^{-1} = \frac{\alpha^{2mn}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}\left(1-\omega+\omega^2-\frac{\omega^3}{1+\omega}\right)$$

$$
\begin{aligned}
&=\frac{\alpha^{2mn}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}\left(1-\omega+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\
&=\frac{\alpha^{2mn}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}\left(1-\frac{2B^{mn}}{\alpha^{2mn}}\cdot\frac{\alpha^{2m}(\alpha^{2m}-1)}{(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\
&=\frac{\alpha^{2mn}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}-\frac{2B^{mn}}{(\alpha-\beta)^2}\cdot\frac{(\alpha^{2m}-1)^2}{(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right)\\
&=\frac{\alpha^{2mn}}{(\alpha-\beta)^2}-\frac{\alpha^{2m(n-1)}}{(\alpha-\beta)^2}-\frac{2B^{mn}}{(\alpha-\beta)^2}\cdot\frac{(\alpha^{2m}-1)^2}{(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right)\\
&=\left(\frac{\alpha^{mn}-\beta^{mn}+\beta^{mn}}{\alpha-\beta}\right)^2-\left(\frac{\alpha^{m(n-1)}-\beta^{m(n-1)}+\beta^{m(n-1)}}{\alpha-\beta}\right)^2-\frac{2B^{mn}}{(\alpha-\beta)^2}\cdot\frac{(\alpha^{2m}-1)^2}{(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right)\\
&=u_{mn}^2-u_{m(n-1)}^2+\frac{2B^{mn}(1-B^m)}{(\alpha-\beta)^2}-\frac{2B^{mn}}{(\alpha-\beta)^2}\cdot\frac{(\alpha^{2m}-1)^2}{(\alpha^{4m}-B^m)}+\frac{\beta^{2mn}(\alpha^{2m}-1)}{(\alpha-\beta)^2}+O\left(\frac{1}{\alpha^{2mn}}\right)\\
&=u_{mn}^2-u_{m(n-1)}^2+B^{mn}C_m+O\left(\frac{1}{\alpha^{2mn}}\right),
\end{aligned}
$$

where

$$C_m=\frac{2(1-B^m)}{(\alpha-\beta)^2}-\frac{2(\alpha^{2m}-1)^2}{(\alpha-\beta)^2\left(\alpha^{4m}-B^m\right)}.$$

Then

$$\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^2}\right)^{-1}-\left(u_{mn}^2-u_{m(n-1)}^2+B^{mn}C_m\right)\right)=0.$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^2}\right)^{-1}\sim u_{mn}^2-u_{m(n-1)}^2+B^{mn}C_m,$$

where $C_m=\frac{2(1-B^m)}{(\alpha-\beta)^2}-\frac{2(\alpha^{2m}-1)^2}{(\alpha-\beta)^2\left(\alpha^{4m}-B^m\right)}$. □

3.3. The proof of Theorem 2.3

In this subsection, we will provide a proof of Theorem 2.3.

Proof. By (3.1) and (3.2d), we have

$$
\begin{aligned}
\frac{1}{u_{mk}^3}&=\frac{(\alpha-\beta)^3}{\alpha^{3mk}}\left(1-\frac{B^{mk}}{\alpha^{2mk}}\right)^{-3}=\frac{(\alpha-\beta)^3}{\alpha^{3mk}}\left(1+\frac{3B^{mk}}{\alpha^{2mk}}+\frac{6}{\alpha^{4mk}}+\frac{10\alpha^{4mk}-15B^{mk}\alpha^{2mk}+6}{\alpha^{4mk}(B^{mk}\alpha^{2mk}-1)^3}\right)\\
&=(\alpha-\beta)^3\left(\frac{1}{\alpha^{3mk}}+\frac{3B^{mk}}{\alpha^{5mk}}+\frac{6}{\alpha^{7mk}}+\frac{10\alpha^{4mk}-15B^{mk}\alpha^{2mk}+6}{\alpha^{7mk}(B^{mk}\alpha^{2mk}-1)^3}\right)\\
&=(\alpha-\beta)^3\left(\frac{1}{\alpha^{3mk}}+\frac{3B^{mk}}{\alpha^{5mk}}+\frac{6}{\alpha^{7mk}}+R_k\right),
\end{aligned}
\tag{3.8}
$$

where

$$R_k=\frac{10\alpha^{4mk}-15B^{mk}\alpha^{2mk}+6}{\alpha^{7mk}(B^{mk}\alpha^{2mk}-1)^3}.$$

Let n be a positive integer. Then it follows from (3.8) that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_{mk}^3} =&(\alpha-\beta)^3\left(\sum_{k=n}^{\infty}\frac{1}{\alpha^{3mk}}+\sum_{k=n}^{\infty}\frac{3B^{mk}}{\alpha^{5mk}}+\sum_{k=n}^{\infty}\frac{6}{\alpha^{7mk}}+\sum_{k=n}^{\infty}R_k\right)\\ =&(\alpha-\beta)^3\left(\frac{\alpha^{3m}}{\alpha^{3mn}(\alpha^{3m}-1)}+\frac{3B^{mn}\alpha^{5m}}{\alpha^{5mn}(\alpha^{5m}-B^m)}+\frac{6\alpha^{7m}}{\alpha^{7mn}(\alpha^{7m}-1)}+\sum_{k=n}^{\infty}R_k\right)\\ =&\frac{(\alpha-\beta)^3\alpha^{3m}}{\alpha^{3mn}(\alpha^{3m}-1)}\left(1+\frac{3B^{mn}\alpha^{2m}(\alpha^{3m}-1)}{\alpha^{2mn}(\alpha^{5m}-B^m)}+\frac{6\alpha^{4m}(\alpha^{3m}-1)}{\alpha^{4mn}(\alpha^{7m}-1)}\right)\\ &+\frac{(\alpha-\beta)^3\alpha^{3m}}{\alpha^{3mn}(\alpha^{3m}-1)}\cdot\frac{\alpha^{3mn}(\alpha^{3m}-1)}{\alpha^{3m}}\sum_{k=n}^{\infty}R_k\\ =&\frac{(\alpha-\beta)^3\alpha^{3m}}{\alpha^{3mn}(\alpha^{3m}-1)}(1+\omega),\end{aligned}\tag{3.9}$$

where

$$\omega=\frac{3B^{mn}}{\alpha^{2mn}}\cdot\frac{\alpha^{2m}(\alpha^{3m}-1)}{(\alpha^{5m}-B^m)}+\frac{6}{\alpha^{4mn}}\cdot\frac{\alpha^{4m}(\alpha^{3m}-1)}{(\alpha^{7m}-1)}+\frac{\alpha^{3mn}(\alpha^{3m}-1)}{\alpha^{3m}}\sum_{k=n}^{\infty}R_k.$$

Note that

$$\omega=\frac{3B^{mn}}{\alpha^{2mn}}\cdot\frac{\alpha^{2m}(\alpha^{3m}-1)}{(\alpha^{5m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right).$$

Then we have

$$\omega^2-\frac{\omega^3}{1+\omega}=O\left(\frac{1}{\alpha^{4mn}}\right).\tag{3.10}$$

