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Research article

α-robust error analysis of two nonuniform schemes for Caputo-Hadamard fractional reaction sub-diffusion problems

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Abstract: In this paper, we focused on the Caputo-Hadamard fractional reaction sub-diffusion equations. By using the nonuniform L1 scheme and nonuniform Alikhanov scheme in the temporal domain, we formulated two efficient numerical schemes, where the second order difference method was used in the spatial dimension. Furthermore, we derived the stability and convergence of these proposed schemes. Remarkably, both derived numerical methods exhibited α-robustness, that is, it remained valid when $\alpha \to 1^-$. Numerical experiments were given to demonstrate the theoretical statements.

Keywords: Caputo-Hadamard derivative; reaction subdiffusion equations; the nonuniform L1 scheme; the nonuniform Alikhanov scheme; α-robust error estimates

1. Introduction

In the past, researchers have conducted extensive studies in the fields of fractional calculus and fractional differential equations (FDEs) [1–4]. To date, numerous scholars have extensively explored the realm of fractional integrals and derivatives in numerous forms, including Riemann-Liouville, Caputo, and Riesz integrals and derivatives, among others. However, there exists another type of fractional derivative that incorporates a logarithmic function in its definition, known as the Hadamard fractional derivative, which is defined as [2]

$$ {}^{CH}D^{\alpha}_{a,t}v(x,t) = \int_a^t \omega_{1-\alpha}(\log t - \log s)\,\delta v(x,s)\frac{ds}{s},\ 0 < a < t, $$

where $0 < \alpha < 1$, $\omega_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $\delta v(x,s) = \left(s\frac{\partial}{\partial s}\right)v(x,s)$, and $\Gamma(\cdot)$ representing the Gamma function.

Compared to the Riemann-Liouville and Caputo derivatives, the Caputo-Hadamard derivative, first introduced in 1892, more accurately captures some complex processes in practical applications. This includes Lomnitz logarithmic creep law [5, 6] and ultra-slow mechanics [7, 8] , among others. In particular, the logarithmic increase in the mean square displacement of particles during ultra-slow diffusion has been demonstrated in [9–11]. As a result, the Hadamard fractional operator, whose kernel is a logarithmic function, has emerged as a natural model for ultra-slow diffusion processes and has garnered significant attention.

Let $\Omega = (x_l, x_r), \Lambda = (a, T)$ with $a > 0$, and we will pay attention to the numerical approximation for the following Caputo-Hadamard fractional reaction sub-diffusion equation:

$$^{CH}D^{\alpha}_{a,t}u(x,t) - \partial^2_x u(x,t) + \kappa u(x,t) = f(x,t), \quad (x,t) \in \Omega \times \Lambda, \tag{1.1}$$

$$u(x,a) = \varphi(x), \qquad x \in \Omega, \tag{1.2}$$

$$u(x_l,t) = u(x_r,t) = 0, \quad t \in \Lambda. \tag{1.3}$$

Here, the real constant $\kappa \in \mathbb{R}$ is the reaction coefficient, and the source term $f(x,t)$ and the initial data $\varphi(x)$ are given functions.

Recent studies have explored various numerical methods for tackling Caputo-Hadamard FDEs, encompassing the L1 scheme [12], L1-2 scheme, and L2-1σ scheme [13]. Moreover, Zhao et al. [14] introduced a spectral collocation method utilizing mapped Jacobi log orthogonal functions as basis functions, resulting in an efficient algorithm for solving Hadamard-type FDEs. Based on block-by-block approach, Ye et al. [15] proposed and analyzed a high order time stepping scheme having the convergence order more than three for the Caputo-Hadamard fractional differential equations.

Actually, the aforementioned studies mainly concentrate on the Caputo-Hadamard FDEs that possess smooth solutions. Recently, numerous efficient numerical methods have been developed for Caputo FDEs with weakly singular solutions, including the nonuniform L1 scheme [16, 17], the nonuniform Alikhanov scheme [18–20], convolution quadrature method [21], and the spectral method [22]. Interested readers can also consult some recent references [23–27] for more numerical methods about FDE, such as the Alternating Direction Implicit method, extrapolation method, meshless method, and so on.

Nevertheless, numerical simulations for Caputo-Hadamard fractional reaction sub-diffusion equation (1.1)–(1.3) with weakly singular solutions remain relatively limited. For Eqs (1.1)–(1.3) without reaction term, Li et al. [28] proposed an L1 scheme on nonuniform meshes to approximate the time Caputo-Hadamard fractional derivative and employed the local discontinuous Galerkin method to approximate the spatial derivative. Later, the Alikhanov scheme with nonuniform time meshes for Caputo-Hadamard fractional sub-diffusion equations with an initial singularity was investigated in [29]. The stability and convergence of the resulting discrete scheme were analyzed, but the error bounds generally contain a constant factor $\Gamma(1-\alpha)$ or $1/(1-\alpha)$ which will blow up as α approaches 1^-.

Zhang et al. [30] derived a novel α-robust error analysis for convolution-type schemes with general nonuniform time step for Caputo fractional reaction sub-diffusion equations. By virtue of the ideas derived in [30], this paper will extend the nonuniform L1 and Alikhanov scheme presented in [28] and [29] to Caputo-Hadamard fractional sub-diffusion equations with reaction term, and then we will consider the α-robust error analysis of the proposed schemes. This means that the derived error bounds will not contain any blowup factor and will remain valid as $\alpha \to 1^-$.

Throughout this paper, we employ C to represent a generic constant that is independent of the mesh, and it may take different values at different places. Additionally, C exhibits α-robustness, meaning it is influenced by α, yet as α approaches 1, the value of C remains finite, avoiding any potential explosion. The proposed method in this paper is analyzed under the following regularity assumptions: for all $(x, t) \in \Omega \times \Lambda$, it holds

$$\left|\frac{\partial^l u(x,t)}{\partial x^l}\right| \leq C, \quad l = 0,1,2,3,4. \tag{1.4}$$

$$|\delta^l u(x,t)| \leq C\left(1 + \left(\log\frac{t}{a}\right)^{\sigma-l}\right), \quad l = 0,1,2, \tag{1.5}$$

with the regularity parameter $\sigma \in (0,1) \cup (1,2)$, and $\delta^l u(x,t) = (t\frac{\partial}{\partial t})^l u(x,t)$.

The remainder of this paper is organized as follows. In Section 2, we describe the detailed construction of the general convolution-type scheme. After that, the abstract result for graded mesh is applied to two typical numerical schemes, i.e., the widely used L1 scheme and Alikhanov's scheme. In Sections 3 and 4, we give a rigorous analysis of the stability and convergence of the L1 scheme and Alikhanov's scheme, and derive α-robust error estimates under specific regularity conditions imposed on the exact solution. In Section 5, some numerical examples are provided to support the theoretical statement. Some concluding remarks are given in the final section.

2. Preliminaries

To develop a finite difference scheme for solving (1.1)–(1.3), we first divide the spatial interval $[x_l, x_r]$ into M subintervals with grid size $h = \frac{x_r - x_l}{M}$. Set discrete grid $\Omega_h = \{x_i | 0 \leq i \leq M\}$ with $x_i = x_l + ih$. For any grid function $w = \{w_i | w_i = w(x_i),\ x_i \in \Omega_h\}$, let $\delta_x^2 w_i$ be the standard second-order approximation of $\partial_x^2 w(x_i)$, i.e.,

$$\partial_x^2 w(x_i) \approx \delta_x^2 w_i := \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}.$$

Denote the space of grid functions $\mathcal{W} = \{w | w_0 = w_M = 0\}$. For any $w, v \in \mathcal{W}$, the discrete L^2 inner product and the associate L^2 norm are given as

$$\langle w, v\rangle := h\sum_{i=0}^{M} w_i v_i, \quad \|w\| := \sqrt{\langle w, w\rangle}.$$

Denote

$$\nabla_x w_i = \frac{w_i - w_{i-1}}{h}, \ 1 \leq i \leq M,$$

for any grid functions $w, v \in \mathcal{W}$, and it holds

$$-\langle \delta_x^2 w, v\rangle = \langle \nabla_x w, \nabla_x v\rangle. \tag{2.1}$$

We now proceed to the discretization of time. First, we partition the interval $[a, T]$ arbitrarily with $a = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots < t_N = T$, and set

$$\begin{aligned} \tau_k &= \log t_k - \log t_{k-1}, \quad 1 \leq k \leq N, \\ \rho_k &= \frac{\tau_k}{\tau_{k+1}}, \qquad\qquad\quad 1 \leq k \leq N-1. \end{aligned}$$

Based on this partition, we derive L1 and Alikhanov's scheme, which are given in (3.1) and (4.1), respectively. Second, we further study and discuss these two formulas in the following divisions on the interval $[a, T]$:

$$t_k = a\left(\frac{T}{a}\right)^{(k/N)^r}, \quad k = 0, 1, \cdots, N, \ r \geq 1,$$

and, correspondingly,

$$\log t_k = \log a + \left(\log \frac{T}{a}\right)\left(\frac{k}{N}\right)^r, \quad k = 0, 1, \cdots, N. \tag{2.2}$$

Define $v^n := v(x, t_n), t_{n-\theta} := \theta t_{n-1} + (1-\theta)t_n$, and $v^{n-\theta} := \theta v^{n-1} + (1-\theta)v^n$ with an offset parameter $\theta \in [0, 1)$. Then, the Caputo-Hadamard derivative operator in the problem (1.1) can be approximated as a convolution as detailed in the succeeding article [28]:

$${}^{CH}D^{\alpha}_{a,t}v(x, t_{n-\theta}) \approx {}^{CH}D^{\alpha}_{a,\tau}v^{n-\theta} = \sum_{k=1}^{n} A^{(n)}_{n-k}\nabla_\tau v^k, \tag{2.3}$$

where the difference operator is $\nabla_\tau v^k = v^k - v^{k-1}$ for $k \geq 1$. To conduct our error analysis, we require the following three assumptions.

A1. The discrete kernels are monotone, meaning that

$$A^{(n)}_0 \geq A^{(n)}_1 \geq A^{(n)}_2 \geq ... \geq A^{(n)}_{n-1} \ \text{ for } 1 \leq n \leq N.$$

A2. There exists a constant $\pi_A > 0$ such that the discrete kernels satisfy a lower bound

$$A^{(n)}_{n-k} \geq \frac{1}{\pi_A \tau_k}\int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(\log t_n - \log s)\frac{ds}{s} \ \text{ for } 1 \leq k \leq n \leq N.$$

A3. There exists a constant $\rho > 0$ for which the step size ratios ρ_k satisfy

$$\rho_k \leq \rho \ \text{ for } 1 \leq k \leq N-1.$$

For the graded mesh, it can be checked that the step size ratio is $\rho_k < 1$. In fact, by using the mean value theorem, we have

$$\rho_k = \frac{\log t_k - \log t_{k-1}}{\log t_{k+1} - \log t_k} = \frac{k^r - (k-1)^r}{(k+1)^r - k^r} = \frac{\eta_1^{r-1}}{\eta_2^{r-1}} < 1,$$

with $\eta_1 \in (k-1, k), \ \ \eta_2 \in (k, k+1)$.

