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Research article

Global solutions to the Cauchy problem of BNSP equations in some classes of large data

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Abstract: In this paper, we obtained the global existence and large time behavior of the solution for bipolar Navier-Stokes-Poisson equations under the partially smallness assumption of the initial data. Due to the complexity of bipolar Navier-Stokes-Poisson equations, we chose Green's function method instead of the classical energy method and thus discussed the regularity criterion under decaying structures in time instead of only integrability of time variable. It made the whole proof more simple and clear, meanwhile, resulted in the large time decaying estimates of the solution. It also showed the advantage of Green's function method in the study of global existence in the large perturbation framework.

Keywords: regularity criterion; bootstrap argument; Green's function; global existence; decay rate of solution

1. Introduction

The bipolar Navier-Stokes-Poisson system has been used to simulate the transport of charged particles under the influence of electrostatic force governed by the self-consistent Poisson. In this paper, we are concerned with the Cauchy problem of the bipolar Navier-Stokes-Poisson system in 3 dimensions:

$$\begin{cases} \partial_t\rho_1 + div(\rho_1 u_1) = 0, \\ \partial_t u_1 + div(\rho_1 u_1 \otimes u_1) + \nabla P_1(\rho_1) = \mu_1 \Delta u_1 + \mu_2 \nabla div u_1 + \rho_1 \nabla\Phi, \\ \partial_t\rho_2 + div(\rho_2 u_2) = 0, \\ \partial_t u_2 + div(\rho_2 u_2 \otimes u_2) + \nabla P_2(\rho_2) = \mu_1 \Delta u_2 + \mu_2 \nabla div u_2 - \rho_2 \nabla\Phi, \\ \Delta\Phi = \rho_1 - \rho_2, \lim_{|x|\to\infty} \Phi(x,t) = 0, \end{cases} \tag{1.1}$$

with initial data

$$(\rho_1, u_1, \rho_2, u_2, \nabla\Phi)(x, 0) = (\rho_{1,0}, u_{1,0}, \rho_{2,0}, u_{2,0}, \nabla\Phi_0)(x), x \in \mathbb{R}^3. \tag{1.2}$$

Here $\rho_i(x,t), u_i(x,t), \Phi(x,t)$, and $P_i(\rho)((x,t)$ represent the fluid density, velocity, self-consistent electric potential and pressure. The viscosity coefficients satisfy the usual physical conditions $\mu_1 > 0,\ 3\mu_1 + 2\mu_2 > 0$. We assume that $P_i(\rho)$ satisfies $P_i'(\rho) > 0$ for all $\rho > 0$ and $P_i'(\bar{\rho}) = 1$, where $\overline{\rho} > 0$ denotes the prescribed density of positive charged background ions, and in this paper is taken as a positive constant. Without loss of generality, we take $\overline{\rho}$ to be 1. For the initial data $(\rho_{1,0}, u_{1,0}, \rho_{2,0}, u_{2,0})$, we consider small perturbations of $(\overline{\rho}, 0, \overline{\rho}, 0)$, in which $\overline{\rho}$ is defined as before and taken to be 1, and we assume that $\rho_{1,0}, \rho_{2,0}$ has positive lower bound and upper bound.

Now, we review some previous works on the Cauchy problem for some related models. There has been a lot of studies for the compressible Navier-Stokes system (CNS) for either isentropic or non-isentropic cases on the existence, stability, and L^p-decay rates with $p \geq 2$. For the results of small solutions, see [1, 2] and [3–7], where the authors use the (weighted) energy method together with spectrum analysis, and for the results of partial small solutions (under the setting that the initial data is of small energy but possibly large oscillations), see [8, 9] and the references therein. On the other hand, many scholars also use the method of Green's function to analyze the asymptotic behavior of a specific system, for example, by using the method of Green's function, Liu and Zeng [10] first studied the point-wise estimates of solutions to the general hyperbolic-parabolic equations in one dimension. Later, Liu and Wang [11] give the point-wise estimates of diffusion wave for the Navier-Stokes systems in odd multi-dimension and explain the generalized Huygens' principle for the Navier-Stokes systems.

For the unipolar Navier-Stokes-Poisson system (NSP), there is also a mass of results for the Cauchy problem when the initial data (ρ_0, u_0) is a small perturbation around the constant state $(\bar{\rho}, 0)$. For instance, the global existence of weak solutions was obtained by [12, 13]; the framework of Matsumura and Nishda [14,15] shows the global existence of small strong solutions in H^N Sobolev spaces. In [16,17], the authors obtain the global existence of small solutions in some Besov spaces. For the solutions which are of small energy but possibly large oscillations, see [18,19] and the references therein. In fact, the NSP system is a hyperbolic-parabolic system with a nonlocal term arising from the electric field $\nabla\Phi$. From the analysis of Green's function, the symbol of this nonlocal term is singular in the long wave of the Green's function and it destroys the time-decay rate for the velocity. As we know, for the CNS system, when the initial perturbation $\rho_0 - 1, u_0 \in L^p \cap H^N$, with p near 1, and $N \geq 3$ is a large enough integer for the nonlinear system, then the solutions have L^2 optimal decay rate

$$\|(\rho - 1, u)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}.$$

In [15,20,21], the authors survey the decay rate of solutions for the NSP system and they observed that the electric field destroys the decay rate of the solutions, i.e., when the initial perturbation $\rho_0 - 1, u_0 \in L^p \cap H^N$, with $p \in [1, 2]$, then the solutions have L^2 optimal decay rate

$$\|(\rho - 1)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})},\ \ \|u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}.$$

In [22], the author gives another comprehension toward the effect of the electric field on the decay rate of the solutions for the NSP system. The author believes that it is natural to assume that $\nabla\Phi_0 \in L^2$, and with this condition in hand, we can obtain the L^2 optimal decay rate for the linear NSP system as follows:

$$\|(\rho-1)(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}, \ \ \|u(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}.$$

In this sense, we see that the electric force enhances the decay rate of the density with the factor $\frac{1}{2}$ compared with the CNS system.

For the bipolar Navier-Stokes-Poisson system (BNSP), Wang and Xu [23] obtain the global existence of small solutions and the decay rate of the solutions. [24–26] observed the large time behavior of the BNSP system. The global existence of the solutions for the BNSP system under the partial smallness of the initial value is still an open problem, and the aim of this paper is to obtain the global existence of the solution to the system (1.1) and (1.2) provided that the initial data is partially small, which means that we require the initial value itself to be small and its derivative only to be bounded. In other words, its initial value is large in some classes, that is, its derivative can be large except for the initial value itself. In this paper, we first establish a regularity criterion to obtain the uniform boundedness of the solutions, and then combine abstract bootstrap argument to extend the local solutions. In order to prove the global existence of solution under the condition of initial data is partially small, the general regularity hypothesis requires that the solution is bounded with respect to time in the sense of some Sobolev norm, and another important condition is that the solution is integrable with respect to the time variable, which is necessary to obtain a consistent estimate of the solution through Gronwall's inequality.

However, in this paper, it is difficult to obtain the regularity condition of integrability due to the more complex nonlinear structure in BNSP equations. The classical regularity criterion is established based on the energy method and the decay property in time variable is usually captured by integrability. It is always difficult and not applicable for the bootstrap argument. In fact, in [27] for the global existence of solutions for shallow water equations with partially large initial data, we applied Green's function method and replaced the integrability condition by detailed decaying structure in time variable, which makes the bootstrap argument more clear and concise.

The linear and nonlinear structures of BNSP equations discussed in this paper are far more complicated, and we can imagine that it could be quite difficult and complex for the energy method. Even in the construction of Green's function, compared with the previous case for shallow water equations, BNSP equations are hyperbolic-parabolic-elliptic coupled ones and the structure is very complicated. The elliptic structure provides a nonlocal operator and also causes the lack of symmetry. These are all troubles we need to overcome in this paper.

Before we list the main result, we introduce some notations. Throughout this paper, ∂_t stands for the derivative with respect to time variable and $f_t = \partial_t f$. The symbol $\partial_i f (i = 1, 2, 3)$ means partial derivative with respect to x_i,

$$\partial_i f = \frac{\partial f}{\partial x_i}.$$

We use the notation $\nabla^k f$ to mean the partial derivative of order k. That is, if k is a nonnegative integer, then

$$\nabla^k f := \{\nabla^\alpha f | \alpha = (\alpha_1, \alpha_2, \alpha_3), |\alpha| = k\}$$

is a set of all partial derivatives of order k, endowed with the norm

$$\|\nabla^k f\|_{L^2}^2 = \sum_{|\alpha|=k} \|\nabla^\alpha f\|_{L^2}^2,$$

where

$$\nabla^\alpha f := \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} f, \ \ \alpha_i \geq 0.$$

Let Λ be a quasi-differential operator defined as follows:

$$\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}.$$

The main result of this paper is the following:

Theorem 1.1. *Let* $(\rho_{1,0}-1, u_{1,0}, \rho_{2,0}-1, u_{2,0}, \nabla\Phi_0) \in H^{s+1}(s \geq 4)$, *and*

$$\|(\rho_{1,0}-1, u_{1,0}, \rho_{2,0}-1, u_{2,0}, \nabla\Phi_0)\|_{L^2(\mathbb{R}^3)\bigcap L^1(\mathbb{R}^3)} \leq E_0,$$

where E_0 *is sufficiently small, then the Cauchy problem (1.1) and (1.2) has a global solution in time that satisfies*

$$(\rho_1-1, u_1, \rho_2-1, u_2) \in L^\infty([0,\infty); H^{s+1}), \ \ \nabla(u_1, u_2) \in L^2([0,\infty); H^{s+1}),$$

For $2 \leq p \leq +\infty$ *and* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| \leq s-1$, *it holds*

$$\|D^\alpha(\rho_1-1, u_1, \rho_2-1, u_2)(\cdot, t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}.$$

Remark 1.1. *In Theorem 1.1, we only assume the norm of the initial datum* $\rho_{1,0}, \rho_{2,0}, u_{1,0}, u_{2,0}$ *and* $\nabla\Phi_0$ *are small enough, but for the derivatives of* $\rho_{1,0}, \rho_{2,0}, u_{1,0}, u_{2,0}$, *and* $\nabla\Phi_0$, *we assume that they are bounded.*

The rest of this paper is organized as follows. In Section 2, we establish the uniform time estimate of solutions. In Section 3, we analyze the Green's function of the linear BNSP system, and the different properties of the Green function at high and low frequencies are obtained. In Section 4, we complete the partial proof of Theorem 1.1, i.e., we mainly obtain the existence of the solutions. In this paper, we also obtain the decay rate of the solutions, and the reader can see Sections 4 and 5 for details.

Throughout this paper, we denote by C a positive constant that varies from line to line.

2. Uniform estimation of solutions

We reformulate the Cauchy problem (1.1) and (1.2) about constant state $(1,0,1,0)$ as follows:

$$\begin{cases} \partial_t\rho_1 + divu_1 = -div(\rho_1 u_1), \\ \partial_t u_1 - \mu_1\Delta u_1 - \mu_2\nabla divu_1 + \nabla\rho_1 - \nabla\Phi \\ = -u_1\cdot\nabla u_1 - (\frac{P_1'(1+\rho_1)}{1+\rho_1} - 1)\nabla\rho_1 - \frac{\rho_1}{1+\rho_1}(\mu_1\Delta u_1 + \mu_2\nabla divu_1), \\ \partial_t\rho_2 + divu_2 = -div(\rho_2 u_2), \\ \partial_t u_2 - \mu_1\Delta u_2 - \mu_2\nabla divu_2 + \nabla\rho_2 + \nabla\Phi \\ = -u_2\cdot\nabla u_2 - (\frac{P_2'(1+\rho_2)}{1+\rho_2} - 1)\nabla\rho_2 - \frac{\rho_2}{1+\rho_2}(\mu_1\Delta u_2 + \mu_2\nabla divu_2), \\ \Delta\Phi = \rho_1 - \rho_2, \lim_{|x|\to\infty}\Phi(x,t) = 0, \end{cases} \tag{2.1}$$

where we also note $(\rho_1 - 1, u_1, \rho_2 - 1, u_2)$ as $(\rho_1, u_1, \rho_2, u_2)$ without causing confusion. In this section, we will get the estimates of the solutions for the system (2.1) under the assumption that for any fixed $0 < T < +\infty,\ t \in [0, T]$,

$$(\|\nabla\rho_1(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla u_1(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla\rho_2(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla u_2(t)\|_{L^\infty(\mathbb{R}^3)}) \le C(1+t)^{-2}, \tag{2.2}$$

and

$$\|(\rho_{1,0}, \rho_{2,0}, u_{1,0}, u_{2,0}, \nabla\Phi_0)\|_{L^2 \cap L^1} \le E_0, \tag{2.3}$$

where E_0 is a positive constant which is sufficiently small. Similar to reference [28, 29], we call Eq (2.2) the regularity criterion. In this section, based on this regularity criterion, we get a consistent estimate of the solutions of the equations and its derivatives.

We first define

$$F_1(\rho) = \frac{P_1'(1+\rho)}{1+\rho} - 1, \quad F_2(\rho) = \frac{P_2'(1+\rho)}{1+\rho} - 1, \quad H(\rho) = \frac{\rho}{1+\rho}.$$

To do this end, from (2.2), (2.3), and Gagliardo-Nirenberg's inequality, we know that there exists a sufficiently small $\varepsilon_1 > 0$ that satisfies

$$\|\rho_1\|_{L^\infty(\mathbb{R}^3)} \le \varepsilon_1 \text{ and } \|\rho_2\|_{L^\infty(\mathbb{R}^3)} \le \varepsilon_1. \tag{2.4}$$

First of all, from (2.4) , we obtain

$$\frac{2}{3} \le 1 + \rho_1 \le \frac{4}{3}, \quad \frac{2}{3} \le 1 + \rho_2 \le \frac{4}{3}.$$

Hence, by the definition of $F_1(\rho), F_2(\rho)$, and $H(\rho)$, we immediately have

$$\begin{aligned} &|F_1(\rho_1)|, |H(\rho_1)| \le C|\rho_1|, \quad |F_2(\rho_2)|, |H(\rho_2)| \le C|\rho_2|, \\ &|F_1^{(k)}(\rho_1)|, |F_2^{(k)}(\rho_2)|, |H^{(k)}(\rho_1)|, |H^{(k)}(\rho_2)| \le C \ \ for\ any\ k \ge 1. \end{aligned} \tag{2.5}$$

Let us start with a lemma that will be frequently used later.

Lemma 2.1. *[22] Assume that $\|\rho\|_{L^\infty(\mathbb{R}^3)} \le 1$, and $f(\rho)$ is a smooth function of ρ with bounded derivatives of any order, then for any integer $k \ge 1$, we have*

$$\|\nabla^k(f(\rho))\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla^k\rho\|_{L^\infty(\mathbb{R}^3)}.$$

We obtain the estimates of the low order derivatives of the system (2.1) first.

Lemma 2.2. *Under the assumption (2.2) and (2.4), we have*

$$\|(\rho_1, u_1, \rho_2, u_2, \nabla\Phi)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}, \|(\nabla u_1, \nabla u_2)\|_{L^2(0,T;L^2(\mathbb{R}^3))} \le C,$$

where C is a constant depending on $\|(\rho_{1,0}, \rho_{2,0}, u_{1,0}, u_{2,0}, \nabla\Phi_0)\|_{L^2}$.

Proof. Multiplying the Eqs $(2.1)_1$, $(2.1)_2$, $(2.1)_3$, and $(2.1)_4$ by ρ_1, u_1, ρ_2, u_2, respectively, and integrating the equations over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t \int_{\mathbb{R}^3} (\rho_1^2 + |u_1|^2 + \rho_2^2 + |u_2|^2)dx \\
&+ \mu_1 \int_{\mathbb{R}^3} (|\nabla u_1|^2 + |\nabla u_2|^2)dx + \mu_2 \int_{\mathbb{R}^3} (|divu_1|^2 + |divu_2|^2)dx \\
&= - \int_{\mathbb{R}^3} \nabla\Phi(u_2 - u_1)dx - \int_{\mathbb{R}^3} div(\rho_1 u_1)\rho_1 dx - \int_{\mathbb{R}^3} div(\rho_2 u_2)\rho_2 dx - \int_{\mathbb{R}^3} u_1 \cdot \nabla u_1 \cdot u_1 dx \\
&- \int_{\mathbb{R}^3} u_2 \cdot \nabla u_2 \cdot u_2 dx - \int_{\mathbb{R}^3} F_1(\rho_1)u_1 \cdot \nabla\rho_1 dx - \int_{\mathbb{R}^3} F_2(\rho_2)u_2 \cdot \nabla\rho_2 dx \\
&- \int_{\mathbb{R}^3} H(\rho_1)(\mu_1\Delta u_1 + \mu_2\nabla divu_1)u_1 dx - \int_{\mathbb{R}^3} H(\rho_2)(\mu_1\Delta u_2 + \mu_2\nabla divu_2)u_2 dx \\
&=: \sum_{i=1}^{9} A_i.
\end{aligned} \tag{2.6}$$

Now we estimate A_i one by one. Because the self-consistent potential $\Phi(x,t)$ is coupled with the density through the Poisson equation, using Hölder's inequality and Cauchy's inequality, for A_1, it holds

$$\begin{aligned}
A_1 &= - \int_{\mathbb{R}^3} \nabla\Phi(u_2 - u_1)dx \\
&= - \int_{\mathbb{R}^3} \Phi div(u_1 - u_2)dx \\
&= - \int_{\mathbb{R}^3} \Phi(-\partial_t\rho_1 - div(\rho_1 u_1) + \partial_t\rho_2 + div(\rho_2 u_2))dx \\
&= \int_{\mathbb{R}^3} \Phi\partial_t(\rho_1 - \rho_2)dx + \int_{\mathbb{R}^3} \Phi(div(\rho_1 u_1) - div(\rho_2 u_2))dx \\
&= \int_{\mathbb{R}^3} \Phi\partial_t\Delta\Phi dx + \int_{\mathbb{R}^3} \Phi(div(\rho_1 u_1) - div(\rho_2 u_2))dx \\
&= -\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\nabla\Phi|^2 dx - \int_{\mathbb{R}^3} \nabla\Phi(\rho_1 u_1 - \rho_2 u_2))dx \\
&\leq -\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\nabla\Phi|^2 dx + (\|\nabla\Phi\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2)(\|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}).
\end{aligned} \tag{2.7}$$

For A_4 and A_5, by Hölder's inequality, we easily check that

$$|A_4| + |A_5| \leq \|\nabla u_1\|_{L^\infty}\|u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty}\|u_2\|_{L^2}^2. \tag{2.8}$$

Integrating by parts, and using Hölder's inequality and Cauchy's inequality, it holds

$$|A_2| + |A_3| \leq \|\nabla\rho_1\|_{L^\infty}(\|u_1\|_{L^2}^2 + \|\rho_1\|_{L^2}^2) + \|\nabla\rho_2\|_{L^\infty}(\|u_2\|_{L^2}^2 + \|\rho_2\|_{L^2}^2). \tag{2.9}$$

By the definition of $F_1(\rho)$ and $F_2(\rho)$, using Hölder's inequality and Cauchy's inequality, and from (2.5), we have

$$|A_6| + |A_7| \leq \|\nabla\rho_1\|_{L^\infty}(\|u_1\|_{L^2}^2 + \|\rho_1\|_{L^2}^2) + \|\nabla\rho_2\|_{L^\infty}(\|u_2\|_{L^2}^2 + \|\rho_2\|_{L^2}^2). \tag{2.10}$$

Also integrating by parts, using Hölder's inequality and Cauchy's inequality, and by (2.5) and Lemma (2.1), we have

$$
\begin{aligned}
|A_8|+|A_9| &= |-\int_{\mathbb{R}^3}\nabla H(\rho_1)(\mu_1\nabla u_1+\mu_2 divu_1)u_1dx-\int_{\mathbb{R}^3}H(\rho_1)(\mu_1\nabla u_1+\mu_2 divu_1)\nabla u_1dx|\\
&+|-\int_{\mathbb{R}^3}\nabla H(\rho_2)(\mu_1\nabla u_2+\mu_2 divu_2)u_2dx-\int_{\mathbb{R}^3}H(\rho_2)(\mu_1\nabla u_2+\mu_2 divu_2)\nabla u_2dx|\\
&\le \|\nabla\rho_1\|_{L^\infty}^2\|u_1\|_{L^2}^2+\epsilon\|\nabla u_1\|_{L^2}^2+\|\nabla\rho_2\|_{L^\infty}^2\|u_2\|_{L^2}^2+\epsilon\|\nabla u_2\|_{L^2}^2,
\end{aligned}
\tag{2.11}
$$

where we take ϵ small enough such that $\epsilon \ll 1$. Plugging the estimates for A_1–A_9, i.e., (2.7)–(2.11) into (2.6), we get

$$
\begin{aligned}
&\frac{1}{2}\partial_t\int_{\mathbb{R}^3}(\rho_1^2+|u_1|^2+\rho_2^2+|u_2|^2+|\nabla\Phi|^2)dx\\
&+\mu_1\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2)dx+\mu_2\int_{\mathbb{R}^3}(|divu_1|^2+|divu_2|^2)dx\\
&\le(\|\rho_1\|_{L^\infty}+\|\rho_2\|_{L^\infty})(\|\nabla\Phi\|_{L^2}^2+\|u_1\|_{L^2}^2+\|u_2\|_{L^2}^2)\\
&+(\|\nabla\rho_1\|_{L^\infty}+\|\nabla\rho_1\|_{L^\infty}^2+\|\nabla u_1\|_{L^\infty})(\|u_1\|_{L^2}^2+\|\rho_1\|_{L^2}^2)\\
&+(\|\nabla\rho_2\|_{L^\infty}+\|\nabla\rho_2\|_{L^\infty}^2+\|\nabla u_2\|_{L^\infty})(\|u_2\|_{L^2}^2+\|\rho_2\|_{L^2}^2).
\end{aligned}
\tag{2.12}
$$

Using Gronwall's inequality, we complete the proof of the lemma. □

In the following, we would like to give the high regularity estimates of the solutions.