From (3.2a), (3.9), and (3.10), it follows that

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^3}\right)^{-1} =&\left(\frac{(\alpha-\beta)^3\alpha^{3m}}{\alpha^{3mn}(\alpha^{3m}-1)}\right)^{-1}(1+\omega)^{-1}=\frac{\alpha^{3mn}(\alpha^{3m}-1)}{(\alpha-\beta)^3\alpha^{3m}}\left(1-\omega+\omega^2-\frac{\omega^3}{1+\omega}\right)\\ =&\frac{\alpha^{3mn}(\alpha^{3m}-1)}{(\alpha-\beta)^3\alpha^{3m}}\left(1-\omega+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\ =&\frac{\alpha^{3mn}(\alpha^{3m}-1)}{(\alpha-\beta)^3\alpha^{3m}}\left(1-\frac{3B^{mn}}{\alpha^{2mn}}\cdot\frac{\alpha^{2m}(\alpha^{3m}-1)}{(\alpha^{5m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\ =&\frac{\alpha^{3mn}(\alpha^{3m}-1)}{(\alpha-\beta)^3\alpha^{3m}}-\frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\cdot\frac{(\alpha^{3m}-1)^2}{\alpha^m(\alpha^{5m}-B^m)}+O\left(\frac{1}{\alpha^{mn}}\right)\\ =&\frac{\alpha^{3mn}}{(\alpha-\beta)^3}-\frac{\alpha^{3m(n-1)}}{(\alpha-\beta)^3}-\frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\cdot\frac{(\alpha^{3m}-1)^2}{\alpha^m(\alpha^{5m}-B^m)}+O\left(\frac{1}{\alpha^{mn}}\right)\\ =&\left(\frac{\alpha^{mn}-\beta^{mn}+\beta^{mn}}{\alpha-\beta}\right)^3-\left(\frac{\alpha^{m(n-1)}-\beta^{m(n-1)}+\beta^{m(n-1)}}{\alpha-\beta}\right)^3-\frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\cdot\frac{(\alpha^{3m}-1)^2}{\alpha^m(\alpha^{5m}-B^m)}+O\left(\frac{1}{\alpha^{mn}}\right)\\ =&u_{mn}^3-u_{m(n-1)}^3+\delta+O\left(\frac{1}{\alpha^{mn}}\right),\end{aligned}$$

where

$$\begin{aligned}\delta &= \frac{3(B\alpha)^{mn}(1-\beta^m)+3(B\beta)^{mn}(\alpha^m-1)+\beta^{3mn}(1-(B\alpha)^{3m})}{(\alpha-\beta)^3} - \frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\cdot\frac{(\alpha^{3m}-1)^2}{\alpha^m(\alpha^{5m}-B^m)}\\ &= \frac{3(B\alpha)^{mn}(1-\beta^m)}{(\alpha-\beta)^3} - \frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\cdot\frac{(\alpha^{3m}-1)^2}{\alpha^m(\alpha^{5m}-B^m)} + O\left(\frac{1}{\alpha^{mn}}\right)\\ &= \frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\left(\frac{-(\alpha^{3m}-1)^2+(1-\beta^m)\alpha^m(\alpha^{5m}-B^m)}{\alpha^m(\alpha^{5m}-B^m)}\right) + O\left(\frac{1}{\alpha^{mn}}\right)\\ &= -\frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\left(\frac{\alpha^{4m}-2B^m\alpha^{2m}+B^{2m}}{(B\alpha)^{5m}-1}\right) + O\left(\frac{1}{\alpha^{mn}}\right)\\ &= \frac{3(B\alpha)^{mn}}{(\alpha-\beta)^3}\cdot\frac{\alpha^{2m}\,(\alpha^m-\beta^m)^2}{1-B^m\alpha^{5m}} + O\left(\frac{1}{\alpha^{mn}}\right)\\ &= \left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)^2\cdot\frac{3(B\alpha)^{m(n+2)}}{\alpha-\beta}\cdot\frac{1}{1-(B\alpha)^{5m}} + O\left(\frac{1}{\alpha^{mn}}\right).\end{aligned}$$

Let $Q_m = \frac{u_m^2}{(1-(B\alpha)^{5m})(1-(B\beta)^{5m})}$. Then we can obtain

$$\begin{aligned}\delta &= 3B^{mn}Q_m\frac{\alpha^{m(n+2)}\left(1-(B\beta)^{5m}\right)}{\alpha-\beta} + O\left(\frac{1}{\alpha^{mn}}\right)\\ &= 3B^{mn}Q_m\left(\frac{\alpha^{m(n+2)}}{\alpha-\beta} - \frac{\alpha^{m(n-3)}}{\alpha-\beta}\right) + O\left(\frac{1}{\alpha^{mn}}\right)\\ &= 3B^{mn}Q_m\left(\frac{\alpha^{m(n+2)}-\beta^{m(n+2)}+\beta^{m(n+2)}}{\alpha-\beta} - \frac{\alpha^{m(n-3)}-\beta^{m(n-3)}+\beta^{m(n-3)}}{\alpha-\beta}\right) + O\left(\frac{1}{\alpha^{mn}}\right)\\ &= 3B^{mn}Q_m\left(u_{m(n+2)} - u_{m(n-3)} + \frac{\beta^{m(n+2)}-\beta^{m(n-3)}}{\alpha-\beta}\right) + O\left(\frac{1}{\alpha^{mn}}\right).\end{aligned}$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^3}\right)^{-1} - \left(u_{mn}^3 - u_{m(n-1)}^3 + 3B^{mn}Q_m\left(u_{m(n+2)}-u_{m(n-3)}\right)\right)\right)\\ &= \lim_{n\to\infty}\left(3B^{mn}Q_m\frac{\beta^{m(n+2)}-\beta^{m(n-3)}}{\alpha-\beta} + O\left(\frac{1}{\alpha^{mn}}\right)\right) = 0.\end{aligned}$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^3}\right)^{-1} \sim u_{mn}^3 - u_{m(n-1)}^3 + 3B^{mn}Q_m\left(u_{m(n+2)}-u_{m(n-3)}\right),$$

where $Q_m = \frac{u_m^2}{(1-(B\alpha)^{5m})(1-(B\beta)^{5m})}$. □

3.4. The proof of Theorem 2.4

In this subsection, we will provide a proof of Theorem 2.4.

Proof. By (3.1) and (3.2e), we have

$$\begin{aligned}\frac{1}{u_{mk}^4} = \frac{(\alpha-\beta)^4}{\alpha^{4mk}}\left(1-\frac{B^{mk}}{\alpha^{2mk}}\right)^{-4} &= \frac{(\alpha-\beta)^4}{\alpha^{4mk}}\left(1+\frac{4B^{mk}}{\alpha^{2mk}}+\frac{10}{\alpha^{4mk}}+R_k\right)\\ &= (\alpha-\beta)^4\left(\frac{1}{\alpha^{4mk}}+\frac{4B^{mk}}{\alpha^{6mk}}+\frac{10}{\alpha^{8mk}}+\frac{R_k}{\alpha^{4mk}}\right),\end{aligned} \tag{3.11}$$

where

$$R_k = \frac{20B^{mk}\alpha^{6mk}-45\alpha^{4mk}+36B^{mk}\alpha^{2mk}-10}{\alpha^{4mk}(B^{mk}\alpha^{2mk}-1)^4}.$$

