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Research article

On the global well-posedness and exponential stability of 3D heat conducting incompressible Navier-Stokes equations with temperature-dependent coefficients and vacuum

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Abstract: This paper focuses on investigating the initial-boundary value problem of incompressible heat conducting Navier-Stokes equations with variable coefficients over bounded domains in $\mathbb{R}^3$, where the viscosity coefficient and heat conduction coefficient are powers of temperature. We obtain the global well-posedness of a strong solution under the assumption that the initial data and the measure of the initial vacuum region are sufficiently small. It is worth mentioning that the initial density is allowed to contain vacuum, and there are no restrictions on the power index of the temperature-dependent viscosity coefficient and heat conductivity coefficient. At the same time, the exponential decay-in-time results are also obtained.

Keywords: Navier-Stokes equations; degenerate and temperature-dependent transport coefficients; strong solution; vacuum

1. Introduction

In this paper, we consider the nonhomogeneous incompressible heat conducting Navier-Stokes equations that can be written as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu D(u)) + \nabla P = 0, \\ c_v[(\rho\theta)_t + \operatorname{div}(\rho u\theta)] - \operatorname{div}(\kappa\nabla\theta) = 2\mu|D(u)|^2, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

supplemented with the initial data:

$$(\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary condition:

$$u = 0, \;\; \nabla\theta \cdot \mathbf{n} = 0, \;\; \text{on } \partial\Omega. \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. ρ, u, θ, and P stand for the density, velocity, absolute temperature, and pressure of the fluid, respectively. $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the deformation tensor. The coefficients μ, c_v, and κ denote the viscosity, specific heat at constant volume, and heat conductivity, respectively.

From a physical perspective, viscosity and heat conductivity coefficients depend on temperature. In fact, when deriving Navier-Stokes equations from the Chapman-Enskog expansion based on the Boltzmann equation, Chapman and Cowling [1], Liu, Xin and Yang [2] confirmed that viscosity and heat conductivity coefficients depend on temperature. For more details, one can refer to [3–6]. This played a guiding role in later mathematical research, and scholars began to focus on investigating the case of variable coefficients. Specifically, there is a special case

$$\mu = c_1\theta^b, \;\; \kappa = c_2\theta^b,$$

where c_1, c_2, b are positive constants. For this situation, if the intermolecular potential of the cut-off reverse power force model varies with $r^{-a}(a \in (0, +\infty], r$ represents the intermolecular distance), then there is the following interesting discovery in [1]

$$b = \frac{a+4}{2a}.$$

Furthermore, $b = \frac{1}{2}$ corresponds to rigid elastic spherical molecules; $b = 1$ corresponds to Maxwellian molecules; $b = \frac{5}{2}$ corresponds to ionized gas. For these reasons, we mainly concentrate on the case that μ, κ satisfy the following physical restrictions:

$$\mu = \mu(\theta) = \theta^\alpha, \;\; \kappa = \kappa(\theta) = \theta^\beta, \;\; \text{for } \alpha, \beta \geq 0. \tag{1.4}$$

The incompressible heat conduction Navier-Stokes equations with constant coefficients have been widely studied, and a large amount of literature can be found on their well-posedness. The weak solvability of homogeneous systems was proved by Lions [7]. For inhomogeneous fluids, Zhang and Tan [8] obtained the global well-posedness of a strong solution for the initial density with a positive lower bound in the whole 3D space. When the initial density includes vacuum, the unique local strong solution was guaranteed by [9] with the help of compatibility conditions. With it, Zhong [10] extended this local solution to the global solution under the condition that $\mu^{-4}\|\sqrt{\rho_0}u_0\|_{L^2}^2\|\nabla u_0\|_{L^2}^2$ is small in 3D bounded domain. Wang, Yu and Zhang [11] studied problems with general external force and also obtained a similar result. The existence and uniqueness of the global strong solution to the Cauchy problem were proved by Zhong [12, 13], that is, 2D allows for large initial data, while 3D required $\|\rho_0\|_{L^\infty}\|\sqrt{\rho_0}u_0\|_{L^2}^2\|\nabla u_0\|_{L^2}^2$ to be sufficiently small. Furthermore, Zhong [14] also studied the 3D initial-boundary value problem with special boundaries, namely the Navier-slip boundary with velocity and Neumann boundary with temperature, and obtained a globally strong solution under the assumption that $\|\sqrt{\rho_0}u_0\|_{L^2}^2\|u_0\|_{L^2}^2$ is small. Without thermal conductivity, it is the classic incompressible Navier-Stokes equations. Abidi et al. [15, 16], and Craig et al. [17] studied the well-posedness and decay behavior in critical space. There were many important research results on the well-posedness of solutions to nonhomogeneous incompressible Navier-Stokes equations, which can be referred to [18–22], and their references.

