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https://www.aimspress.com/journal/era

ERA, 32(9): 5542–5556.
DOI: 10.3934/era.2024256
Received: 18 August 2024
Revised: 13 September 2024
Accepted: 23 September 2024
Published: 26 September 2024

Research article

Existence of nodal solutions of nonlinear Lidstone boundary value problems

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Abstract: We investigate the existence of nodal solutions for the nonlinear Lidstone boundary value problem

$$\begin{cases} (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \lambda a(t)f(u), & t \in (0, r), \\ u^{(2i)}(0) = u^{(2i)}(r) = 0, \quad i = 0, 1, \cdots, m-1, \end{cases} \tag{P}$$

where $\lambda > 0$ is a parameter, c is a constant, $m \geq 1$ is an integer, $a : [0, r] \to [0, \infty)$ is continuous with $a \not\equiv 0$ on the subinterval within $[0, r]$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. We analyze the spectrum structure of the corresponding linear eigenvalue problem via the disconjugacy theory and Elias's spectrum theory. As applications of our spectrum results, we show that problem (P) has nodal solutions under some suitable conditions. The bifurcation technique is used to obtain the main results of this paper.

Keywords: Lidstone; nodal solutions; spectrum; bifurcation; disconjugacy theory

1. Introduction

We investigate the existence of nodal solutions for nonlinear Lidstone boundary value problem

$$\begin{cases} (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \lambda a(t)f(u), & t \in (0, r), \\ u^{(2i)}(0) = u^{(2i)}(r) = 0, \quad i = 0, 1, \cdots, m-1, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, c is a constant, $m \geq 1$ is an integer, $a : [0, r] \to [0, \infty)$ is continuous with $a \not\equiv 0$ on the subinterval within $[0, r]$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Problem (1.1) is called a *Lidstone* boundary value problem. Lidstone boundary value problems arise in a lot of various fields of applied mathematics and physics. If $m = 1$, problem (1.1) are Newton's equations of motion under Dirichlet boundary conditions. If $m = 2$, problem (1.1) is the elastic beam equation with simple support at both ends. If $m = 3$, problem (1.1) can be used to describe the speed of a flagellate protozoan [1]. From this point of view, it is not only of great theoretical significance to discuss such problems, but also of practical application value.

Lidstone boundary value problems in a comparable form to (1.1) have been taken into account in many papers, particularly in second and fourth-order cases, see for example [2–6] and [7–10] and references therein. Specifically, Lazer and McKenna [10] considered the existence for nodal solutions of (1.1) with $m = 2$, $a(t) \equiv 1$, and a "jumping" nonlinearity, i.e.,

$$\begin{cases} u''''(t) + cu'' = \lambda[(u+1)^+ - 1], & t \in (0, r), \\ u(0) = u(r) = u''(0) = u''(r) = 0. \end{cases} \tag{1.2}$$

The eigenvalues of the linear eigenvalue problem corresponding to (1.2) can be obtained directly by ordinary differential equation calculation, but for the linear eigenvalue problem with weighted function $a(t)$, the eigenvalues cannot be calculated. In addition, it is worth noting that a complicated method was used in [10] (see Lemma 2.2) to certify that all zeros of the solutions are simple, moreover, this method does not seem to apply in the case of $m > 2$.

Existence and multiplicity of positive solutions for $2m$th-order Lidstone boundary value problems have been extensively studied by several authors, see [11–17]. For example, Yuan et al. [17] considered the existence of a positive solution for the $2m$th-order Lidstone boundary value problem

$$\begin{cases} (-1)^m u^{(2m)}(t) = \lambda f(t, u(t)), & t \in (0, 1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \ \ i = 0, 1, \cdots, m-1 \end{cases} \tag{1.3}$$

by the fixed point theorem of mixed monotone operators. However, relatively little is known about the existence of nodal solutions for $2m$th-order Lidstone boundary value problems, see [18, 19]. In [19], Xu and Han dealt with the existence of nodal solutions of Lidstone boundary value problems with the assumption that the nonlinearity f is asymptotically linear

$$\begin{cases} (-1)^m u^{(2m)}(t) = \mu a(t) f(u), & t \in (0, 1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \ \ i = 0, 1, \cdots, m-1 \end{cases} \tag{1.4}$$

under these suppositions:

(A1) $a \in C([0, 1], [0, \infty))$, $a \not\equiv 0$ on subinterval within $[0, 1]$;

(A2) $f \in C(\mathbb{R}, \mathbb{R})$, $f(s)s > 0$ for $s \neq 0$;

(A3) there exist $f_0, f_\infty \in (0, \infty)$ such that

$$\lim_{|s|\to 0} \frac{f(s)}{s} = f_0, \quad \lim_{|s|\to\infty} \frac{f(s)}{s} = f_\infty.$$