Next, to derive the global consistency error, we introduce the discrete complementary convolution (DCC) kernels [16]

$$P^{(n)}_{n-k} = \frac{1}{A^{(k)}_0}\begin{cases} 1, & k = n, \\ \sum_{j=k+1}^{n}(A^{(j)}_{j-k-1} - A^{(i)}_{j-k})P^{(n)}_{n-j}, & 1 \leq k \leq n-1, \end{cases} \tag{2.4}$$

which are specifically chosen to enforce the identity

$$\sum_{j=m}^{n} P^{(n)}_{n-j}A^{(j)}_{j-m} \equiv 1, \quad \text{for } \ 1 \leq m \leq n \leq N. \tag{2.5}$$

Futher, a discrete fractional Gronwall inequality is given as follows.

Lemma 2.1. *Assume that* **A1**-**A3** *hold,* $\theta \in [0,1)$*, and the nonnegative sequences* $(g^n)_{n=1}^N$, $(\lambda^l)_{l=0}^{N-1}$, $(v^k)_{k=0}^N$ *satisfy*

$$\sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_\tau (v^k)^2 \le \sum_{k=1}^{n} \lambda_{n-k} (v^{k-\theta})^2 + v^{n-\theta} g^n, \quad 1 \le n \le N. \tag{2.6}$$

If a constant Π *satisfies* $\Pi \ge \sum_{l=0}^{N-1} \lambda_l$ *and the maximum step size satisfies*

$$\max_{1\le n\le N} \tau_n \le \frac{1}{\sqrt[\alpha]{2\max\{1,\rho\}\pi_A \Gamma(2-\alpha)\Pi}}, \tag{2.7}$$

then it holds that

$$v^n \le E_\alpha\left(2\max\{1,\rho\}\pi_A \Pi (\log t_n)^\alpha\right)\left(v^0 + \max_{1\le k\le n} \sum_{j=1}^{k} P_{k-j}^{(k)} g^n\right), \tag{2.8}$$

where $E_\alpha(\cdot) = \sum\limits_{j=0}^{\infty} \frac{(\cdot)^j}{\Gamma(j\alpha+1)}$ *represents the special Mittag-Leffler function.*

The proof of this lemma is similar to that in [19].

Lemma 2.2. *[31] For* $v > 0, \beta > 0, b > a > 0$, *there holds*

$$\int_a^b \left(\log \frac{b}{s}\right)^{\nu-1} \left(\log \frac{s}{a}\right)^{\beta-1} \frac{\mathrm{d}s}{s} = \frac{\Gamma(\nu)\Gamma(\beta)}{\Gamma(\nu+\beta)} \left(\log \frac{b}{a}\right)^{\nu+\beta-1}.$$

Lemma 2.3. *[19] If g is monotone increasing and h is monotone decreasing on the interval* $[a,b]$*, and if both functions are integrable, then*

$$(b-a)\int_a^b g(s)h(s)\mathrm{d}s \le \int_a^b g(t)\mathrm{d}t \int_a^b h(s)\mathrm{d}s.$$

In the following, we derive two important lemmas which are useful in the convergence analysis later on.

Lemma 2.4. *Assume that* **A1**-**A2** *hold, then we have*

$$\sum_{j=1}^{n} P_{n-j}^{(n)} \le \pi_A \omega_{1+\alpha}(\log t_n - \log a).$$

Proof. Denote $h(t) = \omega_{1+\alpha}(\log t - \log a)$, then it holds $\delta h(t) = \omega_\alpha(\log t - \log a)$. In one side, it follows from the definition of the Caputo-Hadamard derivative that

$$\begin{aligned} {}^{CH}D_{a,t}^{\alpha} h(t_j) &= \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(\log t_j - \log s)\omega_\alpha(\log s - \log a)\frac{ds}{s} \\ &= \sum_{k=1}^{j} \int_{\log t_{k-1}}^{\log t_k} \omega_{1-\alpha}(\log t_j - \tau)\omega_\alpha(\tau - \log a)\mathrm{d}\tau. \end{aligned}$$

It is easy to check that $\omega_{1-\alpha}(\log t_j-\tau)$ is monotone increasing and $\omega_\alpha(\tau-\log a)$ is monotone decreasing on the interval $[\log t_{k-1}, \log t_k]$. Thus, by Lemma 2.3 and **A2**, we obtain

$$\begin{aligned}
{}^{CH}D^{\alpha}_{a,t}h(t_j) &\le \sum_{k=1}^{j}\frac{1}{\tau_k}\int_{\log t_{k-1}}^{\log t_k}\omega_{1-\alpha}(\log t_j-\tau)\mathrm{d}\tau\int_{\log t_{k-1}}^{\log t_k}\omega_\alpha(\tau-\log a)\mathrm{d}\tau\\
&=\sum_{k=1}^{j}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\omega_{1-\alpha}(\log t_j-\log t)\frac{\mathrm{d}t}{t}\int_{t_{k-1}}^{t_k}\omega_\alpha(\log s-\log a)\frac{\mathrm{d}s}{s}\\
&\le \pi_A\sum_{k=1}^{j}A^{(j)}_{j-k}\int_{t_{k-1}}^{t_k}\omega_\alpha(\log s-\log a)\frac{\mathrm{d}s}{s}=\pi_A\sum_{k=1}^{j}A^{(j)}_{j-k}\int_{t_{k-1}}^{t_k}\delta h(s)\frac{\mathrm{d}s}{s}. \qquad (2.9)
\end{aligned}$$

In another side, by the definition of the Caputo-Hadamard derivative and Lemma 2.2, we get again that

$$\begin{aligned}
{}^{CH}D^{\alpha}_{a,t}h(t_j) &= \int_a^{t_j}\omega_{1-\alpha}(\log t_j-\log s)\omega_\alpha(\log s-\log a)\frac{\mathrm{d}s}{s}\\
&=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_a^{t_j}\left(\log\frac{t_j}{s}\right)^{-\alpha}\left(\log\frac{s}{a}\right)^{\alpha-1}\frac{\mathrm{d}s}{s}=1. \qquad (2.10)
\end{aligned}$$

Using (2.9), (2.10), and (2.5), we have

$$\begin{aligned}
\sum_{j=1}^{n}P^{(n)}_{n-j} &= \sum_{j=1}^{n}P^{(n)}_{n-j}\cdot{}^{CH}D^{\alpha}_{a,t}h(t_j)\\
&\le \sum_{j=1}^{n}P^{(n)}_{n-j}\pi_A\sum_{k=1}^{j}A^{(j)}_{j-k}\int_{t_{k-1}}^{t_k}\delta h(s)\frac{ds}{s}\\
&=\pi_A\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\delta h(s)\frac{ds}{s}\sum_{j=k}^{n}P^{(n)}_{n-j}A^{(j)}_{j-k}\\
&=\pi_A\int_{t_0}^{t_n}\delta h(s)\frac{ds}{s}=\pi_A\omega_{1+\alpha}(\log t_n-\log a).
\end{aligned}$$

□

Lemma 2.5. *Assume that* **A1**–**A2** *hold, and for any positive sequence* $(\upsilon^k)_{k=1}^n$ *,one has*

$$\sum_{k=1}^{n}P^{(n)}_{n-k}\upsilon_k\le\Gamma(2-\alpha)\pi_A\sum_{j=1}^{n}\tau_j\max_{j\le k\le n}\left(\upsilon_k\left(\log\frac{t_k}{a}\right)^{\alpha-1}\right). \qquad (2.11)$$

Proof. It follows from **A2** that

$$\sum_{j=1}^{k}A^{(k)}_{k-j}\tau_j\ge\frac{1}{\pi_A}\int_{t_0}^{t_k}\omega_{1-\alpha}(\log t_k-\log s)\frac{ds}{s}=\frac{(\log t_k-\log a)^{1-\alpha}}{\pi_A\Gamma(2-\alpha)}. \qquad (2.12)$$

Thus, using (2.12) and (2.5), we have

$$\sum_{k=1}^{n}P^{(n)}_{n-k}\upsilon_k\cdot 1\le\sum_{k=1}^{n}P^{(n)}_{n-k}\upsilon_k\pi_A\Gamma(2-\alpha)\sum_{j=1}^{k}(\log t_k-\log a)^{\alpha-1}A^{(k)}_{k-j}\tau_j$$

$$\leq \pi_A \Gamma(2-\alpha) \sum_{j=1}^{n} \tau_j \sum_{k=j}^{n} P_{n-k}^{(n)} A_{k-j}^{(k)} (\log t_k - \log a)^{\alpha-1} \upsilon_k$$

$$\leq \Gamma(2-\alpha)\pi_A \sum_{j=1}^{n} \tau_j \max_{j\leq k\leq n} \left(\upsilon_k \left(\log \frac{t_k}{a} \right)^{\alpha-1} \right) \cdot \sum_{k=j}^{n} P_{n-k}^{(n)} A_{k-j}^{(k)}$$

$$= \Gamma(2-\alpha)\pi_A \sum_{j=1}^{n} \tau_j \max_{j\leq k\leq n} \left(\upsilon_k \left(\log \frac{t_k}{a} \right)^{\alpha-1} \right).$$

□

3. α-robust error analysis of the L1 scheme

We are now in a position to consider the α-robust error estimate of the L1 scheme. Following [28], the L1 approximation to the Caputo-Hadamard derivative ${}^{CH}D_{a,t}^{\alpha} v^n$ is given by

$$ {}^{CH}D_{a,\tau}^{\alpha} v^n = \sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_\tau v^k, \tag{3.1}$$

where the discrete coefficients $A_{n-k}^{(n)}$ are defined by

$$A_{n-k}^{(n)} = \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(\log t_n - \log s) \frac{ds}{s}. \tag{3.2}$$

It is easy to see that Assumption **A2** holds with $\pi_A = 1$. Using the integral mean-value theorem, one has

$$\begin{aligned} A_{n-k-1}^{(n)} - A_{n-k}^{(n)} &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{1}{\tau_{k+1}} \int_{t_k}^{t_{k+1}} \left(\log \frac{t_n}{s} \right)^{-\alpha} \frac{ds}{s} - \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_n}{s} \right)^{-\alpha} \frac{ds}{s} \right) \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\left(\log \frac{t_n}{\xi_{k+1}} \right)^{-\alpha} - \left(\log \frac{t_n}{\xi_k} \right)^{-\alpha} \right] > 0, \end{aligned} \tag{3.3}$$

with $\xi_{k+1} \in (t_k, t_{k+1}), \quad \xi_k \in (t_{k-1}, t_k)$. Thus Assumption **A1** holds.

Let u_i^n be the discrete approximation of solution $u(x_i, t_n)$ for $x_i \in \Omega_h, 0 \leq n \leq N$. The fully discrete scheme for problem (1.1)–(1.3) is given as

$$ {}^{CH}\mathcal{D}_{a,\tau}^{\alpha} u_i^n - \delta_x^2 u_i^n + \kappa u_i^n = f_i^n, \qquad 1 \leq i \leq M-1, 1 \leq n \leq N, \tag{3.4}$$

$$u_i^0 = \varphi(x_i), \qquad 0 \leq i \leq M, \tag{3.5}$$

$$u_0^n = u_M^n = 0, \qquad 1 \leq n \leq N, \tag{3.6}$$

where $f_i^n = f(x_i, t_n)$.