Lemma 2.3. *Under the assumption (2.2) and (2.4), we have*

$$
\|(\nabla\rho_1,\nabla u_1,\nabla\rho_2,\nabla u_2,\nabla^2\Phi)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))},\|(\nabla^2u_1,\nabla^2u_2)\|_{L^2(0,T;L^2(\mathbb{R}^3))}\le C,
$$

where C is a constant depending on $\|(\nabla\rho_{1,0},\nabla\rho_{2,0},\nabla u_{1,0},\nabla u_{2,0},\nabla^2\Phi_0)\|_{L^2}$.

Proof. We operate each equation of (2.1) with operator ∇ to derive

$$
\begin{cases}
\partial_t\nabla\rho_1+\nabla divu_1=-\nabla div(\rho_1u_1),\\
\partial_t\nabla u_1-\mu_1\nabla\Delta u_1-\mu_2\nabla\nabla divu_1+\nabla\nabla\rho_1-\nabla\nabla\Phi\\
=-\nabla(u_1\cdot\nabla u_1)-\nabla((\frac{P_1'(1+\rho_1)}{1+\rho_1}-1)\nabla\rho_1)-\nabla(\frac{\rho_1}{1+\rho_1}(\mu_1\Delta u_1+\mu_2\nabla divu_1)),\\
\partial_t\nabla\rho_2+\nabla divu_2=-\nabla div(\rho_2u_2),\\
\partial_t\nabla u_2-\mu_1\nabla\Delta u_2-\mu_2\nabla\nabla divu_2+\nabla\nabla\rho_2+\nabla\nabla\Phi\\
=-\nabla(u_2\cdot\nabla u_2)-\nabla((\frac{P_2'(1+\rho_2)}{1+\rho_2}-1)\nabla\rho_2)-\nabla(\frac{\rho_2}{1+\rho_2}(\mu_1\Delta u_2+\mu_2\nabla divu_2)),
\end{cases}
\tag{2.13}
$$

and multiplying the above equations by $\nabla\rho_1,\nabla u_1,\nabla\rho_2, and \nabla u_2$, respectively, and integrating over $\mathbb{R}^3$

yields

$$
\begin{aligned}
&\frac{1}{2}\partial_t \int_{\mathbb{R}^3} (|\nabla\rho_1|^2 + |\nabla u_1|^2 + |\nabla\rho_2|^2 + |\nabla u_2|^2)dx + \mu_1 \int_{\mathbb{R}^3} (|\nabla^2 u_1|^2 + |\nabla^2 u_2|^2)dx \\
&+ \mu_2 \int_{\mathbb{R}^3} (|\nabla divu_1|^2 + |\nabla divu_2|^2)dx + \int_{\mathbb{R}^3} \nabla^2\Phi\nabla(u_2 - u_1)dx \\
&= -\int_{\mathbb{R}^3} \nabla div(\rho_1 u_1)\nabla\rho_1 dx - \int_{\mathbb{R}^3} \nabla div(\rho_2 u_2)\nabla\rho_2 dx - \int_{\mathbb{R}^3} \nabla(u_1\cdot\nabla u_1)\nabla u_1 dx \\
&- \int_{\mathbb{R}^3} \nabla(u_2\cdot\nabla u_2)\nabla u_2 dx - \int_{\mathbb{R}^3} \nabla(F_1(\rho_1)\nabla\rho_1)\nabla u_1 dx - \int_{\mathbb{R}^3} \nabla(F_2(\rho_2)\nabla\rho_2)\nabla u_2 dx \\
&- \int_{\mathbb{R}^3} \nabla(H(\rho_1)(\mu_1\Delta u_1 + \mu_2\nabla divu_1))\nabla u_1 dx - \int_{\mathbb{R}^3} \nabla(H(\rho_2)(\mu_1\Delta u_2 + \mu_2\nabla divu_2))\nabla u_2 dx \\
&=: \sum_{i=1}^{8} B_i.
\end{aligned} \tag{2.14}
$$

For the last term on the left-hand side of (2.14), since $\Phi(x,t)$ satisfies the Poisson equation, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^2\Phi\nabla(u_2 - u_1)dx \\
&= \int_{\mathbb{R}^3} \nabla\Phi\nabla div(u_1 - u_2)dx \\
&= -\int_{\mathbb{R}^3} \nabla^2\Phi(-\partial_t(\rho_1 - \rho_2) + div(\rho_2 u_2 - \rho_1 u_1)dx \\
&= \frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\nabla^2\Phi|^2 dx + \int_{\mathbb{R}^3} \nabla^2\Phi div(\rho_1 u_1 - \rho_2 u_2)dx.
\end{aligned} \tag{2.15}
$$

For the term $\int_{\mathbb{R}^3} \nabla^2\Phi div(\rho_1 u_1 - \rho_2 u_2)dx$, using Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^3} \nabla^2\Phi div(\rho_1 u_1 - \rho_2 u_2)dx| \\
&\le \|\rho_1\|_{L^\infty}(\|\nabla u_1\|_{L^2}^2 + \|\nabla^2\Phi\|_{L^2}^2) + \|u_1\|_{L^\infty}(\|\nabla\rho_1\|_{L^2}^2 + \|\nabla^2\Phi\|_{L^2}^2) \\
&+ \|\rho_2\|_{L^\infty}(\|\nabla u_2\|_{L^2}^2 + \|\nabla^2\Phi\|_{L^2}^2) + \|u_2\|_{L^\infty}(\|\nabla\rho_2\|_{L^2}^2 + \|\nabla^2\Phi\|_{L^2}^2).
\end{aligned} \tag{2.16}
$$

Now, we estimate each term on the righthand side of (2.14). Hölder's inequality and Young's inequality gives

$$
\begin{aligned}
|B_1| &\le \int_{\mathbb{R}^3} |\rho_1||\nabla^2 u_1||\nabla\rho_1|dx + \int_{\mathbb{R}^3} |\nabla u_1||\nabla\rho_1|^2 dx \\
&\le C(\epsilon)\|\rho_1\|_{L^\infty}^2\|\nabla\rho_1\|_{L^2}^2 + \epsilon\|\nabla^2 u_1\|_{L^2}^2 + \|\nabla u_1\|_{L^\infty}\|\nabla\rho_1\|_{L^2}^2,
\end{aligned} \tag{2.17}
$$

where ϵ is a positive number that is small enough to be determined, as ϵ appears in the following inequalities. Similar to the estimate of B_1, we obtain

$$
\begin{aligned}
|B_2| &\le \int_{\mathbb{R}^3} |\rho_2||\nabla^2 u_2||\nabla\rho_2|dx + \int_{\mathbb{R}^3} |\nabla u_2||\nabla\rho_2|^2 dx \\
&\le C(\epsilon)\|\rho_2\|_{L^\infty}^2\|\nabla\rho_2\|_{L^2}^2 + \epsilon\|\nabla^2 u_2\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty}\|\nabla\rho_2\|_{L^2}^2.
\end{aligned} \tag{2.18}
$$

Simple computation gives

$$|B_3| + |B_4| \le \|\nabla u_1\|_{L^\infty}\|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty}\|\nabla u_2\|_{L^2}^2.$$

By the definition of F_1 and F_2, integrating by parts, and using Hölder's inequality and Young's inequality, from (2.5) we have

$$\begin{aligned}|B_5| + |B_6| &= |-\int_{\mathbb{R}^3} f_1(\rho_1)\nabla\rho_1 div\nabla u_1 dx| + |-\int_{\mathbb{R}^3} f_2(\rho_2)\nabla\rho_2 div\nabla u_2 dx| \\ &\le C(\epsilon)\|\rho_1\|_{L^\infty}^2\|\nabla\rho_1\|_{L^2}^2 + \epsilon\|\nabla^2 u_1\|_{L^2}^2 + C(\epsilon)\|\rho_2\|_{L^\infty}^2\|\nabla\rho_2\|_{L^2}^2 + \epsilon\|\nabla^2 u_2\|_{L^2}^2.\end{aligned} \tag{2.19}$$

Integrating by parts, by (2.5) and Lemma 2.1, we obtain

$$\begin{aligned}&|B_7| + |B_8| \\ &= |-\mu_1\int_{\mathbb{R}^3} H(\rho_1)(|\nabla^2 u_1|^2 + \nabla^2 u_1\nabla divu_1)dx| + |-\mu_2\int_{\mathbb{R}^3} H(\rho_2)(|\nabla^2 u_2|^2 + \nabla^2 u_2\nabla divu_2)dx| \\ &\le \epsilon(\|\nabla^2 u_1\|_{L^2}^2 + \|\nabla divu_2\|_{L^2}^2).\end{aligned} \tag{2.20}$$

Consequently, by (2.14)–(2.20) and taking $\epsilon \ll 1$, we deduce

$$\begin{aligned}&\frac{1}{2}\partial_t\int_{\mathbb{R}^3}(|\nabla\rho_1|^2 + |\nabla u_1|^2 + |\nabla\rho_2|^2 + |\nabla u_2|^2 + |\nabla^2\Phi|^2)dx \\ &+ \mu_1\int_{\mathbb{R}^3}(|\nabla^2 u_1|^2 + |\nabla^2 u_2|^2)dx + \mu_2\int_{\mathbb{R}^3}(|\nabla divu_1|^2 + |\nabla divu_2|^2)dx \\ \le&(\|\rho_1\|_{L^\infty}^2 + \|\nabla u_1\|_{L^\infty} + \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty})\|\nabla\rho_1\|_{L^2}^2 \\ &+ (\|\rho_2\|_{L^\infty}^2 + \|\nabla u_2\|_{L^\infty} + \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty})\|\nabla\rho_2\|_{L^2}^2 \\ &+ (\|\nabla u_1\|_{L^\infty} + \|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty})\|\nabla u_1\|_{L^2}^2 \\ &+ (\|\nabla u_2\|_{L^\infty} + \|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty})\|\nabla u_2\|_{L^2}^2 \\ &+ (\|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty} + \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty})\|\nabla^2\Phi\|_{L^2}^2.\end{aligned} \tag{2.21}$$

With the help of Gronwall's inequality, we complete the proof of the lemma. □

Lemma 2.4. *Under the assumption (2.2) and (2.4), we have*

$$\|(\nabla^2\rho_1, \nabla^2 u_1, \nabla^2\rho_2, \nabla^2 u_2, \nabla^3\Phi)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}, \|(\nabla^3 u_1, \nabla^3 u_2)\|_{L^2(0,T;L^2(\mathbb{R}^3))} \le C,$$

where C is a constant depending on $\|(\nabla^2\rho_{1,0}, \nabla^2\rho_{2,0}, \nabla^2 u_{1,0}, \nabla^2 u_{2,0}, \nabla^3\Phi_0)\|_{L^2}$.

Proof. Operating ∇^2 on each equation of (2.1) gives

$$\begin{cases}\partial_t\nabla^2\rho_1 + \nabla^2 divu_1 = -\nabla^2 div(\rho_1 u_1), \\ \partial_t\nabla^2 u_1 - \mu_1\nabla^2\Delta u_1 - \mu_2\nabla^2\nabla divu_1 + \nabla^2\nabla\rho_1 - \nabla^2\nabla\Phi \\ = -\nabla^2(u_1\cdot\nabla u_1) - \nabla^2((\frac{P_1'(1+\rho_1)}{1+\rho_1} - 1)\nabla\rho_1) - \nabla^2(\frac{\rho_1}{1+\rho_1}(\mu_1\Delta u_1 + \mu_2\nabla divu_1)), \\ \partial_t\nabla^2\rho_2 + \nabla^2 divu_2 = -\nabla^2 div(\rho_2 u_2), \\ \partial_t\nabla^2 u_2 - \mu_1\nabla^2\Delta u_2 - \mu_2\nabla^2\nabla divu_2 + \nabla^2\nabla\rho_2 + \nabla^2\nabla\Phi \\ = -\nabla^2(u_2\cdot\nabla u_2) - \nabla^2((\frac{P_2'(1+\rho_2)}{1+\rho_2} - 1)\nabla\rho_2) - \nabla^2(\frac{\rho_2}{1+\rho_2}(\mu_1\Delta u_2 + \mu_2\nabla divu_2)),\end{cases} \tag{2.22}$$

and multiplying the above equations by $\nabla^2\rho_1, \nabla^2 u_1, \nabla^2\rho_2, and\ \nabla^2 u_2$, respectively, and integrating over $\mathbb{R}^3$ gives

$$\begin{aligned}
&\frac{1}{2}\partial_t\int_{\mathbb{R}^3}(|\nabla^2\rho_1|^2+|\nabla^2 u_1|^2+|\nabla^2\rho_2|^2+|\nabla^2 u_2|^2)dx+\mu_1\int_{\mathbb{R}^3}(|\nabla^3 u_1|^2+|\nabla^3 u_2|^2)dx\\
&+\mu_2\int_{\mathbb{R}^3}(|\nabla^2 divu_1|^2+|\nabla^2 divu_2|^2)dx+\int_{\mathbb{R}^3}\nabla^2\nabla\Phi\nabla^2(u_2-u_1)dx\\
=&-\int_{\mathbb{R}^3}\nabla^2 div(\rho_1 u_1)\nabla^2\rho_1 dx-\int_{\mathbb{R}^3}\nabla^2 div(\rho_2 u_2)\nabla^2\rho_2 dx-\int_{\mathbb{R}^3}\nabla^2(u_1\cdot\nabla u_1)\nabla^2 u_1 dx\\
&-\int_{\mathbb{R}^3}\nabla^2(u_2\cdot\nabla u_2)\nabla^2 u_2 dx-\int_{\mathbb{R}^3}\nabla^2(F_1(\rho_1)\nabla\rho_1)\nabla^2 u_1 dx-\int_{\mathbb{R}^3}\nabla^2(F_2(\rho_2)\nabla\rho_2)\nabla^2 u_2 dx\\
&-\int_{\mathbb{R}^3}\nabla^2(H(\rho_1)(\mu_1\Delta u_1+\mu_2\nabla divu_1))\nabla^2 u_1 dx-\int_{\mathbb{R}^3}\nabla^2(H(\rho_2)(\mu_1\Delta u_2+\mu_2\nabla divu_2))\nabla^2 u_2 dx\\
=:&\sum_{i=1}^{8}C_i.
\end{aligned}\tag{2.23}$$

Similar to the proof of Lemma 2.3, for the last term on the left-hand side of (2.23), we get

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^2\nabla\Phi\nabla^2(u_2-u_1)dx\\
=&\int_{\mathbb{R}^3}\nabla^2\Phi\nabla^2 div(u_1-u_2)dx\\
=&\int_{\mathbb{R}^3}\nabla^2\Phi\nabla^2(-\partial_t(\rho_1-\rho_2)+div(\rho_2 u_2-\rho_1 u_1))dx\\
=&-\int_{\mathbb{R}^3}\nabla^2\Phi\nabla^2\partial_t\nabla^2\Phi dx+\int_{\mathbb{R}^3}\nabla^3\Phi\nabla^2(\rho_1 u_1-\rho_2 u_2)dx\\
=&\frac{1}{2}\partial_t\int_{\mathbb{R}^3}|\nabla^3\Phi|^2 dx+\int_{\mathbb{R}^3}\nabla^3\Phi\nabla^2(\rho_1 u_1-\rho_2 u_2)dx.
\end{aligned}\tag{2.24}$$

For the term $\int_{\mathbb{R}^3}\nabla^3\Phi\nabla^2(\rho_1 u_1-\rho_2 u_2)dx$, we can easily check that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^3\Phi\nabla^2(\rho_1 u_1-\rho_2 u_2)dx\\
\le&(\|\rho_1\|_{L^\infty}+\|u_1\|_{L^\infty})(\|\nabla^3\Phi\|_{L^2}^2+\|\nabla^2\rho_1\|_{L^2}^2+\|\nabla^2 u_1\|_{L^2}^2)+\|\nabla\rho_1\|_{L^\infty}(\|\nabla^3\Phi\|_{L^2}^2+\|\nabla u_1\|_{L^2}^2)\\
&+(\|\rho_2\|_{L^\infty}+\|u_2\|_{L^\infty})(\|\nabla^3\Phi\|_{L^2}^2+\|\nabla^2\rho_2\|_{L^2}^2+\|\nabla^2 u_2\|_{L^2}^2)+\|\nabla\rho_2\|_{L^\infty}(\|\nabla^3\Phi\|_{L^2}^2+\|\nabla u_2\|_{L^2}^2).
\end{aligned}\tag{2.25}$$

Now we estimate C_i. We hereby declare that ϵ occurring in the following inequalities is a sufficiently small positive number to be determined. To begin, it is easy to check that

$$\begin{aligned}
|C_3|+|C_4|=&|-\int_{\mathbb{R}^3}\nabla(u_1\cdot\nabla u_1)\nabla^3 u_1 dx|+|-\int_{\mathbb{R}^3}\nabla(u_2\cdot\nabla u_2)\nabla^3 u_2 dx|\\
\le&\epsilon\|\nabla^3 u_1\|_{L^2}^2+C(\epsilon)\|\nabla u_1\|_{L^\infty}^2\|\nabla u_1\|_{L^2}^2+\|\nabla u_1\|_{L^\infty}\|\nabla^2 u_1\|_{L^2}^2\\
&+\epsilon\|\nabla^3 u_2\|_{L^2}^2+C(\epsilon)\|\nabla u_2\|_{L^\infty}^2\|\nabla u_2\|_{L^2}^2+\|\nabla u_2\|_{L^\infty}\|\nabla^2 u_2\|_{L^2}^2.
\end{aligned}\tag{2.26}$$

Using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}|C_1| =&|\int_{\mathbb{R}^3}\nabla^2(\rho_1 divu_1+u_1\cdot\nabla\rho_1)\nabla^2\rho_1dx|\\ \le&\int_{\mathbb{R}^3}|\nabla^2\rho_1||\nabla u_1||\nabla^2\rho_1|dx+\int_{\mathbb{R}^3}\rho_1|\nabla^3u_1||\nabla^2\rho_1|dx+\int_{\mathbb{R}^3}|\nabla\rho_1||\nabla^2u_1||\nabla^2\rho_1|dx\\ \le&\|\nabla u_1\|_{L^\infty}\|\nabla^2\rho_1\|_{L^2}^2+\epsilon\|\nabla^3u_1\|_{L^2}^2+C(\epsilon)\|\rho_1\|_{L^\infty}^2\|\nabla^2\rho_1\|_{L^2}^2\\ &+\|\nabla\rho_1\|_{L^\infty}(\|\nabla^2\rho_1\|_{L^2}^2+\|\nabla^2u_1\|_{L^2}^2).\end{aligned} \tag{2.27}$$