Let μ_k be the kth eigenvalue of the linear eigenvalue problem associated with (1.4). By using bifurcation techniques, they obtained

Theorem A. [19, Theorem 1.1.] Let (A1)–(A3) hold. Assume that for some $k \in \mathbb{N}$, either

$$\frac{\mu_k}{f_\infty} < \mu < \frac{\mu_k}{f_0},$$

or

$$\frac{\mu_k}{f_0} < \mu < \frac{\mu_k}{f_\infty}.$$

Then, (1.4) possesses two solutions, namely u_k^+ and u_k^-. The solution u_k^+ has precisely $k-1$ simple zeros within the interval $(0,1)$ and is positive in the vicinity of 0. Similarly, u_k^- also has exactly $k-1$ simple zeros in $(0,1)$ and is negative near 0.

However, the more general operator $(-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t))$ and "jumping" nonlinearity are not considered in [19].

Inspired by [10] and [19], the main purpose of this paper is to analyze the existence of nodal solutions for problem (1.1). One of the contributions of this paper is to employ the disconjugacy theory to examine a sufficient condition that guarantees the disconjugacy of $(-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = 0$ on the interval $[0,r]$. In addition, it is probably the first time that we utilize Elias's spectrum theory to explore the spectrum structure of the linear operator $(-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \lambda a(t)u,\ t \in (0,r)$ coupled with the boundary conditions $u^{(2i)}(0) = u^{(2i)}(r) = 0,\ \ i = 0,1,\cdots,m-1$. Moreover, we use a bifurcation technique to obtain the existence of nodal solutions for problem (1.1). It is worth noting that a novel method is employed to prove that all zeros of the solution of problem (1.1) are simple, which is based on the well-known Uri Elias formula, see [20, 21].

The assumptions of this paper are as follows:

(H1) c is a constant with $c < \dfrac{\pi^2}{r^2}$;
(H2) $a \in C([0,r],[0,\infty))$ and $a \not\equiv 0$ on subinterval within $[0,r]$;
(H3) $f \in C(\mathbb{R},\mathbb{R})$, $f(s)s > 0$ for $s \neq 0$,

$$\lim_{s\to-\infty}\frac{f(s)}{s} = 0,\quad \lim_{s\to+\infty}\frac{f(s)}{s} = f_{+\infty},\quad \lim_{s\to 0}\frac{f(s)}{s} = f_0$$

for some $f_{+\infty}, f_0 \in (0,\infty)$.

Using the disconjugacy theory and Elias's spectrum theory, we first consider the spectrum structure of the linear eigenvalue problem

$$\begin{cases} (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \lambda a(t)u, & t \in (0,r), \\ u^{(2i)}(0) = u^{(2i)}(r) = 0, \ \ i = 0,1,\cdots,m-1. \end{cases} \tag{1.5}$$

Theorem 1.1. *Assume that (H1) is satisfied, and a satisfies (H2). Then,*
(i) There is an infinite series of positive eigenvalues for problem (1.5)

$$\lambda_1[c] < \cdots < \lambda_k[c] < \cdots.$$

(ii) $\lambda_k[c] \to \infty$ *as* $k \to \infty$.
(iii) For each eigenvalue $\lambda_k[c]$*, there is an essentially unique eigenfunction* φ_k*. This eigenfunction* φ_k *has exactly* $k-1$ *simple zeros within the interval* $(0,r)$ *and is positive in the vicinity of 0.*
(iv) For an arbitrary subinterval within $[0,r]$*, an eigenfunction corresponding to a sufficiently large eigenvalue will undergo a sign change in that subinterval.*
(v) For every $k \in \mathbb{N}$*, the algebraic multiplicity for* $\lambda_k[c]$ *is* 1.

According to Theorem 1.1, we employ bifurcation theory to acquire our main results.

Theorem 1.2. *Let (H1), (H2), (A2), and (A3) hold. Suppose that for $k \in \mathbb{N}$, either*

$$\frac{\lambda_k[c]}{f_\infty} < \lambda < \frac{\lambda_k[c]}{f_0},$$

or

$$\frac{\lambda_k[c]}{f_0} < \lambda < \frac{\lambda_k[c]}{f_\infty}.$$

Then, (1.1) possesses two solutions, namely u_k^+ and u_k^-. The solution u_k^+ has precisely $k-1$ simple zeros within the interval $(0, r)$ and is positive in the vicinity of 0. Similarly, u_k^- also has exactly $k-1$ simple zeros in $(0, r)$ and is negative near 0.

Remark 1.1. *If $c = 0$ and $r = 1$, then Theorem 1.2 is transformed into Theorem 1.1 in [19].*

Furthermore, we consider the case of the "jumping" nonlinearity.