We aim to demonstrate the stability and convergence of the scheme (3.4)–(3.6). The stability is established in the following theorem.

Theorem 3.1. *Assume that the assumptions* (1.4)*,* (1.5)*, and* **A3** *hold. Let* $\kappa_+ := \max\{-\kappa, 0\}$*, and* v_i^n *be the solutions of the following difference equation:*

$$\left({}^{CH}\mathcal{D}_{a,\tau}^{\alpha} - \delta_x^2 + \kappa \right) v_i^n = g_i^n, \qquad 1 \leq i \leq M-1, 1 \leq n \leq N, \tag{3.7}$$

$$v_i^0 = \varphi(x_i), \qquad 0 \le i \le M, \tag{3.8}$$
$$v_0^n = v_M^n = 0, \qquad 1 \le n \le N. \tag{3.9}$$

If the maximum step size is $\tau \le \mathfrak{B}_1$ with

$$\mathfrak{B}_1 = \frac{1}{\sqrt[\alpha]{4\kappa_+ \max\{1,\rho\}\Gamma(2-\alpha)}}, \tag{3.10}$$

we have

$$\|v^n\| \le 2E_\alpha(4\kappa_+ \max\{1,\rho\}(\log t_n)^\alpha)\left(\|v^0\| + 2\max_{1\le k\le n}\sum_{j=1}^{k} P_{k-j}^{(k)}\|g^j\|\right), 1 \le n \le N. \tag{3.11}$$

Proof. After taking the inner product with $2v^n$ on both sides of (3.7), we obtain

$$2({}^{CH}D_{a,\tau}^{\alpha} v^n, v^n) - 2(\delta_x^2 v^n, v^n) + 2\kappa(v^n, v^n) = 2(g^n, v^n),$$

By employing the definition of the discrete Caputo-Hadamard derivative given in (3.1) and utilizing the monotone property given in (3.3), it can be inferred that

$$\begin{aligned}({}^{CH}D_{a,\tau}^{\alpha} v^n, v^n) &= 2A_0^{(n)}\|v^n\|^2 - 2\sum_{k=1}^{n-1}(A_{n-k-1}^{(n)} - A_{n-k}^{(n)})(v^k, v^n) - 2A_{n-1}^{(n)}(v^0, v^n)\\ &\ge 2A_0^{(n)}\|v^n\|^2 - \sum_{k=1}^{n-1}(A_{n-k-1}^{(n)} - A_{n-k}^{(n)})\|v^n\|^2 - A_{n-1}^{(n)}\|v^n\|^2\\ &- \sum_{k=1}^{n-1}(A_{n-k-1}^{(n)} - A_{n-k}^{(n)})\|v^k\|^2 - A_{n-1}^{(n)}\|v^k\|^2\\ &= \sum_{k=1}^{n} A_{n-k}^{(n)}\nabla_\tau\|v^k\|^2. \end{aligned} \tag{3.12}$$

It follows that

$$\sum_{k=1}^{n} A_{n-k}^{(n)}\nabla_\tau\|v^k\|^2 - 2(\delta_x^2 v^n, v^n) + 2\kappa(v^n, v^n) \le 2(g^n, v^n),$$

Noting (2.1), and using the Schwarz inequality, we obtain

$$\sum_{k=1}^{n} A_{n-k}^{(n)}\nabla_\tau\|v^k\|^2 \le 2\kappa_+\|v^n\|^2 + 2\|g^n\|\|v^n\|.$$

By applying Lemma 2.1 to the above inequality, we obtain the desired estimate (3.11), thereby completing the proof. □

We now consider the convergence of the scheme. To this end, let $u(x,t)$ be the exact solution of (1.1)–(1.3), and let $\{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ be the solution of problem (3.4)–(3.6). Set

$$e_i^n = u(x_i, t_n) - u_i^n, \;\; 0 \le i \le M, \; 0 \le n \le N.$$

It is straightforward to obtain the following error equation:

$$
\begin{aligned}
&{}^{CH}\mathcal{D}^{\alpha}_{a,\tau}e^n_i - \delta^2_x e^n_i + \kappa e^n_i = (R_t)^n_i + (R_s)^n_i, \quad 1 \le i \le M-1, 1 \le n \le N, && (3.13)\\
&e^0_i = 0, \qquad 0 \le i \le M, && (3.14)\\
&e^n_0 = e^n_M = 0, \qquad 1 \le n \le N, && (3.15)
\end{aligned}
$$

where

$$
\begin{aligned}
(R_t)^n_i &= {}^{CH}\mathcal{D}^{\alpha}_{a,t}u(x_i,t_n) - {}^{CH}\mathcal{D}^{\alpha}_{a,\tau}u^n_i, && (3.16)\\
(R_s)^n_i &= \partial^2_x u(x_i,t_n) - \delta^2_x u^n_i. && (3.17)
\end{aligned}
$$

After conducting stability analysis in Theorem 3.1, if the maximum step size satisfies (3.10), one has, with $\bar{C} = 2E_\alpha(4\kappa_+ \max\{1,\rho\}(\log t_n)^\alpha)$,

$$
\|e^n\| \le \bar{C}\left(\|e^0\| + 2\max_{1\le k\le n}\sum_{j=1}^{k} P^{(k)}_{k-j}(\|(R_t)^j\| + \|(R_s)^j\|)\right), \quad 1 \le n \le N. \tag{3.18}
$$

Under the assumption of spatial regularity given by (1.4), the Taylor expansion provides a direct demonstration that $\|(R_s)^n\| \le Ch^2$. When combined with Lemma 2.4, we can further deduce that

$$
\sum_{j=1}^{n} P^{(n)}_{n-j}\|(R_s)^n\| \le C\left(\log\frac{t_n}{a}\right)^{\alpha} h^2. \tag{3.19}
$$

Now, we only need to estimate the term $\sum_{j=1}^{n} P^{(n)}_{n-j}\|(R_t)^n\|$, which will be achieved through the following lemmas.

Lemma 3.1. *[29, Lemma 3.1] Suppose that $f(x)$ has a continuous δ-derivative of $n+1$ order in some field of point x_0. A Taylor-like formula with integral remainder is given by*

$$
\begin{aligned}
f(x) &= f(x_0) + \delta f(x_0)(\log x - \log x_0) + \frac{\delta^2 f(x_0)}{2!}(\log x - \log x_0)^2 + \cdots\\
&\quad + \frac{\delta^n f(x_0)}{n!}(\log x - \log x_0)^n + \frac{1}{n!}\int_{x^0}^{x} \delta^{n+1} f(s)(\log x - \log s)^n \frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Lemma 3.2. *For any function $u \in C^3((a,T])$, the local truncation errors $(R_t)^k_i$ satisfy*

$$
(R_t)^k_i \le A^{(k)}_0 G^k + \sum_{j=1}^{k-1}(A^{(k)}_{k-j-1} - A^{(k)}_{k-j})G^j. \tag{3.20}
$$

where

$$
G^k = \int_{t_{k-1}}^{t_k} (\log s - \log t_{k-1})|\delta^2 u(x_i,s)|\frac{\mathrm{d}s}{s}, \quad 1 \le k \le n. \tag{3.21}
$$

Lemma 3.3. *Assume that* **A3** *holds, and $u(x,t)$ satisfies the regularity assumption* (1.5)*, then*

$$
\sum_{k=1}^{n} P^{(n)}_{n-k}\|(R_t)^k\| \le C\left(\frac{\tau_1^\sigma}{\sigma} + \sum_{j=2}^{n}\tau_j \max_{j\le k\le n}\left(\log\frac{t_k}{a}\right)^{\alpha-1}\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-2}\tau_k^{2-\alpha}\right).
$$

where $C = (1+\rho)\Gamma(2-\alpha)$.

The proofs of Lemmas 3.2 and 3.3 are left to Appendix 7.1 and 7.2 for brevity.

Theorem 3.2. *Let* $u(x,t)$ *be the exact solution of* (1.1)–(1.3)*, and let* $\{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ *be the solution of* (3.4)–(3.6)*. Suppose the regularity assumptions* (1.4)–(1.5) *hold. Set*

$$e_i^n = u(x_i, t_n) - u_i^n, \;\; 0 \le i \le M, \; 0 \le n \le N.$$

If the maximum step size is $\tau \le \mathfrak{B}_1$*, then the discrete solution is convergent in the* L^2*-norm with* $\bar{C} = 2E_\alpha(4\kappa_+ \max\{1,\rho\} (\log t_n)^\alpha)$ *such that*

$$\|e^n\| \le C\bar{C}\left(\|e^0\| + \left(\log\frac{t_n}{a}\right)^\alpha h^2 + (1+\rho)\Gamma(2-\alpha)\max_{1\le k\le n} E_t^k\right), \tag{3.22}$$

where

$$E_t^k = \frac{\tau_1^\sigma}{\sigma} + \sum_{j=2}^k \tau_j \max_{j\le l\le k}\left(\log\frac{t_l}{a}\right)^{\alpha-1}\left(\log\frac{t_{l-1}}{a}\right)^{\sigma-2}\tau_l^{2-\alpha}. \tag{3.23}$$

In particular, if graded mesh is used, then it holds

$$\|e^n\| \le 2E_\alpha(4\kappa_+(\log t_n)^\alpha)\left(\|e^0\| + \left(\log\frac{t_n}{a}\right)^\alpha h^2 + 2\Gamma(2-\alpha)\left(\frac{1}{\sigma} + 4^r r^{3-\alpha}\vartheta\right)N^{-\min\{r\sigma, 2-\alpha\}}\right),$$

with

$$\vartheta = \begin{cases} \ln\, n, & \sigma r \ge 2-\alpha, \\ \frac{1}{2-\alpha-r\sigma}, & \sigma r < 2-\alpha. \end{cases} \tag{3.24}$$

Proof. By combining (3.18), (3.19) and Lemma 3.3, we arrive at (3.22). We now proceed to examine the global approximation error on the graded mesh. It is easy to verify $\tau_k \le r\log\frac{T}{a}N^{-r}k^{r-1}$ as follows:

$$\begin{aligned}\tau_k &= \log t_k - \log t_{k-1} \\ &= \log a + \log\frac{T}{a}\left(\log\frac{k}{N}\right)^r - \log a - \log\frac{T}{a}\left(\log\frac{k-1}{N}\right)^r \\ &= \log\frac{T}{a}N^{-r}\left(k^r - (k-1)^r\right) \\ &\le \log\frac{T}{a}N^{-r}rk^{r-1}. \end{aligned} \tag{3.25}$$