Let n be a positive integer. Then it follows from (3.11) that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_{mk}^4} =&(\alpha-\beta)^4\left(\sum_{k=n}^{\infty}\frac{1}{\alpha^{4mk}}+\sum_{k=n}^{\infty}\frac{4B^{mk}}{\alpha^{6mk}}+\sum_{k=n}^{\infty}\frac{10}{\alpha^{8mk}}+\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{4mk}}\right)\\ =&(\alpha-\beta)^4\left(\frac{\alpha^{4m}}{\alpha^{4mn}(\alpha^{4m}-1)}+\frac{4B^{mn}\alpha^{6m}}{\alpha^{6mn}(\alpha^{6m}-B^m)}+\frac{10\alpha^{8m}}{\alpha^{8mn}(\alpha^{8m}-1)}+\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{4mk}}\right)\\ =&\frac{(\alpha-\beta)^4\alpha^{4m}}{\alpha^{4mn}(\alpha^{4m}-1)}\left(1+\frac{4B^{mn}\alpha^{2m}(\alpha^{4m}-1)}{\alpha^{2mn}(\alpha^{6m}-B^m)}+\frac{10\alpha^{4m}(\alpha^{4m}-1)}{\alpha^{4mn}(\alpha^{8m}-1)}\right)\\ &+\frac{(\alpha-\beta)^4\alpha^{4m}}{\alpha^{4mn}(\alpha^{4m}-1)}\cdot\frac{\alpha^{4mn}(\alpha^{4m}-1)}{\alpha^{4m}}\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{4mk}}\\ =&\frac{(\alpha-\beta)^4\alpha^{4m}}{\alpha^{4mn}(\alpha^{4m}-1)}(1+\omega),\end{aligned} \tag{3.12}$$

where

$$\omega = \frac{4B^{mn}\alpha^{2m}(\alpha^{4m}-1)}{\alpha^{2mn}(\alpha^{6m}-B^m)}+\frac{10\alpha^{4m}(\alpha^{4m}-1)}{\alpha^{4mn}(\alpha^{8m}-1)}+\frac{\alpha^{4mn}(\alpha^{4m}-1)}{\alpha^{4m}}\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{4mk}}.$$

Note that

$$\omega = \frac{4B^{mn}\alpha^{2m}(\alpha^{4m}-1)}{\alpha^{2mn}(\alpha^{6m}-B^m)}+\frac{10\alpha^{4m}(\alpha^{4m}-1)}{\alpha^{4mn}(\alpha^{8m}-1)}+O\left(\frac{1}{\alpha^{6mn}}\right).$$

Then we have

$$\omega^2-\frac{\omega^3}{1+\omega} = \frac{16\alpha^{4m}(\alpha^{4m}-1)^2}{\alpha^{4mn}(\alpha^{6m}-B^m)^2}+O\left(\frac{1}{\alpha^{6mn}}\right). \tag{3.13}$$

By (3.2a), (3.12), and (3.13), we have

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^4}\right)^{-1} =&\left(\frac{(\alpha-\beta)^4\alpha^{4m}}{\alpha^{4mn}(\alpha^{4m}-1)}\right)^{-1}(1+\omega)^{-1} = \frac{\alpha^{4mn}(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}\left(1-\omega+\omega^2-\frac{\omega^3}{1+\omega}\right)\\ =&\frac{\alpha^{4mn}(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}\left(1-\omega+\frac{16\alpha^{4m}(\alpha^{4m}-1)^2}{\alpha^{4mn}(\alpha^{6m}-B^m)^2}+O\left(\frac{1}{\alpha^{6mn}}\right)\right)\\ =&\frac{\alpha^{4mn}(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}\left(1-\frac{4B^{mn}\alpha^{2m}(\alpha^{4m}-1)}{\alpha^{2mn}(\alpha^{6m}-B^m)}+\frac{1}{\alpha^{4mn}}C'_m+O\left(\frac{1}{\alpha^{6mn}}\right)\right)\\ =&\frac{\alpha^{4mn}(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}-\frac{4B^{mn}\alpha^{2mn}(\alpha^{4m}-1)^2}{(\alpha-\beta)^4\alpha^{2m}(\alpha^{6m}-B^m)}+\frac{(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}C'_m+O\left(\frac{1}{\alpha^{2mn}}\right)\end{aligned}$$

$$
\begin{aligned}
=&\frac{\alpha^{4mn}}{(\alpha-\beta)^4}-\frac{\alpha^{4m(n-1)}}{(\alpha-\beta)^4}-\frac{4B^{mn}\alpha^{2mn}(\alpha^{4m}-1)^2}{(\alpha-\beta)^4\alpha^{2m}(\alpha^{6m}-B^m)}+\frac{(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}C'_m+O\left(\frac{1}{\alpha^{2mn}}\right)\\
=&\left(\frac{\alpha^{mn}-\beta^{mn}+\beta^{mn}}{\alpha-\beta}\right)^4-\left(\frac{\alpha^{m(n-1)}-\beta^{m(n-1)}+\beta^{m(n-1)}}{\alpha-\beta}\right)^4\\
&-\frac{4B^{mn}\alpha^{2mn}(\alpha^{4m}-1)^2}{(\alpha-\beta)^4\alpha^{2m}(\alpha^{6m}-B^m)}+\frac{(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}C'_m+O\left(\frac{1}{\alpha^{2mn}}\right)\\
=&u_{mn}^4-u_{m(n-1)}^4+\delta+V_m+O\left(\frac{1}{\alpha^{2mn}}\right),
\end{aligned}
$$

where

$$
C'_m=\frac{16\alpha^{4m}(\alpha^{4m}-1)^2}{(\alpha^{6m}-B^m)^2}-\frac{10\alpha^{4m}(\alpha^{4m}-1)}{\alpha^{8m}-1},
$$

$$
\begin{aligned}
V_m=\frac{(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}C'_m&=\frac{(\alpha^{4m}-1)}{(\alpha-\beta)^4\alpha^{4m}}\left(\frac{16\alpha^{4m}(\alpha^{4m}-1)^2}{(\alpha^{6m}-B^m)^2}-\frac{10\alpha^{4m}(\alpha^{4m}-1)}{\alpha^{8m}-1}\right)\\
&=\frac{(\alpha^{4m}-1)^2}{(\alpha-\beta)^4}\left(\frac{16(\alpha^{4m}-1)}{(\alpha^{6m}-B^m)^2}-\frac{10}{\alpha^{8m}-1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta&=\frac{4B^{mn}\alpha^{2mn}}{(\alpha-\beta)^4}\left(\frac{(1-B^m\beta^{2m})\alpha^{2m}(\alpha^{6m}-B^m)-(\alpha^{4m}-1)^2}{\alpha^{2m}(\alpha^{6m}-B^m)}\right)=\frac{4B^{mn}\alpha^{2mn}}{(\alpha-\beta)^4}\left(\frac{\alpha^{4m}-2B^m\alpha^{2m}+B^{2m}}{1-B^m\alpha^{6m}}\right)\\
&=\frac{4B^{mn}\alpha^{2mn}}{(\alpha-\beta)^4}\cdot\frac{\alpha^{2m}(\alpha^m-\beta^m)^2}{1-B^m\alpha^{6m}}=\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)^2\cdot\frac{4B^{mn}}{(1-B^m\alpha^{6m})(1-B^m\beta^{6m})}\cdot\frac{\alpha^{2m(n+1)}(1-B^m\beta^{6m})}{(\alpha-\beta)^2}\\
&=\frac{4B^{mn}u_m^2}{(1-B^m\alpha^{6m})(1-B^m\beta^{6m})}\left(\frac{\alpha^{2m(n+1)}}{(\alpha-\beta)^2}-\frac{B^m\alpha^{2m(n-2)}}{(\alpha-\beta)^2}\right).
\end{aligned}
$$

