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Research article

Differential Harnack estimates for the semilinear parabolic equation with three exponents on $\mathbb{R}^n$

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Abstract: In this paper, we thought about the positive solutions to the semilinear parabolic equation with three exponents, and obtained several differential Harnack estimates of the positive solutions to the equation. As applications of the main theorems, we found blow-up solutions for the equation and classical Harnack inequalities. Our results generalize some recent works in this direction.

Keywords: differential Harnack inequality; semilinear parabolic equations; maximum principle

1. Introduction

The differential Harnack estimate is a fundamental and powerful technique in the study of partial differential equations on $\mathbb{R}^n$ (see [1, 2]). Gaussian bounds for the heat kernel follow immediately from the differential Harnack estimate. The Hölder continuity is also a direct consequence of the differential Harnack estimate. Numerous other conclusions about the fundamental geometry of space can also be deduced by differential Harnack estimates. Many mathematicians have paid attention to the study on this topic (see, for example, [3–5] and the references therein).

In this paper, we consider differential Harnack estimates for the following Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t} f(x,t) = \Delta f + h_1(x,t) f^p + h_2(x,t) f^q + h_3(x,t) f^s \quad in \quad \mathbb{R}^n \times [0,\infty), \\ f(x,0) = f_0(x) \quad in \quad \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the functions h_1, h_2, and h_3 are C^2 in x and C^0 in t with $h_1 > 0$, $h_2 > 0$, and $h_3 > 0$, and p, q, and s are positive constants with $p \geq q \geq s > 1$. Equation (1.1) arises from many classical equations (see [6–8]) and there are many questions related to Eq (1.1) (see [9, 10]).

Now, let us recall some relevant work with the above Eq (1.1). In the case where $h_1(x,t) = 1$ and $h_2(x,t) = h_3(x,t) = 0$, Eq (1.1) reduces to the endangered species equation. Cao et al. [8] proved a differential Harnack estimate for positive solutions of the Cauchy problem for the endangered species equation. In the case where $h_1(x,t) = c$, $h_2(x,t) = -c$, $h_3(x,t) = 0$, $p = 1$, and $q = 2$, Eq (1.1) reduces to the Fisher-Kolmogorov-Petrovsky-Piskunov (Fisher-KPP) equation. Cao et al. [9] proved a differential Harnack estimate for positive solutions of the Fisher-KPP equation on an n-dimensional Riemannian manifold M with non-negative Ricci curvature, where c is a positive constant. If $h_1(x,t) = -1$, $h_2(x,t) = 1$, $h_3(x,t) = 0$, $p = 3$, and $q = 1$, then Eq (1.1) reduces to the parabolic Allen-Cahn equation. Băileşteanu [6] proved a differential Harnack estimate for the solution of the parabolic Allen-Cahn equation on a closed n-dimensional manifold. When $h_1(x,t) = a$, $h_2(x,t) = -b$, $h_3(x,t) = 0$, $p = 1$, and $q = 3$, where a and b are two constants, Eq (1.1) reduces to the Newell-Whitehead-Segel equation. The differential Harnack estimate for the Newell-Whitehead-Segel equation was obtained by the authors in [7]. Hou [10] proved a differential Harnack estimate for positive solutions of equation (1.1) when $h_3(x,t) = 0$. For more results on differential Harnack estimates of Eq (1.1), see [11–15].

The motivation of this article is to develop some differential Harnack estimates for positive solutions to Eq (1.1) on $\mathbb{R}^n$. The method we employ is the parabolic maxinum principle. We are now ready to state our main results.

Theorem 1.1. *Assume that $f(x,t)$ is a positive solution of Eq (1.1) and $u = \ln f$. If α, β, c, d, k, a, and h_i $(i = 1,2,3)$ satisfy*

$$\alpha \geq 2\beta \geq 0, \quad \alpha > 0, \tag{1.2}$$

$$\begin{cases} \frac{\alpha(p-1)+2\beta}{p} \geq c \geq \max\left\{\frac{(p-1)n\alpha^2}{4(\alpha-\beta)}, \frac{\alpha(p-1)+\beta}{p}\right\}, \\ \frac{\alpha(q-1)+2\beta}{q} \geq d \geq \max\left\{\frac{(q-1)n\alpha^2}{4(\alpha-\beta)}, \frac{\alpha(q-1)+\beta}{q}\right\}, \\ \frac{\alpha(s-1)+2\beta}{s} \geq k \geq \max\left\{\frac{(s-1)n\alpha^2}{4(\alpha-\beta)}, \frac{\alpha(s-1)+\beta}{s}\right\}, \end{cases} \tag{1.3}$$

$$c \geq d \geq k \geq \beta, \quad a \geq \frac{n\alpha^2}{2(\alpha-\beta)} > 0, \tag{1.4}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) h_i \geq 0, \quad \Delta h_i \geq 0, \quad i = 1,2,3, \tag{1.5}$$

then we have

$$H_0 \equiv \alpha\Delta u + \beta|\nabla u|^2 + ch_1 e^{u(p-1)} + dh_2 e^{u(q-1)} + kh_3 e^{u(s-1)} + \frac{a}{t} \geq 0 \tag{1.6}$$

for all t.

Remark 1.1. (1) Compared with the previous work established in [7, 8, 11], here we do not assume the coefficients of equations are constant, and therefore our results can be regarded as an extension of several classical estimates.

(2) When $h_3(x,t) = 0$, the estimate (1.6) above can be reduced to the formulas (1.6) in Theorem 1.1 of [10]. Hence the above Theorem 1.1 generalizes the result in [10].

As applications of this estimate (1.6), we derive the blow-up of the solutions for Eq (1.1) and a classical Harnack inequality by integrating along space-time paths.

Corollary 1.2. *Let f be a positive solution of equation (1.1) with h_i $(i = 1,2,3)$ satisfying (1.5), and c is a constant satisfying $0 < n(p-1) \leq c < 2$ and $c \geq d \geq k \geq 1$. Then f blows up in finite time provided that*

$$f(x_0,t_0) \geq \left(\frac{4n}{(2-c)h_1(x_0,t_0)}\right)^{\frac{1}{p-1}} \tag{1.7}$$

at some point (x_0, t_0).

Corollary 1.3. *Let f be a positive solution of Eq* (1.1) *with h_i $(i = 1,2,3)$ satisfying* (1.5) *and $u = \ln f$. Let $\gamma(t) = (x(t), t)$, $t \in [t_1, t_2]$, be a space-time curve joining two given points (x_1, t_1), $(x_2, t_2) \in \mathbb{R}^n \times [0, \infty)$ with $0 < t_1 < t_2$. Assume further that $a = \frac{n\alpha^2}{2(\alpha-\beta)} \leq n\alpha$. Then we get*

$$f(x_1, t_1) \leq f(x_2, t_2)\left(\frac{t_2}{t_1}\right)^n \exp\left[\frac{|x_2 - x_1|^2}{2(t_2 - t_1)}\right]. \tag{1.8}$$

We also get the following differential Harnack estimate, which is different from (1.6).