Similarly, we also have

$$\begin{aligned}|C_2| \le&\|\nabla u_2\|_{L^\infty}\|\nabla^2\rho_2\|_{L^2}^2+\epsilon\|\nabla^3u_2\|_{L^2}^2+C(\epsilon)\|\rho_2\|_{L^\infty}^2\|\nabla^2\rho_2\|_{L^2}^2\\ &+\|\nabla\rho_2\|_{L^\infty}(\|\nabla^2\rho_2\|_{L^2}^2+\|\nabla^2u_2\|_{L^2}^2).\end{aligned} \tag{2.28}$$

Also, integrating by parts, using Hölder's inequality and Cauchy's inequality, and by (2.5) and Lemma 2.1, we have

$$\begin{aligned}|C_5| =& |\int_{\mathbb{R}^3}\nabla(F_1(\rho_1)\nabla\rho_1)\nabla^3u_1dx|\\ =& |\int_{\mathbb{R}^3}\nabla F_1\nabla\rho_1\nabla^3u_1dx+\int_{\mathbb{R}^3}F_1\nabla^2\rho_1\nabla^3u_1dx|\\ \le& \epsilon\|\nabla^3u_1\|_{L^2}^2+C(\epsilon)\|\nabla\rho_1\|_{L^\infty}^2\|\nabla\rho_1\|_{L^2}^2+C(\epsilon)\|\rho_1\|_{L^\infty}^2\|\nabla^2\rho_1\|_{L^2}^2.\end{aligned} \tag{2.29}$$

Similarly to the estimate of C_5, we obtain that

$$|C_6| \le \epsilon\|\nabla^3u_2\|_{L^2}^2+C(\epsilon)\|\nabla\rho_2\|_{L^\infty}^2\|\nabla\rho_2\|_{L^2}^2+C(\epsilon)\|\rho_2\|_{L^\infty}^2\|\nabla^2\rho_2\|_{L^2}^2. \tag{2.30}$$

For the rest estimates of C_i, it is easy to check that

$$\begin{aligned}|C_7| =& |\int_{\mathbb{R}^3}\nabla(H(\rho_1)(\mu_1\Delta u_1+\mu_2\nabla divu_1))\nabla^3u_1dx|\\ =& |\int_{\mathbb{R}^3}\nabla H(\rho_1)(\mu_1\Delta u_1+\mu_2\nabla divu_1)\nabla^3u_1dx+\int_{\mathbb{R}^3}H(\rho_1)\nabla(\mu_1\Delta u_1+\mu_2\nabla divu_1)\nabla^3u_1dx|\\ \le& \epsilon\|\nabla^3u_1\|_{L^2}^2+C(\epsilon)\|\nabla\rho_1\|_{L^\infty}^2\|\nabla^2u_1\|_{L^2}^2,\end{aligned} \tag{2.31}$$

and

$$|C_8| \le \epsilon\|\nabla^3u_2\|_{L^2}^2+C(\epsilon)\|\nabla\rho_2\|_{L^\infty}^2\|\nabla^2u_2\|_{L^2}^2. \tag{2.32}$$

Combining (2.23)–(2.32), we have

$$\begin{aligned}
&\frac{1}{2}\partial_t \int_{\mathbb{R}^3}(|\nabla^2\rho_1|^2+|\nabla^2u_1|^2+|\nabla^2\rho_2|^2+|\nabla^2u_2|^2+|\nabla^3\Phi|^2)dx+\mu_1\int_{\mathbb{R}^3}(|\nabla^3u_1|^2+|\nabla^3u_2|^2)dx\\
&+\mu_2\int_{\mathbb{R}^3}(|\nabla^2 divu_1|^2+|\nabla^2 divu_2|^2)dx\\
&\le C(\|\rho_1\|_{L^\infty}+\|u_1\|_{L^\infty}+\|\rho_2\|_{L^\infty}+\|u_2\|_{L^\infty}+\|\nabla\rho_1\|_{L^\infty}+\|\nabla\rho_2\|_{L^\infty})\|\nabla^3\Phi\|_{L^2}^2\\
&+C(\|\rho_1\|_{L^\infty}+\|u_1\|_{L^\infty}+\|\rho_1\|_{L^\infty}^2+\|\nabla\rho_1\|_{L^\infty}+\|\nabla u_1\|_{L^\infty})\|\nabla^2\rho_1\|_{L^2}^2\\
&+C(\|\rho_1\|_{L^\infty}+\|u_1\|_{L^\infty}+\|\nabla\rho_1\|_{L^\infty}+\|\nabla u_1\|_{L^\infty}+\|\nabla\rho_1\|_{L^\infty}^2)\|\nabla^2u_1\|_{L^2}^2\\
&+C(\|\rho_2\|_{L^\infty}+\|u_2\|_{L^\infty}+\|\rho_2\|_{L^\infty}^2+\|\nabla\rho_2\|_{L^\infty}+\|\nabla u_2\|_{L^\infty})\|\nabla^2\rho_2\|_{L^2}^2\\
&+C(\|\rho_2\|_{L^\infty}+\|u_2\|_{L^\infty}+\|\nabla\rho_2\|_{L^\infty}+\|\nabla u_2\|_{L^\infty}+\|\nabla\rho_2\|_{L^\infty}^2)\|\nabla^2u_2\|_{L^2}^2\\
&+C(\|\nabla\rho_1\|_{L^\infty}^2\|\nabla\rho_1\|_{L^2}^2+\|\nabla\rho_2\|_{L^\infty}^2\|\nabla\rho_2\|_{L^2}^2)+C(\|\nabla\rho_1\|_{L^\infty}+\|\nabla u_1\|_{L^\infty}^2)\|\nabla u_1\|_{L^2}^2\\
&+C(\|\nabla\rho_2\|_{L^\infty}+\|\nabla u_2\|_{L^\infty}^2)\|\nabla u_2\|_{L^2}^2.
\end{aligned}\tag{2.33}$$

By Gronwall's inequality, we complete the proof of the lemma. □

Lemma 2.5. *Under the assumption (2.2) and (2.4), for* $3\le l\le s+1$*, we have*

$$\|(\nabla^l\rho_1,\nabla^lu_1,\nabla^l\rho_2,\nabla^lu_2,\nabla^{l+1}\Phi)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))},\|(\nabla^{l+1}u_1,\nabla^{l+1}u_2)\|_{L^2(0,T;L^2(\mathbb{R}^3))}\le C,$$

where C is a constant that depends only on $\|(\nabla^l\rho_{1,0},\nabla^l\rho_{2,0},\nabla^lu_{1,0},\nabla^lu_{2,0},\nabla^{l+1}\Phi_0)\|_{L^2}$.

Proof. Similar to the proof of Lemma 2.4, we can obtain the conclusion of the lemma. So we omit it. □

3. Green's function of the linearized system

In order to see the Green's function of the linear part of the system better, we reformulate the system (2.1) slightly. Let

$$n=\rho_1+\rho_2, m=\rho_2-\rho_1, v=u_1+u_2, w=u_2-u_1,\tag{3.1}$$

which equivalently gives

$$\rho_1=\frac{n-m}{2},\rho_2=\frac{n+m}{2},u_1=\frac{v-w}{2},u_2=\frac{v+w}{2}.\tag{3.2}$$

Then, the Cauchy problem (2.1) can be reformulated into the following form:

$$\begin{cases}
\partial_tn+divv=Q_1(n,v,m,w),\\
\partial_tv-\mu_1\Delta v-\mu_2\nabla divv+\nabla n=Q_2(n,v,m,w),\\
\partial_tm+divw=Q_3(n,v,m,w),\\
\partial_tw-\mu_1\Delta w-\mu_2\nabla divw+\nabla m+2\nabla\Phi=Q_4(n,v,m,w),\\
\Delta\Phi=-m,\\
(n,v,m,w,\nabla\Phi)(x,0)=(n_0,v_0,m_0,w_0,\nabla\Phi_0)(x),
\end{cases}\tag{3.3}$$

where $(n_0, v_0, m_0, w_0, \nabla\Phi_0)(x) = (\rho_{1,0} + \rho_{2,0}, u_{1,0} + u_{2,0}, \rho_{2,0} - \rho_{1,0}, u_{2,0} - u_{1,0}, \nabla\Phi_0)(x)$ and

$$Q_1(n, v, m, w) = -div(\frac{n-m}{2}\frac{v-w}{2}) - div(\frac{n+m}{2}\frac{v+w}{2}) \tag{3.4}$$

$$\begin{aligned} Q_2 \;&= -\tfrac{v-w}{2} \cdot \nabla(\tfrac{v-w}{2}) - \tfrac{v+w}{2} \cdot \nabla(\tfrac{v+w}{2}) \\ &-(\tfrac{P_1'(1+\frac{n-m}{2})}{1+\frac{n-m}{2}} - 1)\nabla(\tfrac{n-m}{2}) - (\tfrac{P_2'(1+\frac{n+m}{2})}{1+\frac{n+m}{2}} - 1)\nabla(\tfrac{n+m}{2}) \\ &-\tfrac{\frac{n-m}{2}}{1+\frac{n-m}{2}}(\mu_1\Delta(\tfrac{v-w}{2}) + \mu_2\nabla div(\tfrac{v-w}{2})) - \tfrac{\frac{n+m}{2}}{1+\frac{n+m}{2}}(\mu_1\Delta(\tfrac{v+w}{2}) + \mu_2\nabla div(\tfrac{v+w}{2})), \end{aligned} \tag{3.5}$$

$$Q_3(n, v, m, w) = div(\frac{n-m}{2}\frac{v-w}{2}) - div(\frac{n+m}{2}\frac{v+w}{2}) \tag{3.6}$$

$$\begin{aligned} Q_4 &= \frac{v-w}{2} \cdot \nabla(\frac{v-w}{2}) - \frac{v+w}{2} \cdot \nabla(\frac{v+w}{2}) \\ &+ (\frac{P_1'(1+\frac{n-m}{2})}{1+\frac{n-m}{2}} - 1)\nabla(\frac{n-m}{2}) - (\frac{P_2'(1+\frac{n+m}{2})}{1+\frac{n+m}{2}} - 1)\nabla(\frac{n+m}{2}) \\ &+ \frac{\frac{n-m}{2}}{1+\frac{n-m}{2}}(\mu_1\Delta(\frac{v-w}{2}) + \mu_2\nabla div(\frac{v-w}{2})) - \frac{\frac{n+m}{2}}{1+\frac{n+m}{2}}(\mu_1\Delta(\frac{v+w}{2}) + \mu_2\nabla div(\frac{v+w}{2})). \end{aligned} \tag{3.7}$$

The linearized system of (3.3) is

$$\begin{cases} \partial_t n + divv = 0, \\ \partial_t v - \mu_1\Delta v - \mu_2\nabla divv + \nabla n = 0, \\ \partial_t m + divw = 0, \\ \partial_t w - \mu_1\Delta w - \mu_2\nabla divw + \nabla m + 2\nabla\Phi = 0, \\ \Delta\Phi = -m. \end{cases} \tag{3.8}$$

We can also rewrite (3.8) as

$$(\partial_t + A(D))V = 0, \tag{3.9}$$

where

$$A(D) = \begin{pmatrix} 0 & div & 0 & 0 \\ \nabla & -\mu_1\Delta - \mu_2\nabla div & 0 & 0 \\ 0 & 0 & 0 & div \\ 0 & 0 & \nabla + 2\nabla(-\Delta)^{-1} & -\mu_1\Delta - \mu_2\nabla div \end{pmatrix}, \; V = \begin{pmatrix} n \\ v \\ m \\ w \end{pmatrix}. \tag{3.10}$$

Consider the Green's function $\mathbb{G}$ of (3.9), i.e., the solution to the following Cauchy problem

$$\begin{cases} (\partial_t + A(D))\mathbb{G}(x, t) = 0, \\ \mathbb{G}(x, 0) = \delta(x)I_{8\times 8}, \end{cases} \tag{3.11}$$

where $\delta(x)$ denotes the Dirac function and $I_{8\times 8}$ denotes the unit matrix. By direct calculation, we obtain the Fourier transform of the Green's function $\mathbb{G}$ as

$$\widehat{\mathbb{G}} = \begin{pmatrix} \widehat{\mathbb{G}}_1 & 0 \\ 0 & \widehat{\mathbb{G}}_2 \end{pmatrix}, \tag{3.12}$$

where

$$\widehat{\mathbb{G}}_1(\xi,t) = \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & \sqrt{-1}\frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\xi^T \\ \sqrt{-1}\frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\xi & -e^{-\mu_1|\xi|^2 t} I + (\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} + e^{-\mu_1|\xi|^2 t})\frac{\xi\xi^T}{|\xi|^2} \end{pmatrix},$$

$$\widehat{\mathbb{G}}_2(\xi,t) = \begin{pmatrix} \frac{\tilde{\lambda}_+ e^{\tilde{\lambda}_- t} - \tilde{\lambda}_- e^{\tilde{\lambda}_+ t}}{\tilde{\lambda}_+ - \tilde{\lambda}_-} & \sqrt{-1}\frac{e^{\tilde{\lambda}_- t} - e^{\tilde{\lambda}_+ t}}{\tilde{\lambda}_+ - \tilde{\lambda}_-}\xi^T \\ \sqrt{-1}(1+|\xi|^{-2})\frac{e^{\tilde{\lambda}_- t} - e^{\tilde{\lambda}_+ t}}{\tilde{\lambda}_+ - \tilde{\lambda}_-}\xi & -e^{-\mu_1|\xi|^2 t} I + (\frac{\tilde{\lambda}_+ e^{\tilde{\lambda}_+ t} - \tilde{\lambda}_- e^{\tilde{\lambda}_- t}}{\tilde{\lambda}_+ - \tilde{\lambda}_-} + e^{-\mu_1|\xi|^2 t})\frac{\xi\xi^T}{|\xi|^2} \end{pmatrix},$$

and

$$\lambda_\pm = \frac{-\mu|\xi|^2 \pm \sqrt{\mu^2|\xi|^4 - 4|\xi|^2}}{2},$$

$$\tilde{\lambda}_\pm = \frac{-\mu|\xi|^2 \pm \sqrt{\mu^2|\xi|^4 - 4(|\xi|^2+2)}}{2}, \quad \mu = \mu_1 + \mu_2.$$

For the convenience of writing, we also give the following definition,

$$\widehat{\mathbf{G}} = \begin{pmatrix} \widehat{\mathbb{G}}_1 & 0 \\ 0 & \widehat{\mathbb{G}}_1 \end{pmatrix}, \tag{3.13}$$

where $\widehat{\mathbf{G}}$ is the Fourier transform of the Green's function $\mathbf{G}$. In this paper, we divide Green's function into a high frequency part and a low frequency part since Green's function has different properties in high and low frequency. Let $\chi(\xi)$ be a smooth cutoff function

$$\chi(\xi) = \begin{cases} 1, |\xi| \le \frac{1}{2}, \\ 0, |\xi| > 1. \end{cases} \tag{3.14}$$

We denote $\mathbb{G} = \mathbb{G}_L + \mathbb{G}_{RH} + S$, where $\mathbb{G}_L$ stands for the lower frequency part, $\mathbb{G}_{RH}$ stands for the regular part of the high frequency, and the S stands for the sigular part. $\mathbb{G}_L, \mathbb{G}_{RH}$, and S have the following forms:

$$\mathbb{G}_L = \chi(D)\mathbb{G}, \quad \mathbb{G}_{RH} = (1-\chi(D))\mathbb{G} - S,$$

$$S = e^{-\frac{1}{\mu}t}\delta(x)\begin{pmatrix} 1-\chi(D) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\chi(D) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.15}$$

Here, $\mu = \mu_1 + \mu_2$. For the convenience of the description in the fifth part of this paper, we redefine smooth cutoff functions $\chi_1(\xi)$ and $\chi_2(\xi)$

$$\chi_1(\xi) = \begin{cases} 1, |\xi| \le 1, \\ 0, |\xi| > \frac{7}{6}. \end{cases}$$

$$\chi_2(\xi) = \begin{cases} 1, |\xi| \le \frac{1}{3}, \\ 0, |\xi| > \frac{1}{2}. \end{cases} \tag{3.16}$$

Let us redefine the low frequency of $\mathbb{G}$ and the high frequency of $\mathbf{G}$,

$$\mathbb{G}_{\widetilde{L}} = \chi_1(D)\mathbb{G}, \quad \mathbf{G}_{\widetilde{H}} = (1 - \chi_2(D))\mathbf{G} =: \mathbf{G}_{\widetilde{RH}} + \mathbf{G}_{\widetilde{S}},$$

where

$$\mathbf{G}_{\widetilde{S}} = e^{-\frac{1}{\mu}t}\delta(x)\begin{pmatrix} 1-\chi_2(D) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\chi_2(D) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

Here, $\mu = \mu_1 + \mu_2$. From the definition of $\chi(\xi)$, $\chi_1(\xi)$, and $\chi_2(\xi)$ in (3.14) and (3.16), we can obtain

$$\chi(D)\mathbb{G}_{\widetilde{L}} = \chi(D)\mathbb{G}, \quad (1 - \chi(D))\mathbf{G}_{\widetilde{H}} = (1 - \chi(D))\mathbf{G}.$$

In this paper, we use $\mathbb{G}^{i,j}$ to represent the element in row i and column j of $\mathbb{G}$.

Below, we list some properties of Green's function, and the readers can refer to [11, 21, 30, 31] for details.

Lemma 3.1. *If $\epsilon_1 > 0$ is small enough, then for $|\xi| < \epsilon_1$, we have*

$$\lambda_\pm = -\frac{\mu}{2}|\xi|^2 \pm \sqrt{-1}|\xi|(1 + \sum_{j=1}^{\infty} d_j|\xi|^{2j}),$$

and

$$\tilde{\lambda}_\pm = -\frac{\mu}{2}|\xi|^2 + \sum_{j=2}^{\infty} a_j|\xi|^{2j} \pm \sqrt{-1}(\sqrt{2} + \sum_{j=1}^{\infty} b_j|\xi|^{2j}).$$

Proof. The readers can refer to [11, 21, 30, 31] for details, so we omit the proof. □

Lemma 3.2. *If $1 \le p \le +\infty$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \ge 0$, $x \in \mathbb{R}^3$, we have*

$$\|D^\alpha \mathbb{G}_L(\cdot, t)\|_{L^p} \le C(1 + t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}.$$

Proof. By the representation of $\mathbb{G}$ and Lemma 3.1, we can obtain the proof of the lemma, so we omit it. □

Lemma 3.3. *If $K > 0$ is large enough, then for $|\xi| > K$, we have*

$$\lambda_+ = -\frac{1}{\mu} + \frac{\mu}{2}\sum_{j=1}^{\infty} e_j|\xi|^{-2j}, \quad \lambda_- = -\mu|\xi|^2 + \frac{1}{\mu} - \sum_{j=1}^{\infty} e_j|\xi|^{-2j},$$

$$\tilde{\lambda}_+ = -\frac{1}{\mu} + \sum_{j=1}^{\infty} l_j|\xi|^{-2j}, \quad \tilde{\lambda}_- = -\mu|\xi|^2 + \frac{1}{\mu} - \sum_{j=1}^{\infty} l_j|\xi|^{-2j}.$$

Here, $\mu = \mu_1 + \mu_2$, and all e_j, l_j are real constants.

Proof. See [11,21,30,31] for details, and we omit the proof. □

Remark: This lemma states that in $\mathbb{G}$, only the terms related to

$$\frac{\lambda_-(\xi)}{\lambda_+(\xi)-\lambda_-(\xi)}e^{\lambda_+(\xi)t} \quad or \quad \frac{\tilde{\lambda}_-(\xi)}{\tilde{\lambda}_+(\xi)-\tilde{\lambda}_-(\xi)}e^{\tilde{\lambda}_+(\xi)t}$$

will occur in a singular part, and the other terms will bear at least a first derivative.

Lemma 3.4. *If* $1 \leq p \leq +\infty$ *and* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \geq 0$, $|\alpha| \leq 1$, *we have*

$$\|D^\alpha \mathbb{G}_{RH}(\cdot,t)\|_{L^p} \leq Ce^{-C_0 t},$$
$$\|D^\alpha \mathbf{G}_{\widetilde{RH}}(\cdot,t)\|_{L^p} \leq Ce^{-C_0 t},$$
$$\|S(\cdot,t) * V(\cdot,t)\|_{L^p} \leq Ce^{-C_0 t}\|V(\cdot,t)\|_{L^p},$$
$$\|\mathbf{G}_{\widetilde{S}}(\cdot,t) * V(\cdot,t)\|_{L^p} \leq Ce^{-C_0 t}\|V(\cdot,t)\|_{L^p},$$

where $C_0 > 0$ *is the fixed normal number associated with* μ.