Note that $\sigma > 0$, it follows from (3.23) and (3.25) that

$$\begin{aligned} E_t^n \le& \left(\frac{\left(\log\frac{T}{a}\right)^\sigma\left(\frac{1}{N}\right)^{r\sigma}}{\sigma}\right) + \sum_{j=2}^n \tau_j \max_{j\le k\le n}\left(\log\frac{T}{a}\right)^{\sigma-2}\left(\frac{k-1}{N}\right)^{r(\sigma-2)} \\ &\times\left(\log\frac{T}{a}\right)^{\alpha-1}\left(\frac{k}{N}\right)^{r(\alpha-1)}\left(\log\frac{T}{a}\right)^{2-\alpha}N^{-r(2-\alpha)}r^{2-\alpha}k^{(2-\alpha)(r-1)} \\ =&\frac{\left(\log\frac{T}{a}\right)^\sigma\left(\frac{1}{N}\right)^{r\sigma}}{\sigma} + \sum_{j=2}^n \tau_j \max_{j\le k\le n}\left(\log\frac{T}{a}\right)^{\sigma-1} r^{2-\alpha}\left(\frac{k}{N}\right)^{-r} \end{aligned}$$

$$\times \left(\frac{k}{k-1} \right)^{r(2-\sigma)} N^{-\sigma r} k^{r\sigma-(2-\alpha)}. \tag{3.26}$$

Taking into account that

$$\left(\frac{k}{k-1} \right)^{r(2-\sigma)} \le \left(1 + \frac{1}{k-1} \right)^{2r} \le 2^{2r} = 4^r,$$

(3.26) can be simplified to

$$\begin{aligned} E_t^n &\le \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + \frac{r^{2-\alpha}}{\log \frac{T}{a}} \sum_{j=2}^{n} \tau_j \max_{j \le k \le n} \left(\frac{k}{N} \right)^{-r} 4^r k^{\sigma r-(2-\alpha)} \right) \\ &\le \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{2-\alpha} \left(\log \frac{T}{a} \right)^{-1} \sum_{j=2}^{n} \tau_j \max_{j \le k \le n} \left(\frac{k}{N} \right)^{-r} k^{\sigma r-(2-\alpha)} \right). \end{aligned} \tag{3.27}$$

For $\sigma r \ge 2-\alpha$, we have from (3.27) that

$$\begin{aligned} E_t^n &\le \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{2-\alpha} \left(\log \frac{T}{a} \right)^{-1} \sum_{j=2}^{n} \tau_j \left(\frac{j}{N} \right)^{-r} n^{\sigma r-(2-\alpha)} \right) \\ &= \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{2-\alpha} \left(\log \frac{T}{a} \right)^{-1} \sum_{j=2}^{n} \left(\log \frac{T}{a} \right) N^{-r} r j^{r-1} j^{-r} N^r n^{\sigma r-(2-\alpha)} \right) \\ &= \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{3-\alpha} n^{\sigma r-(2-\alpha)} \sum_{j=2}^{n} j^{-1} \right) \\ &\le \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{3-\alpha} N^{\sigma r-(2-\alpha)} \ln\ n \right) \\ &\le \left(\log \frac{T}{a} \right)^{\sigma} \left(\frac{1}{\sigma} + 4^r r^{3-\alpha} \ln\ n \right) N^{-\min\{r\sigma, 2-\alpha\}}. \end{aligned} \tag{3.28}$$

For $\sigma r < 2-\alpha$, it follows from (3.27) that

$$\begin{aligned} E_t^n &\le \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{2-\alpha} \left(\log \frac{T}{a} \right)^{-1} \sum_{j=2}^{n} \tau_j \left(\frac{j}{N} \right)^{-r} j^{\sigma r-(2-\alpha)} \right) \\ &\le \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{2-\alpha} \left(\log \frac{T}{a} \right)^{-1} \sum_{j=2}^{n} r \left(\log \frac{T}{a} \right) N^{-r} j^{r-1} \left(\frac{j}{N} \right)^{-r} j^{\sigma r-(2-\alpha)} \right) \\ &= \left(\log \frac{T}{a} \right)^{\sigma} N^{-r\sigma} \left(\frac{1}{\sigma} + 4^r r^{3-\alpha} \sum_{j=2}^{n} j^{\sigma r-(3-\alpha)} \right). \end{aligned} \tag{3.29}$$

Note that

$$\sum_{j=2}^{n} j^{\sigma r-(3-\alpha)} \le \int_1^n s^{\sigma r-(3-\alpha)} ds = \frac{1}{r\sigma-(2-\alpha)} \left(n^{r\sigma-(2-\alpha)} - 1 \right) \le \frac{1}{2-\alpha-r\sigma},$$

then we get from (3.29) that

$$E_t^n \leq \left(\log \frac{T}{a}\right)^{\sigma} \left(\frac{1}{\sigma} + \frac{4^r r^{3-\alpha}}{2-\alpha-r\sigma}\right) N^{-r\sigma}. \tag{3.30}$$

By utilizing (3.28), (3.30), and (3.23), we achieve the desired global approximation error on the graded mesh. □

4. α-robust error analysis of the Alikhanov's scheme

In this section, we delve into the α-robust error estimate pertaining to the Alikhanov scheme. Referring to [28], the Alikhanov scheme for the Caputo-Hadamard derivative is expressed as follows:

$$^{CH}D_{a,t}^{\alpha} v(t_{n-\theta}) \approx {}^{CH}D_{a,\tau}^{\alpha} v^{n-\theta} = \sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_\tau v^k \quad for \;\; 1 \leq n \leq N, \tag{4.1}$$

with $\theta = \frac{\alpha}{2}$. Here the discrete convolution kernel $A_{n-k}^{(n)}$ is defined as: $A_0^{(n)} = a_0^{(n)}$ if $n = 1$ and, for $n \geq 2$,

$$A_{n-k}^{(n)} = \begin{cases} a_{n-1}^{(n)} - b_{n-1}^{(n)}, & k = 1, \\ a_{n-k}^{(n)} + \rho_{k-1} b_{n-k+1}^{(n)} - b_{n-k}^{(n)}, & k = 2, ..., n-1, \\ a_0^{(n)} + \rho_{n-1} b_1^{(n)}, & k = n. \end{cases}$$

Moreover, the discrete coefficients $a_{n-k}^{(n)}$ and $b_{n-k}^{(n)}$ are determined by:

$$a_0^{(n)} = \frac{1}{\tau_n} \int_{t_{n-1}}^{t_{n-\theta}} \delta\varpi_n(s) \frac{ds}{s}, \;\; a_{n-k}^{(n)} = \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \delta\varpi_n(s) \frac{ds}{s}, \;\; 1 \leq k \leq n-1,$$
$$b_{n-k}^{(n)} = \frac{2}{\tau_k(\tau_k + \tau_{k+1})} \int_{t_{k-1}}^{t_k} (\log s - \log t_{k-1/2}) \delta\varpi_n(s) \frac{ds}{s}, \;\; 1 \leq k \leq n-1,$$

where

$$\varpi_n(s) = -\omega_{2-\alpha}(\log t_{n-\theta} - \log s), \; \log t_{k-1/2} = \frac{1}{2}(\log t_{k-1} + \log t_k).$$

As proved in [29], one can verify that Assumptions **A1** and **A2** hold with $\pi_A = 11/4$.

Next, the difference scheme is established. Considering (1.1) at mesh point $(x_i, t_{n-\theta})$, the fully discrete scheme for problem (1.1) is given as

$$^{CH}D_{a,\tau}^{\alpha} u_i^{n-\theta} - \delta_x^2 u_i^{n-\theta} + \kappa u_i^{n-\theta} = f_i^{n-\theta}, \quad 1 \leq i \leq M-1, 1 \leq k \leq N, \tag{4.2}$$
$$u_i^0 = \varphi(x_i), \quad 0 \leq i \leq M, \tag{4.3}$$
$$u_0^n = u_M^n = 0, \quad 1 \leq k \leq N. \tag{4.4}$$

We proceed to assess the stability and convergence of Alikhanov's scheme.

Theorem 4.1. *Assume that the assumptions* (1.5) *and* **A3** *hold. Let* $\kappa_+ := \max\{-\kappa, 0\}$ *and* v_i^n *be the solution of the following differential equation:*

$$\left({}^{CH}D_{a,\tau}^{\alpha} - \delta_x^2 + \kappa\right) v_i^{n-\theta} = g_i^{n-\theta}, \quad 1 \leq i \leq M-1, 1 \leq n \leq N, \tag{4.5}$$

$$v_i^0 = \varphi(x_i), \qquad 0 \le i \le M, \tag{4.6}$$
$$v_0^n = v_M^n = 0, \qquad 1 \le n \le N. \tag{4.7}$$

If the maximum step size is $\tau \le \mathfrak{B}_2$ *with*

$$\mathfrak{B}_2 = \frac{1}{\sqrt[\alpha]{11\kappa_+ \max\{1,\rho\}\Gamma(2-\alpha)}}, \tag{4.8}$$

we have

$$\|v^n\| \le E_\alpha\left(11\kappa_+ \max\{1,\rho\}(\log t_n)^\alpha\right)\left(\|v^0\| + 2\max_{1\le k\le n}\sum_{j=1}^{k} P_{k-j}^{(k)}\|g^{j-\theta}\|\right). \tag{4.9}$$

Proof. Taking the inner products with $2v_i^{n-\theta}$ on both sides of (4.5), we obtain

$$2({}^{CH}D_{a,\tau}^{\alpha}v^{n-\theta}, v^{n-\theta}) - 2(\delta_x^2 v^{n-\theta}, v^{n-\theta}) + 2\kappa(v^{n-\theta}, v^{n-\theta}) = 2(f^{n-\theta}, v^{n-\theta}),$$

Utilizing (3.12), we can further derive

$$\sum_{k=1}^{n} A_{n-k}^{(n)}\nabla_\tau\|v^k\|^2 - 2(\delta_x^2 v^{n-\theta}, v^{n-\theta}) + 2\kappa(v^{n-\theta}, v^{n-\theta}) \le 2(f^{n-\theta}, v^{n-\theta}).$$

Taking into account (2.1) and applying the Schwarz inequality, we arrive at

$$\sum_{k=1}^{n} A_{n-k}^{(n)}\nabla_\tau\|v^k\|^2 \le 2\kappa_+\|v^{n-\theta}\|^2 + 2\|f^{n-\theta}\|\|v^{n-\theta}\|.$$

By applying Lemma 2.1 to the aforementioned inequality, we successfully derive the desired estimate (4.9), thus, conclusively completing the proof. □

We now consider the convergence of the scheme. To this end, let $u(x,t)$ be the exact solution of (1.1)–(1.3), and let $\{u_i^{n-\theta} | 0 \le i \le M, 0 \le n \le N\}$ be the solution of problem (4.2)–(4.4). Set

$$e_i^{n-\theta} = u(x_i, t_{n-\theta}) - u_i^{n-\theta}.$$

It is straightforward to obtain the following error equation:

$$({}^{CH}D_{a,\tau}^{\alpha} - \delta_x^2 + \kappa)e_i^{n-\theta} = R_i^n, \quad 1 \le i \le M-1, 1 \le n \le N, \tag{4.10}$$
$$e_i^0 = 0, \quad 0 \le i \le M, \tag{4.11}$$
$$e_0^n = e_M^n = 0, \quad 1 \le n \le N, \tag{4.12}$$

where $R_i^n = (R_t)_i^n + (R_s)_i^n + (R_l)_i^n$ with

$$(R_t)_i^n = {}^{CH}D_{a,t}^{\alpha}u(x_i, t_{n-\theta}) - {}^{CH}D_{a,\tau}^{\alpha}u_i^{n-\theta}, \tag{4.13}$$
$$(R_s)_i^n = \partial_x^2 u(x_i, t_{n-\theta}) - \delta_x^2 u(x_i, t_{n-\theta}), \tag{4.14}$$
$$(R_l)_i^n = \delta_x^2(u(x_i, t_{n-\theta}) - u_i^{n-\theta}) - \kappa(u(x_i, t_{n-\theta}) - u_i^{n-\theta}). \tag{4.15}$$

By the stability analysis in Theorem 4.1, if the maximum step size fulfills the condition stated in (4.8), one can derive the following inequality:

$$\|e^n\| \le \tilde{C}\left(\|e^0\| + 2\max_{1\le k\le n}\sum_{j=1}^{k} P_{k-j}^{(k)}(\|(R_t)^j\| + \|(R_s)^j\| + \|(R_l)^j\|)\right) \tag{4.16}$$

with $\tilde{C} = E_\alpha\left(11\kappa_+ \max\{1,\rho\}(\log t_n)^\alpha\right)$.