Let $U_m=\frac{u_m^2}{(1-B^m\alpha^{6m})(1-B^m\beta^{6m})}$. Then we can obtain

$$
\begin{aligned}
\delta&=4B^{mn}U_m\left(\left(\frac{\alpha^{m(n+1)}-\beta^{m(n+1)}+\beta^{m(n+1)}}{\alpha-\beta}\right)^2-B^m\left(\frac{\alpha^{m(n-2)}-\beta^{m(n-2)}+\beta^{m(n-2)}}{\alpha-\beta}\right)^2\right)\\
&=4B^{mn}U_m\left(u_{m(n+1)}^2-B^mu_{m(n-2)}^2+\frac{B^m\beta^{2m(n-2)}-\beta^{2m(n-2)}}{(\alpha-\beta)^2}\right)\\
&=4B^{mn}U_m\left(u_{m(n+1)}^2-B^mu_{m(n-2)}^2\right)+O\left(\frac{1}{\alpha^{2m(n-2)}}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^4}\right)^{-1}-\left(u_{mn}^4-u_{m(n-1)}^4+4B^{mn}U_m\left(u_{m(n+1)}^2-B^mu_{m(n-2)}^2\right)+V_m\right)\right)\\
&=\lim_{n\to\infty}\left(O\left(\frac{1}{\alpha^{2m(n-2)}}\right)+O\left(\frac{1}{\alpha^{2mn}}\right)\right)=0.
\end{aligned}
$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}^4}\right)^{-1}\sim u_{mn}^4-u_{m(n-1)}^4+4B^{mn}U_m\left(u_{m(n+1)}^2-B^m u_{m(n-2)}^2\right)+V_m,$$

where $U_m=\frac{u_m^2}{(1-B^m\alpha^{6m})(1-B^m\beta^{6m})}$ and $V_m=\frac{(\alpha^{4m}-1)^2}{(\alpha-\beta)^4}\left(\frac{16(\alpha^{4m}-1)}{(\alpha^{6m}-B^m)^2}-\frac{10}{\alpha^{8m}-1}\right)$. □

3.5. The proof of Theorem 2.5

In this subsection, we will provide a proof of Theorem 2.5.

Proof. By (1.2) and (3.2b), we have

$$\begin{aligned}\frac{1}{u_{mk}+u_{mk+l}}&=\left(\frac{\alpha^{mk}-\beta^{mk}}{\alpha-\beta}+\frac{\alpha^{mk+l}-\beta^{mk+l}}{\alpha-\beta}\right)^{-1}=\left(\frac{\alpha^{mk}(1+\alpha^l)}{\alpha-\beta}\right)^{-1}\left(1-\frac{B^{mk}(1+\beta^l)}{\alpha^{2mk}(1+\alpha^l)}\right)^{-1}\\&=\frac{\alpha-\beta}{\alpha^{mk}(1+\alpha^l)}\left(1+\frac{B^{mk}(1+\beta^l)}{\alpha^{2mk}(1+\alpha^l)}+\frac{(1+\beta^l)^2}{\alpha^{4mk}(1+\alpha^l)^2}+R_k\right)\\&=\frac{\alpha-\beta}{1+\alpha^l}\left(\frac{1}{\alpha^{mk}}+\frac{B^{mk}(1+\beta^l)}{\alpha^{3mk}(1+\alpha^l)}+\frac{(1+\beta^l)^2}{\alpha^{5mk}(1+\alpha^l)^2}+\frac{R_k}{\alpha^{mk}}\right),\end{aligned}\tag{3.14}$$

where

$$R_k=\frac{B^{mk}(1+\beta^l)}{\alpha^{4mk}(1+\alpha^l)\left(\alpha^{2mk}(1+\alpha^l)-B^{mk}(1+\beta^l)\right)}.$$

Let n be a positive integer. Then it follows from (3.14) that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_{mk}+u_{mk+l}}&=\frac{\alpha-\beta}{1+\alpha^l}\left(\sum_{k=n}^{\infty}\frac{1}{\alpha^{mk}}+\sum_{k=n}^{\infty}\frac{B^{mk}(1+\beta^l)}{\alpha^{3mk}(1+\alpha^l)}+\sum_{k=n}^{\infty}\frac{(1+\beta^l)^2}{\alpha^{5mk}(1+\alpha^l)^2}+\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{mk}}\right)\\&=\frac{\alpha-\beta}{1+\alpha^l}\left(\frac{\alpha^m}{\alpha^{mn}(\alpha^m-1)}+C_m\right)=\frac{\alpha^m(\alpha-\beta)}{\alpha^{mn}(\alpha^m-1)(1+\alpha^l)}\left(1+\frac{\alpha^{mn}(\alpha^m-1)}{\alpha^m}C_m\right),\end{aligned}\tag{3.15}$$

where

$$C_m=\frac{B^{mn}\alpha^{3m}(1+\beta^l)}{\alpha^{3mn}(\alpha^{3m}-B^m)(1+\alpha^l)}+\frac{\alpha^{5m}(1+\beta^l)^2}{\alpha^{5mn}(\alpha^{5m}-1)(1+\alpha^l)^2}+\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{mk}}.$$

Then we have

$$\frac{\alpha^{mn}(\alpha^m-1)}{\alpha^m}C_m=O\left(\frac{1}{\alpha^{2mn}}\right).\tag{3.16}$$

By (3.2a), (3.15), and (3.16), we have

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+u_{mk+l}}\right)^{-1} &=\left(\frac{\alpha^m(\alpha-\beta)}{\alpha^{mn}(1+\alpha^l)(\alpha^m-1)}\right)^{-1}\left(1+O\left(\frac{1}{\alpha^{2mn}}\right)\right)^{-1}\\ &=\frac{\alpha^{mn}(1+\alpha^l)(\alpha^m-1)}{\alpha^m(\alpha-\beta)}\left(1+O\left(\frac{1}{\alpha^{2mn}}\right)\right)\\ &=\frac{\alpha^{mn}+\alpha^{mn+l}-\alpha^{m(n-1)}-\alpha^{m(n-1)+l}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right)\\ &=\frac{\alpha^{mn+l}-\beta^{mn+l}+\beta^{mn+l}}{\alpha-\beta}-\frac{\alpha^{m(n-1)+l}-\beta^{m(n-1)+l}+\beta^{m(n-1)+l}}{\alpha-\beta}\\ &\quad+\frac{\alpha^{mn}-\beta^{mn}+\beta^{mn}}{\alpha-\beta}-\frac{\alpha^{m(n-1)}-\beta^{m(n-1)}+\beta^{m(n-1)}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right)\\ &=u_{mn+l}-u_{m(n-1)+l}+u_{mn}-u_{m(n-1)}+\frac{\beta^{mn}+\beta^{mn+l}-\beta^{m(n-1)}-\beta^{m(n-1)+l}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right).\end{aligned}$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+u_{mk+l}}\right)^{-1}-(u_{mn+l}-u_{m(n-1)+l}+u_{mn}-u_{m(n-1)})\right)\\ &=\lim_{n\to\infty}\left(\frac{\beta^{mn}+\beta^{mn+l}-\beta^{m(n-1)}-\beta^{m(n-1)+l}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right)\right)=0.\end{aligned}$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+u_{mk+l}}\right)^{-1}\sim u_{mn+l}-u_{m(n-1)+l}+u_{mn}-u_{m(n-1)}.$$

□

3.6. The proof of Theorem 2.6

In this subsection, we will provide a proof of Theorem 2.6.