Proof. The readers can refer to [11,21,30,31] for details, so we omit the proof. □

Below we derive an estimation method that combines the advantages of the Green's function and energy estimate. We consider the system:

$$\begin{cases}(\partial_t + B(D))U(x,t) = R(U(x,t)),\\ U(x,0) = U_0(x),\end{cases} \tag{3.18}$$

where $B(D)$ is an operator and $R(U)$ is nonlinear terms. The Green's function $\mathcal{G}(x,t)$ corresponding to the system (3.18) is the fundamental solution of the Cauchy problem of the linear equations of its corresponding system, i.e., $\mathcal{G}(x,t)$ is the solution of the following Cauchy problem:

$$\begin{cases}(\partial_t + B(D))\mathcal{G}(x,t) = 0,\\ \mathcal{G}(x,0) = \delta(x)I_{n\times n}.\end{cases} \tag{3.19}$$

where $\delta(x)$ denotes the Dirac function and $I_{n\times n}$ denotes the unit matrix.

The solution of (3.18) is usually discussed in terms of energy estimate or the Green's function. The following lemma combines the advantages of Green's function and energy estimate, which we may call the G-E estimate. The advantage of this estimate is that on the one hand, the fine decaying estimate of the solution can be obtained with the Green's function; on the other hand, the derivative in the singular part of the high frequency can be shared through integration by parts similar to the energy estimate.

Lemma 3.5. *[27] If* $B(\xi)$ *is a complex normal matrix (i.e.,* $B^*B = BB^*,\ B^* = \overline{B}^T$ *), then it holds*

$$\begin{aligned}\|U(\cdot,t)\|_{L^2}^2 = &\int_{\mathbb{R}^n}(\mathcal{G}^T(\cdot,t) * U_0)^T\mathcal{G}(\cdot,t) * U_0 dx\\ &+ 2\int_0^t\int_{\mathbb{R}^n}(\mathcal{G}^T(x-\cdot,t-\tau) * U(\cdot,\tau))^T\mathcal{G}(x-\cdot,t-\tau) * R(U(\cdot,\tau))dxd\tau,\end{aligned} \tag{3.20}$$

where $\mathcal{G}$ *is Green's function about* (3.19).

Remark 3.1. *If $\mathcal{G}^T = \mathcal{G}$, then we have*

$$\|U(\cdot,t)\|_{L^2}^2 = \|\mathcal{G}(\cdot,t) * U_0\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}^n} \mathcal{G}(x-\cdot, t-\tau) * U(\cdot,\tau) \cdot \mathcal{G}(x-\cdot,t-\tau) * R(U(\cdot,\tau)) dx d\tau. \quad (3.21)$$

The Eq (3.20) is in the form of row vector times column vector, while the Eq (3.21) is in the form of the inner product of vectors.

Remark 3.2. *As can be seen from (3.10),* $\mathbf{G}$*, which we defined in (3.13), satisfies* $\mathbf{G}^T = \mathbf{G}$.

4. The global existence of solutions

In this section, we first give the local existence theory.

4.1. The local existence of solutions

According to (2.1), we construct the approximate solution sequence $\{\widetilde{V}^n(t)\}$ by the following linearized iteration scheme:

$$\begin{cases} \partial_t \rho_1^{n+1} + div u_1^{n+1} + div(\rho_1^{n+1} u_1^n) = 0, \\ \partial_t u_1^{n+1} - \mu_1 \Delta u_1^{n+1} - \mu_2 \nabla div u_1^{n+1} + \nabla \rho_1^{n+1} - \nabla \Phi^{n+1} \\ = -u_1^n \cdot \nabla u_1^n - (\frac{P_1'(1+\rho_1^n)}{1+\rho_1^n} - 1)\nabla \rho_1^n - \frac{\rho_1^n}{1+\rho_1^n}(\mu_1 \Delta u_1^n + \mu_2 \nabla div u_1^n), \\ \partial_t \rho_2^{n+1} + div u_2^{n+1} + div(\rho_2^{n+1} u_2^n) = 0, \\ \partial_t u_2^{n+1} - \mu_1 \Delta u_2^{n+1} - \mu_2 \nabla div u_2^{n+1} + \nabla \rho_2^{n+1} + \nabla \Phi^{n+1} \\ = -u_2^n \cdot \nabla u_2^n - (\frac{P_2'(1+\rho_2^n)}{1+\rho_2^n} - 1)\nabla \rho_2^n - \frac{\rho_2^n}{1+\rho_2^n}(\mu_1 \Delta u_2^n + \mu_2 \nabla div u_2^n), \\ \Delta \Phi^{n+1} = \rho_1^{n+1} - \rho_2^{n+1}, \lim_{|x|\to\infty} \Phi^{n+1}(x,t) = 0, \end{cases} \quad (4.1)$$

where $\{\widetilde{V}^n(t)\}$ is defined as $\widetilde{V}^n(t) = (\rho_1^n(t), u_1^n(t), \rho_2^n(t), u_2^n(t), \nabla\Phi^n(t)), n \geq 0$, and $\widetilde{V}^0(t) = (0,0,0,0,0)$. For any given integer $s \geq [\frac{3}{2}] + 3$, we define

$$X_{T,E}^s = \{\widetilde{V}(t) | \|\widetilde{V}\|_{X^s} < E\}$$

as the suitable space for the solutions, where

$$\|\widetilde{V}\|_{X^s} = \sup_{0\leq t\leq T} \|\widetilde{V}(t)\|_{H^s}.$$

It is easy to show that $X_{T,E}^s$, equipped with the norm $\|\cdot\|_{X^s}$, is a nonempty Banach space. To obtain the local soluiton, we need the following two lemmas. To do this end, we first give a prior assumption. For sufficiently small $\varepsilon_1 > 0$, we have

$$\|\rho_1\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1 \text{ and } \|\rho_2\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1. \quad (4.2)$$

Lemma 4.1. *Under the assumption (4.2), when T is small enough, there exists a constant $E > 0$ such that $\{\widetilde{V}^n(x,t)\} \subseteq X_{T,E}^{s+1}$.*

Proof. We imply the inductive method to accomplish the proof. To start, when $n = 0$, we have

$$\begin{cases} \partial_t \rho_1^1 + divu_1^1 = 0, \\ \partial_t u_1^1 - \mu_1 \Delta u_1^1 - \mu_2 \nabla divu_1^1 + \nabla \rho_1^1 - \nabla \Phi^1 = 0, \\ \partial_t \rho_2^1 + divu_2^1 = 0, \\ \partial_t u_2^1 - \mu_1 \Delta u_2^1 - \mu_2 \nabla divu_2^1 + \nabla \rho_2^1 + \nabla \Phi^1 = 0, \\ \Delta \Phi^1 = \rho_1^1 - \rho_2^1, \lim_{|x|\to\infty} \Phi^1(x,t) = 0, \end{cases} \tag{4.3}$$

By the energy estimate, we have

$$\frac{1}{2}\partial_t (\|\rho_1^1\|_{H^s}^2 + \|u_1^1\|_{H^s}^2 + \|\rho_2^1\|_{H^s}^2 + \|u_2^1\|_{H^s}^2 + \|\nabla\Phi^1\|_{H^s}^2 \le 0.$$

We take $E = 2\|(\|\rho_1(x,0), u_1(x,0), \rho_2(x,0), u_2(x,0))\|_{H^s}$, then we get $\widetilde{V}^1(x,t) \in X_{T,E}^s$. Now, assuming that $\{\widetilde{V}^j(x,t)\} \in X_{T,E}^s$ for all $j \le n$, we need to prove it holds for $j = n+1$.
Applying the energy method to (4.1), we have

$$\begin{aligned} &\frac{1}{2}\partial_t \int_{\mathbb{R}^3} (|\rho_1^{n+1}|^2 + |u_1^{n+1}|^2 + |\rho_2^{n+1}|^2 + |u_2^{n+1}|^2) dx \\ &+ \mu_1 \int_{\mathbb{R}^3} (|\nabla u_1^{n+1}|^2 + |\nabla u_2^{n+1}|^2) dx + \mu_2 \int_{\mathbb{R}^3} (|divu_1^{n+1}|^2 + |divu_2^{n+1}|^2) dx \\ &= -\int_{\mathbb{R}^3} \nabla\Phi^{n+1}(u_2^{n+1} - u_1^{n+1}) dx - \int_{\mathbb{R}^3} div(\rho_1^{n+1} u_1^n)\rho_1^{n+1} dx - \int_{\mathbb{R}^3} div(\rho_2^{n+1} u_2^n)\rho_2^{n+1} dx \\ &- \int_{\mathbb{R}^3} u_1^n \cdot \nabla u_1^n \cdot u_1^{n+1} dx - \int_{\mathbb{R}^3} u_2^n \cdot \nabla u_2^n \cdot u_2^{n+1} dx - \int_{\mathbb{R}^3} F_1(\rho_1^n) u_1^{n+1} \cdot \nabla \rho_1^n dx \\ &- \int_{\mathbb{R}^3} F_2(\rho_2^n) u_2^{n+1} \cdot \nabla \rho_2^n dx - \int_{\mathbb{R}^3} H(\rho_1^n)(\mu_1 \Delta u_1^n + \mu_2 \nabla divu_1^n) u_1^{n+1} dx \\ &- \int_{\mathbb{R}^3} H(\rho_2^n)(\mu_1 \Delta u_2^n + \mu_2 \nabla divu_2^n) u_2^{n+1} dx \\ &=: \sum_{i=1}^{9} G_i. \end{aligned} \tag{4.4}$$

Similar to the proof of Lemma 2.2, we have the following estimates for each G_i, $1 \le i \le 9$.

$$\begin{gathered} |G_1| \le -\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\nabla\Phi|^2 dx + C\|\nabla\Phi^{n+1}\|_{L^2}^2 (\|u_1^n\|_{L^\infty}^2 + \|u_2^n\|_{L^\infty}^2) + C\|\rho_1^{n+1}\|_{L^2}^2 + \|\rho_1^{n+1}\|_{L^2}^2, \\ |G_2 + G_3| \le C\|divu_1^n\|_{L^\infty}\|\rho_1^{n+1}\|_{L^2}^2 + C\|divu_2^n\|_{L^\infty}\|\rho_2^{n+1}\|_{L^2}^2, \\ |G_4 + G_5| \le C\|\nabla u_1^n\|_{L^\infty}^2 \|u_1^n\|_{L^2}^2 + C\|\nabla u_2^n\|_{L^\infty}^2 \|u_2^n\|_{L^2}^2 + C\|u_1^{n+1}\|_{L^2}^2 + C\|u_2^{n+1}\|_{L^2}^2, \\ |G_6 + G_7| \le C\|\nabla\rho_1^n\|_{L^\infty}^2 \|\rho_1^n\|_{L^2}^2 + C\|u_1^{n+1}\|_{L^2}^2 + C\|\nabla\rho_2^n\|_{L^\infty}^2 \|\rho_2^n\|_{L^2}^2 + C\|u_2^{n+1}\|_{L^2}^2, \end{gathered}$$

$$\begin{aligned} |G_8 + G_9| \le\ & C\|\nabla\rho_1^n\|_{L^\infty}^2 \|u_1^{n+1}\|_{L^2}^2 + C(1 + \|\rho_1^n\|_{L^\infty}^2)\|\nabla u_1^n\|_{L^2}^2 + \epsilon\|\nabla u_1^{n+1}\|_{L^2}^2 \\ &+ C\|\nabla\rho_2^n\|_{L^\infty}^2 \|u_2^{n+1}\|_{L^2}^2 + C(1 + \|\rho_2^n\|_{L^\infty}^2)\|\nabla u_2^n\|_{L^2}^2 + \epsilon\|\nabla u_2^{n+1}\|_{L^2}^2. \end{aligned}$$

where we take ϵ small enough such that $\epsilon \ll 1$. Plugging the estimates for $G_1 - G_9$ into (4.4), and integrating with respect to t, we obtain

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\rho_1^{n+1}(t)\|_{L^2}^2+\|u_1^{n+1}(t)\|_{L^2}^2+\|\rho_2^{n+1}(t)\|_{L^2}^2+\|u_2^{n+1}(t)\|_{L^2}^2+\|\nabla\Phi^{n+1}(t)\|_{L^2}^2)\\
&+\mu_1\int_0^t\int_{\mathbb{R}^3}(|\nabla u_1^{n+1}|^2+|\nabla u_2^{n+1}|^2)dx+\mu_2\int_0^t\int_{\mathbb{R}^3}(|divu_1^{n+1}|^2+|divu_2^{n+1}|^2)dx\\
&\le(\frac{1}{2}E)^2+CTE^2\sup_{0\le t\le T}\|\nabla\Phi^{n+1}\|_{L^2}^2+CT(1+E)\sup_{0\le t\le T}\|\rho_1^{n+1}\|_{L^2}^2\\
&+CT(1+E)\sup_{0\le t\le T}\|\rho_2^{n+1}\|_{L^2}^2+CT(1+E^2)\sup_{0\le t\le T}\|u_1^{n+1}\|_{L^2}^2\\
&+CT(1+E^2)\sup_{0\le t\le T}\|u_2^{n+1}\|_{L^2}^2+CTE^2(1+E^2).
\end{aligned}\tag{4.5}$$

Taking T small enough, we can get the following estimate from (4.5),

$$\sup_{0\le t\le T}\|\tilde{V}^{n+1}(t)\|_{L^2}\le E.\tag{4.6}$$

To derive higher-order estimates, similar to the proof of Lemmas 2.3–2.5, we can obtain

$$\sup_{0\le t\le T}\|D^\alpha\tilde{V}^{n+1}(t)\|_{L^2}\le E,\quad 1\le\alpha\le s+1.\tag{4.7}$$

Combining (4.6) and (4.7) yields

$$\sup_{0\le t\le T}\|\tilde{V}^{n+1}(t)\|_{H^{s+1}}\le E,$$

which means $\tilde{V}^{n+1}(x,t)\in X_{T,E}^{s+1}$. We complete the proof of the lemma. □

Lemma 4.2. *Under the assumption (4.2), when T is small enough, $\{\widetilde{V}^n(x,t)\}$ is a Cauchy sequence in $X_{T,E}^s$.*

Proof. We set

$$f_1^{n+1}=\rho_1^{n+1}-\rho_1^n,\ f_2^{n+1}=\rho_2^{n+1}-\rho_2^n,\ g_1^{n+1}=u_1^{n+1}-u_1^n,$$

$$g_2^{n+1}=u_2^{n+1}-u_2^n,\ \Psi^{n+1}=\Phi^{n+1}-\Phi^n,$$

and define

$$Y^{n+1}=(f_1^{n+1},f_2^{n+1},g_1^{n+1},g_2^{n+1},\Psi^{n+1}),$$

then we only need to verify that

$$\|Y^{n+1}\|_{H^s}\le\kappa\|Y^n\|_{H^s},$$

where $0 < \kappa < 1$. From (4.1), we get Y^{n+1} satisfies

$$\begin{cases}\partial_t f_1^{n+1} + divg_1^{n+1} + div(\rho_1^{n+1}u_1^n - \rho_1^n u_1^{n-1}) = 0,\\ \partial_t g_1^{n+1} - \mu_1\Delta g_1^{n+1} - \mu_2\nabla divg_1^{n+1} + \nabla f_1^{n+1} - \nabla\Psi^{n+1}\\ = -(u_1^n\cdot\nabla u_1^n - u_1^{n-1}\cdot\nabla u_1^{n-1}) - ((\frac{P_1'(1+\rho_1^n)}{1+\rho_1^n} - 1)\nabla\rho_1^n - (\frac{P_1'(1+\rho_1^{n-1})}{1+\rho_1^{n-1}} - 1)\nabla\rho_1^{n-1})\\ -(\frac{\rho_1^n}{1+\rho_1^n}(\mu_1\Delta u_1^n + \mu_2\nabla divu_1^n) - \frac{\rho_1^{n-1}}{1+\rho_1^{n-1}}(\mu_1\Delta u_1^{n-1} + \mu_2\nabla divu_1^{n-1})),\\ \partial_t f_2^{n+1} + divg_2^{n+1} + div(\rho_2^{n+1}u_2^n - \rho_2^n u_2^{n-1}) = 0,\\ \partial_t g_2^{n+1} - \mu_1\Delta g_2^{n+1} - \mu_2\nabla divg_2^{n+1} + \nabla f_2^{n+1} + \nabla\Psi^{n+1}\\ = -(u_2^n\cdot\nabla u_2^n - u_2^{n-1}\cdot\nabla u_2^{n-1}) - ((\frac{P_2'(1+\rho_2^n)}{1+\rho_2^n} - 1)\nabla\rho_2^n - (\frac{P_2'(1+\rho_2^{n-1})}{1+\rho_2^{n-1}} - 1)\nabla\rho_2^{n-1})\\ -(\frac{\rho_2^n}{1+\rho_2^n}(\mu_1\Delta u_2^n + \mu_2\nabla divu_2^n) - \frac{\rho_2^{n-1}}{1+\rho_2^{n-1}}(\mu_1\Delta u_2^{n-1} + \mu_2\nabla divu_2^{n-1})),\\ \Delta\Psi^{n+1} = f_1^{n+1} - f_2^{n+1}, \lim_{|x|\to\infty}\Psi^{n+1}(x,t) = 0.\end{cases} \tag{4.8}$$

Applying the energy method to (4.8), we have

$$\begin{aligned}&\frac{1}{2}\partial_t\int_{\mathbb{R}^3}(|f_1^{n+1}|^2 + |g_1^{n+1}|^2 + |f_2^{n+1}|^2 + |g_2^{n+1}|^2)dx\\ &+\mu_1\int_{\mathbb{R}^3}(|\nabla g_1^{n+1}|^2 + |\nabla g_2^{n+1}|^2)dx + \mu_2\int_{\mathbb{R}^3}(|divg_1^{n+1}|^2 + |divg_2^{n+1}|^2)dx\\ &= -\int_{\mathbb{R}^3}\nabla\Psi^{n+1}(g_2^{n+1} - g_1^{n+1})dx - \int_{\mathbb{R}^3}div(\rho_1^{n+1}u_1^n - \rho_1^n u_1^{n-1})f_1^{n+1}dx\\ &-\int_{\mathbb{R}^3}div(\rho_2^{n+1}u_2^n - \rho_2^n u_2^{n-1})f_2^{n+1}dx - \int_{\mathbb{R}^3}(u_1^n\cdot\nabla u_1^n - u_1^{n-1}\cdot\nabla u_1^{n-1})\cdot g_1^{n+1}dx\\ &-\int_{\mathbb{R}^3}(u_2^n\cdot\nabla u_2^n - u_2^{n-1}\cdot\nabla u_2^{n-1})\cdot g_2^{n+1}dx - \int_{\mathbb{R}^3}(F_1(\rho_1^n)\cdot\nabla\rho_1^n - F_1(\rho_1^{n-1})\cdot\nabla\rho_1^{n-1})g_1^{n+1}dx\\ &-\int_{\mathbb{R}^3}(H(\rho_1^n)(\mu_1\Delta u_1^n + \mu_2\nabla divu_1^n) - H(\rho_1^{n-1})(\mu_1\Delta u_1^{n-1} + \mu_2\nabla divu_1^{n-1}))g_1^{n+1}dx\\ &-\int_{\mathbb{R}^3}(H(\rho_2^n)(\mu_1\Delta u_2^n + \mu_2\nabla divu_2^n) - H(\rho_2^{n-1})(\mu_1\Delta u_2^{n-1} + \mu_2\nabla divu_2^{n-1}))g_2^{n+1}dx\\ &-\int_{\mathbb{R}^3}(F_2(\rho_2^n)\cdot\nabla\rho_2^n - F_2(\rho_2^{n-1})\cdot\nabla\rho_2^{n-1})g_2^{n+1}dx\\ &=:\sum_{i=1}^{9}J_i.\end{aligned} \tag{4.9}$$

Similar to the proof of Lemma 2.2, we have the following estimates for each J_i, $1 \le i \le 9$.
For J_1, we have

$$\begin{aligned}J_1 &= -\frac{1}{2}\partial_t\|\nabla\Psi^{n+1}\|_{L^2}^2 + \int_{\mathbb{R}^3}\nabla\Psi^{n+1}(f_2^{n+1}u_2^n + \rho_2^n g_2^n - f_1^{n+1}u_1^n - \rho_1^n g_1^n)dx\\ &\le -\frac{1}{2}\partial_t\|\nabla\Psi^{n+1}\|_{L^2}^2 + 4\|\nabla\Psi^{n+1}\|_{L^2}^2 + \|u_2^n\|_{L^\infty}^2\|f_2^{n+1}\|_{L^2}^2\\ &+ \|u_1^n\|_{L^\infty}^2\|f_1^{n+1}\|_{L^2}^2 + \|\rho_2^n\|_{L^\infty}^2\|g_2^n\|_{L^2}^2 + \|\rho_1^n\|_{L^\infty}^2\|g_1^{n+1}\|_{L^2}^2.\end{aligned}$$