Now, our sole task is to estimate the terms $\sum_{j=1}^{n} P_{n-j}^{(n)}\|(R_l)^n\|$ and $\sum_{j=1}^{n} P_{n-j}^{(n)}\|(R_t)^n\|$, which we will accomplish by utilizing the lemmas outlined below.

Lemma 4.1. *Assume that* **A3** *holds, and $u(x,t)$ satisfies the regularity assumption* (1.5)*, then*

$$\sum_{j=1}^{n} P_{n-j}^{(n)}\|(R_l)^j\| \le C\left(\frac{\tau_1^{\sigma+\alpha}}{\sigma} + \left(\log\frac{t_n}{a}\right)^{\alpha}\max_{2\le k\le n}\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-2}\tau_k^2\right). \tag{4.17}$$

Proof. Set $v(t) = (\delta_x^2 - \kappa)u(x_i,t)$. Using the Taylor-like formula given in Lemma 3.1, we derive

$$v(t_j) = v(t_{j-\theta}) + \delta v(t_{j-\theta})(\log t_j - \log t_{j-\theta}) + \int_{t_{j-\theta}}^{t_j} \delta^2 v(s)(\log t_j - \log s)\frac{ds}{s}, \tag{4.18}$$

$$v(t_{j-1}) = v(t_{j-\theta}) + \delta v(t_{j-\theta})(\log t_{j-1} - \log t_{j-\theta}) + \int_{t_{j-\theta}}^{t_{j-1}} \delta^2 v(s)(\log t_{j-1} - \log s)\frac{ds}{s}. \tag{4.19}$$

By combining (4.18) and (4.19), we obtain

$$\begin{aligned}\theta v(t_{j-1}) + \theta(1-\theta)v(t_j) = v(t_{j-\theta}) &+ \theta\int_{t_{j-1}}^{t_{j-\theta}} \delta^2 v(s)(\log s - \log t_{j-1})\frac{ds}{s}\\ &+ (1-\theta)\int_{t_{j-\theta}}^{t_j} \delta^2 v(s)(\log t_j - \log s)\frac{ds}{s}.\end{aligned}$$

This leads to the following expression for $(R_l)_i^j$:

$$(R_l)_i^j = -\theta\int_{t_{j-1}}^{t_{j-\theta}} \delta^2 v(s)(\log s - \log t_{j-1})\frac{ds}{s} - (1-\theta)\int_{t_{j-\theta}}^{t_j} \delta^2 v(s)(\log t_j - \log s)\frac{ds}{s}. \tag{4.20}$$

For $j = 1$, utilizing the regularity assumption (1.5), we deduce from (4.20) that

$$\begin{aligned}\|(R_l)^1\| &\le \theta\int_{t_0}^{t_{1-\theta}} \left|\delta^2 v(s)\right|(\log s - \log t_0)\frac{ds}{s} + (1-\theta)\int_{t_{1-\theta}}^{t_1} \left|\delta^2 v(s)\right|(\log t_1 - \log s)\frac{ds}{s}\\ &\le C\int_{t_0}^{t_{1-\theta}}\left(\log\frac{s}{a}\right)^{\sigma-1}\frac{ds}{s} + C\int_{t_{1-\theta}}^{t_1}(\log t_1 - \log s)\left(\log\frac{s}{a}\right)^{\sigma-2}\frac{ds}{s}\\ &\le C\int_{t_0}^{t_{1-\theta}}\left(\log\frac{s}{a}\right)^{\sigma-1}\frac{ds}{s} \le C\frac{\tau_1^{\sigma}}{\sigma}.\end{aligned} \tag{4.21}$$

Analogously, for $2 \le j \le N$, we have

$$\|(R_l)^j\| \le \theta\int_{t_{j-1}}^{t_{j-\theta}} \left|\delta^2 v(s)\right|(\log s - \log t_{j-1})\frac{ds}{s} + (1-\theta)\int_{t_{j-\theta}}^{t_j} \left|\delta^2 v(s)\right|(\log t_j - \log s)\frac{ds}{s}$$

$$\leq C\left(\log\frac{t_{j-1}}{a}\right)^{\sigma-2}\left(\int_{t_{j-1}}^{t_{j-\theta}}(\log s-\log t_{j-1})\frac{ds}{s}+\int_{t_{j-\theta}}^{t_j}(\log t_j-\log s)\frac{ds}{s}\right)$$
$$\leq C\left(\log\frac{t_{j-1}}{a}\right)^{\sigma-2}\tau_j^2. \tag{4.22}$$

It follows from (2.4), **A1**, and **A2** that

$$P_{n-1}^{(n)}\leq 1/A_0^{(1)}\leq\frac{11}{4}\Gamma(2-\alpha)\tau_1^{\alpha}. \tag{4.23}$$

By integrating the information from (4.21) to (4.23) along with Lemma 2.4, we arrive at the conclusion stated in (4.17). □

Lemma 4.2. *Assume that* **A3** *holds and $u(x,t)$ satisfies the regularity assumption* (1.5)*, then*

$$\sum_{j=1}^{n}P_{n-j}^{(n)}\|(R_t)^j\|\leq C\left(\frac{\tau_1^{\sigma}}{\sigma}+\tau_1^{\sigma-3}\tau_2^3\right)$$
$$+C\sum_{j=2}^{n}\tau_j\max_{j\leq k\leq n}\left(\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-3}\left(\log\frac{t_k}{a}\right)^{\alpha-1}\tau_k^{3-\alpha}+\left(\log\frac{t_k}{a}\right)^{\sigma+\alpha-4}\frac{\tau_{k+1}^3}{\tau_k^{\alpha}}\right). \tag{4.24}$$

Proof. It follows from [29] that the local truncation error defined in (4.13) can be bounded by

$$\left\|(R_t)^k\right\|\leq A_0^{(k)}G_{loc}^k+\sum_{j=1}^{k-1}(A_{k-j-1}^{(k)}-A_{k-j}^{(k)})G_{his}^j, \tag{4.25}$$

where

$$G_{loc}^k=\frac{3}{2}\int_{t_{k-1}}^{t_{k-1/2}}(\log s-\log t_{k-1})^2|\delta^3u(s)|\frac{ds}{s}+\frac{3\tau_k}{2}\int_{t_{k-1/2}}^{t_k}(\log t_k-\log s)|\delta^3u(s)|\frac{ds}{s},$$
$$G_{his}^k=\frac{5}{2}\int_{t_{k-1}}^{t_k}(\log s-\log t_{k-1})^2|\delta^3u(s)|\frac{ds}{s}+\frac{5}{2}\int_{t_k}^{t_{k+1}}(\log t_{k+1}-\log s)|\delta^3u(s)|\frac{ds}{s}.$$

By the regularity assumption (1.5), it is easy to obtain

$$G_{loc}^1\leq C\frac{\tau_1^{\sigma}}{\sigma},\ \ G_{loc}^k\leq C\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-3}\tau_k^3,\ k\geq 2, \tag{4.26}$$

and

$$G_{his}^1\leq C\left(\frac{\tau_1^{\sigma}}{\sigma}+\left(\log\frac{t_1}{a}\right)^{\sigma-3}\tau_2^3\right), \tag{4.27}$$
$$G_{his}^k\leq C\left(\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-3}\tau_k^3+\left(\log\frac{t_k}{a}\right)^{\sigma-3}\tau_{k+1}^3\right),\ k\geq 2. \tag{4.28}$$

Similar to (7.8), we have

$$\sum_{k=1}^{n}P_{n-k}^{(n)}\|(R_t)^k\|\leq\sum_{k=1}^{n}P_{n-k}^{(n)}A_0^{(k)}(G_{loc}^k+G_{his}^k):=\sum_{k=1}^{n}P_{n-k}^{(n)}\mathcal{G}^k. \tag{4.29}$$

Applying Lemma 2.5 by taking $\upsilon_k = \mathcal{G}^k$ in (2.11) to (4.29), we get

$$\begin{aligned}
&\sum_{k=1}^{n} P_{n-k}^{(n)} \|(R_t)^k\| \\
\leq &\frac{11\Gamma(2-\alpha)}{4} \sum_{j=1}^{n} \tau_j \max_{j\leq k\leq n} \left(\log \frac{t_k}{a}\right)^{\alpha-1} \mathcal{G}^k \\
= &\frac{11\Gamma(2-\alpha)}{4} \left[\sum_{j=2}^{n} \tau_j \max_{j\leq k\leq n} \left(\log \frac{t_k}{a}\right)^{\alpha-1} \mathcal{G}^k + \max\left\{\tau_1 \max_{2\leq k\leq n} \left(\log \frac{t_k}{a}\right)^{\alpha-1} \mathcal{G}^k, \tau_1^{\alpha}\mathcal{G}^1\right\}\right] \\
\leq &\frac{11\Gamma(2-\alpha)}{4} \left(\sum_{j=2}^{n} \tau_j \max_{j\leq k\leq n} \left(\log \frac{t_k}{a}\right)^{\alpha-1} \mathcal{G}^k + \rho\tau_2 \max_{2\leq k\leq n} \left(\log \frac{t_k}{a}\right)^{\alpha-1} \mathcal{G}^k + \tau_1^{\alpha}\mathcal{G}^1\right) \\
\leq &\frac{11}{4}(1+\rho)\Gamma(2-\alpha) \left(\sum_{j=2}^{n} \tau_j \max_{j\leq k\leq n} \left(\left(\log \frac{t_k}{a}\right)^{\alpha-1} \cdot \mathcal{G}^k\right) + \tau_1^{\alpha}\mathcal{G}^1\right).
\end{aligned} \tag{4.30}$$