Theorem 1.3. *Assume (H1)–(H3) hold. When $\lambda > \dfrac{\lambda_k[c]}{f_0}$, there exist at least $2k-1$ non-trivial solutions of the boundary value problem (1.1). Indeed, there are solutions $w_1^-, \cdots, w_k^-$, such that, for each $1 \le j \le k$, w_j^- has exactly $j-1$ simple zeros on the open interval $(0, r)$ and is negative close to 0. Also, there are solutions $z_2^+, \cdots, z_k^+$ such that for each $2 \le j \le k$, z_j^+ has precisely $j-1$ simple zeros on the open interval $(0, r)$ and is positive close to 0.*

Remark 1.2. *If $m = 2$, $a(t) \equiv 1$, $f(u) = (u+1)^+ - 1$, then Theorem 1.3 is transformed into Theorem 1 in [10].*

Example 1.1. *Let $a(t) = \sin \dfrac{\pi}{r} t$ and $c < \dfrac{\pi^2}{r^2}$. We consider the existence of nodal solutions for sixth order boundary value problem*

$$\begin{cases} -u''''''(t) - cu''''(t) = \lambda \sin \dfrac{\pi}{r} t f(u), & t \in (0, r), \\ u(0) = u(r) = u''(0) = u''(r) = u''''(0) = u''''(r) = 0, \end{cases} \tag{1.6}$$

where

$$f(u) = \begin{cases} -4, & u < -2, \\ 2u, & -2 \le u \le 2, \\ 6u - 8, & u > 2. \end{cases}$$

It is easy to verify that

$$\lim_{u \to -\infty} \frac{f(u)}{u} = 0, \ \lim_{u \to +\infty} \frac{f(u)}{u} = 6, \ \lim_{u \to 0} \frac{f(u)}{u} = 2.$$

Then, the conditions of Theorem 1.3 are fulfilled. Therefore, when $\lambda > \dfrac{\lambda_k[c]}{f_0}$, there exist at least $2k-1$ non-trivial solutions of the boundary value problem (1.6). Indeed, there are solutions $w_1^-, \cdots, w_k^-$, such that, for each $1 \le j \le k$, w_j^- has exactly $j-1$ simple zeros on the open interval $(0, r)$ and is negative close to 0. Also, there are solutions $z_2^+, \cdots, z_k^+$ such that for each $2 \le j \le k$, z_j^+ has precisely $j-1$ simple zeros on the open interval $(0, r)$ and is positive close to 0.

The remainder of this paper is structured as follows. Section 2 is dedicated to demonstrating the spectrum results of the linear eigenvalue problem (1.5). In Section 3, we study the existence of nodal solutions for the nonlinear problem (1.1) under some suitable conditions via bifurcation theory.

2. Spectrum results and maximum principle

Definition 2.1. *[22] Let $p_k \in C[a,b]$ for $k = 1, ..., n$. A linear differential equation of order n*

$$Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{2.1}$$

is regarded as disconjugate on an interval $[a,b]$ if every non-trivial solution possesses less than n zeros on $[a,b]$, where multiple zeros are accounted for in accordance with their multiplicity.

Definition 2.2. *[22] The functions $y_1, \cdots, y_n \in C^n[a,b]$ are claimed to form a Markov system when the n Wronskians*

$$W_k := W[y_1, \cdots, y_k] = \begin{vmatrix} y_1 & \cdots & y_k \\ \cdots & \cdots & \cdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix}, \ (k = 1, \cdots, n)$$

are positive on $[a,b]$.

Lemma 2.1. *[22] Equation (2.1) possesses a Markov fundamental system of solutions on $[a,b]$ when and only when it is disconjugate on $[a,b]$.*

Lemma 2.2. *[22] Equation (2.1) possesses a Markov fundamental system of solutions when and only when L has a representation*

$$Ly \equiv v_1 v_2 \cdots v_n D \frac{1}{v_n} D \cdots D \frac{1}{v_2} D \frac{1}{v_1} y,$$

where $D = d/dt$, and

$$1 = W_0, \ v_1 = W_1, \ v_k = W_k W_{k-2}/W_{k-1}^2, \ (k = 2, \cdots, n).$$

Example 2.1. *For every $M \in (-m_0^4, m_1^4)$, the equation $u''''(t) + Mu(t) = 0$ is disconjugate on the interval $[0,1]$, where m_0, m_1 are the first positive solution of the equations*

$$\cos(m)\cosh(m) = 1, \quad \tanh\frac{m}{\sqrt{2}} = \tan\frac{m}{\sqrt{2}}$$

respectively. See [23] for details.