For J_2, we have

$$\begin{aligned} J_2 &= \int_{\mathbb{R}^3} div(f_1^{n+1}u_1^n + \rho_1^n g_1^n) f_1^{n+1} dx \\ &\le \|\nabla u_1^n\|_{L^\infty} \|f_1^{n+1}\|_{L^2}^2 + \|\nabla \rho_1^n\|_{L^\infty} \|f_1^{n+1}\|_{L^2}^2 + \|g_1^n\|_{L^2}^2 \\ &+ \|\nabla g_1^n\|_{L^2}^2 + \|\rho_1^n\|_{L^\infty} \|f_1^{n+1}\|_{L^2}^2. \end{aligned}$$

Similar to the proof of J_2, for J_3, we can obtain

$$\begin{aligned} J_3 &= \int_{\mathbb{R}^3} div(f_2^{n+1}u_2^n + \rho_2^n g_2^n) f_2^{n+1} dx \\ &\le \|\nabla u_2^n\|_{L^\infty} \|f_2^{n+1}\|_{L^2}^2 + \|\nabla \rho_2^n\|_{L^\infty} \|f_2^{n+1}\|_{L^2}^2 + \|g_2^n\|_{L^2}^2 \\ &+ \|\nabla g_2^n\|_{L^2}^2 + \|\rho_2^n\|_{L^\infty} \|f_2^{n+1}\|_{L^2}^2. \end{aligned}$$

For J_4, we have

$$\begin{aligned} J_4 &= -\int_{\mathbb{R}^3} (g_1^n \cdot \nabla u_1^n + u_1^{n-1} \cdot \nabla g_1^n) g_1^{n+1} dx \\ &\le \|g_1^n\|_{L^2}^2 + (\|\nabla u_1^n\|_{L^\infty}^2 + \|\nabla u_1^{n-1}\|_{L^\infty}^2) \|g_1^{n+1}\|_{L^2}^2 \\ &+ \epsilon \|\nabla g_1^{n+1}\|_{L^2}^2 + C\|g_1^n\|_{L^2}^2 \|u_1^{n-1}\|_{L^\infty}^2. \end{aligned}$$

Similar to the estimate of J_4, for J_5, we can obtain

$$\begin{aligned} J_5 &= -\int_{\mathbb{R}^3} (g_2^n \cdot \nabla u_2^n + u_2^{n-1} \cdot \nabla g_2^n) g_2^{n+1} dx \\ &\le \|g_2^n\|_{L^2}^2 + (\|\nabla u_2^n\|_{L^\infty}^2 + \|\nabla u_2^{n-1}\|_{L^\infty}^2) \|g_2^{n+1}\|_{L^2}^2 \\ &+ \epsilon \|\nabla g_2^{n+1}\|_{L^2}^2 + C\|g_2^n\|_{L^2}^2 \|u_2^{n-1}\|_{L^\infty}^2. \end{aligned}$$

For J_6, we have

$$\begin{aligned} J_6 &= -\int_{\mathbb{R}^3} ((F_1(\rho_1^n) - F_1(\rho_1^{n-1}) \nabla \rho_1^n + F_1(\rho_1^{n-1}) \nabla f_1^n) g_1^{n+1} dx \\ &\le \|\nabla \rho_1^n\|_{L^\infty}^2 \|g_1^{n+1}\|_{L^2}^2 + 2\|\rho_1^n\|_{L^2}^2 + 2\|\rho_1^{n-1}\|_{L^2}^2 \\ &+ \|f_1^n\|_{L^2}^2 + \|\nabla \rho_1^{n-1}\|_{L^\infty}^2 \|g_1^{n+1}\|_{L^2}^2 + \epsilon \|\nabla g_1^{n+1}\|_{L^2}^2 + C\|\rho_1^{n-1}\|_{L^\infty}^2 \|f_1^n\|_{L^2}^2. \end{aligned}$$

Similar to the estimate of J_6, for J_9, we can obtain

$$\begin{aligned} J_9 &= -\int_{\mathbb{R}^3} ((F_2(\rho_2^n) - F_2(\rho_2^{n-1}) \nabla \rho_2^n + F_2(\rho_2^{n-1}) \nabla f_2^n) g_2^{n+1} dx \\ &\le \|\nabla \rho_2^n\|_{L^\infty}^2 \|g_2^{n+1}\|_{L^2}^2 + 2\|\rho_2^n\|_{L^2}^2 + 2\|\rho_2^{n-1}\|_{L^2}^2 \\ &+ \|f_2^n\|_{L^2}^2 + \|\nabla \rho_2^{n-1}\|_{L^\infty}^2 \|g_2^{n+1}\|_{L^2}^2 + \epsilon \|\nabla g_2^{n+1}\|_{L^2}^2 + C\|\rho_2^{n-1}\|_{L^\infty}^2 \|f_2^n\|_{L^2}^2. \end{aligned}$$

For J_7, we have

$$\begin{aligned} J_7 &= -\int_{\mathbb{R}^3} ((H(\rho_1^n) - H(\rho_1^{n-1}))(\mu_1 \Delta u_1^n + \mu_2 \nabla div u_1^n) + H(\rho_1^{n-1})(\mu_1 \Delta g_1^n + \mu_2 \nabla div g_1^n)) g_1^{n+1} dx \\ &\le C(\|\nabla \rho_1^n\|_{L^\infty}^2 + \|\nabla \rho_1^{n-1}\|_{L^\infty}^2) \|g_1^{n+1}\|_{L^2}^2 + C\|\nabla u_1^n\|_{L^2}^2 + \epsilon \|\nabla g_1^{n+1}\|_{L^2}^2 \\ &+ C(\rho_1^n\|_{L^\infty}^2 + \|\rho_1^{n-1}\|_{L^\infty}^2) \|\nabla u_1^n\|_{L^2}^2 + C\|\nabla g_1^n\|_{L^2}^2 + C\|\nabla \rho_1^{n-1}\|_{L^\infty}^2 \|g_1^{n+1}\|_{L^2}^2. \end{aligned}$$

Similar to the estimate of J_7, for J_8, we have

$$\begin{aligned}J_8 &= -\int_{\mathbb{R}^3}((H(\rho_2^n)-H(\rho_2^{n-1}))(\mu_1\Delta u_2^n+\mu_2\nabla divu_2^n)+H(\rho_2^{n-1})(\mu_1\Delta g_2^n+\mu_2\nabla divg_2^n))g_2^{n+1}dx\\ &\le C(\|\nabla\rho_2^n\|_{L^\infty}^2+\|\nabla\rho_2^{n-1}\|_{L^\infty}^2)\|g_2^{n+1}\|_{L^2}^2+C\|\nabla u_2^n\|_{L^2}^2+\epsilon\|\nabla g_2^{n+1}\|_{L^2}^2\\ &+C(\rho_2^n\|_{L^\infty}^2+\|\rho_2^{n-1}\|_{L^\infty}^2)\|\nabla u_2^n\|_{L^2}^2+C\|\nabla g_2^n\|_{L^2}^2+C\|\nabla\rho_2^{n-1}\|_{L^\infty}^2\|g_2^{n+1}\|_{L^2}^2.\end{aligned}$$

Plugging the estimates for J_1–J_9 into (4.9), and integrating with respect to t over [0,T], we obtain

$$\sup_{0\le t\le T}\|Y^{n+1}(t)\|_{L^2}^2\le\frac{1}{2}\sup_{0\le t\le T}\|Y^n(t)\|_{L^2}^2, \tag{4.10}$$

where T is small enough. To derive higher-order estimates, similar to the proof of Lemma (2.3)–(2.5), we can obtain

$$\sup_{0\le t\le T}\|D^\alpha Y^{n+1}(t)\|_{L^2}\le\frac{1}{2}\sup_{0\le t\le T}\|D^\alpha Y^n(t)\|_{L^2}^2,\quad 1\le\alpha\le s. \tag{4.11}$$

Combining (4.10) and (4.11) yields

$$\sup_{0\le t\le T}\|Y^{n+1}(t)\|_{H^s}\le\frac{1}{2}\sup_{0\le t\le T}\|Y^n(t)\|_{H^s},$$

so we complete the proof of the lemma. □

So far, we complete the proof of local existence.

Lemma 4.3. *Let* $(\rho_{1,0}-1,u_{1,0},\rho_{2,0}-1,u_{2,0},\nabla\Phi_0)\in H^{s+1}(s\ge4)$, *and*

$$\|(\rho_{1,0}-1,u_{1,0},\rho_{2,0}-1,u_{2,0},\nabla\Phi_0)\|_{L^2(\mathbb{R}^3)\bigcap L^1(\mathbb{R}^3)}\le E_0,$$

where E_0 is sufficiently small, then there exists a time $T>0$, such that the Cauchy problem (1.1) and (1.2) admits a unique classical solution in [0,T) that satisfies

$$(\rho_1-1,u_1,\rho_2-1,u_2)\in L^\infty([0,T);H^{s+1}),\quad\nabla(u_1,u_2)\in L^2([0,T);H^{s+1}).$$

4.2. The global existence of solutions

In this subsection, we will establish the global solution to the systems (1.1) and (1.2) by using the bootstrap argument if the initial data satisfies

$$\|(\rho_{1,0},\rho_{2,0},u_{1,0},u_{2,0},\nabla\Phi_0)\|_{L^2\bigcap L^1}\le E_0, \tag{4.12}$$

where E_0 is sufficiently small. However, for the derivatives of $\rho_{1,0},\rho_{2,0},u_{1,0},u_{2,0}$, and $\nabla\Phi_0$, we only assume that they are bounded. Now, we first give the following abstract bootstrap argument.

Lemma 4.4. *[32] Let $T>0$. Assume that two statements C(t) and H(t) with $t\in[0,T]$ satisfy the following conditions:*
1) If H(t) holds for some $t\in[0,T]$, then C(t) holds for the same t;
2) If C(t) holds for some $t_0\in[0,T]$, then H(t) holds for t in a neighborhood of t_0;
3) If C(t) holds for $t_m\in[0,T]$ and $t_m\to t$, then C(t) holds;
4) C(t) holds for at least one $t_1\in[0,T]$.
Then, C(t) holds on $[0,T]$.

For any fixed $0 < T < \infty$, $t \in [0, T]$, through the regularity criterion in the Section 2, we know that if $(\rho_1, \rho_2, u_1, u_2, \nabla\Phi)(x, t)$ satisfies

$$(\|\nabla\rho_1(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla u_1(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla\rho_2(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla u_2(t)\|_{L^\infty(\mathbb{R}^3)}) \leq C(1+t)^{-2},$$

and

$$\|\rho_1(t)\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1 \text{ and } \|\rho_2(t)\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1,$$

where $\varepsilon_1 > 0$ is small enough, then

$$\|(\rho_1, u_1, \rho_2, u_2, \nabla\Phi)\|_{L^\infty([0,T];H^{s+1})} + \|(\nabla u_1, \nabla u_2)\|_{L^2([0,T];H^{s+1})} \leq C,$$

where $s \geq 4$. From (3.1) and (3.2), we get

$$\|(\rho_{1,0}, u_{1,0}, \rho_{2,0}, u_{2,0}, \nabla\Phi_0)\|_{L^2 \cap L^1} \leq E_0,$$

is equal to

$$\|(n_0, v_0, m_0, w_0, \nabla\Phi_0)\|_{L^2 \cap L^1} \leq E_0, \tag{4.13}$$

and

$$(\|\nabla\rho_1(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla u_1(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla\rho_2(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla u_2(t)\|_{L^\infty(\mathbb{R}^3)}) \leq C(1+t)^{-2},$$

is equal to

$$(\|\nabla n(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla m(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla w(t)\|_{L^\infty(\mathbb{R}^3)}) \leq C(1+t)^{-2}, \tag{4.14}$$

as well as

$$\|\rho_1\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1 \text{ and } \|\rho_2\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1,$$

is equal to

$$\|n\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1 \text{ and } \|m\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon_1,$$

For convenience, in this subsection, we use the condition (4.13) and the assumption (4.14). Let δ be a fixed positive number, say,

$$2\|V_0\|_{L^1\cap L^2} + 2\|\nabla V_0\|_{L^\infty} \leq \frac{\delta}{2},$$

where V_0 is defined as $V_0 = (n_0, v_0, m_0, w_0, \nabla\Phi_0)^T$. For any fixed $0 < T < \infty, t \in [0, T]$, let us denote

$$C(T) : (\|\nabla n(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla m(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla w(t)\|_{L^\infty(\mathbb{R}^3)}) \leq \frac{\delta}{2}(1+t)^{-2}$$

and

$$H(T) : (\|\nabla n(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla m(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla w(t)\|_{L^\infty(\mathbb{R}^3)}) \leq \delta(1+t)^{-2}, \tag{4.15}$$

Based on the local existence of solutions, we only need to verify the condition 1 in Lemma (4.4) under the condition (4.12), i.e., given $H(T)$ as the condition, to derive $C(T)$ is valid. Before we check the condition 1, we need some lemmas.

Lemma 4.5. *Under the assumption (4.15) and the condition (4.13), the following estimate holds:*

$$\|(n, v, m, w, \nabla\Phi)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq CE_0, \quad T \in (0, +\infty). \tag{4.16}$$

Proof. By (4.15) and Lemma 2.2, we complete the proof of the lemma. □

Lemma 4.6. *Under the assumption (4.15) and the condition (4.13), we have*

$$(\|(n,v,m,w)(t)\|_{L^\infty(\mathbb{R}^3)} \leq CE_0^{\frac{2}{5}}(1+t)^{-\frac{6}{5}},$$

$$\|\nabla(n,v,m,w)(t)\|_{L^2(\mathbb{R}^3)} \leq CE_0^{\frac{17}{28}}(1+t)^{-\frac{3}{4}},$$

$$\|\nabla(v,m)(t)\|_{L^4(\mathbb{R}^3)} \leq C(1+t)^{-\frac{11}{8}}, \quad t\in[0,T].$$

Proof. By the Gagliardo-Nirenberg inequality, we know

$$\begin{aligned}&\|(n,v,m,w)(t)\|_{L^\infty(\mathbb{R}^3)}\\ &\leq C\|(n,v,m,w)(t)\|_{L^2}^{\frac{2}{5}}\|\nabla(n,v,m,w)(t)\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{5}}\\ &\leq CE_0^{\frac{2}{5}}(1+t)^{-\frac{6}{5}},\end{aligned}$$

$$\|\nabla v(t)\|_{L^2} \leq C\|\Lambda^{1+\epsilon}v(t)\|_{L^\infty}^{\theta}\|v(t)\|_{L^2}^{1-\theta}, \tag{4.17}$$

and

$$\|\Lambda^{1+\epsilon}v(t)\|_{L^\infty} \leq C\|\nabla v(t)\|_{L^\infty}^{1-\beta}\|\Lambda^{1+k\epsilon}v(t)\|_{L^\infty}^{\beta}, \tag{4.18}$$

where $\theta=\frac{2}{2\epsilon+5}, \beta=\frac{1}{k}$. Then, from (4.17) and (4.18), we have

$$\begin{aligned}\|\nabla v(t)\|_{L^2} &\leq C\|v(t)\|_{L^2}^{1-\frac{2}{2\epsilon+5}}\|\nabla v(t)\|_{L^\infty}^{(1-\frac{1}{k})\frac{2}{2\epsilon+5}}\\ &\leq CE_0^{1-\frac{2}{2\epsilon+5}}(1+t)^{-(1-\frac{1}{k})\frac{4}{2\epsilon+5}}.\end{aligned}$$

Taking $k=\frac{1}{\epsilon},\ \epsilon=\frac{1}{22}$ in the above inequality, we obtain

$$\|\nabla v(t)\|_{L^2(\mathbb{R}^2)} \leq CE_0^{\frac{17}{28}}(1+t)^{-\frac{3}{4}}.$$

The same computation also gives

$$\|\nabla(n,m,w)(t)\|_{L^2(\mathbb{R}^2)} \leq CE_0^{\frac{17}{28}}(1+t)^{-\frac{3}{4}}.$$

By the interpolation inequality, it holds

$$\begin{aligned}\|\nabla(v,w)(t)\|_{L^4} &\leq C\|\nabla(v,w)\|_{L^2}^{\frac{1}{2}}\|\nabla(v,w)\|_{L^\infty}^{\frac{1}{2}}.\\ &\leq C(1+t)^{-(\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot 2)}\\ &= C(1+t)^{-\frac{11}{8}}.\end{aligned}$$

Thus we complete the proof of the lemma. □

Lemma 4.7. *Under the condition (4.13), we have*

$$\|(n(t),m(t))\|_{L^1(\mathbb{R}^3)} \leq CE_0.$$

Proof. By $(3.3)_1$ and $(3.3)_3$, and using the condition (4.13), we can obtain the result. □

Now let us use Green's function to prove some lemmas. To begin, by Duhamel's principle, we know

$$V(t) = \mathbb{G}(t) * V_0 + \int_0^t \mathbb{G}(t-s) * N(V)(s)ds, \tag{4.19}$$

where

$$V = \begin{pmatrix} n \\ v \\ m \\ w \end{pmatrix},\ V_0 = \begin{pmatrix} n_0 \\ v_0 \\ m_0 \\ w_0 \end{pmatrix} =: \begin{pmatrix} V_0^1 \\ V_0^2 \\ V_0^3 \\ V_0^4 \end{pmatrix},\ N(V) =: \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}. \tag{4.20}$$

where the $Q_i(i = 1, 2, 3, 4)$ are defined by (3.4)–(3.7). From (4.19), for the component v and w of V, we have

$$\begin{aligned} v &= \begin{pmatrix} \sum_{k=1}^4 \mathbb{G}^{2,k}(t)) * V_0^k + \int_0^t \sum_{k=1}^4 \mathbb{G}^{2,k}(t-s) * Q^k(s))ds \\ \sum_{k=1}^4 \mathbb{G}^{3,k}(t)) * V_0^k + \int_0^t \sum_{k=1}^4 \mathbb{G}^{3,k}(t-s) * Q^k(s))ds \\ \sum_{k=1}^4 \mathbb{G}^{4,k}(t)) * V_0^k + \int_0^t \sum_{k=1}^4 \mathbb{G}^{3,k}(t-s) * Q^k(s))ds \end{pmatrix} \\ &=: \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} w &= \begin{pmatrix} \sum_{k=5}^8 \mathbb{G}^{6,k}(t)) * V_0^k + \int_0^t \sum_{k=5}^8 \mathbb{G}^{6,k}(t-s) * Q^k(s))ds \\ \sum_{k=5}^8 \mathbb{G}^{7,k}(t)) * V_0^k + \int_0^t \sum_{k=5}^8 \mathbb{G}^{7,k}(t-s) * Q^k(s))ds \\ \sum_{k=5}^8 \mathbb{G}^{8,k}(t)) * V_0^k + \int_0^t \sum_{k=5}^8 \mathbb{G}^{8,k}(t-s) * Q^k(s))ds \end{pmatrix} \\ &=: \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}, \end{aligned} \tag{4.22}$$

where $\mathbb{G}(t) * V_0,\ \mathbb{G}(t-s) * N(V)(s)$ obey matrix multiplication.

Lemma 4.8. *Under the assumption (4.15) and the condition (4.13), it holds that*

$$\|(v(t), w(t))\|_{L^1(\mathbb{R}^3)} \le CE_0^{\frac{2}{5}},\ \ t \in [0, T].$$

Proof. Because we can't directly get the estimate of $\|(v(t), w(t))\|_{L^1}$, we first obtain the boundedness of $\|(v(t), w(t))\|_{L^r}$ for some $r \in (1, 2)$. In the following, we take $r = \frac{4}{3}$.