Theorem 1.4. *Assume $f(x,t)$ is a positive solution of Eq* (1.1) *and $u = \ln f$. If $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{c}$, $\tilde{d}$, $\tilde{k}$, $\tilde{a}$, m, and h_i $(i = 1,2,3)$ satisfy*

$$\tilde{\alpha} \geq 2\tilde{\beta} \geq 0, \quad m > 0, \tag{1.9}$$

$$\begin{cases} \frac{\tilde{\alpha}(p-1)+2\tilde{\beta}}{p} \geq \tilde{c} \geq \max\left\{\frac{(p-1)n\tilde{\alpha}^2}{4(\tilde{\alpha}-\tilde{\beta})}, \frac{\tilde{\alpha}(p-1)+\tilde{\beta}}{p}\right\}, \\ \frac{\tilde{\alpha}(q-1)+2\tilde{\beta}}{q} \geq \tilde{d} \geq \max\left\{\frac{(q-1)n\tilde{\alpha}^2}{4(\tilde{\alpha}-\tilde{\beta})}, \frac{\tilde{\alpha}(q-1)+\tilde{\beta}}{q}\right\}, \\ \frac{\tilde{\alpha}(s-1)+2\tilde{\beta}}{s} \geq \tilde{k} \geq \max\left\{\frac{(s-1)n\tilde{\alpha}^2}{4(\tilde{\alpha}-\tilde{\beta})}, \frac{\tilde{\alpha}(s-1)+\tilde{\beta}}{s}\right\}, \end{cases} \tag{1.10}$$

$$\tilde{c} \geq \tilde{d} \geq \tilde{k} \geq \tilde{\beta}, \quad \tilde{a} \geq \frac{nm\tilde{\alpha}^2}{2(\tilde{\alpha} - \tilde{\beta})} > 0, \tag{1.11}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right)h_i \geq 0, \quad \Delta h_i \geq 0, \quad i = 1,2,3, \tag{1.12}$$

then we have

$$\widetilde{H}_0 \equiv \tilde{\alpha}\Delta u + \tilde{\beta}|\nabla u|^2 + \tilde{c}h_1 e^{u(p-1)} + \tilde{d}h_2 e^{u(q-1)} + \tilde{k}h_3 e^{u(s-1)} + \frac{\tilde{a}}{1 - e^{-mt}} \geq 0 \tag{1.13}$$

for all t.

Remark 1.2. (1) When $h_1(x,t) = e^{\gamma t}$ with a constant γ and $h_2(x,t) = h_3(x,t) = 0$, Theorem 1.4 reduces to Theorem 1 in [14]. Hence the above Theorem 1.4 generalizes the result in [14].

(2) The case of $n = 1$, $p = 2$, and $h_2 = h_3 = 0$ was studied by Hamilton in [16]. Particularly, we apply Theorem 1.4 with $n = 1$ and $p = 2$, and by picking $\tilde{\alpha} = 1, \tilde{\beta} = 0$, $h_1 = 1$, $h_2 = h_3 = 0$, $\tilde{a} = \frac{m}{2}$, and $\tilde{c} = \frac{1}{4}$, we conclude that

$$u_{xx} + \frac{1}{4}e^u + \frac{m}{2(1 - e^{-mt})} \geq 0,$$

yielding

$$f_t + \frac{m}{2(1 - e^{-mt})}f \geq \frac{f_x^2}{f} + \frac{3}{4}f^2.$$

If m is small enough, the estimate in [16] will be improved.

Corollary 1.5. *Let f be a positive solution of Eq* (1.1) *with h_i $(i = 1,2,3)$ satisfying* (1.12) *and $u = \ln f$. Let $\tau(t) = (x(t), t)$, $t \in [t_1, t_2]$, be a space-time curve joining two given points (x_1, t_1), $(x_2, t_2) \in \mathbb{R}^n \times [0, \infty)$ with $0 < t_1 < t_2$. Assume further that $\tilde{a} = \frac{nm\tilde{\alpha}^2}{2(\tilde{\alpha}-\tilde{\beta})} \leq nm\tilde{\alpha}$. Then we get*

$$f(x_1, t_1) \leq f(x_2, t_2)\left(\frac{e^{mt_2} - 1}{e^{mt_1} - 1}\right)^n \exp\left[\frac{|x_2 - x_1|^2}{2(t_2 - t_1)}\right]. \tag{1.14}$$

The paper is structured as follows. In Section 2, we prove Theorem 1.1, Corollary 1.2 and Corollary 1.3. In Section 3, we prove Theorem 1.4 and Corollary 1.5.

2. The proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3

Using the parabolic maximum principle, we will first derive our differential Harnack estimate in this section. We always write u_t for the partial derivative of u with respect to t and omit the time variable t for simplicity.

Let $f(x,t) \in C^\infty(\mathbb{R}^n \times [0,\infty))$ be a positive solution of (1.1) and $u = \ln f$. Substituting $f = e^u$ into Eq (1.1), we have

$$u_t = \Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}. \tag{2.1}$$

Based on this observation, a Harnack quantity H is defined as

$$H := \alpha\Delta u + \beta|\nabla u|^2 + ch_1 e^{u(p-1)} + dh_2 e^{u(q-1)} + kh_3 e^{u(s-1)} + \phi, \tag{2.2}$$

where $\alpha, \beta, c, d, k \in \mathbb{R}$ and $\phi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ will be determined later. To support our primary findings, we first assert and prove a technical lemma.

Lemma 2.1. $u = \ln f$ *and* H *are defined as in* (2.2). *Assume that* $f(x,t)$ *is a positive solution of Eq* (1.1). *Then we have*

$$\begin{aligned} H_t = {} & \Delta H + 2\nabla H \cdot \nabla u + (p-1)h_1 e^{u(p-1)} H + (q-1)h_2 e^{u(q-1)} H + (s-1)h_3 e^{u(s-1)} H \\ & + 2(\alpha-\beta)|\nabla\nabla u|^2 + [\alpha(p-1)+\beta-cp](p-1)h_1 e^{u(p-1)}|\nabla u|^2 \\ & + [\alpha(q-1)+\beta-dq](q-1)h_2 e^{u(q-1)}|\nabla u|^2 \\ & + [\alpha(s-1)+\beta-ks](s-1)h_3 e^{u(s-1)}|\nabla u|^2 \\ & + [(\alpha-c)\Delta h_1 + 2(\alpha(p-1)+\beta-cp)\nabla h_1 \cdot \nabla u + c(h_1)_t - h_1(p-1)\phi]e^{u(p-1)} \\ & + [(\alpha-d)\Delta h_2 + 2(\alpha(q-1)+\beta-dq)\nabla h_2 \cdot \nabla u + d(h_2)_t - h_2(q-1)\phi]e^{u(q-1)} \\ & + [(\alpha-k)\Delta h_3 + 2(\alpha(s-1)+\beta-ks)\nabla h_3 \cdot \nabla u + k(h_3)_t - h_3(s-1)\phi]e^{u(s-1)} \\ & + [(c-d)(p-q)]h_1h_2 e^{u(p-1)}e^{u(q-1)} \\ & + [(c-k)(p-s)]h_1h_3 e^{u(p-1)}e^{u(s-1)} \\ & + [(d-k)(q-s)]h_2h_3 e^{u(q-1)}e^{u(s-1)} - 2\nabla\phi\cdot\nabla u - \Delta\phi + \phi_t. \end{aligned} \tag{2.3}$$