Proof. By (1.2), we have

$$\begin{aligned}\sum_{i=0}^{l}u_{mk+i}&=\frac{1}{\alpha-\beta}\left(\alpha^{mk}\sum_{i=0}^{l}\alpha^i-\beta^{mk}\sum_{i=0}^{l}\beta^i\right)=\frac{1}{\alpha-\beta}\left(\frac{\alpha^{mk}(1-\alpha^{l+1})}{1-\alpha}-\frac{\beta^{mk}(1-\beta^{l+1})}{1-\beta}\right)\\ &=\frac{\alpha^{mk}(1-\alpha^{l+1})}{(\alpha-\beta)(1-\alpha)}\left(1-\frac{B^{mk}(1-\alpha)(1-\beta^{l+1})}{\alpha^{2mk}(1-\beta)(1-\alpha^{l+1})}\right).\end{aligned}\tag{3.17}$$

From (3.2b) and (3.17), it follows that

$$
\begin{aligned}
\frac{1}{\sum_{i=0}^{l} u_{mk+i}} =&\left(\frac{\alpha^{mk}(1-\alpha^{l+1})}{(\alpha-\beta)(1-\alpha)}\right)^{-1}\left(1-\frac{B^{mk}(1-\alpha)(1-\beta^{l+1})}{\alpha^{2mk}(1-\beta)(1-\alpha^{l+1})}\right)^{-1}\\
=&\frac{(\alpha-\beta)(1-\alpha)}{\alpha^{mk}(1-\alpha^{l+1})}\left(1+\frac{B^{mk}(1-\alpha)(1-\beta^{l+1})}{\alpha^{2mk}(1-\beta)(1-\alpha^{l+1})}+\frac{(1-\alpha)^2(1-\beta^{l+1})^2}{\alpha^{4mk}(1-\beta)^2(1-\alpha^{l+1})^2}+R_k\right)\\
=&\frac{(\alpha-\beta)(1-\alpha)}{(1-\alpha^{l+1})}\left(\frac{1}{\alpha^{mk}}+\frac{B^{mk}(1-\alpha)(1-\beta^{l+1})}{\alpha^{3mk}(1-\beta)(1-\alpha^{l+1})}+\frac{(1-\alpha)^2(1-\beta^{l+1})^2}{\alpha^{5mk}(1-\beta)^2(1-\alpha^{l+1})^2}\right)\\
&+\frac{(\alpha-\beta)(1-\alpha)}{(1-\alpha^{l+1})}\cdot\frac{R_k}{\alpha^{mk}},
\end{aligned}
\tag{3.18}
$$

where

$$
R_k=\frac{(1-\beta^{l+1})^3(1-\alpha)^3B^{mk}\left(4\alpha^{mk}(1-\beta)(1-\alpha^{l+1})-3B^{mk}(1-\alpha)(1-\beta^{l+1})\right)}{\alpha^{4mk}(1-\beta)^2(1-\alpha^{l+1})^2\left(\alpha^{2mk}(1-\beta)(1-\alpha^{l+1})-B^{mk}(1-\alpha)(1-\beta^{l+1})\right)}.
$$

Let n be a positive integer. Then it follows from (3.18) that

$$
\begin{aligned}
\sum_{k=n}^{\infty}\frac{1}{\sum_{i=0}^{l} u_{mk+i}} &= \frac{(\alpha-\beta)(1-\alpha)}{(1-\alpha^{l+1})}\left(\sum_{k=n}^{\infty}\frac{1}{\alpha^{mk}}+C_m\right)=\frac{(\alpha-\beta)(1-\alpha)}{(1-\alpha^{l+1})}\left(\frac{\alpha^m}{\alpha^{mn}(\alpha^m-1)}+C_m\right)\\
&=\frac{\alpha^m(\alpha-\beta)(1-\alpha)}{\alpha^{mn}(\alpha^m-1)(1-\alpha^{l+1})}\left(1+\frac{\alpha^{mn}(\alpha^m-1)}{\alpha^m}C_m\right),
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
C_m &= \sum_{k=n}^{\infty}\frac{B^{mk}(1-\alpha)(1-\beta^{l+1})}{\alpha^{3mk}(1-\beta)(1-\alpha^{l+1})}+\sum_{k=n}^{\infty}\frac{(1-\alpha)^2(1-\beta^{l+1})^2}{\alpha^{5mk}(1-\beta)^2(1-\alpha^{l+1})^2}+\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{mk}}\\
&=\frac{B^{mn}\alpha^{3m}(1-\alpha)(1-\beta^{l+1}))}{\alpha^{3mn}(\alpha^{3m}-B^m)(1-\beta)(1-\alpha^{l+1})}+\frac{\alpha^{5m}(1-\alpha)^2(1-\beta^{l+1})^2}{\alpha^{5mn}(\alpha^{5m}-1)(1-\beta)^2(1-\alpha^{l+1})^2}+\sum_{k=n}^{\infty}\frac{R_k}{\alpha^{mk}}\\
&=O\left(\frac{1}{\alpha^{3mn}}\right).
\end{aligned}
$$

Then we have

$$
\frac{\alpha^{mn}(\alpha^m-1)}{\alpha^m}C_m=O\left(\frac{1}{\alpha^{2mn}}\right).
\tag{3.20}
$$

By (3.2a), (3.19), and (3.20), we have

$$\begin{aligned}
\left(\sum_{k=n}^{\infty}\frac{1}{\sum_{i=0}^{l}u_{mk+i}}\right)^{-1} =&\left(\frac{\alpha^{m}(\alpha-1)(\alpha-\beta)}{\alpha^{mn}(\alpha^{l+1}-1)(\alpha^{m}-1)}\right)^{-1}\left(1+O\left(\frac{1}{\alpha^{2mn}}\right)\right)^{-1}\\
=&\frac{\alpha^{mn}(\alpha^{l+1}-1)(\alpha^{m}-1)}{\alpha^{m}(\alpha-1)(\alpha-\beta)}\left(1+O\left(\frac{1}{\alpha^{2mn}}\right)\right)\\
=&\frac{\alpha^{mn+l+1}-\alpha^{m(n-1)+l+1}-\alpha^{mn}+\alpha^{m(n-1)}}{(\alpha-1)(\alpha-\beta)}+O\left(\frac{1}{\alpha^{mn}}\right)\\
=&\frac{\alpha^{m(n-1)}-\beta^{m(n-1)}+\beta^{m(n-1)}}{(\alpha-1)(\alpha-\beta)}-\frac{\alpha^{m(n-1)+l+1}-\beta^{m(n-1)+l+1}+\beta^{m(n-1)+l+1}}{(\alpha-1)(\alpha-\beta)}\\
&-\frac{\alpha^{mn}-\beta^{mn}+\beta^{mn}}{(\alpha-1)(\alpha-\beta)}+\frac{\alpha^{mn+l+1}-\beta^{mn+l+1}+\beta^{mn+l+1}}{(\alpha-1)(\alpha-\beta)}+O\left(\frac{1}{\alpha^{mn}}\right)\\
=&\frac{1}{\alpha-1}\left(u_{mn+l+1}-u_{m(n-1)+l+1}-u_{mn}+u_{m(n-1)}\right)\\
&+\frac{\beta^{mn+l+1}-\beta^{mn}+\beta^{m(n-1)}-\beta^{m(n-1)+l+1}}{(\alpha-1)(\alpha-\beta)}+O\left(\frac{1}{\alpha^{mn}}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{\sum_{i=0}^{l}u_{mk+i}}\right)^{-1}-\frac{1}{\alpha-1}\left(u_{mn+l+1}-u_{m(n-1)+l+1}-u_{mn}+u_{m(n-1)}\right)\right)\\
=&\lim_{n\to\infty}\left(\frac{\beta^{mn+l+1}-\beta^{mn}+\beta^{m(n-1)}-\beta^{m(n-1)+l+1}}{(\alpha-1)(\alpha-\beta)}+O\left(\frac{1}{\alpha^{mn}}\right)\right)=0.
\end{aligned}$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{\sum_{i=0}^{l}u_{mk+i}}\right)^{-1}\sim\frac{1}{\alpha-1}\left(u_{mn+l+1}-u_{m(n-1)+l+1}-u_{mn}+u_{m(n-1)}\right).$$

□

3.7. The proof of Theorem 2.7

In this subsection, we will provide a proof of Theorem 2.7.