We first consider $\|(v,w)(t)\|_{L^r}$. By (4.21) and the representation (3.15) of S, we have

$$
\begin{aligned}
&\|M_1\|_{L^r}\\
&\le \sum_{k=1}^{4}\|\mathbb{G}^{2,k}(t))*V_0^k\|_{L^r}+\int_0^t\sum_{k=1}^{4}\|\mathbb{G}^{2,k}(t-s)*Q^k(s))\|_{L^r}ds\\
&\le \sum_{k=1}^{4}\|\mathbb{G}_L^{2,k}(t))*V_0^k\|_{L^r}+\sum_{k=1}^{4}\|\mathbb{G}_{RH}^{2,k}(t))*V_0^k\|_{L^r}\\
&+\int_0^t\sum_{k=1}^{4}\|\mathbb{G}_L^{2,k}(t-s)*Q^k(s))\|_{L^r}ds+\int_0^t\sum_{k=1}^{4}\|\mathbb{G}_{RH}^{2,k}(t-s)*Q^k(s))\|_{L^r}ds\\
&=:\sum_{i=1}^{4}P_i.
\end{aligned}
\tag{4.23}
$$

For the estimate of the linear part, it is easy to check that

$$
P_1\le C\|V_0\|_{L^1}\le CE_0,\ \ P_2\le Ce^{-C_0t}\|V_0\|_{L^1}\le CE_0. \tag{4.24}
$$

For the estimate of the nonlinear part, utilizing the definitions (3.4)–(3.7) of Q_i, we have

$$
\begin{aligned}
P_3&\le \int_0^t\sum_{k=1}^{4}\|\mathbb{G}_L^{2,k}(t-s)*Q^k(s))\|_{L^{\frac{1}{r}}}ds\\
&\le \int_0^t(1+t-s)^{-\frac{3}{2}(1-\frac{1}{r})-\frac{1}{2}}(\|n(s)\|_{L^1}+\|m(s)\|_{L^1})(\|v(s)\|_{L^\infty}+\|w(s)\|_{L^\infty})ds\\
&+\int_0^t(1+t-s)^{-\frac{3}{2}\cdot(1-\frac{1}{r})}(\|v(s)\|_{L^2}+\|w(s)\|_{L^2})(\|\nabla v(s)\|_{L^2}+\|\nabla w(s)\|_{L^2})ds\\
&+\int_0^t(1+t-s)^{-\frac{3}{2}\cdot(1-\frac{1}{r})}(\|n(s)\|_{L^2}+\|m(s)\|_{L^2})(\|\nabla n(s)\|_{L^2}+\|\nabla m(s)\|_{L^2})ds\\
&+\int_0^t(1+t-s)^{-\frac{3}{2}\cdot(1-\frac{1}{r})}(\|\nabla^2 v(s)\|_{L^2}+\|\nabla^2 w(s)\|_{L^2})(\|n(s)\|_{L^\infty}+\|m(s)\|_{L^\infty})ds\\
&\le CE_0\int_0^t(1+t-s)^{-\frac{7}{8}}(1+s)^{-\frac{6}{5}}ds+CE_0\int_0^t(1+t-s)^{-\frac{3}{8}}(1+s)^{-\frac{3}{4}}ds\\
&+CE_0^{\frac{2}{5}}\int_0^t(1+t-s)^{-\frac{3}{8}}(1+s)^{-\frac{6}{5}}ds\\
&\le CE_0^{\frac{2}{5}},
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
P_4 &\le \int_0^t \sum_{k=1}^{4} \|\mathbb{G}_{RH}^{2,k}(t-s) * Q^k(s))\|_{L^{\frac{1}{r}}} ds \\
&\le C\int_0^t e^{-C_0(t-s)}(\|n(s)\|_{L^1} + \|m(s)\|_{L^1})(\|v(s)\|_{L^\infty} + \|w(s)\|_{L^\infty})ds \\
&+ C\int_0^t e^{-C_0(t-s)}(\|v(s)\|_{L^2} + \|w(s)\|_{L^2})(\|\nabla v(s)\|_{L^2} + \|\nabla w(s)\|_{L^2})ds \\
&+ C\int_0^t e^{-C_0(t-s)}(\|n(s)\|_{L^2} + \|m(s)\|_{L^2})(\|\nabla n(s)\|_{L^2} + \|\nabla m(s)\|_{L^2})ds \\
&+ C\int_0^t e^{-C_0(t-s)}(\|\nabla^2 v(s)\|_{L^2} + \|\nabla^2 w(s)\|_{L^2})(\|n(s)\|_{L^2} + \|m(s)\|_{L^2})ds \\
&\le CE_0^{\frac{7}{5}}\int_0^t e^{-C_0(t-s)}(1+s)^{-\frac{6}{5}}ds + CE_0\int_0^t e^{-C_0(t-s)}(1+s)^{-\frac{3}{4}}ds \\
&+ CE_0\int_0^t e^{-C_0(t-s)}ds \\
&\le CE_0^{\frac{2}{5}}.
\end{aligned} \tag{4.26}
$$

Combining (4.24)–(4.26), we obtain

$$\|M_1\|_{L^{\frac{4}{3}}} \le CE_0^{\frac{2}{5}}.$$

The same procedure gives

$$\sum_{i=1}^{3}(\|M_i\|_{L^{\frac{4}{3}}} + \|N_i\|_{L^{\frac{4}{3}}}) \le CE_0^{\frac{2}{5}}.$$

So far, we can obtain

$$\|(v(t), w(t))\|_{L^{\frac{4}{3}}} \le CE_0^{\frac{2}{5}}.$$

Now, we can obtain the estimate of $\|(v,w)\|_{L^1}$ by using $\|(v,w)\|_{L^{\frac{4}{3}}}$. For M_1,

$$
\begin{aligned}
\|M_1\|_{L^1} \le& \sum_{k=1}^{4}\|\mathbb{G}_L^{2,k}(t)) * V_0^k\|_{L^1} + \sum_{k=1}^{4}\|\mathbb{G}_{RH}^{2,k}(t)) * V_0^k\|_{L^1} \\
&+ \int_0^t \sum_{k=1}^{4}\|\mathbb{G}_L^{2,k}(t-s) * N^k(s))\|_{L^1}ds + \int_0^t \sum_{k=1}^{4}\|\mathbb{G}_{RH}^{2,k}(t-s) * N^k(s))\|_{L^1}ds \\
&=: \sum_{i=1}^{4} F_i.
\end{aligned}
$$

A simple check gives us that

$$F_1 \le C\|V_0\|_{L^1} \le CE_0,$$

$$F_2 \le Ce^{-C_0 t}\|V_0\|_{L^1} \le CE_0.$$

Then, by using Lemma 4.6 and Young's inequality for convolution, we have

$$
\begin{aligned}
F_3 &\le \int_0^t \sum_{k=1}^4 \|\mathbb{G}_L^{2,k}(t-s) * Q^k(s))\|_{L^1} ds \\
&\le \int_0^t (1+t-s)^{-\frac{3}{2}(1-1)-\frac{1}{2}} (\|n(s)\|_{L^1} + \|m(s)\|_{L^1})(\|v(s)\|_{L^\infty} + \|w(s)\|_{L^\infty}) ds \\
&+ \int_0^t (1+t-s)^{-\frac{3}{2}\cdot(1-1)} (\|v(s)\|_{L^{\frac{4}{3}}} + \|w(s)\|_{L^{\frac{4}{3}}})(\|\nabla v(s)\|_{L^4} + \|\nabla w(s)\|_{L^4}) ds \\
&+ \int_0^t (1+t-s)^{-\frac{3}{2}\cdot(1-1)} (\|n(s)\|_{L^1} + \|m(s)\|_{L^1})(\|\nabla n(s)\|_{L^\infty} + \|\nabla m(s)\|_{L^\infty}) ds \\
&+ \int_0^t (1+t-s)^{-\frac{3}{2}\cdot(1-1)} (\|\nabla^2 v(s)\|_{L^2} + \|\nabla^2 w(s)\|_{L^2})(\|n(s)\|_{L^\infty} + \|m(s)\|_{L^\infty}) ds \\
&\le CE_0 \int_0^t (1+t-s)^{-\frac{1}{2}} (1+s)^{-\frac{6}{5}} ds + CE_0^{\frac{2}{5}} \int_0^t (1+s)^{-\frac{11}{8}} ds + CE_0 \int_0^t (1+s)^{-2} ds \\
&+ CE_0^{\frac{2}{5}} \int_0^t (1+s)^{-\frac{6}{5}} ds \\
&\le CE_0^{\frac{2}{5}},
\end{aligned} \tag{4.27}
$$

and

$$
\begin{aligned}
F_4 &\le \int_0^t \sum_{k=1}^4 \|\mathbb{G}_{RH}^{2,k}(t-s) * Q^k(s))\|_{L^1} ds \\
&\le C \int_0^t e^{-C_0(t-s)} (\|n(s)\|_{L^1} + \|m(s)\|_{L^1})(\|v(s)\|_{L^\infty} + \|w(s)\|_{L^\infty}) ds \\
&+ C \int_0^t e^{-C_0(t-s)} (\|v(s)\|_{L^2} + \|w(s)\|_{L^2})(\|\nabla v(s)\|_{L^2} + \|\nabla w(s)\|_{L^2}) ds \\
&+ C \int_0^t e^{-C_0(t-s)} (\|n(s)\|_{L^2} + \|m(s)\|_{L^2})(\|\nabla n(s)\|_{L^2} + \|\nabla m(s)\|_{L^2}) ds \\
&+ C \int_0^t e^{-C_0(t-s)} (\|\nabla^2 v(s)\|_{L^2} + \|\nabla^2 w(s)\|_{L^2})(\|n(s)\|_{L^2} + \|m(s)\|_{L^2}) ds \\
&\le CE_0^{\frac{7}{5}} \int_0^t e^{-C_0(t-s)} (1+s)^{-\frac{6}{5}} ds + CE_0 \int_0^t e^{-C_0(t-s)} (1+s)^{-\frac{3}{4}} ds \\
&+ CE_0 \int_0^t e^{-C_0(t-s)} ds \\
&\le CE_0^{\frac{2}{5}}.
\end{aligned} \tag{4.28}
$$

Combining the estimate of each F_i, we obtain

$$\|M_1\|_{L^1} \le CE_0^{\frac{2}{5}}.$$

The same procedure gives

$$\sum_{i=1}^3 (\|M_i\|_{L^1} + \|N_i\|_{L^1}) \le CE_0^{\frac{2}{5}}.$$

So far, we can obtain

$$\|(v(t), w(t))\|_{L^1} \leq CE_0^{\frac{2}{5}},$$

and we complete the proof of the lemma. □

Lemma 4.9. *Under the assumption (4.15) and the condition (4.13), we have*

$$\|\nabla^2 V(t)\|_{L^\infty(\mathbb{R}^3)} \leq CE_0^{\frac{1}{8}}(1+t)^{-\frac{6}{5}}, \quad t \in [0, T]. \tag{4.29}$$

Proof. Applying Duhamel's principle, we have

$$D^2 V(t) = D^2(\mathbb{G}(t) * V_0) + \int_0^t D^2(\mathbb{G}(t-s) * N(V)(s))ds.$$

Then,

$$\|D^2 V(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|D^2(\mathbb{G}(t) * V_0)\|_{L^\infty(\mathbb{R}^3)} + \int_0^t \|D^2(\mathbb{G}(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds.$$

Now we estimate the righthand side of the inequality. Simple computation yields

$$\begin{aligned}
&\|D^2(\mathbb{G}(t) * V_0)\|_{L^\infty(\mathbb{R}^3)} \\
&\leq \|D^2(\mathbb{G}_L(t) * V_0)\|_{L^\infty(\mathbb{R}^3)} + \|D^2(\mathbb{G}_{RH}(t) * V_0)\|_{L^\infty(\mathbb{R}^3)} + \|D^2(S(t) * V_0)\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C\|V_0\|_{L^1(\mathbb{R}^3)}(1+t)^{-\frac{5}{2}} + C\|D\mathbb{G}_{RH}\|_{L^2(\mathbb{R}^3)}\|DV_0\|_{L^2(\mathbb{R}^3)} + C\|D^2 V_0\|_{L^\infty(\mathbb{R}^3)} e^{-C_0 t} \\
&\leq C\|V_0\|_{L^1(\mathbb{R}^3)}(1+t)^{-\frac{5}{2}} + C\|V_0\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|D^2 V_0\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} e^{-C_0 t} + C\|V_0\|_{L^2(\mathbb{R}^3)}^{\frac{1}{8}}\|D^4 V_0\|_{L^2(\mathbb{R}^3)}^{\frac{7}{8}} e^{-C_0 t} \\
&\leq C(\|V_0\|_{L^1(\mathbb{R}^3)} + \|V_0\|_{L^2(\mathbb{R}^3)}^{\frac{1}{8}})(1+t)^{-\frac{6}{5}} \\
&\leq CE_0^{\frac{1}{8}}(1+t)^{-\frac{6}{5}},
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
&\int_0^t \|D^2(\mathbb{G}(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&\leq \int_0^t \|D^2(\mathbb{G}_L(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds + \int_0^t \|D^2(\mathbb{G}_{RH}(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&+ \int_0^t \|D^2(S(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&=: \sum_{i=1}^3 O_i.
\end{aligned}$$

Now we turn to estimate each O_i. By the definition (4.20) of $N(V)$, we easily check that

$$\begin{aligned} O_1 \leq & \int_0^t \|D^2\mathbb{G}_L(t-s)\|_{L^2}(\|n(s)\|_{L^2} + \|m(s)\|_{L^2} + \|v(s)\|_{L^2} + \|w(s)\|_{L^2}) \\ & (\|\nabla v(s)\|_{L^\infty} + \|\nabla w(s)\|_{L^\infty} + \|\nabla n(s)\|_{L^\infty} + \|\nabla m(s)\|_{L^\infty})ds \\ & + \int_0^t \|D^2\mathbb{G}_L(t-s)\|_{L^2}(\|v(s)\|_{L^2} + \|w(s)\|_{L^2})(\|\nabla v(s)\|_{L^\infty} + \|\nabla w(s)\|_{L^\infty})ds \\ & + \int_0^t \|D^2\mathbb{G}_L(t-s)\|_{L^2}(\|n(s)\|_{L^2} + \|m(s)\|_{L^2})\|\nabla n(s)\|_{L^\infty} + \|\nabla m(s)\|_{L^\infty})ds \\ & + \int_0^t \|D^2\mathbb{G}_L(t-s)\|_{L^2}(\|\nabla^2 v(s)\|_{L^2} + \|\nabla^2 w(s)\|_{L^2})(\|n(s)\|_{L^\infty} + \|m(s)\|_{L^\infty})ds. \end{aligned} \tag{4.31}$$

Using Sobolev's inequality, we obtain

$$\|(n(s), m(s))\|_{L^\infty} \leq CE_0^{\frac{2}{5}}(1+s)^{-\frac{6}{5}},$$

then for O_1, we have

$$\begin{aligned} O_1 \leq & CE_0 \int_0^t (1+t-s)^{-\frac{3}{4}-1}(1+s)^{-2}ds \\ & + CE_0^{\frac{2}{5}} \int_0^t (1+t-s)^{-\frac{3}{4}-1}(1+s)^{-\frac{6}{5}}ds \\ \leq & CE_0^{\frac{2}{5}}(1+t)^{-\frac{6}{5}}. \end{aligned} \tag{4.32}$$

For O_2, we can get

$$\begin{aligned} O_2 \leq & \int_0^t \|\nabla\mathbb{G}_{RH}(t-s)\|_{L^2}[(\|\nabla^2 v(s)\|_{L^2} + \|\nabla^2 w(s)\|_{L^2})(\|n(s)\|_{L^\infty} + \|m(s)\|_{L^\infty}) \\ & + (\|\nabla v(s)\|_{L^2} + \|\nabla w(s)\|_{L^2})(\|\nabla n(s)\|_{L^\infty} + \|\nabla m(s)\|_{L^\infty}) \\ & + (\|\nabla^2 n(s)\|_{L^2} + \|\nabla^2 m(s)\|_{L^2})(\|v(s)\|_{L^\infty} + \|w(s)\|_{L^\infty})]ds \\ & + \int_0^t \|\nabla\mathbb{G}_{RH}(t-s)\|_{L^2}[(\|\nabla v(s)\|_{L^2} + \|\nabla w(s)\|_{L^2})(\|\nabla v(s)\|_{L^\infty} + \|\nabla w(s)\|_{L^\infty}) \\ & + (\|\nabla^2 v(s)\|_{L^2} + \|\nabla^2 w(s)\|_{L^2})(\|v(s)\|_{L^\infty} + \|w(s)\|_{L^\infty})]ds \\ & + \int_0^t \|\nabla\mathbb{G}_{RH}(t-s)\|_{L^2}[(\|\nabla n(s)\|_{L^2} + \|\nabla m(s)\|_{L^2})(\|\nabla n(s)\|_{L^\infty} + \|\nabla m(s)\|_{L^\infty}) \\ & + (\|\nabla^2 n(s)\|_{L^2} + \|\nabla^2 m(s)\|_{L^2})(\|n(s)\|_{L^\infty} + \|m(s)\|_{L^\infty})]ds \\ & + \int_0^t \|\nabla\mathbb{G}_{RH}(t-s)\|_{L^2}[(\|\nabla^2 v(s)\|_{L^2} + \|\nabla^2 w(s)\|_{L^2})(\|\nabla n(s)\|_{L^\infty} + \|\nabla m(s)\|_{L^\infty}) \\ & + (\|\nabla^3 v(s)\|_{L^2} + \|\nabla^3 w(s)\|_{L^2})(\|n(s)\|_{L^\infty} + \|m(s)\|_{L^\infty})]ds. \end{aligned} \tag{4.33}$$

Also by Sobolev's inequality, it holds

$$\|(\nabla^2 v(s), \nabla^2 w(s))\|_{L^2} \leq C\|(v(s), w(s))\|_{L^2}^{\frac{1}{3}}\|(\nabla^3 v(s), \nabla^3 w(s))\|_{L^2}^{\frac{2}{3}},$$

then we have

$$O_2 \leq CE_0^{\frac{2}{5}} \int_0^t e^{-C_0(t-s)}(1+s)^{-\frac{6}{5}}ds + CE_0^{\frac{1}{4}} \int_0^t e^{-C_0(t-s)}(1+s)^{-2}ds + CE_0^{\frac{1}{3}} \int_0^t e^{-C_0(t-s)}(1+s)^{-2}ds \leq CE_0^{\frac{1}{3}}(1+t)^{-\frac{6}{5}}. \tag{4.34}$$

For O_3, we have

$$\begin{aligned} O_3 \leq C \int_0^t e^{-C_0(t-s)}[&(\|n(s)\|_{L^\infty} + \|m(s)\|_{L^\infty})(\|\nabla^3 v(s)\|_{L^\infty} + \|\nabla^3 w(s)\|_{L^\infty}) \\ &+ (\|\nabla n(s)\|_{L^\infty} + \|\nabla m(s)\|_{L^\infty})(\|\nabla^2 v(s)\|_{L^\infty} + \|\nabla^2 w(s)\|_{L^\infty}) \\ &+ (\|\nabla^2 n(s)\|_{L^\infty} + \|\nabla^2 m(s)\|_{L^\infty})(\|\nabla v(s)\|_{L^\infty} + \|\nabla w(s)\|_{L^\infty}) \\ &+ (\|\nabla^3 n(s)\|_{L^\infty} + \|\nabla^3 m(s)\|_{L^\infty})(\|v(s)\|_{L^\infty} + \|w(s)\|_{L^\infty})]ds. \end{aligned} \tag{4.35}$$

Sobolev's inequality gives

$$\begin{aligned} &\|(\nabla^2 n(s), \nabla^2 v(s), \nabla^2 m(s), \nabla^2 w(s))\|_{L^\infty} \\ \leq &C\|(n(s), v(s), m(s), w(s))\|_{L^2}^{\frac{1}{8}} \|(\nabla^4 n(s), \nabla^4 v(s), \nabla^4 m(s), \nabla^4 w(s))\|_{L^2}^{\frac{7}{8}}, \end{aligned} \tag{4.36}$$

then we have

$$\begin{aligned} O_3 &\leq CE_0^{\frac{2}{5}} \int_0^t e^{-C_0(t-s)}(1+s)^{-\frac{6}{5}}ds + CE_0^{\frac{1}{8}} \int_0^t e^{-C_0(t-s)}(1+s)^{-2}ds \\ &\leq CE_0^{\frac{1}{8}}(1+t)^{-\frac{6}{5}}ds. \end{aligned} \tag{4.37}$$

Combining (4.32), (4.34), and (4.37), we obtain

$$\begin{aligned} &\int_0^t \|D^2(\mathbb{G}(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)}ds \\ \leq &CE_0^{\frac{1}{8}}(1+t)^{-\frac{6}{5}}. \end{aligned} \tag{4.38}$$

Using (4.30) and (4.38), we complete the proof of the lemma. □

With the above lemmas at hand, we now check the condition 1 of Lemma 4.4, that is, the following lemma.