Proof. Using (2.1), we can compute the following evolution equations:

$$\begin{aligned} H_t &= \alpha(\Delta u)_t + \beta(|\nabla u|^2)_t + c(h_1 e^{u(p-1)})_t + d(h_2 e^{u(q-1)})_t + k(h_3 e^{u(s-1)})_t + \phi_t, \\ (\Delta u)_t &= \Delta(u_t) \\ &= \Delta\left(\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}\right) \\ &= \Delta(\Delta u) + \Delta|\nabla u|^2 + \Delta(h_1 e^{u(p-1)}) + \Delta(h_2 e^{u(q-1)}) + \Delta(h_3 e^{u(s-1)}) \end{aligned}$$

and

$$\begin{aligned} (|\nabla u|^2)_t &= 2\nabla(u_t)\cdot\nabla u \\ &= 2\nabla\left(\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}\right)\cdot\nabla u \\ &= \Delta|\nabla u|^2 - 2|\nabla\nabla u|^2 + 2\nabla|\nabla u|^2\cdot\nabla u \\ &\quad + 2\nabla(h_1 e^{u(p-1)})\cdot\nabla u + 2\nabla(h_2 e^{u(q-1)})\cdot\nabla u + 2\nabla(h_3 e^{u(s-1)})\cdot\nabla u, \end{aligned}$$

where we applied the formula

$$\Delta|\nabla u|^2 = 2\nabla u \cdot \nabla\Delta u + 2|\nabla\nabla u|^2. \tag{2.4}$$

Hence we get

$$\begin{aligned} H_t = {} & \alpha[\Delta(\Delta u) + \Delta|\nabla u|^2 + \Delta(h_1 e^{u(p-1)}) + \Delta(h_2 e^{u(q-1)}) + \Delta(h_3 e^{u(s-1)})] \\ & + \beta[\Delta|\nabla u|^2 - 2|\nabla\nabla u|^2 + 2\nabla|\nabla u|^2 \cdot \nabla u \\ & + 2\nabla(h_1 e^{u(p-1)}) \cdot \nabla u + 2\nabla(h_2 e^{u(q-1)}) \cdot \nabla u + 2\nabla(h_3 e^{u(s-1)}) \cdot \nabla u] \\ & + ce^{u(p-1)}(h_1)_t + ch_1(p-1)e^{u(p-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + de^{u(q-1)}(h_2)_t + dh_2(q-1)e^{u(q-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + ke^{u(s-1)}(h_3)_t + kh_3(s-1)e^{u(s-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + \phi_t. \end{aligned} \tag{2.5}$$

A direct calculation gives

$$\begin{aligned} \Delta H &= \Delta(\alpha\Delta u + \beta|\nabla u|^2 + ch_1 e^{u(p-1)} + dh_2 e^{u(q-1)} + kh_3 e^{u(s-1)} + \phi) \\ &= \alpha\Delta(\Delta u) + \beta\Delta|\nabla u|^2 + c\Delta(h_1 e^{u(p-1)}) + d\Delta(h_2 e^{u(q-1)}) + k\Delta(h_3 e^{u(s-1)}) + \Delta\phi \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \nabla H &= \nabla(\alpha\Delta u + \beta|\nabla u|^2 + ch_1 e^{u(p-1)} + dh_2 e^{u(q-1)} + kh_3 e^{u(s-1)} + \phi) \\ &= \alpha\nabla\Delta u + \beta\nabla|\nabla u|^2 + c\nabla(h_1 e^{u(p-1)}) + d\nabla(h_2 e^{u(q-1)}) + k\nabla(h_3 e^{u(s-1)}) + \nabla\phi. \end{aligned} \tag{2.7}$$

Using (2.4), (2.6), and (2.5), we obtain

$$\begin{aligned} H_t = {} & \Delta H + 2(\alpha-\beta)|\nabla\nabla u|^2 + 2\alpha\nabla u \cdot \nabla\Delta u \\ & + (\alpha-c)\Delta(h_1 e^{u(p-1)}) + (\alpha-d)\Delta(h_2 e^{u(q-1)}) + (\alpha-k)\Delta(h_3 e^{u(s-1)}) \\ & + \beta[2\nabla|\nabla u|^2 \cdot \nabla u + 2\nabla(h_1 e^{u(p-1)}) \cdot \nabla u + 2\nabla(h_2 e^{u(q-1)}) \cdot \nabla u \\ & + 2\nabla(h_3 e^{u(s-1)}) \cdot \nabla u] - \Delta\phi \\ & + ce^{u(p-1)}(h_1)_t + ch_1(p-1)e^{u(p-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + de^{u(q-1)}(h_2)_t + dh_2(q-1)e^{u(q-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + ke^{u(s-1)}(h_3)_t + kh_3(s-1)e^{u(s-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + \phi_t. \end{aligned} \tag{2.8}$$

By (2.8) and (2.7), we get

$$\begin{aligned} H_t = {} & \Delta H + 2(\alpha-\beta)|\nabla\nabla u|^2 + 2\nabla H \cdot \nabla u \\ & + (\alpha-c)\Delta(h_1 e^{u(p-1)}) + (\alpha-d)\Delta(h_2 e^{u(q-1)}) + (\alpha-k)\Delta(h_3 e^{u(s-1)}) \\ & + 2(\beta-c)\nabla(h_1 e^{u(p-1)}) \cdot \nabla u + 2(\beta-d)\nabla(h_2 e^{u(q-1)}) \cdot \nabla u \\ & + 2(\beta-k)\nabla(h_3 e^{u(s-1)}) \cdot \nabla u - \Delta\phi - 2\nabla\phi \cdot \nabla u \\ & + ce^{u(p-1)}(h_1)_t + ch_1(p-1)e^{u(p-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + de^{u(q-1)}(h_2)_t + dh_2(q-1)e^{u(q-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + ke^{u(s-1)}(h_3)_t + kh_3(s-1)e^{u(s-1)}[\Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)}] \\ & + \phi_t. \end{aligned} \tag{2.9}$$

Direct computations show that

$$\begin{cases}\Delta(h_1e^{u(p-1)}) = e^{u(p-1)}\Delta h_1 + 2(p-1)e^{u(p-1)}\nabla h_1\cdot\nabla u\\ \qquad\qquad + h_1(p-1)^2e^{u(p-1)}|\nabla u|^2 + h_1(p-1)e^{u(p-1)}\Delta u,\\ \Delta(h_2e^{u(q-1)}) = e^{u(q-1)}\Delta h_2 + 2(q-1)e^{u(q-1)}\nabla h_2\cdot\nabla u\\ \qquad\qquad + h_2(q-1)^2e^{u(q-1)}|\nabla u|^2 + h_2(q-1)e^{u(q-1)}\Delta u,\\ \Delta(h_3e^{u(s-1)}) = e^{u(s-1)}\Delta h_3 + 2(s-1)e^{u(s-1)}\nabla h_3\cdot\nabla u\\ \qquad\qquad + h_3(s-1)^2e^{u(s-1)}|\nabla u|^2 + h_3(s-1)e^{u(s-1)}\Delta u\end{cases} \tag{2.10}$$

and

$$\begin{cases}\nabla(h_1e^{u(p-1)})\cdot\nabla u = e^{u(p-1)}\nabla h_1\cdot\nabla u + (p-1)h_1e^{u(p-1)}|\nabla u|^2,\\ \nabla(h_2e^{u(q-1)})\cdot\nabla u = e^{u(q-1)}\nabla h_2\cdot\nabla u + (q-1)h_2e^{u(q-1)}|\nabla u|^2,\\ \nabla(h_3e^{u(s-1)})\cdot\nabla u = e^{u(s-1)}\nabla h_3\cdot\nabla u + (s-1)h_3e^{u(s-1)}|\nabla u|^2.\end{cases} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we get (2.3). This completes the proof of Lemma 2.1. □

We can now validate Theorem 1.1.