Proof. Let h be a positive integer. By (1.2), we have

$$\begin{aligned}
\frac{1}{u_{mk}u_{mk+h}}&=(\alpha-\beta)^{2}\left((\alpha^{mk}-\beta^{mk})(\alpha^{mk+h}-\beta^{mk+h})\right)^{-1}=\frac{(\alpha-\beta)^{2}}{\alpha^{2mk+h}}\left((1-\frac{B^{mk}}{\alpha^{2mk}})(1-\frac{B^{mk+h}}{\alpha^{2mk+2h}})\right)^{-1}\\
&=\frac{(\alpha-\beta)^{2}}{\alpha^{2mk+h}}\left(1-\frac{B^{mk}}{\alpha^{2mk}}-\frac{B^{mk+h}}{\alpha^{2mk+2h}}+\frac{B^{h}}{\alpha^{4mk+2h}}\right)^{-1}=\frac{(\alpha-\beta)^{2}}{\alpha^{2mk+h}}\left(1-\eta\right)^{-1},
\end{aligned}\tag{3.21}$$

where

$$\eta=\frac{B^{mk}}{\alpha^{2mk}}+\frac{B^{mk+h}}{\alpha^{2mk+2h}}-\frac{B^{h}}{\alpha^{4mk+2h}}=\frac{B^{mk}}{\alpha^{2mk}}+\frac{B^{mk+h}}{\alpha^{2mk+2h}}+O\left(\frac{1}{\alpha^{4mk}}\right).$$

Then we have

$$\eta^{2}+\frac{\eta^{3}}{1-\eta}=O\left(\frac{1}{\alpha^{4mk}}\right).\tag{3.22}$$

By (3.2b), (3.21), and (3.22), we have

$$\begin{aligned}\frac{1}{u_{mk}u_{mk+h}} &= \frac{(\alpha-\beta)^2}{\alpha^{2mk+h}}\left(1+\eta+\eta^2+\frac{\eta^3}{1-\eta}\right) = \frac{(\alpha-\beta)^2}{\alpha^{2mk+h}}\left(1+\eta+O\left(\frac{1}{\alpha^{4mk}}\right)\right)\\ &= \frac{(\alpha-\beta)^2}{\alpha^{2mk+h}}\left(1+\frac{B^{mk}}{\alpha^{2mk}}\left(1+\frac{B^h}{\alpha^{2h}}\right)+O\left(\frac{1}{\alpha^{4mk}}\right)\right)\\ &= \frac{(\alpha-\beta)^2}{\alpha^{h}}\left(\frac{1}{\alpha^{2mk}}+\frac{B^{mk}}{\alpha^{4mk}}\left(1+\frac{B^h}{\alpha^{2h}}\right)+O\left(\frac{1}{\alpha^{6mk}}\right)\right).\end{aligned} \tag{3.23}$$

Let n be a positive integer. Then it follows from (3.23) that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+h}} &= \frac{(\alpha-\beta)^2}{\alpha^{h}}\left(\sum_{k=n}^{\infty}\frac{1}{\alpha^{2mk}}+\left(1+\frac{B^h}{\alpha^{2h}}\right)\sum_{k=n}^{\infty}\frac{B^{mk}}{\alpha^{4mk}}\right)+O\left(\frac{1}{\alpha^{6mn}}\right)\\ &= \frac{(\alpha-\beta)^2}{\alpha^{h}}\left(\frac{\alpha^{2m}}{\alpha^{2mn}(\alpha^{2m}-1)}+\left(1+\frac{B^h}{\alpha^{2h}}\right)\frac{B^{mn}\alpha^{4m}}{\alpha^{4mn}(\alpha^{4m}-B^m)}\right)+O\left(\frac{1}{\alpha^{6mn}}\right)\\ &= \frac{(\alpha-\beta)^2\alpha^{2m}}{\alpha^{2mn+h}(\alpha^{2m}-1)}\left(1+\left(1+\frac{B^h}{\alpha^{2h}}\right)\frac{B^{mn}\alpha^{2m}(\alpha^{2m}-1)}{\alpha^{2mn}(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\ &= \frac{(\alpha-\beta)^2\alpha^{2m}}{\alpha^{2mn+h}(\alpha^{2m}-1)}(1+\omega),\end{aligned} \tag{3.24}$$

where

$$\omega = \left(1+\frac{B^h}{\alpha^{2h}}\right)\frac{B^{mn}\alpha^{2m}(\alpha^{2m}-1)}{\alpha^{2mn}(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right).$$

Then we have

$$\omega^2-\frac{\omega^3}{1+\omega} = O\left(\frac{1}{\alpha^{4mn}}\right). \tag{3.25}$$

By (3.2a), (3.24), and (3.25), we have

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+h}}\right)^{-1} &= \left(\frac{(\alpha-\beta)^2\alpha^{2m}}{\alpha^{2mn+h}(\alpha^{2m}-1)}\right)^{-1}(1+\omega)^{-1}\\ &= \frac{\alpha^{2mn+h}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}\left(1-\omega+\omega^2-\frac{\omega^3}{1+\omega}\right)\\ &= \frac{\alpha^{2mn+h}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}\left(1-\omega+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\ &= \frac{\alpha^{2mn+h}(\alpha^{2m}-1)}{(\alpha-\beta)^2\alpha^{2m}}\left(1-\left(1+\frac{B^h}{\alpha^{2h}}\right)\frac{B^{mn}\alpha^{2m}(\alpha^{2m}-1)}{\alpha^{2mn}(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{4mn}}\right)\right)\\ &= \frac{\alpha^{2mn+h}-\alpha^{2m(n-1)+h}}{(\alpha-\beta)^2}-\left(1+\frac{B^h}{\alpha^{2h}}\right)\frac{B^{mn}(\alpha^{2m}-1)^2\alpha^h}{(\alpha-\beta)^2(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right).\end{aligned} \tag{3.26}$$

(i) If we take $h = 2l$, then it follows from (3.26) that

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l}}\right)^{-1} =&\left(\frac{\alpha^{mn+l}-\beta^{mn+l}+\beta^{mn+l}}{\alpha-\beta}\right)^2-\left(\frac{\alpha^{m(n-1)+l}-\beta^{m(n-1)+l}+\beta^{m(n-1)+l}}{\alpha-\beta}\right)^2\\ &-\left(1+\frac{1}{\alpha^{4l}}\right)\frac{B^{mn}(\alpha^{2m}-1)^2\alpha^{2l}}{(\alpha-\beta)^2(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right)\\ =&u_{mn+l}^2-u_{m(n-1)+l}^2+\delta+O\left(\frac{1}{\alpha^{2mn}}\right),\end{aligned}$$

where

$$\begin{aligned}\delta =& \frac{2(B^{mn+l}-B^{m(n-1)+l})}{(\alpha-\beta)^2}-\left(1+\frac{1}{\alpha^{4l}}\right)\frac{B^{mn}(\alpha^{2m}-1)^2\alpha^{2l}}{(\alpha-\beta)^2(\alpha^{4m}-B^m)}\\ =&-\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}\left(\frac{\alpha^{2l}+\beta^{2l}}{(\alpha-\beta)^2}-\frac{2B^l(1-B^m)(\alpha^{4m}-B^m)}{(\alpha-\beta)^2(\alpha^{2m}-1)^2}\right)\\ =&-\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}\left(u_l^2+\frac{2B^l}{(\alpha-\beta)^2}-\frac{2B^l(1-B^m)(\alpha^{4m}-B^m)}{(\alpha-\beta)^2(\alpha^{2m}-1)^2}\right)\\ =&-\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}\left(u_l^2+\frac{2B^l}{(\alpha-\beta)^2}\left(1-\frac{(1-B^m)(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)\right).\end{aligned}$$

Let $C_{m,l} = u_l^2 + \frac{2B^l}{(\alpha-\beta)^2}\left(1-\frac{(1-B^m)(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)$. Then

$$\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l}}\right)^{-1}-\left(u_{mn+l}^2-u_{m(n-1)+l}^2-\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C_{m,l}\right)\right)=0.$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l}}\right)^{-1}\sim u_{mn+l}^2-u_{m(n-1)+l}^2-\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C_{m,l},$$

where $C_{m,l} = u_l^2 + \frac{2B^l}{A^2-4B}\left(1-\frac{(1-B^m)(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)$.