Example 2.2. *For every $M \in (-\infty, 2\pi)$, $u''' + Mu' = 0$ is disconjugate on $[0,1]$. See [24] for details.*

2.1. Spectrum results

Elias [21] developed a theory on the eigenvalue problem

$$\begin{cases} \mathcal{L}y + \lambda h(x)y = 0, \quad x \in [a,b], \\ (\mathcal{L}_i y)(a) = 0, \quad i \in \{i_1, \cdots, i_k\}, \\ (\mathcal{L}_j y)(b) = 0, \quad i \in \{j_1, \cdots, j_{n-k}\} \end{cases} \tag{2.2}$$

where

$$\begin{aligned} &\mathcal{L}_0 y = \rho_0 y \\ &\mathcal{L}_i y = \rho_i(\mathcal{L}_{i-1}y)', \quad i = 1, \cdots, n \\ &\mathcal{L}y = \mathcal{L}_n y \end{aligned}$$

and $\rho_i \in C^{n-i}[a,b]$ with $\rho_i > 0$ on $[a,b]$. $\mathcal{L}_0 y, \cdots, \mathcal{L}_{n-1}y$ are called the *quasi-derivatives* of $y(x)$. To apply Elias's spectrum theory, we have to prove that problem (1.1) can be rewritten in the form of (2.2), i.e., the linear operator

$$L[u] := u^{(2m)}(t) + cu^{(2m-2)}(t)$$

has a factorization of the form

$$L[u] = v_1 v_2 v_3 \cdots v_{2m}(\frac{1}{v_{2m}} \cdots (\frac{1}{v_3}(\frac{1}{v_2}(\frac{1}{v_1}u)')')' \cdots)'$$

on $[0,r]$, where $v_i \in C^{2m-i}[0,r]$ with $v_i > 0$ on $[0,r]$, and

$$u^{(2i)}(0) = u^{(2i)}(r) = 0, \ \ i = 0, 1, \cdots, m-1$$

if and only if

$$\mathcal{L}_i u(0) = \mathcal{L}_i u(r) = 0, \ \ i = 0, 1, \cdots, m-1.$$

This can be achieved under (H1) by using the disconjugacy theory in [22].

Define a linear differential equation of order $2m$

$$L[u] := u^{(2m)}(t) + cu^{(2m-2)}(t) = 0, \qquad t \in (0,r).$$

Theorem 2.1. *Let (H1) hold. Then,* $L[u]$ *has a factorization*

$$L[u] = v_1 v_2 v_3 \cdots v_{2m}(\frac{1}{v_{2m}} \cdots (\frac{1}{v_3}(\frac{1}{v_2}(\frac{1}{v_1}u)')')' \cdots)', \tag{2.3}$$

where $v_k \in C^{2m-k+1}[0,r]$ *with* $v_k > 0$ $(k = 1, 2, 3, \cdots, 2m)$.

Proof. Take $\phi(t)$ as the unique solution of the initial value problem

$$\begin{aligned} &\phi'' + c\phi = 0, \\ &\phi(0) = 0, \ \ \phi'(0) = 1. \end{aligned}$$

Subsequently, (H1) along with the Sturm Comparison Theorem leads to the conclusion that

$$\phi(t) > 0, \qquad t \in (0, r],$$

and therefore

$$\phi(t+\sigma) > 0, \qquad t \in [0, r],$$

if $\sigma > 0$ is small enough. Take

$$\begin{aligned} v_1 &= 1, & v_2 &= 1, \\ v_3 &= 1, & v_4 &= 1, \\ &\cdots & &\cdots \\ v_{2m-3} &= 1, & v_{2m-2} &= 1, \\ v_{2m-1} &= \phi(t+\sigma), & v_{2m} &= \frac{1}{\phi^2(t+\sigma)}. \end{aligned}$$

One can easily verify that (2.3) holds true. □

At present, we are able to compute

$$\mathcal{L}_0 u = \frac{1}{v_1} u = u$$

$$\mathcal{L}_1 u = \frac{1}{v_2} (\mathcal{L}_0 u)' = u'$$

$$\mathcal{L}_2 u = \frac{1}{v_3} (\mathcal{L}_1 u)' = u''$$

$$\cdots$$

$$\mathcal{L}_{2m-3} u = \frac{1}{v_{2m-2}} (\mathcal{L}_{2m-4} u)' = u^{(2m-3)}$$

$$\mathcal{L}_{2m-2} u = \frac{1}{v_{2m-1}} (\mathcal{L}_{2m-3} u)' = \frac{1}{\phi(t+\sigma)} u^{(2m-2)}$$

$$\mathcal{L}_{2m-1} u = \frac{1}{v_{2m}} (\mathcal{L}_{2m-2} u)' = u^{(2m-1)} \phi(t+\sigma) - u^{(2m-2)} \phi'(t+\sigma)$$

$$\mathcal{L}_{2m} u = v_1 v_2 v_3 \cdots v_{2m} (\mathcal{L}_{2m-1} u)' = u^{(2m)}(t) + c u^{(2m-2)}$$

Therefore, it is easy for us to reach the following conclusion.