Lemma 4.10. *Under the assumption (4.15) and the condition (4.13), we have*

$$(\|\nabla n(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla m(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla w(t)\|_{L^\infty(\mathbb{R}^3)}) \leq \frac{\delta}{2}(1+t)^{-2}.$$

Proof. Duhamel's principle gives rise to

$$\nabla V(t) = \nabla(\mathbb{G} * V_0)(t) + \int_0^t \nabla(\mathbb{G}(t-s) * N(V)(s))ds.$$

Then

$$\|\nabla V(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\nabla(\mathbb{G} * V_0)(t)\|_{L^\infty(\mathbb{R}^3)} + \int_0^t \|\nabla(\mathbb{G}(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds.$$

Similar to the proof of Lemma 4.9, it holds for the first term on the righthand side of the inequality that

$$\|\nabla(\mathbb{G} * V_0)(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{\delta}{4}(1+t)^{-2}. \tag{4.39}$$

For for the second term on the right side of the inequality, it is easy to get that

$$\begin{aligned}
&\int_0^t \|\nabla(\mathbb{G}(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&\leq \int_0^t \|\nabla(\mathbb{G}_L(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds + \int_0^t \|\nabla(\mathbb{G}_{RH}(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&+ \int_0^t \|\nabla(S(t-s) * N(V)(s))\|_{L^\infty(\mathbb{R}^3)} ds.
\end{aligned}$$

For the nonlinear part of lower frequency, Lemma 4.5 gives

$$\begin{aligned}
&\int_0^t \|\nabla(\mathbb{G}_L^{k,\{1\}}(t-s) * N(V)^1(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&= \int_0^{\frac{t}{2}} \|\nabla(\mathbb{G}_L^{k,\{1\}}(t-s) * N(V)^1(s))\|_{L^\infty(\mathbb{R}^3)} ds + \int_{\frac{t}{2}}^t \|\nabla(\mathbb{G}_L^{k,\{1\}}(t-s) * N(V)^1(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&\leq \int_0^{\frac{t}{2}} \|D^2\mathbb{G}_L^{k,\{1\}}(t-s)\|_{L^\infty(\mathbb{R}^3)} \|(n(s), m(s)\|_{L^1(\mathbb{R}^3)} \|(v(s), w(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
&+ \int_{\frac{t}{2}}^t \|\nabla\mathbb{G}_L^{k,\{1\}}(t-s)\|_{L^1(\mathbb{R}^3)} [(\|(n(s), m(s))\|_{L^\infty(\mathbb{R}^3)} \|(\nabla v(s), \nabla w(s))\|_{L^\infty(\mathbb{R}^3)} \\
&+ \|(v(s), w(s))\|_{L^\infty(\mathbb{R}^3)} \|(\nabla n(s), \nabla m(s))\|_{L^\infty(\mathbb{R}^3)})] ds \\
&\leq CE_0 \int_0^{\frac{t}{2}} (1+t-s)^{-\frac{5}{2}} (1+s)^{-\frac{6}{5}} ds \\
&+ CE_0^{\frac{2}{5}} \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{1}{2}} (1+s)^{-\frac{6}{5}-2} ds \\
&\leq CE_0^{\frac{2}{5}} ((1+t)^{-\frac{5}{2}} + (1+t)^{\frac{27}{10}}) \\
&\leq CE_0^{\frac{2}{5}} (1+t)^{-2}.
\end{aligned} \tag{4.40}$$

For the terms of $\int_0^t \|\nabla(\mathbb{G}_L^{k,\{2,3,4\}}(t-s) * N(V)^{\{2,3,4\}}(s))\|_{L^\infty(\mathbb{R}^3)} ds$, Lemma 4.5 gives

$$\begin{aligned}
&\int_0^t \|\nabla(\mathbb{G}_L^{k,\{2,3,4\}}(t-s) * N(V)^{\{2,3,4\}}(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
\leq &\int_0^{\frac{t}{2}} \|\nabla \mathbb{G}_L^{k,\{2,3,4\}}(t-s)\|_{L^\infty(\mathbb{R}^3)} \|(v(s), w(s))\|_{L^1(\mathbb{R}^3)} \|(\nabla v(s), \nabla w(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
+ &\int_{\frac{t}{2}}^t \|\nabla \mathbb{G}_L^{k,\{2,3,4\}}(t-s)\|_{L^1(\mathbb{R}^3)} \|(v(s), w(s))\|_{L^\infty(\mathbb{R}^3)} \|(\nabla v(s), \nabla w(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
+ &\int_0^{\frac{t}{2}} \|\nabla \mathbb{G}_L^{k,\{2,3,4\}}(t-s)\|_{L^\infty(\mathbb{R}^3)} \|(n(s), m(s))\|_{L^1(\mathbb{R}^3)} \|(\nabla n(s), \nabla m(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
+ &\int_{\frac{t}{2}}^t \|\nabla \mathbb{G}_L^{k,\{2,3,4\}}(t-s)\|_{L^1(\mathbb{R}^3)} \|(n(s), m(s))\|_{L^\infty(\mathbb{R}^3)} \|(\nabla n(s), \nabla m(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
+ &\int_0^{\frac{t}{2}} \|\nabla \mathbb{G}_L^{k,\{2,3,4\}}(t-s)\|_{L^\infty(\mathbb{R}^3)} \|(n(s), m(s))\|_{L^1(\mathbb{R}^3)} \|(\nabla^2 v(s), \nabla^2 w(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
+ &\int_{\frac{t}{2}}^t \|\nabla \mathbb{G}_L^{k,\{2,3,4\}}(t-s)\|_{L^2(\mathbb{R}^3)} \|(\nabla n(s), \nabla m(s))\|_{L^\infty(\mathbb{R}^3)} \|(\nabla v(s), \nabla w(s))\|_{L^2(\mathbb{R}^3)} ds \\
+ &\int_{\frac{t}{2}}^t \|\nabla^2 \mathbb{G}_L^{k,\{2,3,4\}}(t-s)\|_{L^2(\mathbb{R}^3)} \|(n(s), m(s))\|_{L^2(\mathbb{R}^3)} \|(\nabla v(s), \nabla w(s))\|_{L^\infty(\mathbb{R}^3)} ds.
\end{aligned}$$

By Lemmas 3.2 and 4.5–4.9, we have

$$\begin{aligned}
&\int_0^t \|\nabla(\mathbb{G}_L^{k,\{2,3,4\}}(t-s) * N(V)^{\{2,3,4\}}(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
\leq &CE_0 \int_0^{\frac{t}{2}} (1+t-s)^{-2}(1+s)^{-2} ds \\
+ &CE_0^{\frac{2}{5}} \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-2-\frac{6}{5}} ds \\
+ &CE_0 \int_0^{\frac{t}{2}} (1+t-s)^{-2}(1+s)^{-2} ds \\
+ &CE_0^{\frac{9}{8}} \int_0^{\frac{t}{2}} (1+t-s)^{-2}(1+s)^{-\frac{6}{5}} ds \\
+ &CE_0^{\frac{2}{5}} \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{1}{2}-\frac{3}{4}}(1+s)^{-2-\frac{6}{5}} ds \\
+ &CE_0 \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{3}{4}-1}(1+s)^{-2} ds \\
\leq &CE_0^{\frac{2}{5}}(1+t)^{-2}.
\end{aligned} \tag{4.41}$$

The same procedure gives

$$\begin{aligned}
&\int_0^t \|\nabla(\mathbb{G}_L^{k,5}(t-s) * N(V)^5(s))\|_{L^\infty(\mathbb{R}^3)} ds \\
\leq &CE_0^{\frac{2}{5}}(1+t)^{-2},
\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}&\int_0^t \|\nabla(\mathbb{G}_L^{k,\{6,7,8\}}(t-s)*N(V)^{\{6,7,8\}}(s))\|_{L^\infty(\mathbb{R}^3)}ds\\&\le CE_0^{\frac{2}{5}}(1+t)^{-2}.\end{aligned}\tag{4.43}$$

Combining (4.40), (4.41), (4.42), and (4.43), we have

$$\begin{aligned}&\int_0^t \|\nabla(\mathbb{G}_L(t-s)*N(V)(s))\|_{L^\infty(\mathbb{R}^3)}ds\\&\le CE_0^{\frac{2}{5}}(1+t)^{-2}.\end{aligned}\tag{4.44}$$

For the nonlinear part of high frequency, Lemma 4.5 gives

$$\begin{aligned}&\int_0^t \|\nabla(\mathbb{G}_{RH}^{k,\{1,5\}}(t-s)*N(V)^{\{1,5\}}(s))\|_{L^\infty(\mathbb{R}^3)}ds\\&\le \int_0^t \|\nabla\mathbb{G}_{RH}^{k,\{1,5\}}(t-s)\|_{L^2(\mathbb{R}^3)}[(\|(n(s),m(s))\|_{L^2(\mathbb{R}^3)}\|(\nabla v(s),\nabla w(s))\|_{L^\infty(\mathbb{R}^3)}\\&+(\|(\nabla n(s),\nabla m(s))\|_{L^\infty(\mathbb{R}^3)}\|(v(s),w(s))\|_{L^2(\mathbb{R}^3)}]ds\\&\le CE_0\int_0^t e^{-C_0(t-s)}(1+s)^{-2}ds\\&\le CE_0(1+t)^{-2},\end{aligned}\tag{4.45}$$

and

$$\begin{aligned}&\int_0^t \|\nabla(\mathbb{G}_{RH}^{k,\{2,3,4,6,7,8\}}(t-s)*N(V)^{\{2,3,4,6,7,8\}}(s))\|_{L^\infty(\mathbb{R}^3)}ds\\&\le \int_0^t \|\nabla\mathbb{G}_{RH}^{k,\{2,3,4,6,7,8\}}(t-s)\|_{L^2(\mathbb{R}^3)}[(\|(v(s),w(s))\|_{L^2(\mathbb{R}^3)}\|(\nabla v(s),\nabla w(s))\|_{L^\infty(\mathbb{R}^3)}\\&+(\|(n(s),m(s))\|_{L^2(\mathbb{R}^3)}\|(\nabla n(s),\nabla m(s))\|_{L^\infty(\mathbb{R}^3)}]ds\\&+\int_0^t \|\nabla\mathbb{G}_{RH}^{k,\{2,3,4,6,7,8\}}(t-s)\|_{L^1(\mathbb{R}^3)}(\|(n(s),m(s))\|_{L^\infty(\mathbb{R}^3)})\|(\nabla^2 v(s),\nabla^2 w(s))\|_{L^\infty(\mathbb{R}^3)}ds\\&\le CE_0\int_0^t e^{-C_0(t-s)}(1+s)^{-2}ds+CE_0^{\frac{21}{40}}\int_0^t e^{-C_0(t-s)}(1+s)^{-\frac{6}{5}-\frac{6}{5}}ds\\&\le CE_0^{\frac{21}{40}}(1+t)^{-2}.\end{aligned}\tag{4.46}$$

(4.45) and (4.46) gives

$$\begin{aligned}&\int_0^t \|\nabla(\mathbb{G}_{RH}(t-s)*N(V)(s))\|_{L^\infty(\mathbb{R}^3)}ds\\&\le CE_0^{\frac{21}{40}}(1+t)^{-2}.\end{aligned}\tag{4.47}$$

For the nonlinear part estimate for the singular part, by Lemmas 4.6 and 4.9, we have

$$\begin{aligned}
&\int_0^t \|\nabla(S(t-s)*N(V)(s))\|_{L^\infty(\mathbb{R}^3)}ds \\
&\leq C\int_0^t e^{-C_0(t-s)}[\|(n(s),m(s))\|_{L^\infty(\mathbb{R}^3)}\|(\nabla^2 v(s),\nabla^2 w(s))\|_{L^\infty(\mathbb{R}^3)} \\
&+\|(\nabla n(s),\nabla m(s))\|_{L^\infty(\mathbb{R}^3)}\|(\nabla v(s),\nabla w(s))\|_{L^\infty(\mathbb{R}^3)} \\
&+\|(\nabla^2 n(s),\nabla^2 m(s))\|_{L^\infty(\mathbb{R}^3)}\|(v(s),w(s))\|_{L^\infty(\mathbb{R}^3)}]ds \\
&\leq CE_0^{\frac{21}{40}}\int_0^t e^{-C_0(t-s)}(1+s)^{-\frac{6}{5}-\frac{6}{5}}ds + CE_0^{\frac{1}{6}}\int_0^t e^{-C_0(t-s)}(1+s)^{-2}ds \\
&\leq CE_0^{\frac{1}{6}}(1+t)^{-2}.
\end{aligned} \tag{4.48}$$

Combining (4.43), (4.47), and (4.48), we obtain

$$\begin{aligned}
&\int_0^t \|\nabla(\mathbb{G}(t-s)*N(V)(s))\|_{L^\infty(\mathbb{R}^3)}ds \\
&\leq CE_0^{\frac{1}{6}}(1+t)^{-2}.
\end{aligned} \tag{4.49}$$

By (4.39) and (4.49), we complete the proof of the lemma. □

Under the condition (4.13), the bootstrap argument and the local existence of the solutions for the syestem (1.1) and (1.2) gives the following result.

Proposition 4.1. *Let* $(\rho_{1,0}-1, u_{1,0}, \rho_{2,0}-1, u_{2,0}, \nabla\Phi_0) \in H^{s+1} (s \geq 4)$, *and*

$$\|\rho_{1,0}-1, u_{1,0}, \rho_{2,0}-1, u_{2,0}, \nabla\Phi_0)\|_{L^2(\mathbb{R}^3)\bigcap L^1(\mathbb{R}^3)} \leq E_0,$$

where E_0 *is sufficiently small, then the Cauchy problem (1.1) and (1.2) has a global solution in time that satisfies*

$$(\rho_1-1, u_1, \rho_2-1, u_2) \in L^\infty([0,\infty); H^{s+1}), \ \ \nabla(u_1,u_2) \in L^2([0,\infty); H^{s+1}),$$

and

$$(\|(\nabla\rho_1, \nabla u_1, \nabla\rho_2, \nabla u_2)(t)\|_{L^\infty(\mathbb{R}^3)} \leq CE_0^{\frac{1}{6}}(1+t)^{-2}, \ \ t\in(0,+\infty). \tag{4.50}$$

5. The decay rate of solutions

In this section, we would like to get the decay rate of solutions. The main result is stated as follows.

Proposition 5.1. *Let* $(\rho_{1,0}-1, u_{1,0}, \rho_{2,0}-1, u_{2,0}, \nabla\Phi_0) \in H^{s+1} (s \geq 4)$, *and*

$$\|\rho_{1,0}-1, u_{1,0}, \rho_{2,0}-1, u_{2,0}, \nabla\Phi_0)\|_{L^2(\mathbb{R}^3)\bigcap L^1(\mathbb{R}^3)} \leq E_0,$$

where E_0 *is sufficiently small,* $(\rho_1, u_1, \rho_2, u_2)$ *is the solutions for the Cauchy problem (1.1) and (1.2), then when* $2 \leq p \leq +\infty$ *and* $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_i \geq 0$, $|\alpha| \leq s-1$, *it holds*

$$\|D^\alpha(\rho_1-1, u_1, \rho_2-1, u_2)(\cdot,t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}.$$

Proof. From (3.1) and (3.2), the decay rate of $(\rho_1 - 1, u_1, \rho_2 - 1, u_2)$ is equivalent to the decay rate of (n, v, m, w). Therefore, we only need to consider the attenuation estimate of (n, v, m, w). Note that if the L^2 decay rate of the higher-order spatial derivatives of the solution are obtained, then the general L^q decay rate of the solution follows by the Sobolev interpolation. For instance, using the Sobolev embedding theorem, we have

$$||D^\alpha V(t)||_{L^\infty} \leq ||D^\alpha V(t)||_{L^2}^{\frac{1}{4}} ||D^{\alpha+2} V(t)||_{L^2}^{\frac{3}{4}},$$

where V(t) is defined as (3.10). So we only consider the decay of $||D^\alpha V(t)||_{L^2}$. Below we will prove the following assertion by induction,

$$||D^\alpha V(\cdot, t)||_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}}, \quad 0 \leq |\alpha| \leq s+1. \tag{5.1}$$

When $|\alpha| = 0$, we have

$$\begin{aligned}
&||V(t)||_{L^2} \\
&\leq ||\mathbb{G}(t) * V_0||_{L^2} + \int_0^t ||\mathbb{G}(t-s) * N(V)(s)||_{L^2} ds. \\
&\leq ||\mathbb{G}_L(t)||_{L^2} ||V_0||_{L^1} + ||\mathbb{G}_{RH}(t)||_{L^2} ||V_0||_{L^1} + ||S * V_0||_{L^2} \\
&+ \int_0^t ||\mathbb{G}_L(t-s) * N(V)(s))||_{L^2} ds + \int_0^t ||\mathbb{G}_{RH}(t-s) * N(V)(s))||_{L^2} ds \\
&+ \int_0^t ||S(t-s) * N(V)(s)||_{L^2} ds.
\end{aligned}$$

For the terms $\int_0^t ||\mathbb{G}_L(t-s) * N(V)(s))||_{L^2} ds$ and $\int_0^t ||\mathbb{G}_{RH}(t-s) * N(V)(s))||_{L^2} ds$, we have

$$\begin{aligned}
&\int_0^t ||\mathbb{G}_L(t-s) * N(V)(s))||_{L^2} + ||\mathbb{G}_{RH}(t-s) * N(V)(s))||_{L^2} ds \\
&\leq \int_0^t (||\mathbb{G}_L(t-s)||_{L^2} + ||\mathbb{G}_{RH}(t-s)||_{L^2})[||(n,m)(s)||_{L^1} ||(\nabla v, \nabla w)(s)||_{L^\infty} \\
&+ ||(v,w)(s)||_{L^1} ||(\nabla n, \nabla m)(s)||_{L^\infty}] ds \\
&+ \int_0^t (||\mathbb{G}_L(t-s)||_{L^2} + ||\mathbb{G}_{RH}(t-s)||_{L^2}) ||(n,m)(s)||_{L^1} ||(\nabla n, \nabla m)(s)||_{L^\infty} ds \\
&+ \int_0^t (||\mathbb{G}_L(t-s)||_{L^2} + ||\mathbb{G}_{RH}(t-s)||_{L^2}) ||(v,w)(s)||_{L^1} ||(\nabla v, \nabla w)(s)||_{L^\infty} ds \\
&+ \int_0^t (||\nabla\mathbb{G}_L(t-s)||_{L^2} + ||\nabla\mathbb{G}_{RH}(t-s)||_{L^2}) ||(n,m)(s)||_{L^1} ||(\nabla v, \nabla w)(s)||_{L^\infty} ds \\
&+ \int_0^t (||\mathbb{G}_L(t-s)||_{L^2} + ||\mathbb{G}_{RH}(t-s)||_{L^2}) ||(\nabla n, \nabla m)(s)||_{L^2} ||(\nabla v, \nabla w)(s)||_{L^2} ds.
\end{aligned}$$

Then, from Proposition (4.1), using Sobolev's inequality, we obtain

$$\begin{aligned}
&\int_0^t \|\mathbb{G}_L(t-s) * N(V)(s))\|_{L^2} ds \\
&\leq C\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-2} ds + C\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{1}{2}}(1+s)^{-2} ds \\
&+ C\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{3}{4}-\frac{3}{4}} ds \\
&\leq C(1+t)^{-\frac{3}{4}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \|\mathbb{G}_{RH}(t-s) * N(V)(s))\|_{L^2} ds \\
&\leq C\int_0^t e^{-C_0(t-s)}(1+s)^{-2} ds + C\int_0^t e^{-C_0(t-s)}(1+s)^{-\frac{3}{4}-\frac{3}{4}} ds \\
&\leq C(1+t)^{-\frac{3}{4}}.
\end{aligned}$$

For the term $\int_0^t \|S(t-s) * N(V)(s)\|_{L^2} ds$, from Proposition (4.1), we have

$$\begin{aligned}
&\int_0^t \|S(t-s) * N(V)(s)\|_{L^2} ds \\
&\leq C\int_0^t e^{-C_0(t-s)}(\|(n,m)(s)\|_{L^2}\|(\nabla v, \nabla w)(s)\|_{L^\infty} + \|(v,w)(s)\|_{L^2}\|(\nabla n, \nabla m)(s)\|_{L^\infty}) ds \\
&\leq C\int_0^t e^{-C_0(t-s)}(1+s)^{-2} ds \\
&\leq C(1+t)^{-\frac{3}{4}}.
\end{aligned}$$