Proof of Theorem 1.1. Define the n-rectangle $R := \Pi_{i=1}^n[p_i, q_i] \subset \mathbb{R}^n$, and set

$$\phi_R(x,t) = \frac{a}{t} + \sum_{k=1}^{n}\left(\frac{b}{(x_k-p_k)^2} + \frac{b}{(q_k-x_k)^2}\right) \tag{2.12}$$

for $t > 0$, $a > 0$, $b > 0$, and $x = (x_1, ..., x_n) \in R$, while $\phi_R \to +\infty$ as $x_i \to p_i, q_i$ or $t \to 0$.

The corresponding Harnack quantity is

$$H_R = \alpha\Delta u + \beta|\nabla u|^2 + ch_1e^{u(p-1)} + dh_2e^{u(q-1)} + kh_3e^{u(s-1)} + \phi_R(x,t).$$

Note that $H_R \to H_0$ as $R \to \mathbb{R}^n$, and $H_R > 0$ for small t.

So as to obtain a contradiction, assume that there is a first time t_0 and point $x_0 \in R$ such that $H_R(x_0, t_0) = 0$. Then at (x_0, t_0), we have

$$(H_R)_t \le 0, \quad \nabla H_R = 0, \quad \Delta H_R \ge 0$$

and

$$\Delta u = -\frac{1}{\alpha}(\beta|\nabla u|^2 + ch_1e^{u(p-1)} + dh_2e^{u(q-1)} + kh_3e^{u(s-1)} + \phi_R).$$

Then using Lemma 2.1 and the Cauchy-Schwarz inequality $|\nabla\nabla u|^2 \ge \frac{1}{n}(\Delta u)^2$, we can get

$$\begin{aligned}0 \ge\ & \frac{2(\alpha-\beta)}{n\alpha^2}(\beta|\nabla u|^2 + ch_1e^{u(p-1)} + dh_2e^{u(q-1)} + kh_3e^{u(s-1)} + \phi_R)^2\\ &+ [\alpha(p-1)+\beta-cp](p-1)h_1e^{u(p-1)}|\nabla u|^2 + [\alpha(q-1)+\beta-dq](q-1)h_2e^{u(q-1)}|\nabla u|^2\\ &+ [\alpha(s-1)+\beta-ks](s-1)h_3e^{u(s-1)}|\nabla u|^2\\ &+ [(\alpha-c)\Delta h_1 + 2(\alpha(p-1)+\beta-cp)\nabla h_1\cdot\nabla u + c(h_1)_t - h_1(p-1)\phi_R]e^{u(p-1)}\\ &+ [(\alpha-d)\Delta h_2 + 2(\alpha(q-1)+\beta-dq)\nabla h_2\cdot\nabla u + d(h_2)_t - h_2(q-1)\phi_R]e^{u(q-1)}\\ &+ [(\alpha-k)\Delta h_3 + 2(\alpha(s-1)+\beta-ks)\nabla h_3\cdot\nabla u + k(h_3)_t - h_3(s-1)\phi_R]e^{u(s-1)}\\ &+ [(c-d)(p-q)]h_1h_2e^{u(p-1)}e^{u(q-1)} + [(c-k)(p-s)]h_1h_3e^{u(p-1)}e^{u(s-1)}\\ &+ [(d-k)(q-s)]h_2h_3e^{u(q-1)}e^{u(s-1)} - 2\nabla\phi_R\cdot\nabla u - \Delta\phi_R + (\phi_R)_t.\end{aligned}$$

Setting $X = e^{u(p-1)}$, $Y = e^{u(q-1)}$, $Z = e^{u(s-1)}$, and $W = |\nabla u|^2$, and using

$$2\nabla h_1 \cdot \nabla u \geq -2|\nabla h_1|^2 - \frac{1}{2}|\nabla u|^2,$$

$$2\nabla h_2 \cdot \nabla u \geq -\frac{1}{2}|\nabla h_2|^2 - 2|\nabla u|^2,$$

and

$$2\nabla h_3 \cdot \nabla u \geq -|\nabla h_3|^2 - |\nabla u|^2,$$

we arrive at

$$\begin{aligned}
0 \geq\ & \frac{2(\alpha-\beta)}{n\alpha^2}(c^2h_1^2X^2 + d^2h_2^2Y^2 + k^2h_3^2Z^2 + \beta^2W^2) \\
&+ \left[(\alpha(p-1)+\beta-cp)\left(1-\frac{1}{h_1(p-1)}\right) + \frac{4(\alpha-\beta)\beta c}{n(p-1)\alpha^2}\right](p-1)h_1XW \\
&+ \left[(\alpha(q-1)+\beta-dq)\left(1-\frac{1}{h_2(q-1)}\right) + \frac{4(\alpha-\beta)\beta d}{n(q-1)\alpha^2}\right](q-1)h_2YW \\
&+ \left[(\alpha(s-1)+\beta-ks)\left(1-\frac{1}{h_3(s-1)}\right) + \frac{4(\alpha-\beta)\beta k}{n(s-1)\alpha^2}\right](s-1)h_3ZW \\
&+ \left[(\alpha-c)\Delta h_1 - 4(\alpha(p-1)+\beta-cp)|\nabla h_1|^2 + c(h_1)_t + \left(\frac{4(\alpha-\beta)c}{n\alpha^2}-(p-1)\right)h_1\phi_R\right]X \\
&+ \left[(\alpha-d)\Delta h_2 - 4(\alpha(q-1)+\beta-dq)|\nabla h_2|^2 + d(h_2)_t + \left(\frac{4(\alpha-\beta)d}{n\alpha^2}-(q-1)\right)h_2\phi_R\right]Y \\
&+ \left[(\alpha-k)\Delta h_3 - 4(\alpha(s-1)+\beta-ks)|\nabla h_3|^2 + k(h_3)_t + \left(\frac{4(\alpha-\beta)k}{n\alpha^2}-(s-1)\right)h_3\phi_R\right]Z \\
&+ \left[(c-d)(p-q) + \frac{4(\alpha-\beta)cd}{n\alpha^2}\right]h_1h_2XY + \left[(c-k)(p-s) + \frac{4(\alpha-\beta)ck}{n\alpha^2}\right]h_1h_3XZ \\
&+ \left[(d-k)(q-s) + \frac{4(\alpha-\beta)dk}{n\alpha^2}\right]h_2h_3YZ - 2\nabla\phi_R \cdot \nabla u - \Delta\phi_R \\
&+ (\phi_R)_t + \frac{2(\alpha-\beta)}{n\alpha^2}\phi_R^2 + \frac{4(\alpha-\beta)\beta\phi_R}{n\alpha^2}W.
\end{aligned} \tag{2.13}$$