Theorem 2.2. *Let (H1) hold. Then,*

$$u^{(2i)}(0) = u^{(2i)}(r) = 0, \;\; i = 0, 1, \cdots, m-1$$

is equivalent to

$$\mathcal{L}_i u(0) = \mathcal{L}_i u(r) = 0, \;\; i = 0, 1, \cdots, m-1.$$

Remark 2.1. *In [20], Rynne considered the boundary value problem*

$$Lu(x) = p(x)u(x) + g(x)u(x), \quad t \in (0, \pi).$$

He assumes that L is a 2mth-order, disconjugate, ordinary differential operator on $(0, \pi)$*, together with separated boundary conditions at* 0 *and* π*. However, we give a constructive proof (Theorem 2.1.) to obtain that* $u^{(2m)}(t) + cu^{(2m-2)}(t) = 0$ *is disconjugate on the interval* $[0, r]$ *if and only if* $c < \frac{\pi^2}{r^2}$ *in Theorem 2.1.*

Currently, we are capable of applying Elias's spectrum theory to investigate the eigenvalue problem.

$$\begin{cases} (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \lambda a(t)u, \quad t \in (0, r), \\ u^{(2i)}(0) = u^{(2i)}(r) = 0, \ \ i = 0, 1, \cdots, m-1. \end{cases} \tag{2.4}$$

The Proofs of Theorem 1.1. Items (i) to (iv) are direct consequences of Elias (as mentioned in Theorems 1–5 in reference [21]) as well as Theorems 2.1 and 2.2. We shall only provide a proof for (v). Let $Y = C[0, r]$ be the Banach space which is equipped with the norm

$$\|u\|_\infty = \max_{t \in [0,r]} |u(t)|.$$

Define $\bar{L} : D(\bar{L}) \to Y$ by setting

$$\bar{L}u := (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)), \qquad u \in D(\bar{L}), \tag{2.5}$$

where

$$D(\bar{L}) = \{u \in C^{2m}[0, r] : \ u^{(2i)}(0) = u^{(2i)}(r) = 0, \ \ i = 0, 1, \cdots, m-1\}.$$

For simplicity, we define $\lambda_k := \lambda_k[c]$. For demonstrating (v), it suffices to prove

$$\ker(\bar{L} - \lambda_k)^2 = \ker(\bar{L} - \lambda_k).$$

Clearly,

$$\ker(\bar{L} - \lambda_k)^2 \supseteq \ker(\bar{L} - \lambda_k).$$

Assume by contradiction that the algebraic multiplicity of λ_k is greater than 1. In that case, there exists a vector u that belongs to $\ker(\bar{L} - \lambda_k)^2$ but not to $\ker(\bar{L} - \lambda_k)$. Subsequently,

$$\bar{L}u - \lambda_k u = \gamma\varphi_k$$

for some $\gamma \neq 0$. After multiplying both sides of the aforementioned equation by φ_k and integrating from 0 to r, we can infer that

$$0 = \gamma \int_0^r [\varphi_k(t)]^2 dt,$$

which is a contradiction. □

Remark 2.2. *Let $a(t) = 1$. We consider the eigenvalue problem*

$$\begin{cases} (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \lambda u(t), \quad t \in (0, r), \\ u^{(2i)}(0) = u^{(2i)}(r) = 0, \;\; i = 0, 1, \cdots, m-1, \end{cases} \tag{2.6}$$

we see that the eigenvalues are $\lambda_k = \underbrace{\eta_k \cdots \eta_k}_{m-1}(\eta_k - c)$ for $k = 1, 2, \cdots$ with corresponding eigenfunctions $\varphi_k(t) = \sin kt,\ k = 1, 2, \cdots$ for $c < \dfrac{\pi^2}{r^2}$, where $\{\eta_k\}_1^\infty$ are the eigenvalues of the problem

$$\begin{cases} -y'' = \eta y, \quad t \in (0, r), \\ y(0) = y(r) = 0. \end{cases}$$

It is worth noting that the first eigenvalue $\eta_1 = \dfrac{\pi^2}{r^2}$. Therefore, the eigenvalues of problem (2.6) are simple, the eigenfunctions have only simple zeros on $(0, r)$, and the kth eigenfunction possesses exactly $k - 1$ simple zeros on the open interval $(0, r)$. If $m = 2$, then problem (2.6) reduces to the situation in [10].

2.2. Maximum principle

Theorem 2.3. *Suppose that condition (H1) is satisfied. Take $e \in C[0, r]$, $e \geq 0$ in $[0, r]$ with $e \not\equiv 0$ in $[0, r]$. If $u \in C^{2m}[0, r]$ satisfies*

$$\begin{cases} (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = e(t), \quad t \in (0, r), \\ u^{(2i)}(0) = u^{(2i)}(r) = 0, \;\; i = 0, 1, \cdots, m-1, \end{cases} \tag{2.7}$$

then $u(t) > 0$ in $(0, r)$.