According to (4.26), (4.27), and (4.28), we have

$$\mathcal{G}^1 \leq CA_0^{(1)} \left(\frac{\tau_1^{\sigma}}{\sigma} + \frac{\tau_1^{\sigma}}{\sigma} + \left(\log \frac{t_1}{a}\right)^{\sigma-3} \tau_2^3\right) \leq \frac{C\tau_1^{-\alpha}}{\Gamma(2-\alpha)} \left(\frac{\tau_1^{\sigma}}{\sigma} + \left(\log \frac{t_1}{a}\right)^{\sigma-3} \tau_2^3\right), \tag{4.31}$$

and

$$\begin{aligned}
\mathcal{G}^k &\leq CA_0^{(k)} \left(\log \frac{t_{k-1}}{a}\right)^{\sigma-3} \tau_k^3 + CA_0^{(k)} \left(\left(\log \frac{t_{k-1}}{a}\right)^{\sigma-3} \tau_k^3 + \left(\log \frac{t_k}{a}\right)^{\sigma-3} \tau_{k+1}^3\right) \\
&\leq \frac{C\tau_1^{-\alpha}}{\Gamma(2-\alpha)} \left(\left(\log \frac{t_{k-1}}{a}\right)^{\sigma-3} \tau_k^{3-\alpha} + \left(\log \frac{t_k}{a}\right)^{\sigma-3} \tau_{k+1}^3 \tau_k^{-\alpha}\right).
\end{aligned} \tag{4.32}$$

Therefore, combining (4.30), (4.31), and (4.32) we obtain the desired result. □

Theorem 4.2. *Let $u(x,t)$ be the exact solution of* (1.1)–(1.3)*, and let $\{u_i^n | 0 \leq i \leq M, 0 \leq n \leq N\}$ be the solution of* (4.2)–(4.4)*. Suppose the regularity assumptions* (1.4)–(1.5) *hold. Let*

$$e_i^n = u(x_i, t_n) - u_i^n, \;\; 0 \leq i \leq M, \;\; 0 \leq n \leq N.$$

If the maximum time step is $\tau \leq \mathfrak{B}_2$, then the discrete solution is convergent in the L^2-norm with $\tilde{C} = E_{\alpha}\left(11\kappa_+ \max\{1,\rho\}(\log t_n)^{\alpha}\right)$ such that

$$\|e^n\| \leq \tilde{C}C \left(\frac{\tau_1^{\sigma+\alpha}}{\sigma} + \tau_1^{\sigma-3}\tau_2^3 + \left(\log \frac{t_n}{a}\right)^{\alpha} \max_{2\leq k\leq n} \left(\log \frac{t_{k-1}}{a}\right)^{\sigma-2} \tau_k^2 + \left(\log \frac{t_n}{a}\right)^{\alpha} h^2 + E_t^n\right), \tag{4.33}$$

where

$$E_t^n = \sum_{j=2}^{n} \tau_j \max_{j\leq k\leq n} \left(\left(\log \frac{t_{k-1}}{a}\right)^{\sigma-3} \left(\log \frac{t_k}{a}\right)^{\alpha-1} \tau_k^{3-\alpha} + \left(\log \frac{t_k}{a}\right)^{\sigma+\alpha-4} \frac{\tau_{k+1}^3}{\tau_k^{\alpha}}\right). \tag{4.34}$$

In particular, if graded mesh is used, then it holds that

$$\|e^n\| \leq CE_{\alpha}\left(11\kappa_+(\log t_n)^{\alpha}\right)\left(N^{-\min\{\sigma r, 2\}} + \left(\log \frac{t_n}{a}\right)^{\alpha} h^2\right). \tag{4.35}$$

Proof. By combining (3.19), (4.16), Lemma 4.1 and Lemma 4.2, we arrive at (4.33). We now proceed to examine the global approximation error on the graded mesh. Due to (2.2) and (3.25), we can estimate the righthand side of (4.33) term by term as follows. It easy to check that

$$\frac{\tau_1^{\sigma+\alpha}}{\sigma} = \frac{1}{\sigma}\left(\log\frac{T}{a}\right)^{\sigma+\alpha}\left(\frac{1}{N}\right)^{r(\sigma+\alpha)} \leq \frac{C}{\sigma}N^{-r(\sigma+\alpha)}. \tag{4.36}$$

Further, it holds that

$$\begin{aligned}\tau_2^3\tau_1^{\sigma-3} = \tau_2^3\left(\log\frac{t_1}{a}\right)^{\sigma-3} &\leq \left(\log\frac{T}{a}\right)^{\sigma-3}\left(\frac{1}{N}\right)^{r(\sigma-3)}\left(\log\frac{T}{a}\right)^{3}N^{-3r}r^3 2^{3(r-1)}\\ &\leq CN^{-r\sigma}.\end{aligned} \tag{4.37}$$

For the third term in the righthand side of (4.33), we have

$$\begin{aligned}&\left(\log\frac{t_n}{a}\right)^{\alpha}\max_{2\leq k\leq n}\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-2}\tau_k^2\\ \leq &C\max_{2\leq k\leq n}\left(\left(\log\frac{T}{a}\right)^{\sigma}\left(\frac{k-1}{N}\right)^{r(\sigma-2)}N^{-2r}r^2k^{2(r-1)}\right)\\ \leq &CN^{-r\sigma}\max_{2\leq k\leq n}\left(\frac{k}{k-1}\right)^{r(2-\sigma)}k^{r\sigma-2} \leq CN^{-r\sigma}\max_{2\leq k\leq n}k^{r\sigma-2}.\end{aligned}$$

If $r\sigma \geq 2$, then $\max\limits_{2\leq k\leq n}k^{r\sigma-2} = n^{r\sigma-2} \leq N^{r\sigma-2}$. If $r\sigma < 2$, then $\max\limits_{2\leq k\leq n}k^{r\sigma-2} = 2^{r\sigma-2} < 1$. Thus, we obtain

$$\left(\log\frac{t_n}{a}\right)^{\alpha}\max_{2\leq k\leq n}\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-2}\tau_k^2 \leq CN^{-\min\{2,r\sigma\}}. \tag{4.38}$$

Moreover, we have

$$\begin{aligned}&\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-3}\left(\log\frac{t_k}{a}\right)^{\alpha-1}\tau_k^{3-\alpha} + \left(\log\frac{t_k}{a}\right)^{\sigma+\alpha-4}\frac{\tau_{k+1}^3}{\tau_k^{\alpha}}\\ \leq &\left(\log\frac{T}{a}\right)^{\sigma-3}\left(\frac{k-1}{N}\right)^{r(\sigma-3)}\left(\log\frac{T}{a}\right)^{\alpha-1}\left(\frac{k}{N}\right)^{r(\alpha-1)}\left(r\log\frac{T}{a}N^{-r}k^{r-1}\right)^{3}\tau_1^{-\alpha}\\ &+\left(\log\frac{T}{a}\right)^{\sigma+\alpha-4}\left(\frac{k}{N}\right)^{r(\sigma+\alpha-4)}\left(r\log\frac{T}{a}N^{-r}(k+1)^{r-1}\right)^{3}\tau_1^{-\alpha}\\ = &\tau_1^{-\alpha}r^3\left(\log\frac{T}{a}\right)^{\sigma+\alpha-1}N^{-r(\sigma+\alpha-1)}k^{r(\sigma-1)-(3-\alpha)}\left(\left(\frac{k}{k-1}\right)^{r(3-\sigma)}+\left(\frac{k+1}{k}\right)^{3(r-1)}\right)\\ \leq &C\tau_1^{-\alpha}N^{-r(\sigma+\alpha-1)}k^{r(\sigma-1)-(3-\alpha)}\\ = &C\left(\log\frac{T}{a}\right)^{-\alpha}\left(\frac{1}{N}\right)^{-r\alpha}N^{-r(\sigma+\alpha-1)}k^{r(\sigma-1)-(3-\alpha)}\\ \leq &Ck^{r(\sigma-1)-(3-\alpha)}N^{-r(\sigma-1)}.\end{aligned} \tag{4.39}$$

Combining (4.34) and (4.39), we obtain

$$E_t^n \leq C\sum_{j=2}^{n}\tau_j\max_{j\leq k\leq n}k^{-r}k^{r\sigma-(3-\alpha)}N^{-r(\sigma-1)}$$

$$\leq C\sum_{j=2}^{n} N^{-r} j^{r-1} \max_{j\leq k\leq n} k^{-r} k^{r\sigma-(3-\alpha)} N^{-r(\sigma-1)}. \tag{4.40}$$

If $r\sigma \geq 3-\alpha$, we have from (4.40) that

$$E_t^n \leq C\sum_{j=2}^{n} N^{-r} j^{-1} n^{r\sigma-(3-\alpha)} N^{-r(\sigma-1)} \leq CN^{-(3-\alpha)} \sum_{j=2}^{n} j^{-1} \leq C\ln n N^{-(3-\alpha)}. \tag{4.41}$$

If $r\sigma < 3-\alpha$, we get from (4.40) that

$$E_t^n \leq C\sum_{j=2}^{n} N^{-r} j^{r-1} j^{r(\sigma-1)-(3-\alpha)} N^{-r(\sigma-1)} \leq CN^{-r\sigma} \sum_{j=2}^{n} j^{r\sigma-(4-\alpha)}. \tag{4.42}$$

Note that

$$\sum_{j=2}^{n} j^{r\sigma-(4-\alpha)} \leq \int_1^n s^{r\sigma-(4-\alpha)} ds = \frac{1}{r\sigma-(3-\alpha)}\left(n^{r\sigma-(3-\alpha)}-1\right) \leq \frac{1}{3-\alpha-r\sigma}, \tag{4.43}$$

then we have from (4.42) that

$$E_t^n \leq \tfrac{1}{3-\alpha-r\sigma} N^{-(3-\alpha)}. \tag{4.44}$$

By combining (4.36), (4.37), (4.38), (4.41), and (4.44), we achieve the desired global approximation error (4.35) on the graded mesh. □

5. Numerical expertiments

In this section, we present several numerical examples to validate the theoretical result stated in Theorem 3.2 and Theorem 4.2. In our computations, the spatial domain $\Omega = [0,\pi]$ is uniformly divided into M parts, and the time interval Λ is divided into N subintervals using graded meshes $t_k = a\left(\frac{T}{a}\right)^{(k/N)^r}$. The grading constant $r \geq 1$ controls the extent to which the time levels are concentrated near $t = a$. As r increases, the initial step sizes become smaller than the later ones, which can be visually observed in Figure 1.

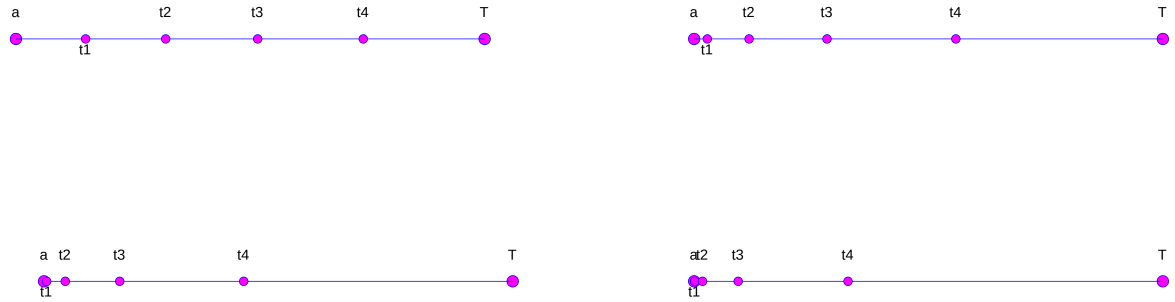
Example 5.1. *Consider the problem* (1.1)–(1.3) *with*

$$a = 1, T = 2, \kappa = 2, f(x,t) = (\sin x)(\Gamma(1+\alpha) + (\log t)^{\alpha} + \kappa \cdot (\log t)^{\alpha}).$$

It can be verified that the corresponding exact solution is $u(x,t) = (\sin x)(\log t)^{\alpha}$ *and* $\sigma = \alpha$.