(ii) If we take $h = 2l-1$, then it follows from (3.26) that

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l-1}}\right)^{-1} =&\frac{\alpha^{2mn+2l-1}-\alpha^{2m(n-1)+2l-1}}{(\alpha-\beta)^2}-\left(1+\frac{B}{\alpha^{4l-2}}\right)\frac{B^{mn}(\alpha^{2m}-1)^2\alpha^{2l-1}}{(\alpha-\beta)^2(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right)\\ =&(\alpha-\alpha\beta^{2m})\frac{\alpha^{2mn+2l-2}}{(\alpha-\beta)^2}-\left(1+\frac{B}{\alpha^{4l-2}}\right)\frac{B^{mn}(\alpha^{2m}-1)^2\alpha^{2l-1}}{(\alpha-\beta)^2(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right)\\ =&(\alpha-\alpha\beta^{2m})\left(\frac{\alpha^{2mn+2l-2}-\beta^{2mn+2l-2}+\beta^{2mn+2l-2}}{(\alpha-\beta)^2}\right)\\ &-\left(1+\frac{B}{\alpha^{4l-2}}\right)\frac{B^{mn}(\alpha^{2m}-1)^2\alpha^{2l-1}}{(\alpha-\beta)^2(\alpha^{4m}-B^m)}+O\left(\frac{1}{\alpha^{2mn}}\right)\\ =&(\alpha-\alpha\beta^{2m})u_{mn+l-1}^2+\delta+O\left(\frac{1}{\alpha^{2mn}}\right),\end{aligned}$$

where

$$\begin{aligned}\delta &= \frac{2B^{mn+l-1}(\alpha-\alpha\beta^{2m})}{(\alpha-\beta)^2} - \left(1+\frac{B^{2l-1}}{\alpha^{4l-2}}\right)\frac{B^{mn}(\alpha^{2m}-1)^2\alpha^{2l-1}}{(\alpha-\beta)^2(\alpha^{4m}-B^m)}\\
&= -\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}\left(\frac{\alpha^l\alpha^{l-1}+\beta^{2l-1}}{(\alpha-\beta)^2} - \frac{2B^{l-1}(\alpha-\alpha\beta^{2m})(\alpha^{4m}-B^m)}{(\alpha-\beta)^2(\alpha^{2m}-1)^2}\right)\\
&= -\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}\left(u_lu_{l-1}+\frac{B^{l-1}(\alpha+\beta)}{(\alpha-\beta)^2} - \frac{2B^{l-1}(\alpha-\alpha\beta^{2m})(\alpha^{4m}-B^m)}{(\alpha-\beta)^2(\alpha^{2m}-1)^2}\right)\\
&= -\frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}\left(u_lu_{l-1}+\frac{B^{l-1}}{(\alpha-\beta)^2}\left((\alpha+\beta) - \frac{2(\alpha-\alpha\beta^{2m})(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)\right).\end{aligned}$$

Let $C'_{m,l} = u_lu_{l-1} + \frac{B^{l-1}}{(\alpha-\beta)^2}\left((\alpha+\beta) - \frac{2(\alpha-\alpha\beta^{2m})(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)$. Then

$$\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l-1}}\right)^{-1} - \left((\alpha-\alpha\beta^{2m})u_{mn+l-1}^2 - \frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C'_{m,l}\right)\right) = 0.$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}u_{mk+2l-1}}\right)^{-1} \sim (\alpha-\alpha\beta^{2m})u_{mn+l-1}^2 - \frac{B^{mn}(\alpha^{2m}-1)^2}{\alpha^{4m}-B^m}C'_{m,l},$$

where $C'_{m,l} = u_lu_{l-1} + \frac{B^{l-1}}{A^2-4B}\left(A - \frac{2(\alpha-\alpha\beta^{2m})(\alpha^{4m}-B^m)}{(\alpha^{2m}-1)^2}\right)$.

□

3.8. The proof of Theorem 2.8

In this subsection, we will provide a proof of Theorem 2.8.

Proof. By (1.2), we have

$$u_{mk} + C = \frac{\alpha^{mk}-\beta^{mk}}{\alpha-\beta} + C = \frac{\alpha^{mk}}{\alpha-\beta}\left(1+\frac{C(\alpha-\beta)}{\alpha^{mk}} - \frac{B^{mk}}{\alpha^{2mk}}\right) = \frac{\alpha^{mk}}{\alpha-\beta}(1+\eta), \tag{3.27}$$

where

$$\eta = \frac{C(\alpha-\beta)}{\alpha^{mk}} - \frac{B^{mk}}{\alpha^{2mk}}.$$

Then we have

$$\eta^2 - \frac{\eta^3}{1+\eta} = O\left(\frac{1}{\alpha^{2mk}}\right). \tag{3.28}$$

From (3.2a), (3.27), and (3.28), it follows that

$$\begin{aligned}\frac{1}{u_{mk}+C} &= \left(\frac{\alpha^{mk}}{\alpha-\beta}\right)^{-1}(1+\eta)^{-1} = \frac{\alpha-\beta}{\alpha^{mk}}\left(1-\eta+\eta^2-\frac{\eta^3}{1+\eta}\right) = \frac{\alpha-\beta}{\alpha^{mk}}\left(1-\eta+O\left(\frac{1}{\alpha^{2mk}}\right)\right)\\
&= \frac{\alpha-\beta}{\alpha^{mk}}\left(1-\frac{C(\alpha-\beta)}{\alpha^{mk}}+O\left(\frac{1}{\alpha^{2mk}}\right)\right) = (\alpha-\beta)\left(\frac{1}{\alpha^{mk}}-\frac{C(\alpha-\beta)}{\alpha^{2mk}}+O\left(\frac{1}{\alpha^{3mk}}\right)\right).\end{aligned} \tag{3.29}$$

Let n be a positive integer. Then it follows from (3.29) that

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_{mk}+C}&=\sum_{k=n}^{\infty}\frac{\alpha-\beta}{\alpha^{mk}}-\sum_{k=n}^{\infty}\frac{C(\alpha-\beta)^2}{\alpha^{2mk}}+O\left(\frac{1}{\alpha^{3mn}}\right)=\frac{\alpha^m(\alpha-\beta)}{\alpha^{mn}(\alpha^m-1)}-\frac{C\alpha^{2m}(\alpha-\beta)^2}{\alpha^{2mn}(\alpha^{2m}-1)}+O\left(\frac{1}{\alpha^{3mn}}\right)\\&=\frac{(\alpha-\beta)\alpha^m}{\alpha^{mn}(\alpha^m-1)}\left(1-\frac{C(\alpha-\beta)\alpha^m}{\alpha^{mn}(\alpha^m+1)}+O\left(\frac{1}{\alpha^{2mn}}\right)\right)=\frac{(\alpha-\beta)\alpha^m}{\alpha^{mn}(\alpha^m-1)}(1-\omega),\end{aligned}\tag{3.30}$$

where

$$\omega=\frac{C(\alpha-\beta)\alpha^m}{\alpha^{mn}(\alpha^m+1)}+O\left(\frac{1}{\alpha^{2mn}}\right).$$