By demonstrating that the right-hand side of (2.13) is positive, we can then obtain a contradiction. The assumption of (1.3) in Theorem 1.1 implies

$$\begin{cases} c \geq \max\left\{\frac{(p-1)n\alpha^2}{4(\alpha-\beta)}, \frac{\alpha(p-1)+\beta}{p}\right\}, \\ d \geq \max\left\{\frac{(q-1)n\alpha^2}{4(\alpha-\beta)}, \frac{\alpha(q-1)+\beta}{q}\right\}, \\ k \geq \max\left\{\frac{(s-1)n\alpha^2}{4(\alpha-\beta)}, \frac{\alpha(s-1)+\beta}{s}\right\}. \end{cases} \tag{2.14}$$

By (2.14), we get

$$c \geq \frac{(p-1)n\alpha^2}{4(\alpha-\beta)}, \quad d \geq \frac{(q-1)n\alpha^2}{4(\alpha-\beta)}, \quad k \geq \frac{(s-1)n\alpha^2}{4(\alpha-\beta)}, \tag{2.15}$$

and by rewriting (2.15), we obtain

$$\frac{4(\alpha-\beta)c}{n\alpha^2} - (p-1) \geq 0, \quad \frac{4(\alpha-\beta)d}{n\alpha^2} - (q-1) \geq 0, \quad \frac{4(\alpha-\beta)k}{n\alpha^2} - (s-1) \geq 0, \tag{2.16}$$

and

$$\frac{4(\alpha-\beta)c}{(p-1)n\alpha^2} \geq 1, \quad \frac{4(\alpha-\beta)d}{(q-1)n\alpha^2} \geq 1, \quad \frac{4(\alpha-\beta)k}{(s-1)n\alpha^2} \geq 1. \tag{2.17}$$

Using (2.14), we have

$$c \geq \frac{\alpha(p-1)+\beta}{p}, \quad d \geq \frac{\alpha(q-1)+\beta}{q}, \quad k \geq \frac{\alpha(s-1)+\beta}{s}, \tag{2.18}$$

and by rewriting (2.18), we obtain

$$\alpha(p-1)+\beta-cp \leq 0, \quad \alpha(q-1)+\beta-dq \leq 0, \quad \alpha(s-1)+\beta-ks \leq 0. \tag{2.19}$$

By combining (2.19) and (2.16), we get

$$\begin{aligned} \alpha(p-1)+\beta-cp \leq 0, &\quad \frac{4(\alpha-\beta)c}{n\alpha^2}-(p-1) \geq 0, \\ \alpha(q-1)+\beta-dq \leq 0, &\quad \frac{4(\alpha-\beta)d}{n\alpha^2}-(q-1) \geq 0, \\ \alpha(s-1)+\beta-ks \leq 0, &\quad \frac{4(\alpha-\beta)k}{n\alpha^2}-(s-1) \geq 0. \end{aligned} \tag{2.20}$$

The requirement of (1.3) in Theorem 1.1 also suggests

$$\begin{cases} \frac{\alpha(p-1)+2\beta}{p} \geq c, \\ \frac{\alpha(q-1)+2\beta}{q} \geq d, \\ \frac{\alpha(s-1)+2\beta}{s} \geq k. \end{cases} \tag{2.21}$$

Then, combining (2.21) and (2.17), we have

$$\begin{aligned} \alpha(p-1)+\beta-cp+\frac{4(\alpha-\beta)\beta c}{n(p-1)\alpha^2} &\geq \alpha(p-1)+2\beta-cp \geq 0, \\ \alpha(q-1)+\beta-dq+\frac{4(\alpha-\beta)\beta d}{n(q-1)\alpha^2} &\geq \alpha(q-1)+2\beta-dq \geq 0, \\ \alpha(s-1)+\beta-ks+\frac{4(\alpha-\beta)\beta k}{n(s-1)\alpha^2} &\geq \alpha(s-1)+2\beta-ks \geq 0. \end{aligned} \tag{2.22}$$

Rewriting (2.22), we obtain

$$\begin{aligned} (\alpha(p-1)+\beta-cp)\left(1-\frac{1}{h_1(p-1)}\right)+\frac{4(\alpha-\beta)\beta c}{n(p-1)\alpha^2} &\geq 0, \\ (\alpha(q-1)+\beta-dq)\left(1-\frac{1}{h_2(q-1)}\right)+\frac{4(\alpha-\beta)\beta d}{n(q-1)\alpha^2} &\geq 0, \\ (\alpha(s-1)+\beta-ks)\left(1-\frac{1}{h_3(s-1)}\right)+\frac{4(\alpha-\beta)\beta k}{n(s-1)\alpha^2} &\geq 0. \end{aligned} \tag{2.23}$$

Note the inequality

$$\frac{4(\alpha-\beta)\beta\phi_R}{n\alpha^2}W - 2\nabla\phi_R\cdot\nabla u \geq \frac{-n\alpha^2|\nabla\phi_R|^2}{4(\alpha-\beta)\beta\phi_R}.$$

Combining (1.2), (1.4), (1.5), (2.20), and (2.23) and removing a number of non-negative terms from the right side of (2.13), we have

$$0 \geq (\phi_R)_t - \Delta\phi_R - \frac{n\alpha^2|\nabla\phi_R|^2}{4(\alpha-\beta)\beta\phi_R} + \frac{2(\alpha-\beta)}{n\alpha^2}\phi_R^2. \tag{2.24}$$

By (2.12), we can compute

$$\Delta\phi_R = \sum_{k=1}^{n}\left(\frac{6b}{(x_k-p_k)^4} + \frac{6b}{(q_k-x_k)^4}\right), \tag{2.25}$$

$$|\nabla\phi_R|^2 = \sum_{k=1}^{n}\left(-\frac{2b}{(x_k-p_k)^3} - \frac{2b}{(q_k-x_k)^3}\right)^2,$$

and

$$\begin{aligned}\frac{|\nabla\phi_R|^2}{\phi_R} &= \sum_{k=1}^{n}\left(-\frac{2b}{(x_k-p_k)^3\sqrt{\phi_R}} - \frac{2b}{(q_k-x_k)^3\sqrt{\phi_R}}\right)^2\\ &\leq \sum_{k=1}^{n}\left(\frac{4b}{(x_k-p_k)^4} + \frac{4b}{(q_k-x_k)^4}\right).\end{aligned} \tag{2.26}$$

For the sake of simplicity, we set

$$A := \frac{2(\alpha-\beta)}{n\alpha^2} > 0, \quad B := \frac{n\alpha^2}{4(\alpha-\beta)\beta} > 0.$$

To arrive at a contradiction, we need

$$A\phi_R^2 - \Delta\phi_R - B\frac{|\nabla\phi_R|^2}{\phi_R} + (\phi_R)_t > 0. \tag{2.27}$$