Proof. Let $\mathcal{D}u = -u''$. Then,

$$(-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \underbrace{\mathcal{D} \cdots \mathcal{D}}_{m-1}(\mathcal{D} - cI)u.$$

Let $z_1 := \underbrace{\mathcal{D} \cdots \mathcal{D}}_{m-2}(\mathcal{D} - cI)u = (-1)^{m-1}(u^{(2m-2)}(t) + cu^{(2m-4)}(t))$. From (2.7), then we have that

$$-z_1'' = e, \quad z_1(0) = z_1(r) = 0,$$

which implies

$$z_1(t) > 0, \quad t \in (0, r).$$

Repeating this procedure $m - 1$ times, we obtain

$$-z_{m-1}'' = z_{m-2}, \quad z_{m-1}(0) = z_{m-1}(r) = 0,$$

which implies

$$z_{m-1}(t) > 0, \quad t \in (0, r),$$

where $z_{m-1} = (\mathcal{D} - cI)u$.

According to $c < \dfrac{\pi^2}{r^2}$, we get

$$-u'' - cu = z_{m-1}, \quad u(0) = u(r) = 0.$$

This together with $z_{m-1}(t) > 0$ implies that $u(t) > 0$ for $t \in (0, r)$. □

3. Existence of a nodal solutions

Let $E = \{u \in C^{2m-2}[0, r] : \ u^{(2i)}(0) = u^{(2i)}(r) = 0, \ \ i = 0, 1, \cdots, m-1\}$ be the Banach space which is equipped with the norm

$$\|u\|_E = \max_{t\in[0,r]}\{\|u^{(2i)}\|_\infty\}, \ \ i = 0, 1, \cdots, m-1.$$

Then, $\bar{L}^{-1} : Y \to E$ is completely continuous. Here, $\bar{L}$ is presented as in (2.5).

Take $\zeta, \ \xi \in C(\mathbb{R}, \mathbb{R})$ such that

$$f(u) = f_0 u + \zeta(u), \qquad f(u) = f_{+\infty} u^+ + \xi(u),$$

where $u^+ = \max\{u, 0\}$. Clearly,

$$\lim_{|u|\to 0} \frac{\zeta(u)}{u} = 0, \quad \lim_{|u|\to\infty} \frac{\xi(u)}{u} = 0. \tag{3.1}$$

Let

$$\tilde{\xi}(u) = \max_{0\le|s|\le u} |\xi(s)|.$$

Then, $\tilde{\xi}$ is nondecreasing and

$$\lim_{u\to\infty} \frac{\tilde{\xi}(u)}{u} = 0. \tag{3.2}$$

Let us consider

$$\bar{L}u - \lambda a(t) f_0 u = \lambda a(t)\zeta(u) \tag{3.3}$$

as a bifurcation problem originating from the trivial solution $u \equiv 0$.

Equation (3.3) can be equivalently transformed into

$$u(x) = \lambda \bar{L}^{-1}[a(\cdot) f_0 u(\cdot)](t) + \lambda \bar{L}^{-1}[a(\cdot)\zeta(u(\cdot))](t). \tag{3.4}$$

Obviously, the compactness of $\bar{L}^{-1}$ combined with (3.1) indicates that

$$\|\bar{L}^{-1}[a(\cdot)\zeta(u(\cdot))]\| = o(\|u\|_E) \quad \text{as } \|u\|_E \to 0.$$

In the following content, we adopt the terminology of Rabinowitz [25].

Let S_k^+ represent the set of functions in E that possess exactly $k-1$ interior nodal (i.e., nondegenerate) zeros within the interval $(0, r)$ and are positive in the vicinity of $t = 0$. Set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. These sets are disjoint and open in E. Let Σ denote the closure of the set consisting of nontrivial solutions of (3.4) in $\mathbb{R} \times E$.

Lemma 3.1. *Suppose that (H1)–(H3) (or (A2) and (A3)) are satisfied. If $u \in D(\bar{L})$ is a nontrivial solution of*

$$\begin{cases} (-1)^m(u^{(2m)}(t) + cu^{(2m-2)}(t)) = \lambda a(t)f(u), \quad t \in (0,r), \\ u^{(2i)}(0) = u^{(2i)}(r) = 0, \ \ i = 0, 1, \cdots, m-1, \end{cases} \tag{3.5}$$

then u has only simple zeros in $(0,r)$. Thus, by definition, $u \in S_k$.