Since the spacial accuracy is standard for the second order central difference scheme, we only explore the convergence rate of the time stepping scheme. To this end, we calculate the L^2 errors between the exact and numerical solutions

$$Error(M,N) = \max_{1\leq k\leq N} \|e^k\|.$$



(c) $r = 3$ (d) $r = 4$

Figure 1. The temporal meshes on $[a, T]$ with $N = 5$.

In Tables 1–3, we list the temporal L^2 errors for the L1 scheme by taking fixed M and increasing N for different α. Based on the obtained numerical errors, we further estimate the order of temporal convergence using the formula as

$$\text{Order} = \log_2 \frac{Error(M, N/2)}{Error(M, N)}.$$

Results are also listed in Tables 1–3. As can be observed, the temporal convergence order for the L1 scheme is close to $\min\{r\sigma, 2 - \alpha\}$ for all cases, aligning with the theoretical analysis presented in Theorem 3.2. In Tables 4–7, we list the temporal L^2 errors for Alikhanov's scheme as a function of N for different α. Also shown are the corresponding decay rates based on graded meshes about Alikhanov's scheme. From Tables 4–7, it is observed that the convergence rate is close to $\min\{r\sigma, 2\}$ in time.

Table 1. Numerical results for Example 5.1 with $r = 2, M = 500$ (L1 scheme).

N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	$Error(M, N)$	Order	$Error(M, N)$	Order	$Error(M, N)$	Order
64	1.1134×10^{-2}	–	2.6984×10^{-3}	–	9.4560×10^{-4}	–
128	7.9395×10^{-3}	0.5153	1.3721×10^{-3}	0.9757	4.0801×10^{-4}	1.2126
256	5.4559×10^{-3}	0.5412	6.9778×10^{-4}	0.9756	1.7224×10^{-4}	1.2442
512	3.7010×10^{-4}	0.5599	3.5227×10^{-4}	0.9861	7.1677×10^{-5}	1.2648
$\min\{r\sigma, 2 - \alpha\}$		0.6		1.0		1.3

Example 5.2. *Consider the problem* (1.1) *with*

$$a = 2, T = 3, \kappa = -1, f(x,t) = \sin(x) \cdot \left(\Gamma(1+\alpha) + (\log\frac{t}{2})^{\alpha} + \kappa \cdot (\log\frac{t}{2})^{\alpha}\right).$$

Table 2. Numerical results for Example 5.1 with $r = 3, M = 500$ (L1 scheme).

N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	$Error(M,N)$	Order	$Error(M,N)$	Order	$Error(M,N)$	Order
64	3.7010×10^{-3}	–	7.0310×10^{-4}	–	6.2060×10^{-4}	–
128	2.0350×10^{-3}	0.8629	2.6509×10^{-4}	1.4072	2.5489×10^{-4}	1.2838
256	1.1093×10^{-3}	0.8797	9.7726×10^{-5}	1.4397	1.0412×10^{-4}	1.2917
512	5.9718×10^{-4}	0.8890	3.5547×10^{-5}	1.4590	4.2399×10^{-5}	1.2961
$\min\{r\sigma, 2-\alpha\}$		0.9		1.5		1.3

Table 3. Numerical results for Example 5.1 with $r = \frac{2-\alpha}{\alpha}, M = 100$ (L1 scheme).

N	$\alpha = 0.5$		$\alpha = 0.7$		$\alpha = 0.9$	
	$Error(M,N)$	Order	$Error(M,N)$	Order	$Error(M,N)$	Order
32	1.7972×10^{-3}	–	2.4367×10^{-3}	–	1.9017×10^{-3}	–
64	7.0306×10^{-4}	1.3541	1.1229×10^{-3}	1.1176	1.0393×10^{-3}	0.8716
128	2.6507×10^{-4}	1.4072	4.9963×10^{-4}	1.1683	5.5426×10^{-4}	0.9070
256	9.7719×10^{-5}	1.4397	2.1701×10^{-4}	1.2031	2.8970×10^{-4}	0.9360
$\min\{r\sigma, 2-\alpha\}$		1.5		1.3		1.1

Table 4. Numerical results for Example 5.1 with $r = 1/\alpha, M = 500$ (Alikhanov's scheme).

N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	$Error(M,N)$	Order	$Error(M,N)$	Order	$Error(M,N)$	Order
32	2.7134×10^{-3}	–	2.7984×10^{-3}	–	2.4560×10^{-3}	–
64	1.4395×10^{-3}	0.9746	1.4721×10^{-3}	0.9757	1.2811×10^{-3}	0.9426
128	7.4559×10^{-4}	0.9871	6.9776×10^{-4}	0.9839	6.7224×10^{-4}	0.9622
256	3.7010×10^{-4}	0.9935	4.5227×10^{-4}	0.9860	3.1677×10^{-4}	0.9809
$\min\{r\sigma, 2\}$		1.0		1.0		1.0

Table 5. Numerical results for Example 5.1 with $r = 1, M = 500$ (Alikhanov's scheme).

N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.8$	
	$Error(M,N)$	Order	$Error(M,N)$	Order	$Error(M,N)$	Order
64	2.2791×10^{-2}	–	1.7493×10^{-2}	–	3.7540×10^{-3}	–
128	2.4622×10^{-2}	0.1838	1.3208×10^{-2}	0.4053	2.2841×10^{-3}	0.7240
256	2.1453×10^{-2}	0.1982	9.8096×10^{-3}	0.4291	1.3224×10^{-3}	0.7552
512	1.8563×10^{-2}	0.2117	7.1924×10^{-4}	0.4477	8.1627×10^{-4}	0.7658
$\min\{r\sigma, 2\}$		0.3		0.5		0.8

Table 6. Numerical results for Example 5.1 with $r = 3, M = 100$ (Alikhanov's scheme).

N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.8$	
	$Error(M, N)$	Order	$Error(M, N)$	Order	$Error(M, N)$	Order
16	5.2002×10^{-3}	–	1.1646×10^{-3}	–	2.2565×10^{-4}	–
32	3.0471×10^{-3}	0.7711	4.3955×10^{-4}	1.4057	5.8962×10^{-5}	1.9362
64	1.7190×10^{-3}	0.8259	1.5959×10^{-4}	1.4616	1.4430×10^{-5}	2.0308
128	9.4796×10^{-4}	0.8587	5.6967×10^{-5}	1.4862	3.3768×10^{-6}	2.0953
$\min\{r\sigma, 2\}$		0.9		1.5		2

Table 7. Numerical results for Example 5.1 with $r = 2/\alpha, M = 300$ (Alikhanov's scheme).

N	$\alpha = 0.5$		$\alpha = 0.7$		$\alpha = 0.9$	
	$Error(M, N)$	Order	$Error(M, N)$	Order	$Error(M, N)$	Order
16	5.9243×10^{-4}	–	4.1200×10^{-4}	–	1.6522×10^{-4}	–
32	1.6398×10^{-4}	1.8531	1.1720×10^{-4}	1.8136	5.0273×10^{-5}	1.7165
64	4.2850×10^{-5}	1.9361	3.3138×10^{-5}	1.9008	1.4495×10^{-5}	1.7942
128	1.0862×10^{-5}	1.9801	8.0773×10^{-6}	1.9582	4.0215×10^{-6}	1.8497
$\min\{r\sigma, 2\}$		2		2		2

It can be verified that the corresponding exact solution is $u(x, t) = (\sin x)(\log \frac{t}{2})^{\alpha}$ *and* $\sigma = \alpha$.

Tables 8 and 9 display the temporal L^2 errors by taking fixed M and increasing N for the L1 scheme and Alikhanov's scheme with $\kappa = -1$, respectively. Additionally, the maximum time step size τ, as well as the conditions $\mathfrak{B}_1$ and $\mathfrak{B}_2$ defined in (3.10) and (4.8), respectively, are also presented. The convergent results displayed show that the rate of convergence in time is in agreement with Theorem 3.2 and Theorem 4.2. It is worth noting that when α approaches 1^{-}, the convergence order in the time direction is also in accordance with the theoretical analysis.

Table 8. Numerical results for Example 5.2 with $r = 3, M = 500$ (L1 scheme).

N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.99$		τ
	$Error(M, N)$	Order	$Error(M, N)$	Order	$Error(M, N)$	Order	
32	6.2201×10^{-3}	–	1.8010×10^{-3}	–	2.6758×10^{-4}	–	3.6837×10^{-2}
64	3.3333×10^{-3}	0.8999	6.4494×10^{-4}	1.4816	1.3421×10^{-4}	0.9954	1.8711×10^{-2}
128	1.7863×10^{-3}	0.9000	2.2925×10^{-4}	.4922	6.5942×10^{-5}	1.0252	9.4290×10^{-3}
256	9.5724×10^{-4}	0.9000	8.1375×10^{-5}	1.49427	3.2376×10^{-5}	1.0262	4.7330×10^{-3}
$\min\{r\sigma, 2 - \alpha\}$		0.9		1.5		1.01	$\mathfrak{B}_1 = 0.2479$

Table 9. Numerical results for Example 5.2 with $r = 1, M = 300$ (Alikhanov's scheme).

N	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.99$		τ
	$Error(M,N)$	Order	$Error(M,N)$	Order	$Error(M,N)$	Order	
32	2.9061×10^{-2}	–	1.0479×10^{-2}	–	4.3943×10^{-5}	–	3.6836×10^{-2}
64	2.5299×10^{-2}	0.1999	7.4103×10^{-2}	0.4999	2.2143×10^{-5}	0.9887	1.8711×10^{-2}
128	2.2025×10^{-2}	0.1999	5.2399×10^{-3}	0.4999	1.1154×10^{-5}	0.9892	9.4290×10^{-3}
256	1.9174×10^{-2}	0.1999	3.7052×10^{-3}	0.4999	5.6176×10^{-6}	0.9896	4.7330×10^{-3}
$\min\{r\sigma, 2\}$		0.2		0.5		0.99	$\mathfrak{B}_2 = 0.0892$

6. Concluding remarks

In this paper, we have proposed L1 and Alikhanov schemes with nonuniform time steps for solving Caputo-Hadamard fractional reaction sub-diffusion equations. We conduct a rigorous analysis of the stability and convergence of these two schemes, and further derive α-robust error estimates under specific regularity conditions imposed on the exact solution. The derivation of these regularity assumptions is currently under active investigation and will be the subject of our forthcoming research.