Then we have

$$\omega^2+\frac{\omega^3}{1-\omega}=O\left(\frac{1}{\alpha^{2mn}}\right).\tag{3.31}$$

By (3.2b), (3.30), and (3.31), we have

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+C}\right)^{-1}&=\left(\frac{(\alpha-\beta)\alpha^m}{\alpha^{mn}(\alpha^m-1)}\right)^{-1}(1-\omega)^{-1}=\frac{\alpha^{mn}(\alpha^m-1)}{(\alpha-\beta)\alpha^m}\left(1+\omega+\omega^2+\frac{\omega^3}{1-\omega}\right)\\&=\frac{\alpha^{mn}(\alpha^m-1)}{(\alpha-\beta)\alpha^m}\left(1+\omega+O\left(\frac{1}{\alpha^{2mn}}\right)\right)=\frac{\alpha^{mn}(\alpha^m-1)}{(\alpha-\beta)\alpha^m}\left(1+\frac{C(\alpha-\beta)\alpha^m}{\alpha^{mn}(\alpha^m+1)}+O\left(\frac{1}{\alpha^{2mn}}\right)\right)\\&=\frac{\alpha^{mn}-\alpha^{m(n-1)}}{\alpha-\beta}+\frac{C(\alpha^m-1)}{\alpha^m+1}+O\left(\frac{1}{\alpha^{mn}}\right)\\&=\frac{\alpha^{mn}-\beta^{mn}+\beta^{mn}}{\alpha-\beta}-\frac{\alpha^{m(n-1)}-\beta^{m(n-1)}+\beta^{m(n-1)}}{\alpha-\beta}+\frac{C(\alpha^m-1)}{\alpha^m+1}+O\left(\frac{1}{\alpha^{mn}}\right)\\&=u_{mn}-u_{m(n-1)}+\frac{\beta^{mn}-\beta^{m(n-1)}}{\alpha-\beta}+\frac{C(\alpha^m-1)}{\alpha^m+1}+O\left(\frac{1}{\alpha^{mn}}\right).\end{aligned}$$

Then

$$\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+C}\right)^{-1}-\left(u_{mn}-u_{m(n-1)}+\frac{C(\alpha^m-1)}{\alpha^m+1}\right)\right)=\lim_{n\to\infty}\left(\frac{\beta^{mn}-\beta^{m(n-1)}}{\alpha-\beta}+O\left(\frac{1}{\alpha^{mn}}\right)\right)=0.$$

So we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{u_{mk}+C}\right)^{-1}\sim u_{mn}-u_{m(n-1)}+C\frac{\alpha^m-1}{\alpha^m+1}.$$

□

4. Conclusions

Let $(u_n)_{n\geq 0}$ be the special Lucas u-sequence defined by $u_{n+2}=Au_{n+1}-Bu_n$, $u_0=0$, $u_1=1$, where $n\geq 0$, $B=\pm 1$, and A is an integer such that $A^2-4B>0$. In this paper, we study the asymptotic behavior of the sequences involving u_n. In Section 1, we give the definition of the asymptotic behavior and introduce the asymptotic behavior of some sequences. In Section 2, we give the asymptotic formulas for $\left(\sum_{k=n}^{\infty} a_k\right)^{-1}$, where

$$a_k = \frac{1}{u_{mk}^s}, \frac{1}{u_{mk}+u_{mk+l}}, \frac{1}{\sum_{i=0}^{l} u_{mk+i}}, \frac{1}{u_{mk}u_{mk+2l}}, \frac{1}{u_{mk}u_{mk+2l-1}}, \frac{1}{u_{mk}+C},$$

m, l are positive integers, $s = 1, 2, 3, 4$, and C is any constant. In Section 3, we give the proof of these results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. H. Ohtsuka, S. Nakamura, On the sum of reciprocal Fibonacci numbers, *Fibonacci Quart.*, **46** (2008), 153–159. https://doi.org/10.1080/00150517.2008.12428174
2. T. Wang, Infinite sums of reciprocal Fibonacci numbers (in Chinese), *Acta Math. Sin.*, **55** (2012), 517–524. https://doi.org/10.12386/A2012sxxb0048
3. W. Hwang, J. D. Park, K. Song, On the reciprocal sum of the fourth power of Fibonacci numbers, *Open Math.*, **20** (2022), 1642–1655. https://doi.org/10.1515/math-2022-0525
4. H. Li, Y. He, The reciprocal sums of the cubes of odd and even terms in the Fibonacci sequence, *Acta Math. Sin.*, **67** (2024), 926–938. https://doi.org/10.12386/A20210193
5. T. Komatsu, On the nearest integer of the sum of reciprocal Fibonacci numbers, *Aportaciones Mat. Investig.*, **20** (2011), 171–184.
6. G. Choi, Y. Choo, On the reciprocal sums of products of Fibonacci and Lucas numbers, *Filomat*, **32** (2018), 2911–2920. https://doi.org/10.2298/FIL1808911C
7. Z. Xu, T. Wang, The infinite sum of the cubes of reciprocal Pell numbers, *Adv. Differ. Equations*, **2013** (2013), 184. https://doi.org/10.1186/1687-1847-2013-184

8. Z. Wu, H. Zhang, On the reciprocal sums of higher-order sequences, *Adv. Differ. Equations*, **2013** (2013), 189. https://doi.org/10.1186/1687-1847-2013-189
9. X. Lin, Partial reciprocal sums of the Mathieu series, *J. Inequal. Appl.*, **2017** (2017), 1–8. https://doi.org/10.1186/s13660-017-1327-x
10. T. Komatsu, V. Laohakosol, On the sum of reciprocals of numbers satisfying a recurrence relation of order s, *J. Integer Seq.*, **13** (2010), 1–9.
11. H. H. Lee, J. D. Park, The limit of reciprocal sum of some subsequential Fibonacci numbers, *AIMS Math.*, **6** (2021), 12379–12394. https://doi.org/10.3934/math.2021716
12. D. Marques, P. Trojovský, The proof of a formula concerning the asymptotic behavior of the reciprocal sum of the square of multiple-angle Fibonacci numbers, *J. Inequal. Appl.*, **2022** (2022), 21. https://doi.org/10.1186/s13660-022-02755-7
13. H. H. Lee, J. D. Park, Asymptotic behavior of reciprocal sum of two products of Fibonacci numbers, *J. Inequal. Appl.*, **2020** (2020), 91. https://doi.org/10.1186/s13660-020-02359-z
14. Z. Pan, Z. Wu, The inverses of tails of the generalized Riemann zeta function within the range of integers, *AIMS Math.*, **8** (2023), 28558–28568. https://doi.org/10.3934/math.20231461
15. P. Trojovsky, On the sum of reciprocal of polynomial applied to higher order recurrences, *Mathematics*, **7** (2019), 638. https://doi.org/10.3390/math7070638
16. P. Yuan, Z. He, J. Zhou, On the sum of reciprocal generalized Fibonacci numbers, *Abstr. Appl. Anal.*, **2014** (2014). http://dx.doi.org/10.1155/2014/402540
17. Z. Sun, *Fibonacci Numbers and Hilbert's Tenth Problem*, Harbin Institute of Technology Press, Harbin, 2024.