Proof. In fact, (3.5) can be rewritten as

$$\bar{L}u = \lambda \hat{a}(t)u$$

where

$$\hat{a}(t) = \begin{cases} a(t)\frac{f(u(t))}{u(t)}, & \text{as } u(t) \neq 0, \\ a(t)f_0, & \text{as } u(t) = 0. \end{cases}$$

Obviously, $\hat{a}(t)$ meets (H2). Therefore, Lemma 2.2 of [20] implies that all zeros of u on the interval $(0,r)$ are simple. □

Remark 3.1. *We say that u is a nodal solution if all of zeros of the solution are simple. It is a challenging problem to prove that all zeros of the solution are simple, see [10, Lemma 2.2]. In this paper, a novel method is employed to prove that all zeros of the solution of problem (1.1) are simple, which is based on the well-known Uri Elias formula [20, 21], see Lemma 3.1.*

By the Rabinowitz global bifurcation theorem [25], there is a continuum $C_k \subset \Sigma$ of solutions for (3.4) bifurcating from $(\frac{\lambda_k}{f_0}, 0)$ which is either unbounded or contains a pair $(\frac{\lambda_j}{f_0}, 0)$ for $j \neq k$. By Lemma 1.24 of [25], it implies that if $(\lambda, u) \in C_k$ and is near $(\frac{\lambda_k}{f_0}, 0)$, $u = \alpha\varphi_k + \omega$ with $\omega = o(|\alpha|)$. Since $S_k^{\pm}$ is open and $\varphi_k \in S_k$, then

$$(C_k \setminus \{(\frac{\lambda_k}{f_0}, 0)\} \cap \mathbb{B}_\varepsilon(\frac{\lambda_k}{f_0}, 0)) \subset \mathbb{R} \times S_k,$$

for all positive ε small enough, where

$$\mathbb{B}_\varepsilon(\frac{\lambda_k}{f_0}, 0) = \{(\lambda, u) \in \mathbb{R} \times E : \|u\| + |\lambda - \frac{\lambda_k}{f_0}| < \varepsilon\}.$$

Define $\hat{C}_k = C_k - \mathbb{B}_\varepsilon(\frac{\lambda_k}{f_0}, 0)$. Then, Lemma 3.1 implies that

$$\hat{C}_k \subset \mathbb{R} \times S_k.$$

Otherwise, there exists $(\bar{\lambda}, \bar{u}) \in \hat{C}_k$ such that $\bar{\lambda} > 0$ and $\bar{u}$ is in the boundary of S_k. If $\bar{u} = 0$, then $\bar{\lambda} = \lambda_j$ for some $j \neq k$, and so all points in $\hat{C}_k^+$ near $(\lambda_j, 0)$ are in S_j, a contradiction. Hence, $\bar{u} \neq 0$. Since all the sets S_j, $j = 1, 2, \cdots$, are open, it follows that there is $t_0 \in [0, r]$ such that $\bar{u}(t_0) = \bar{u}'(t_0) = 0$. But, this contradicts Lemma 3.1. Consequently, one has that

$$C_k \subset (\mathbb{R} \times S_k \cup \{(\frac{\lambda_k}{f_0}, 0)\}).$$

It follows that C_k returning to the set of trivial solution axis is impossible. So, C_k is unbounded.

Furthermore, by Theorem 2 of [26], there exist two continua C_k^+ and C_k^- composed of the bifurcation branch C_k which satisfy that either C_k^+ and C_k^- are both unbounded or $C_k^+ \cap C_k^- \neq \{(\frac{\lambda_k}{f_0}, 0)\}$. We know that $u = \alpha\varphi_k + \omega$ for $(\lambda, u) \in C_k \setminus \{(\frac{\lambda_k}{f_0}, 0)\}$ near $(\frac{\lambda_k}{f_0}, 0)$. Since $\alpha\varphi_k \in S_k^{\pm}$ if $0 \neq \alpha \in \mathbb{R}^{\pm}$ or $\mathbb{R}^{\mp}$, we have that

$$(C_k^{\pm} \setminus \{(\frac{\lambda_1}{f_0}, 0)\} \cap \mathbb{B}_{\varepsilon}(\frac{\lambda_1}{f_0}, 0)) \subset \mathbb{R} \times S_k^{\pm}$$

for all positive ε small enough. Similar to the above argument, we are able to demonstrate that $C_k^{\pm} \setminus \{(\frac{\lambda_k}{f_0}, 0)\}$ cannot depart from $\mathbb{R} \times S_k^{\pm}$ outside a neighborhood of $(\frac{\lambda_k}{f_0}, 0)$. Therefore, we have that $C_k^{\pm} \subset (\mathbb{R} \times S_k^{\pm} \cup \{(\frac{\lambda_k}{f_0}, 0)\})$. It follows that both C_k^+ and C_k^- are unbounded. Otherwise, at the expense of generality, we suppose that C_k^- is bounded. Then, there is $(\lambda_*, u_*) \in C_k^+ \cap C_k^-$ such that $(\lambda_*, u_*) \neq (\frac{\lambda_k}{f_0}, 0)$ and $u_* \in S_k^+ \cap S_k^-$. This contradicts the definitions of S_k^+ and S_k^-.