7. Appendix

7.1. The proof of Lemma 3.2

Proof. Let $v(t) = u(x_i, t)$, for $1 \le k \le n$, and using the Taylor-like formula given in Lemma 3.1, we derive

$$\delta v(t) - \nabla_\tau v^k/\tau_k = \frac{1}{\tau_k}\int_t^{t_{k-1}} \delta^2 v(y)\log\frac{t_{k-1}}{y}\frac{\mathrm{d}y}{y} - \frac{1}{\tau_k}\int_t^{t_k} \delta^2 v(y)\log\frac{t_k}{y}\frac{\mathrm{d}y}{y}.$$

Thus the truncation error at time $t = t_n$ is given as

$$\begin{aligned}(R_t)_i^n =& {}^{CH}\mathcal{D}_{a,t}^{\alpha}u(x_i,t_n) - {}^{CH}\mathcal{D}_{a,\tau}^{\alpha}u_i^n \\ =& \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\omega_{1-\alpha}(\log t_n - \log s)\left(\delta v(s) - \nabla_\tau v^k/\tau_k\right)\frac{\mathrm{d}s}{s} \\ =& \sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\omega_{1-\alpha}(\log t_n - \log s)\int_{t_{k-1}}^{s}\delta^2 v(y)\log\frac{y}{t_{k-1}}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}s}{s} \\ &- \sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\omega_{1-\alpha}(\log t_n - \log s))\int_{s}^{t_k}\delta^2 v(y)\log\frac{t_k}{y}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}s}{s}.\end{aligned}$$

Exchanging the order of integration, we have

$$(R_t)_i^n = \sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{y}{t_{k-1}}\int_{y}^{t_k}\omega_{1-\alpha}(\log t_n - \log s)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}y}{y}$$

$$
\begin{aligned}
&-\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{t_k}{y}\int_{t_{k-1}}^{y}\omega_{1-\alpha}(\log t_n-\log s)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}y}{y}\\
=&\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{y}{t_{k-1}}\omega_{2-\alpha}(\log t_n-\log y)\frac{\mathrm{d}y}{y}\\
&+\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{t_k}{y}\omega_{2-\alpha}(\log t_n-\log y)\frac{\mathrm{d}y}{y}\\
&-\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{y}{t_{k-1}}\omega_{2-\alpha}(\log t_n-\log t_k)\frac{\mathrm{d}y}{y}\\
&-\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{t_k}{y}\omega_{2-\alpha}(\log t_n-\log t_{k-1})\frac{\mathrm{d}y}{y}.
\end{aligned}\tag{7.1}
$$

For the sake of simplicity, we denote $\varpi_n(y)=\omega_{2-\alpha}(\log t_n-\log y)$. Note that the linear logarithmic interpolation function of $\varpi_n(y)$ with respect to the nodes t_{k-1} and t_k is given by

$$
L^k_{\log,1}\varpi_n(y)=\frac{1}{\tau_k}\log\frac{y}{t_{k-1}}\varpi_n(t_k)+\frac{1}{\tau_k}\log\frac{t_k}{y}\varpi_n(t_{k-1}).
$$

Let

$$
\widetilde{L}^k_{\log,1}\varpi_n(y)=\varpi_n(y)-L^k_{\log,1}\varpi_n(y),
$$

then (7.1) can be rewritten as follows:

$$
\begin{aligned}
(R_t)^n_i=&\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{y}{t_{k-1}}\varpi_n(y)\frac{\mathrm{d}y}{y}+\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{t_k}{t_{k-1}}\varpi_n(y)\frac{\mathrm{d}y}{y}\\
&-\sum_{k=1}^{n}\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\log\frac{y}{t_{k-1}}\varpi_n(y)\frac{\mathrm{d}y}{y}-\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\delta^2 v(y)L^k_{log,1}\varpi_n(y)\frac{\mathrm{d}y}{y}\\
=&\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\delta^2 v(y)\widetilde{L}^k_{\log,1}\varpi_n(y)\frac{\mathrm{d}y}{y}\triangleq\sum_{k=1}^{n}(\tilde{R}_t)^n_k,\quad n\geq 1.
\end{aligned}\tag{7.2}
$$

Similar to the proof of Theorem 3.1 in [29], $\widetilde{L}^k_{\log,1}\varpi_n(y)$ can be rewritten as

$$
\widetilde{L}^k_{\log,1}\varpi_n(y)=\int_{t_{k-1}}^{t_k}\xi_k(y,s)\delta^2\varpi_n(s)\frac{\mathrm{d}s}{s},\tag{7.3}
$$

where $\xi_k(y,s)=\max\{\log y-\log s,0\}-(\log y-\log t_{k-1})(\log t_k-\log s)/\tau_k$ such that

$$
-\frac{\log y-\log t_{k-1}}{\tau_k}(\log y-\log s)\leq\xi_k(y,s)\leq 0,\quad\forall y,s\in(t_{k-1},t_k).\tag{7.4}
$$

For $k=n$, since the function $\varpi_n(y)$ is decreasing with respect to y and $\delta^2\varpi_n(y)<0$, one has

$$
0\leq\widetilde{L}^n_{\log,1}\varpi_n(y)\leq\varpi_n(t_{n-1})-L^n_{log,1}\varpi_n(y)
$$

$$= \omega_{2-\alpha}(\log t_n - \log t_{n-1}) - \frac{1}{\tau_n}\log\frac{t_n}{y}\omega_{2-\alpha}(\log t_n - \log t_{n-1})$$
$$= (\log y - \log t_{n-1})A_0^{(n)}.$$

Thus,

$$(\tilde{R}_t)_n^n \le \int_{t_{n-1}}^{t_n} |\delta^2 v(y)| \widetilde{L}_{log,1}^n \varpi_n(y)\frac{\mathrm{d}y}{y}$$
$$\le A_0^{(n)} \int_{t_{n-1}}^{t_n} (\log y - \log t_{n-1})|\delta^2 v(y)|\frac{\mathrm{d}y}{y}. \tag{7.5}$$

For $1 \le k \le n-1$, using (7.3) and (7.4), one has

$$\widetilde{L}_{\log,1}^k \varpi_n(y) \le (\log t_{k-1} - \log y)\int_{t_{k-1}}^{t_k} \delta^2 \omega_{2-\alpha}(\log t_n - \log y)\frac{\mathrm{d}y}{y}$$
$$\le (\log y - \log t_{k-1})(A_{n-k-1}^{(n)} - A_{n-k}^{(n)}). \tag{7.6}$$

Then we obtain for $1 \le k \le n-1$ that

$$(\tilde{R}_t)_k^n \le \int_{t_{k-1}}^{t_k} |\delta^2 v(y)| \widetilde{L}_{\log,1}^k \varpi_n(y)\frac{\mathrm{d}y}{y}$$
$$\le \int_{t_{k-1}}^{t_k} |\delta^2 v(y)|(A_{n-k-1}^{(n)} - A_{n-k}^{(n)})(\log y - \log t_{k-1})\frac{\mathrm{d}y}{y}. \tag{7.7}$$

Combining (7.2), (7.5), and (7.7), the proof is complete. □

7.2. The proof of Lemma 3.3

Proof. After multiplying the inequality (3.20) by $P_{n-k}^{(n)}$ and summing the index k from 1 to n, it is possible to switch the order of summation and apply the definition (2.5) of $P_{n-k}^{(n)}$ to obtain

$$\sum_{k=1}^{n} P_{n-k}^{(n)} \|(R_t)^k\| \le \sum_{k=1}^{n} P_{n-k}^{(n)} A_0^{(k)} G^k + \sum_{k=2}^{n} P_{n-k}^{(n)} \sum_{j=1}^{k-1} (A_{k-j-1}^{(k)} - A_{k-j}^{(k)}) G^j$$
$$= \sum_{k=1}^{n} P_{n-k}^{(n)} A_0^{(k)} G^k + \sum_{j=1}^{n-1} G^j \sum_{k=j+1}^{n} P_{n-k}^{(n)} (A_{k-j-1}^{(k)} - A_{k-j}^{(k)})$$
$$= \sum_{k=1}^{n} P_{n-k}^{(n)} A_0^{(k)} G^k + \sum_{j=1}^{n-1} P_{n-k}^{(n)} A_0^{(k)} G^j$$
$$\le 2\sum_{k=1}^{n} P_{n-k}^{(n)} A_0^{(k)} G^k := \sum_{k=1}^{n} P_{n-k}^{(n)} \mathcal{G}^k. \tag{7.8}$$

Applying Lemma 2.5 by taking $\upsilon_k = \mathcal{G}^k$ in (2.11) to (7.8), one has

$$\sum_{k=1}^{n} P_{n-k}^{(n)} \|(R_t)^k\| \le \sum_{j=1}^{k} P_{k-j}^{(k)} \mathcal{G}^k \le \Gamma(2-\alpha) \sum_{j=1}^{n} \tau_j \max_{j \le k \le n} \left(\left(\log \frac{t_k}{a}\right)^{\alpha-1} \cdot \mathcal{G}^k \right)$$

$$
\begin{aligned}
=&\Gamma(2-\alpha)\sum_{j=2}^{n}\tau_j\max_{j\le k\le n}\left(\left(\log\frac{t_k}{a}\right)^{\alpha-1}\cdot\mathcal{G}^k\right)\\
&+\Gamma(2-\alpha)\max\left\{\tau_1\max_{2\le k\le n}\left(\left(\log\frac{t_k}{a}\right)^{\alpha-1}\cdot\mathcal{G}^k\right),\tau_1^{\alpha}\mathcal{G}^1\right\}\\
\le&\Gamma(2-\alpha)\left(\sum_{j=2}^{n}\tau_j\max_{j\le k\le n}\left(\left(\log\frac{t_k}{a}\right)^{\alpha-1}\cdot\mathcal{G}^k\right)\right)\\
&+\Gamma(2-\alpha)\left(\rho\tau_2\max_{2\le k\le n}\left(\left(\log\frac{t_k}{a}\right)^{\alpha-1}\cdot\mathcal{G}^k\right)+\tau_1^{\alpha}\mathcal{G}^1\right)\\
\le&(1+\rho)\Gamma(2-\alpha)\left(\sum_{j=2}^{n}\tau_j\max_{j\le k\le n}\left(\left(\log\frac{t_k}{a}\right)^{\alpha-1}\cdot\mathcal{G}^k\right)+\tau_1^{\alpha}\mathcal{G}^1\right).
\end{aligned}
\tag{7.9}
$$

On one hand, it follows from (3.2) that

$$
A_0^{(k)}=\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\omega_{1-\alpha}(\log t_k-\log s)\frac{ds}{s}=\frac{1}{\Gamma(2-\alpha)\tau_k^{\alpha}},\quad k=1,...,n. \tag{7.10}
$$

On the other hand, the assumption of regularity (1.5) leads to the conclusions that

$$
G^1\le\int_{t_0}^{t_1}(\log s-\log t_0)\cdot C\left(1+\left(\log\frac{s}{a}\right)^{\sigma-2}\right)\frac{ds}{s}\le C\frac{\tau_1^{\sigma}}{\sigma}, \tag{7.11}
$$

and

$$
G^k\le\int_{t_{k-1}}^{t_k}(\log s-\log t_{k-1})\frac{ds}{s}\cdot C\left(1+\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-2}\right)\le C\tau_k^2\left(\log\frac{t_{k-1}}{a}\right)^{\sigma-2} \tag{7.12}
$$

for $2\le k\le n$. By combining (7.9), (7.10), and (7.11) with (7.12), we obtain the desired result. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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