Next, plugging (2.12), (2.25), and (2.26) into the left-hand side of (2.27), we get

$$\begin{aligned}&A\left[\frac{a}{t} + \sum_{k=1}^{n}\left(\frac{b}{(x_k-p_k)^2} + \frac{b}{(q_k-x_k)^2}\right)\right]^2 - \left[\sum_{k=1}^{n}\left(\frac{6b}{(x_k-p_k)^4} + \frac{6b}{(q_k-x_k)^4}\right)\right]\\ &- B\left[\sum_{k=1}^{n}\left(-\frac{2b}{(x_k-p_k)^3\sqrt{\phi_R}} - \frac{2b}{(q_k-x_k)^3\sqrt{\phi_R}}\right)^2\right] - \frac{a}{t^2}\\ \geq\ &\frac{Aa^2-a}{t^2} + (Ab^2 - 6b - 4bB)\left[\sum_{k=1}^{n}\left(\frac{1}{(x_k-p_k)^4} + \frac{1}{(q_k-x_k)^4}\right)\right].\end{aligned} \tag{2.28}$$

By (1.4), we have $Aa^2 - a \geq 0$. To prove (2.27), we need

$$Ab^2 - b(6+4B) > 0.$$

In summary, a and b satisfy

$$a \geq \frac{n\alpha^2}{2(\alpha-\beta)}, \quad b \geq \frac{n\alpha^2}{2(\alpha-\beta)}\left[6+\frac{n\alpha^2}{(\alpha-\beta)\beta}\right].$$

Then, we can demonstrate that the inequality on the right side is positive. Thus, there is a contradiction.

We obtain $\phi_R \to \frac{a}{t}$, $H_R \to H_0$ if $R \to \mathbb{R}^n$, assuming that the solution is present in the complete space $\mathbb{R}^n$. This suggests $H_0 \geq 0$ and completes the proof.

Proof of Corollary 1.2. We pick $\alpha = 2, \beta = 1, a = 2n$, and c such that $0 < n(p-1) \leq c < 2$ and $c \geq d \geq k \geq \beta$ in Theorem 1.1. Since $u = \ln f$, we get

$$\Delta u = \frac{f\Delta f - |\nabla f|^2}{f^2}, \tag{2.29}$$

$$|\nabla u|^2 = \frac{|\nabla f|^2}{f^2}. \tag{2.30}$$

By substituting (2.29) and (2.30) into (1.6), we can calculate

$$2\frac{\Delta f}{f} - \frac{|\nabla f|^2}{f^2} + ch_1 f^{p-1} + dh_2 f^{q-1} + kh_3 f^{s-1} + \frac{2n}{t} \geq 0,$$

and then

$$2\Delta f - \frac{|\nabla f|^2}{f} + ch_1 f^p + dh_2 f^q + kh_3 f^s + \frac{2n}{t} f \geq 0. \tag{2.31}$$

Noting

$$f_t = \Delta f + h_1 f^p + h_2 f^q + h_3 f^s, \tag{2.32}$$

by (2.31) and (2.32), we have

$$2f_t - \frac{|\nabla f|^2}{f} + \frac{2n}{t} f \geq (2-c)h_1 f^p + (2-d)h_2 f^q + (2-k)h_3 f^s.$$

Furthermore, we observe that

$$2f_t + \frac{2n}{t} f \geq (2-c)h_1 f^p + (2-d)h_2 f^q + (2-k)h_3 f^s,$$

which implies that

$$\begin{aligned} 2\left(\frac{1}{f}\right)_t &= -2\frac{1}{f^2} \cdot f_t \\ &\leq -\frac{1}{f^2}\left(-\frac{2n}{t} f + (2-c)h_1 f^p + (2-d)h_2 f^q + (2-k)h_3 f^s\right) \\ &= \frac{1}{f}\left(\frac{2n - (2-d)th_2 f^{q-1} - (2-k)th_3 f^{s-1}}{t} - (2-c)h_1 f^{p-1}\right) \\ &= \frac{1}{f^{2-p}}\left(\frac{2n - (2-d)th_2 f^{q-1} - (2-k)th_3 f^{s-1}}{tf^{p-1}} - (2-c)h_1\right) \\ &\leq \frac{1}{f^{2-p}}\left(\frac{2n}{tf^{p-1}} - (2-c)h_1\right). \end{aligned} \tag{2.33}$$

We might presume that $f \geq (\frac{4n}{(2-c)h_1})^{\frac{1}{p-1}}$ at the origin $x_0 = 0$ for $t_0 = 1$, and hence we have

$$\frac{1}{f^{p-1}} \leq \frac{(2-c)h_1}{4n},$$

$$\frac{2n}{tf^{p-1}} \leq \frac{(2-c)h_1}{2t}. \tag{2.34}$$

Therefore, for $t \geq 1$, we obtain

$$\begin{aligned} 2\left(\frac{1}{f}\right)_t (0,t) &\leq \frac{1}{f^{2-p}}\left(\frac{2n}{tf^{p-1}} - (2-c)h_1\right) \\ &\leq \frac{(2-c)h_1}{f^{2-p}(0,t)}(\frac{1}{2t} - 1) \\ &< 0, \end{aligned}$$

such that $f(0,t)$ is strictly increasing when $f(0,t)$ is finite.

(i) If $p > 2$, then $f^{p-2}(0,t) \geq f^{p-2}(0,1)$ for $t \geq 1$ and (2.33) simplifies to

$$2\left(\frac{1}{f}\right)_t (0,t) \leq \frac{2n}{tf(0,1)} - (2-c)h_1 f^{p-2}(0,1). \tag{2.35}$$

(ii) If $1 < p \leq 2$, it is easy to obtain that

$$\frac{2}{p-1}\left[\left(\frac{1}{f}\right)^{p-1}\right]_t (0,t) = 2f^{2-p}\left(\frac{1}{f}\right)_t (0,t) \leq \frac{2n}{tf^{p-1}(0,1)} - (2-c)h_1. \tag{2.36}$$

Therefore, there is $\delta > 0$ such that when t is sufficiently large, the right-hand side of (2.35) and (2.36) are smaller than $-\delta < 0$, and therefore $\frac{1}{f} \to 0$ in finite time. This completes the proof.

Proof of Corollary 1.3. We obtain $H_0 \geq 0$ by the differential Harnack estimate (1.6), which indicates that

$$\Delta u \geq \frac{1}{\alpha}\left(-\beta|\nabla u|^2 - ch_1 e^{u(p-1)} - dh_2 e^{u(q-1)} - kh_3 e^{u(s-1)} - \frac{a}{t}\right).$$

Then, combined with (2.1), we calculate the evolution of u along γ, i.e.,

$$\begin{aligned} (u(x(t),t))_t &= \nabla u \cdot \dot{x} + u_t \\ &= \nabla u \cdot \dot{x} + \Delta u + |\nabla u|^2 + h_1 e^{u(p-1)} + h_2 e^{u(q-1)} + h_3 e^{u(s-1)} \\ &\geq \nabla u \cdot \dot{x} + |\nabla u|^2(1 - \frac{\beta}{\alpha}) - \frac{a}{\alpha t} + (1 - \frac{c}{\alpha})h_1 e^{u(p-1)} \\ &\quad + (1 - \frac{d}{\alpha})h_2 e^{u(q-1)} + (1 - \frac{k}{\alpha})h_3 e^{u(s-1)} \\ &\geq |\nabla u|^2(\frac{1}{2} - \frac{\beta}{\alpha}) - \frac{1}{2}|\dot{x}|^2 - \frac{a}{\alpha t} + (1 - \frac{c}{\alpha})h_1 e^{u(p-1)} \\ &\quad + (1 - \frac{d}{\alpha})h_2 e^{u(q-1)} + (1 - \frac{k}{\alpha})h_3 e^{u(s-1)} \\ &\geq -\frac{1}{2}|\dot{x}|^2 - \frac{a}{\alpha t}, \end{aligned}$$

where we have used the assumption $\alpha \geq 2\beta$ and $k \leq d \leq c \leq \alpha$.