The Proofs of Theorem 1.2. By applying a similar method as used to prove [19, Theorem 1.1], with appropriate and obvious modifications, we can obtain the desired result. □

The Proofs of Theorem 1.3. We only need to show that

$$C_j^- \cap (\{\lambda\} \times E) \neq \emptyset, \quad j = 1, 2, \cdots, k,$$

and

$$C_j^+ \cap (\{\lambda\} \times E) \neq \emptyset, \quad j = 2, \cdots, k.$$

Suppose on the contrary that

$$C_j^{\nu} \cap (\{\lambda\} \times E) = \emptyset \text{ for some } (j, \nu) \in \Gamma,$$

where

$$\Gamma := \{(j, \nu) \mid j \in \{2, \cdots, k\} \text{ as } \nu = +, \text{ and } j \in \{1, 2, \cdots, k\} \text{ as } \nu = -\}.$$

As C_j^{ν} joins $(\frac{\lambda_k}{f_0}, 0)$ to infinity in Σ, and since $(\lambda, u) = (0, 0)$ is the sole solution of (3.3) in E, there exists a sequence $\{(\mu_m, u_m)\} \subset C_j^{\nu}$ such that $\mu_m \in (0, \lambda)$ and $||u_m||_E \to \infty$ as $m \to \infty$. We may assume that $\mu_m \to \bar{\mu} \in [0, \lambda]$ as $m \to \infty$. Let $v_m = \frac{u_m}{||u_m||_E}, m \geq 1$. From the fact

$$\bar{L}u_m = \mu_m a(t) f_{+\infty}(u_m)^+ + \mu_m a(t)\xi(u_m),$$

we have that

$$v_m = \mu_m \bar{L}^{-1}(a(t) f_{+\infty}(v_m)^+) + \mu_m \bar{L}^{-1}(a(t)\frac{\xi(u_m)}{||u_m||_E}).$$

Therefore, since $\bar{L}^{-1} : E \to E$ is completely continuous, it can be assumed that there is $v \in E$ together with $||v||_E = 1$ such that $||v_m \to v||_E \to 0$ as $m \to \infty$. According to

$$\frac{|\xi(u_m)|}{||u_m||_E} \leq \frac{\tilde{\xi}(||u_m||_{\infty})}{||u_m||_E} \leq \frac{\tilde{\xi}(||u_m||_E)}{||u_m||_E},$$

and (3.2), we have that

$$v = \bar{\mu}\bar{L}^{-1}(a(t)f_{+\infty}v^{+}). \tag{3.6}$$

i.e.,

$$\begin{cases} (-1)^{m}(v^{(2m)}(t) + cv^{(2m-2)}(t)) = \bar{\mu}a(t)f_{+\infty}v^{+}, & t \in (0, r), \\ v^{(2i)}(0) = v^{(2i)}(r) = 0, \quad i = 0, 1, \cdots, m-1. \end{cases}$$

By (H1), (H2), (3.6), and the fact that $\|v\|_E = 1$, we conclude that $\bar{\mu}a(t)f_{+\infty}v^{+} \not\equiv 0$, and consequently

$$\bar{\mu} > 0, \quad v^{+} \not\equiv 0.$$

According to Theorem 2.3, it is known that $v(t) > 0$, $t \in (0, r)$.

By Theorem 2.3, we know that $v(t) > 0$ in $(0, r)$. This implies that $\bar{\mu}f_{+\infty}$ is the first eigenvalue of $\bar{L}u = \lambda a(t)u$ and v is the corresponding eigenfunction of λ_1. Hence, $v \in S_1^{+}$, and therefore, since S_1^{+} is open and $\|v - v_m\|_E \to 0$ as $m \to \infty$, we have that $v_m \in S_1^{+}$ for m large. But, this contradicts the assumption that $(\lambda_m, v_m) \in C_j^{\nu}$ and $(j, \nu) \in \Gamma$. This proves Theorem 1.3. □

Remark 3.2. *In this paper, we consider the "jumping" nonlinearity, i.e.,*

$$\lim_{s\to-\infty}\frac{f(s)}{s} = 0, \quad \lim_{s\to+\infty}\frac{f(s)}{s} = f_{+\infty}, \quad \lim_{s\to 0}\frac{f(s)}{s} = f_0$$

for some $f_{+\infty}, f_0 \in (0, \infty)$*, which complements the main result in [19].*

Remark 3.3. *For the existence result of solutions for (1.1) with* $m = 2$*,* $n \geq 1$ *(PDE case), see [27–30]. For other results on the Lidstone BVPs, see [31–33].*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This paper was supported by National Natural Science Foundation of China (No.12061064 and No.12361040). The authors would like to thank the editor and the anonymous reviewers for their helpful suggestions.

Conflict of interest

The authors declare there is no conflicts of interest.

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