We assume that (5.1) holds when $|\alpha| = k-1$, and later we shall prove that (5.1) holds when $|\alpha| = k$. To get the estimate of $\|D^k V(t)\|_{L^2}$, we perform high and low frequency decomposition of the solution itself. The low frequency part of the solution V is

$$V_L = \chi(D)V,$$

and the high frequency part of the solution V is

$$V_H = (1-\chi(D))V,$$

where $\chi(D)$ is a pseudo-differential operator with symbol $\chi(\xi)$, we can see (3.14) for the definition of $\chi(\xi)$. Then, we can decompose $D^k V(t)$ as follows:

$$D^k V(t)_L = D^k(\mathbb{G} * (V_0)_L) + \int_0^t D^k(\mathbb{G}(t-s) * (N(V)(s))_L) ds, \tag{5.2}$$

$$D^k V(t)_H = D^k(\mathbf{G} * (V_0)_H) + \int_0^t D^k(\mathbf{G}(t-s) * (\widetilde{N(V)}(s))_H) ds, \tag{5.3}$$

where $\mathbb{G}$, $\mathbf{G}$, and $N(V)$ are defined in (3.12), (3.13), and (4.20), respectively. We define $\widetilde{N(V)}$ as follows:

$$\widetilde{N(V)} =: \begin{pmatrix} Q_1 \\ Q_2 \\ \widetilde{Q_3} \\ Q_4 \end{pmatrix}. \tag{5.4}$$

Here, $\widetilde{Q_3} = Q_3 + 2\nabla\triangle^{-1}m$, and Q_1, Q_2, Q_3, and Q_4 are defined in (3.4)–(3.7). For $D^kV(t)$, we have

$$D^kV(t) = D^kV(t)_L + D^kV(t)_H.$$

Then, (5.2) and (5.3) are equal to

$$D^kV(t)_L = D^k(\mathbb{G}_{\widetilde{L}} * (V_0)_L) + \int_0^t D^k(\mathbb{G}_{\widetilde{L}}(t-s) * (N(V)(s))_L)ds, \tag{5.5}$$

$$D^kV(t)_H = D^k(\mathbf{G}_{\widetilde{H}} * (V_0)_H) + \int_0^t D^k(\mathbf{G}_{\widetilde{H}}(t-s) * (\widetilde{N(V)}(s))_H)ds, \tag{5.6}$$

Now let us estimate (5.5) and (5.6) separately. For the low frequency part, we have

$$\begin{aligned} \|D^kV(t)_L\|_{L^2} \le \|D^k(\mathbb{G}_{\widetilde{L}} * (V_0)_L)\|_{L^2} &+ \int_0^{\frac{t}{2}} \|D^k(\mathbb{G}_{\widetilde{L}}(t-s) * (N(V)(s))_L)\|_{L^2}ds \\ &+ \int_{\frac{t}{2}}^t \|D^k(\mathbb{G}_{\widetilde{L}}(t-s) * (N(V)(s))_L)\|_{L^2}ds \\ &=: \sum_{i=1}^3 H_i. \end{aligned}$$

For H_1, we have

$$H_1 \le C\|D^k\mathbb{G}_{\widetilde{L}}\|_{L^2}\|(V_0)_L)\|_{L^1} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}.$$

For H_2, we first estimate $\|(N(V))_L\|_{L^1}$,

$$\begin{aligned} \|(N(V))_L\|_{L^1} &\le C\|(n,v,m,w)(t)\|_{L^1}\|(\nabla n, \nabla v, \nabla m, \nabla w)(t)\|_{L^\infty} \\ &+ \|(n,m)(t)\|_{L^1}\|(\nabla^2 v, \nabla^2 w)(t)\|_{L^\infty}. \\ &\le C(1+t)^{-2} + C(1+t)^{-\frac{5}{2}}. \end{aligned}$$

Then, for H_2, we have

$$\begin{aligned} H_2 &\le C\int_0^{\frac{t}{2}} \|D^k(\mathbb{G}_{\widetilde{L}})(t-s)\|_{L^2}\|(N(V)(s))_L\|_{L^1}ds \\ &\le C\int_0^{\frac{t}{2}} (1+t-s)^{-\frac{3}{4}-\frac{k}{2}}((1+s)^{-2} + (1+s)^{-\frac{5}{2}})ds \\ &\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}. \end{aligned}$$

For H_3, we have

$$
\begin{aligned}
H_3 &\le C\int_{\frac{t}{2}}^{t} \|D\mathbb{G}_{\widetilde{L}}(t-s)\|_{L^1}\|D^{k-1}((N(V)(s))_L)\|_{L^2}ds\\
&\le C\int_{\frac{t}{2}}^{t} \|D\mathbb{G}_{\widetilde{L}}(t-s)\|_{L^1}\|(n,v,m,w)(t)\|_{L^\infty}\|D^{k-1}(n,v,m,w)(t)\|_{L^2}ds\\
&+ C\int_{\frac{t}{2}}^{t} \|D\mathbb{G}_{\widetilde{L}}(t-s)\|_{L^1}\|\nabla(v,w)(t)\|_{L^\infty}\|D^{k-2}(n,v,m,w)(t)\|_{L^2}ds\\
&+ C\int_{\frac{t}{2}}^{t} \|D\mathbb{G}_{\widetilde{L}}(t-s)\|_{L^1}\|\nabla^2(v,w)(t)\|_{L^\infty}\|D^{k-3}(n,v,m,w)(t)\|_{L^2}ds\\
&\le C\int_{\frac{t}{2}}^{t} (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{4}-\frac{k-1}{2}}ds\\
&+ C\int_{\frac{t}{2}}^{t} (1+t-s)^{-\frac{1}{2}}(1+s)^{-2}(1+s)^{-\frac{3}{4}-\frac{k-2}{2}}ds\\
&+ C\int_{\frac{t}{2}}^{t} (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{5}{2}}(1+s)^{-\frac{3}{4}-\frac{k-3}{2}}ds\\
&\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}},
\end{aligned}
$$

where we use the Gagliardo-Nirenberg inequality to obtain the estimate for $\|(n,v,m,w)(t)\|_{L^\infty}$ and $\|\nabla^2(v,w)(t)\|_{L^\infty}$. Combining the estimate of H_1, H_2, and H_3, we obtain the low frequency part estimate of $D^kV(t)$

$$\|D^kV(t)_L\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}.$$

For the high frequency part of $D^kV(t)$, by Lemma 3.5, we have

$$
\begin{aligned}
&\|D^k(V(\cdot,t))_H\|_{L^2}^2\\
&= \|D^k(\mathbf{G}(\cdot,t)*(V_0)_H)\|_{L^2}^2\\
&+ 2\int_0^t\int_{\mathbb{R}^3} \mathbf{G}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}(x-\cdot,t-s)*D^k(N(\widetilde{V(\cdot,s)}))_H dxds.
\end{aligned}
\tag{5.7}
$$

By the definition of $\chi_2(\xi)$ in (3.16), we know (5.7) is equal to

$$
\begin{aligned}
&\|D^k(V(\cdot,t))_H\|_{L^2}^2\\
&= \|D^k(\mathbf{G}_{\widetilde{H}}(\cdot,t)*(V_0)_H)\|_{L^2}^2\\
&+ 2\int_0^t\int_{\mathbb{R}^3} \mathbf{G}_{\widetilde{H}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{H}}(x-\cdot,t-s)*D^k(N(\widetilde{V(\cdot,s)}))_H dxds\\
&= \sum_{i=1}^{2} I_i.
\end{aligned}
\tag{5.8}
$$

Now we turn to estimate each I_i. To start, for I_1,

$$
\begin{aligned}
I_1 &= |D^k(\mathbf{G}_{\widetilde{H}}(\cdot,t) * V_0)\|_{L^2}^2 \\
&\leq C\|D\mathbb{G}_{\widetilde{HR}}\|_{L^1}^2\|D^{k-1}V_0\|_{L^2}^2 + Ce^{-2C_0t}\|D^kV_0\|_{L^2}^2 \\
&\leq Ce^{-2C_0t} \\
&\leq C(1+t)^{-\frac{3}{2}-k}.
\end{aligned} \tag{5.9}
$$

From the definition of $N(V)$ and $\widetilde{N(V)}$ in (4.20), (5.4), we have

$$
\widetilde{N(V)} = N(V) + (0,0,2\nabla\triangle^{-1}m,0)^T =: \widetilde{N(V)}_1 + \widetilde{N(V)}_2. \tag{5.10}
$$

For the nonlinear item, we can check without difficulty that

$$
\begin{aligned}
&\|D^{k-1}(\widetilde{N(V)}_1)_H(t)\|_{L^2} \\
&\leq C\|(n,v,m,w)(t)\|_{L^\infty}\|D^kV(t)\|_{L^2} + C[\|(\nabla n,\nabla v,\nabla m,\nabla w)(t)\|_{L^\infty} \\
&+ \|(\nabla^2 v,\nabla^2 w)(t)\|_{L^\infty}]\|D^{k-1}V(t)\|_{L^2} + \|(n,m)(t)\|_{L^\infty}\|D^{k+1}(v,w)(t)\|_{L^2}.
\end{aligned} \tag{5.11}
$$

From the conclusion of Proposition (4.1) , we know (4.14) holds, i.e.,

$$
(\|\nabla n(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla m(t)\|_{L^\infty(\mathbb{R}^3)}, \|\nabla w(t)\|_{L^\infty(\mathbb{R}^3)}) \leq C(1+t)^{-2}, \tag{5.12}
$$

and by the Sobolev embedding theorem, we have

$$
(\|n(t)\|_{L^\infty(\mathbb{R}^3)}, \|v(t)\|_{L^\infty(\mathbb{R}^3)}, \|m(t)\|_{L^\infty(\mathbb{R}^3)}, \|w(t)\|_{L^\infty(\mathbb{R}^3)}) \leq CE_0^{\frac{2}{5}}(1+t)^{-\frac{6}{5}}. \tag{5.13}
$$

Now for I_2, we can obtain

$$\begin{aligned}
I_2 \le\, & C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{H}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{H}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_1)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{H}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{H}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_2)_H dxds\\
&\le C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_1)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{S}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_1)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{S}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_1)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{S}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{S}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_1)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_2)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{S}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_2)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{S}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_2)_H dxds\\
&+C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{S}}(x-\cdot,t-\tau)*D^k(V(\cdot,s))_H\cdot\mathbf{G}_{\widetilde{S}}(x-\cdot,t-s)*D^k((N(\widetilde{V(\cdot,s)}))_2)_H dxds\\
&=:\sum_{i=1}^{8}K_i.
\end{aligned}$$

For K_1, we have

$$\begin{aligned}
K_1 =\, & C\int_0^t\int_{\mathbb{R}^3}\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-s)*D^k(V(\cdot,s))_H\cdot D\mathbf{G}_{\widetilde{RH}}(x-\cdot,t-s)*D^{k-1}(N(V(\cdot,s))_1)_H dxds\\
&\le C\|D^kV(s)\|^2_{L^\infty(0,t;L^2)}\int_0^t e^{-2C_0(t-s)}\|(n,v,m,w)(t)\|_{L^\infty}ds\\
&+C\int_0^t e^{-2C_0(t-s)}(\|(\nabla n,\nabla v,\nabla m,\nabla w)(t)\|_{L^\infty}\\
&+\|(\nabla^2v,\nabla^2w)(t)\|_{L^\infty})\|D^{k-1}u(s)\|_{L^2}\|D^kV(s)\|_{L^2}ds\\
&+C\int_0^t e^{-2C_0(t-s)}\|(n,m)(t)\|_{L^\infty}\|D^{k+1}(v,w)(t)\|_{L^2}\|D^kV(s)\|_{L^2}ds.
\end{aligned}$$

With the help of (5.11)–(5.13), we can get the estimates of K_1 as follows. By using Lemma 3.4, we can get the estimate for K_1,

$$
\begin{aligned}
K_1 \le\ & CE_0^{\frac{2}{5}}\|D^kV(s)\|^2_{L^\infty(0,t;L^2)}\int_0^t e^{-2C_0(t-s)}(1+s)^{-\frac{6}{5}}ds + C(E_0^{\frac{1}{8}})^2\|D^kV(s)\|^2_{L^\infty(0,t;L^2)}\\
&+ C\|D^{k-1}V(s)\|^2_{L^\infty(0,t;L^2)}(\int_0^t e^{-2C_0(t-s)}(1+s)^{-\frac{6}{5}}ds)^2\\
&+ CE_0^{\frac{2}{5}}\|D^kV(s)\|^2_{L^\infty(0,t;L^2)}\\
&+ CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|^2_{L^\infty(0,t;L^2)}(\int_0^t e^{-2C_0(t-s)}(1+s)^{-\frac{6}{5}}ds)^2\\
\le\ & CE_0^{\frac{1}{4}}\|D^kV(s)\|^2_{L^\infty(0,t;L^2)} + C(1+t)^{-1}\|D^{k-1}V(s)\|^2_{L^\infty(0,t;L^2)}\\
&+ CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|^2_{L^\infty(0,t;L^2)}\\
\le\ & CE_0^{\frac{1}{4}}\|D^kV(s)\|^2_{L^\infty(0,t;L^2)} + C(1+t)^{-\frac{3}{2}-|k|} + CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|^2_{L^\infty(0,t;L^2)},
\end{aligned}
$$

where we used the (5.1) when $\alpha = k-1$ in the last inequality of the above. The estimate of K_3 is parallel to K_1, i.e.,

$$
K_3 \le CE_0^{\frac{1}{4}}\|D^kV(s)\|^2_{L^\infty(0,t;L^2)} + C(1+t)^{-\frac{3}{2}-|k|} + CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|^2_{L^\infty(0,t;L^2)}.
$$

By integrating by parts, the estimate of K_2 is similar to K_1, then we have

$$
K_2 \le CE_0^{\frac{1}{4}}\|D^kV(s)\|^2_{L^\infty(0,t;L^2)} + C(1+t)^{-\frac{3}{2}-|k|} + CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|^2_{L^\infty(0,t;L^2)}.
$$

As for K_4, by the defination of $\mathbf{G}_{\widetilde{S}}$, and using (5.12) and (5.13), we have

$$
\begin{aligned}
K_4 \le\ & C\int_0^t e^{-2C_0(t-s)}(\|(\nabla v,\nabla w)(s)\|_{L^\infty}\|D^k(n,m)(s)\|^2_{L^2}\\
&+ \|(n,m)(s)\|_{L^\infty}\|D^{k+1}(v,w)(s)\|_{L^2}\|D^k(n,m)(s)\|_{L^2})ds\\
\le\ & CE_0^{\frac{1}{6}}\|D^k(n,m)(s)\|^2_{L^\infty(0,t;L^2)} + CE_0^{\frac{2}{5}}\|D^k(n,m)(s)\|^2_{L^\infty(0,t;L^2)}\\
&+ CE_0^{\frac{2}{5}}\|D^{k+1}(v,m)(s)\|^2_{L^\infty(0,t;L^2)}\\
\le\ & CE_0^{\frac{1}{6}}\|D^k(n,m)(s)\|^2_{L^\infty(0,t;L^2)} + CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|^2_{L^\infty(0,t;L^2)}\\
\le\ & CE_0^{\frac{1}{6}}\|D^kV(s)\|^2_{L^\infty(0,t;L^2)} + CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|^2_{L^\infty(0,t;L^2)}.
\end{aligned}
$$

By the Gagliardo-Nirenberg inequality, we have

$$
\|\nabla(n,v,m,w)(t)\|_{L^2(\mathbb{R}^3)} \le CE_0^{\frac{17}{28}}(1+t)^{-\frac{3}{4}},
$$

$$
\|\nabla^{k-1}m(t)\|_{L^2(\mathbb{R}^3)} \le C\|\nabla m(t)\|^{\frac{1}{k-1}}_{L^2(\mathbb{R}^3)}\|\nabla^k m(t)\|^{\frac{k-2}{k-1}}_{L^2(\mathbb{R}^3)}.
$$

Then, for K_5, it holds

$$\begin{aligned}K_5 &\le C\int_0^t e^{-2C_0(t-s)}\|D^kV(s)\|_{L^2}\|D^{k-1}m(s)\|_{L^2}ds\\&\le CE_0^{\frac{17}{28}\cdot\frac{1}{k-1}}\|D^km(s)\|_{L^\infty(0,t;L^2)}^{\frac{k-2}{k-1}}\|D^kV(s)\|_{L^\infty(0,t;L^2)}\\&\le CE_0^{\frac{17}{28}\cdot\frac{1}{k-1}}\|D^kV(s)\|_{L^\infty(0,t;L^2)}^{1+\frac{k-2}{k-1}}\\&\le CE_0^{\frac{17}{28}\cdot\frac{1}{k-1}}\|D^kV(s)\|_{L^\infty(0,t;L^2)}^2.\end{aligned}$$

By the same analysis as K_5, it holds

$$K_7 \le CE_0^{\frac{17}{28}\cdot\frac{1}{k-1}}\|D^kV(s)\|_{L^\infty(0,t;L^2)}^2.$$

By the definition of $\mathbf{G}_{\widetilde{S}}$ in (3.17) and $\widetilde{N(V)}_2$ in (5.10), we have

$$K_6 + K_8 = 0.$$

Combining the estimate for each K_i, and from (5.7), it holds,

$$\begin{aligned}&\|D^k(V(\cdot,t))_H\|_{L^2}^2\\&\le C(E_0^{\frac{1}{4}}+E_0^{\frac{17}{28}\cdot\frac{1}{k-1}})\|D^kV(s)\|_{L^\infty(0,t;L^2)}^2 + C(1+t)^{-\frac{3}{2}-|k|} + CE_0^{\frac{2}{5}}\|D^{k+1}(v,w)(s)\|_{L^\infty(0,t;L^2)}^2.\end{aligned}\tag{5.14}$$

To close the estimate, we take our attention to the estimate of $\|D^{k+1}(v,m)(t)\|_{L^2}$. From (4.14), we know

$$D^{k+1}v = \begin{pmatrix}\sum_{j=1}^8 D^{k+1}(\mathbb{G}^{2,j}(t)*V_0^j) + \int_0^t\sum_{j=1}^8 D^{k+1}(\mathbb{G}^{2,j}(t-s)*N(V)^j(s))ds\\ \sum_{j=1}^8 D^{k+1}(\mathbb{G}^{3,j}(t)*V_0^j) + \int_0^t\sum_{j=1}^8 D^{k+1}(\mathbb{G}^{3,j}(t-s)*N(V)^j(s))ds\\ \sum_{j=1}^8 D^{k+1}(\mathbb{G}^{4,j}(t)*V_0^j) + \int_0^t\sum_{j=1}^8 D^{k+1}(\mathbb{G}^{4,j}(t-s)*N(V)^j(s))ds\end{pmatrix}.$$

For the linear partition, we have

$$\|D^{k+1}(\mathbb{G}^{i,j}(t)*V_0^j)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}-(|k|+1)}.$$

For the nonlinear partition, we still adopt the method of high and low frequency decomposition of the solution. Here we omit the details of the calculation since the analysis is parallel to the estimate of $D^kV(t)$.

For $\|D^{k+1}v(t)\|_{L^2}$, we have

$$\|D^{k+1}v(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}-(k+1)} + CE_0^{\frac{1}{4}}(\|D^kV(s)\|_{L^\infty(0,t;L^2)}^2 + \|D^{k+1}v(s)\|_{L^\infty(0,t;L^2)}^2).\tag{5.15}$$

The same way also gives

$$\|D^{k+1}w(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}-(k+1)} + CE_0^{\frac{1}{4}}(\|D^kV(s)\|_{L^\infty(0,t;L^2)}^2 + \|D^{k+1}w(s)\|_{L^\infty(0,t;L^2)}^2).\tag{5.16}$$

Combing (5.14)–(5.16), since the E_0 is small enough, we have

$$\begin{aligned}&\|D^kV(\cdot,t)\|_{L^\infty(0,t;L^2)}^2 + \|D^{k+1}(v,w)(t)\|_{L^\infty(0,t;L^2)}^2\\&\le C(1+t)^{-\frac{3}{2}-k} + C(1+t)^{-\frac{3}{2}-(k+1)} \le C(1+t)^{-\frac{3}{2}-k}.\end{aligned}$$

Now we proved (5.1) when $\alpha = k$. The proposition is proved. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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