Hence we have

$$\begin{aligned}(-u(x(t),t))_t &\leq \frac{1}{2}|\dot{x}|^2 + \frac{a}{\alpha t}\\ &\leq \frac{1}{2}|\dot{x}|^2 + \frac{n}{t}.\end{aligned} \tag{2.37}$$

Integrating the previously mentioned quality (2.37) along γ, and taking the infimum of all such space-time pathways, we get

$$\int_{t_1}^{t_2} d(-u(x(t),t)) \leq \inf_{\gamma(t)=(x(t),t)} \int_{t_1}^{t_2} (\frac{1}{2}|\dot{x}|^2 + \frac{n}{t})dt,$$

and then

$$u(x_1,t_1) - u(x_2,t_2) \leq \inf_{\gamma(t)=(x(t),t)} \int_{t_1}^{t_2} (\frac{1}{2}|\dot{x}|^2 + \frac{n}{t})dt.$$

Using $u = \ln f$, we have

$$\frac{f(x_1,t_1)}{f(x_2,t_2)} \leq \exp\left[\inf_{\gamma(t)=(x(t),t)} \int_{t_1}^{t_2} (\frac{1}{2}|\dot{x}|^2 + \frac{n}{t})dt\right].$$

Hence we can arrive at (1.8). This finishes the proof.

3. The proofs of Theorem 1.4 and Corollary 1.5

Estimating the following Harnack quantity is our main method of research:

$$\widetilde{H} := \tilde{\alpha}\Delta u + \tilde{\beta}|\nabla u|^2 + \tilde{c}h_1 e^{u(p-1)} + \tilde{d}h_2 e^{u(q-1)} + \tilde{k}h_3 e^{u(s-1)} + \theta, \tag{3.1}$$

where $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{c}$, $\tilde{d}$, $\tilde{k} \in \mathbb{R}$ and $\theta : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ will be determined later. We now derive the derivation of $\widetilde{H}$ in t.

Next, similar to the proof of Lemma 2.1, we can get Lemma 3.1.

Lemma 3.1. *Suppose that $f(x,t)$ is a positive solution of* (1.1)*, $u = \ln f$, and the definition of $\widetilde{H}$ is stated in* (3.1)*. Then we obtain*

$$\begin{aligned}\widetilde{H}_t = {} & \Delta\widetilde{H} + 2\nabla\widetilde{H}\cdot\nabla u + (p-1)h_1 e^{u(p-1)}\widetilde{H} + (q-1)h_2 e^{u(q-1)}\widetilde{H} + (s-1)h_3 e^{u(s-1)}\widetilde{H}\\ & + 2(\tilde{\alpha} - \tilde{\beta})|\nabla\nabla u|^2 + [\tilde{\alpha}(p-1) + \tilde{\beta} - \tilde{c}p](p-1)h_1 e^{u(p-1)}|\nabla u|^2\\ & + [\tilde{\alpha}(q-1) + \tilde{\beta} - \tilde{d}q](q-1)h_2 e^{u(q-1)}|\nabla u|^2\\ & + [\tilde{\alpha}(s-1) + \tilde{\beta} - \tilde{k}s](s-1)h_3 e^{u(s-1)}|\nabla u|^2\\ & + [(\tilde{\alpha} - \tilde{c})\Delta h_1 + 2(\tilde{\alpha}(p-1) + \tilde{\beta} - \tilde{c}p)\nabla h_1\cdot\nabla u + \tilde{c}(h_1)_t - h_1(p-1)\phi]e^{u(p-1)}\\ & + [(\tilde{\alpha} - \tilde{d})\Delta h_2 + 2(\tilde{\alpha}(q-1) + \tilde{\beta} - \tilde{d}q)\nabla h_2\cdot\nabla u + \tilde{d}(h_2)_t - h_2(q-1)\phi]e^{u(q-1)}\\ & + [(\tilde{\alpha} - \tilde{k})\Delta h_3 + 2(\tilde{\alpha}(s-1) + \tilde{\beta} - \tilde{k}s)\nabla h_3\cdot\nabla u + \tilde{k}(h_3)_t - h_3(s-1)\phi]e^{u(s-1)}\\ & + [(\tilde{c} - \tilde{d})(p-q)]h_1 h_2 e^{u(p-1)}e^{u(q-1)}\\ & + [(\tilde{c} - \tilde{k})(p-s)]h_1 h_3 e^{u(p-1)}e^{u(s-1)}\\ & + [(\tilde{d} - \tilde{k})(q-s)]h_2 h_3 e^{u(q-1)}e^{u(s-1)} - 2\nabla\theta\cdot\nabla u - \Delta\theta + \theta_t.\end{aligned} \tag{3.2}$$

Proof of Theorem 1.4. Define the n-rectangle $R := \Pi_{i=1}^{n}[p_i, q_i] \subset \mathbb{R}^n$, and set

$$\theta_R(x,t) = \frac{\tilde{a}}{1-e^{-mt}} + \sum_{k=1}^{n}\left(\frac{\tilde{b}}{(x_k-p_k)^2} + \frac{\tilde{b}}{(q_k-x_k)^2}\right) \tag{3.3}$$

for $t > 0$, $\tilde{a} > 0$, $\tilde{b} > 0$, $m \geq \frac{n\tilde{\alpha}^2}{2(\tilde{\alpha}-\tilde{\beta})}[6 + \frac{n\tilde{\alpha}^2}{(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}}]$, and $x = (x_1, ..., x_n) \in R$, while $\theta_R \to +\infty$ as $x_i \to p_i, q_i$ or $t \to 0$.

The corresponding Harnack quantity is defined as

$$\widetilde{H}_R = \tilde{\alpha}\Delta u + \tilde{\beta}|\nabla u|^2 + \tilde{c}h_1e^{u(p-1)} + \tilde{d}h_2e^{u(q-1)} + \tilde{k}h_3e^{u(s-1)} + \theta_R(x,t).$$

Note that $\widetilde{H}_R \to \widetilde{H}_0$ as $R \to \mathbb{R}^n$, and $\widetilde{H}_R > 0$ for small t.

In order to obtain a contradiction, assume that there is a first time t_0 and point $x_0 \in R$ such that $\widetilde{H}_R(x_0, t_0) = 0$. Then at (x_0, t_0), we have

$$(\widetilde{H}_R)_t \leq 0, \quad \nabla\widetilde{H}_R = 0, \quad \Delta\widetilde{H}_R \geq 0,$$

and

$$\Delta u = -\frac{1}{\tilde{\alpha}}(\tilde{\beta}|\nabla u|^2 + \tilde{c}h_1e^{u(p-1)} + \tilde{d}h_2e^{u(q-1)} + \tilde{k}h_3e^{u(s-1)} + \theta_R).$$

Similar to the proof of (2.13), we can obtain

$$\begin{aligned}
0 \geq\ & \frac{2(\tilde{\alpha}-\tilde{\beta})}{n\tilde{\alpha}^2}(\tilde{c}^2h_1^2X^2 + \tilde{d}^2h_2^2Y^2 + \tilde{k}^2h_3^2Z^2 + \tilde{\beta}^2W^2)\\
& + \left[(\tilde{\alpha}(p-1)+\tilde{\beta}-\tilde{c}p)\left(1-\frac{1}{h_1(p-1)}\right) + \frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}\tilde{c}}{n(p-1)\tilde{\alpha}^2}\right](p-1)h_1XW\\
& + \left[(\tilde{\alpha}(q-1)+\tilde{\beta}-\tilde{d}q)\left(1-\frac{1}{h_2(q-1)}\right) + \frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}\tilde{d}}{n(q-1)\tilde{\alpha}^2}\right](q-1)h_2YW\\
& + \left[(\tilde{\alpha}(s-1)+\tilde{\beta}-\tilde{k}s)\left(1-\frac{1}{h_3(s-1)}\right) + \frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}\tilde{k}}{n(s-1)\tilde{\alpha}^2}\right](s-1)h_3ZW\\
& + \left[(\tilde{\alpha}-\tilde{c})\Delta h_1 - 4(\tilde{\alpha}(p-1)+\tilde{\beta}-\tilde{c}p)|\nabla h_1|^2 + \tilde{c}(h_1)_t + \left(\frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{c}}{n\tilde{\alpha}^2} - (p-1)\right)h_1\theta_R\right]X\\
& + \left[(\tilde{\alpha}-\tilde{d})\Delta h_2 - 4(\tilde{\alpha}(q-1)+\tilde{\beta}-\tilde{d}q)|\nabla h_2|^2 + \tilde{d}(h_2)_t + \left(\frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{d}}{n\tilde{\alpha}^2} - (q-1)\right)h_2\theta_R\right]Y\\
& + \left[(\tilde{\alpha}-\tilde{k})\Delta h_3 - 4(\tilde{\alpha}(s-1)+\tilde{\beta}-\tilde{k}s)|\nabla h_3|^2 + \tilde{k}(h_3)_t\left(\frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{k}}{n\tilde{\alpha}^2} - (s-1)\right)h_3\theta_R\right]Z\\
& + \left[(\tilde{c}-\tilde{d})(p-q) + \frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{c}\tilde{d}}{n\tilde{\alpha}^2}\right]h_1h_2XY\\
& + \left[(\tilde{c}-\tilde{k})(p-s) + \frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{c}\tilde{k}}{n\tilde{\alpha}^2}\right]h_1h_3XZ\\
& + \left[(\tilde{d}-\tilde{k})(q-s) + \frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{d}\tilde{k}}{n\tilde{\alpha}^2}\right]h_2h_3YZ - 2\nabla\theta_R\cdot\nabla u - \Delta\theta_R\\
& + (\theta_R)_t + \frac{2(\tilde{\alpha}-\tilde{\beta})}{n\tilde{\alpha}^2}\theta_R^2 + \frac{4(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}\theta_R}{n\tilde{\alpha}^2}W,
\end{aligned} \tag{3.4}$$

where $X = e^{u(p-1)}$, $Y = e^{u(q-1)}$, $Z = e^{u(s-1)}$, and $W = |\nabla u|^2$.

By demonstrating that the right-hand side of (3.4) is positive, we can then obtain a contradiction. Similar to the proof of (2.24), we can get

$$0 \geq (\theta_R)_t - \Delta\theta_R - \frac{n\tilde{\alpha}^2|\nabla\theta_R|^2}{4(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}\theta_R} + \frac{2(\tilde{\alpha}-\tilde{\beta})}{n\tilde{\alpha}^2}\theta_R^2. \tag{3.5}$$

For the sake of simplicity, we set

$$\tilde{A} := \frac{2(\tilde{\alpha}-\tilde{\beta})}{n\tilde{\alpha}^2} > 0, \quad \tilde{B} := \frac{n\tilde{\alpha}^2}{4(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}} > 0.$$

In order to obtain a contradiction, we need

$$\tilde{A}\theta_R^2 - \Delta\theta_R - \tilde{B}\frac{|\nabla\theta_R|^2}{\theta_R} + (\theta_R)t > 0. \tag{3.6}$$

Next, similar to the calculation of (2.29), we can get

$$\begin{aligned}
&\tilde{A}\left[\frac{\tilde{a}}{1-e^{-mt}} + \sum_{k=1}^{n}\left(\frac{\tilde{b}}{(x_k-p_k)^2} + \frac{\tilde{b}}{(q_k-x_k)^2}\right)\right]^2 - \left[\sum_{k=1}^{n}\left(\frac{6\tilde{b}}{(x_k-p_k)^4} + \frac{6\tilde{b}}{(q_k-x_k)^4}\right)\right]\\
&- \tilde{B}\left[\sum_{k=1}^{n}\left(-\frac{2\tilde{b}}{(x_k-p_k)^3\sqrt{\theta_R}} - \frac{2\tilde{b}}{(q_k-x_k)^3\sqrt{\theta_R}}\right)^2\right] - \frac{m\tilde{a}}{(1-e^{-mt})^2e^{mt}}\\
\geq\ & \frac{\tilde{A}\tilde{a}^2e^{mt} - m\tilde{a}}{(1-e^{-mt})^2e^{mt}} + (\tilde{A}\tilde{b}^2 - 6\tilde{b} - 4\tilde{b}\tilde{B})\left[\sum_{k=1}^{n}\left(\frac{1}{(x_k-p_k)^4} + \frac{1}{(q_k-x_k)^4}\right)\right].
\end{aligned}$$

By (1.9), we have $\tilde{A}\tilde{a}^2e^{mt} - m\tilde{a} \geq 0$. To prove (3.6), we need

$$\tilde{A}\tilde{b}^2 - \tilde{b}(6 + 4\tilde{B}) > 0.$$

In summary, $\tilde{a}$ and $\tilde{b}$ satisfy

$$\tilde{a} \geq \frac{nm\tilde{\alpha}^2}{2(\tilde{\alpha}-\tilde{\beta})}, \quad \tilde{b} \geq \frac{n\tilde{\alpha}^2}{2(\tilde{\alpha}-\tilde{\beta})}\left[6 + \frac{n\tilde{\alpha}^2}{(\tilde{\alpha}-\tilde{\beta})\tilde{\beta}}\right].$$

Then, we can demonstrate that the inequality on the right side is positive. Thus, we obtain a contradiction.

Assuming that the solution exists in the whole space $\mathbb{R}^n$, we get $\theta_R \to \frac{\tilde{a}}{1-e^{-mt}}$, $\widetilde{H}_R \to \widetilde{H}_0$ if $R \to \mathbb{R}^n$. This implies $\widetilde{H}_0 \geq 0$ and completes the proof.

Proof of Corollary 1.5. Corollary 1.5 follows immediately from Theorem 1.4 by using a similar method to that in the proof of Corollary 1.3. We omit the proof of Corollary 1.5.

4. Conclusions

In this paper, some new types of differential Harnack estimates were established for positive solutions of the semilinear parabolic equation with three exponents on $\mathbb{R}^n$. Additionally, as applications, we found the blow-up of the solutions and classical Harnack inequalities for this equation. Our results generalize some known results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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