In general, the Navier-Stokes equations with variable coefficients can more accurately describe the motion of fluids in practice. When considering that viscosity and heat conduction depend on temperature or density, the strong coupling makes the study of system (1.1) more difficult. For 3D homogeneous incompressible heat conduction Navier-Stokes equations with $\mu = \mu(\theta)$, $\kappa = \kappa(\theta)$, weak solutions were established by Naumann [23] in bounded domains and by Feireisl and Málek [24] in periodic domains, respectively. Amann [25] considered the initial-boundary value problem with coefficients depending on $D(u)$ and θ, and showed the unique solvability under the assumption of small data through the linearization method. Later, Frehse et al. [26] proved the existence of a global weak solution for shear-thickening heat conducting incompressible fluids with $\mu = \mu(\rho, \theta, |D(u)|^2)$, $\kappa = \kappa(\rho, \theta)$ in bounded regions. When the initial vacuum is taken into account, one needs to put more effort into the structure of the equation. In the 3D domain (both bounded or unbounded), Cho and Kim [9] considered the cases $\mu = \mu(\rho, \rho\theta)$, $\kappa = \kappa(\rho, \rho\theta)$ and obtained the unique local strong solution under the compatibility conditions

$$\begin{cases} -\operatorname{div}(2\mu_0 D(u_0)) + \nabla P_0 = \sqrt{\rho_0} g_1, \\ -\operatorname{div}(\kappa_0 \nabla\theta_0) - 2\mu_0 |D(u_0)|^2 = \sqrt{\rho_0} g_2, \end{cases}$$

for some $P_0 \in H^1$, $(g_1, g_2) \in L^2$. On this basis, for the case $\mu = \mu(\rho, \theta)$, $\kappa = const.$, Xu and Yu [27, 28] extended this strong solution globally under the small initial energy. Specifically, the global strong solution to the initial-boundary value problem was established in [28], and the global existence and algebraic decay of the strong solution to the Cauchy problem were established in [27]. For the 2D case, if $\mu = \mu(\rho) \geq \underline{\mu} > 0$, $\kappa = \kappa(\rho) \geq \underline{\kappa} > 0$, the global strong solution was established by Zhong [29] under the smallness on $\|\nabla\mu(\rho_0)\|_{L^q} (2 < q < \infty)$. Although these research results considered variable coefficients, it is necessary that the viscosity or heat conductivity does not degenerate. Recently, for the degenerate compressible Naiver-Stokes equations for non-isentropic flow, Xin and his collaborators [30, 31] obtained several important progresses on the local-in-time well-posedness of regular solutions with vacuum. Meanwhile, Zhang and Fang [32] also established the unique local strong solution and blow-up criterion for the 3D case. In the absence of vacuum, Guo and Li [33] showed the existence and uniqueness of a global strong solution to the problem (1.1)–(1.4) and also obtained the large-time behavior.

Based on the above research results, we are very interested in the problem of nonhomogeneous heat conduction incompressible Navier-Stokes equations with temperature-dependent viscosity and heat conductivity. Little is known about the global solvability of the initial-boundary value problem (1.1)–(1.4) with vacuum. The problem becomes quite difficult when vacuum is taken into account. To study this issue, we must deal with the following main obstacles:

(i) Strong coupling caused by temperature-dependent viscosity. Compared with the constant coefficients equation, the temperature-dependent equations have a stronger coupling between the momentum equations and the temperature equations, which makes the structure of the equations more complex. The temperature-dependent viscosity makes it difficult to estimate the velocity and temperature in the higher order, and even the lower order estimates cannot be obtained directly. Based on the regularity theory of the Stokes equations (see [21]), to get an H^2-estimate of u, we need to first deal with

$\|\nabla\mu(\theta)\|_{L^q}$. Similarly, to get the estimate of θ, we need to deal with $\|\kappa(\theta)\|_{L^\infty}$ first. Dealing with $\|\nabla\mu(\theta)\|_{L^q}$ and $\|\kappa(\theta)\|_{L^\infty}$ is essentially a matter of getting a consistent bound on temperature. It is precisely because we have no restrictions on the power index of viscosity μ and thermal conductivity κ that this creates a certain hindrance to energy estimation. To overcome this difficulty, we need to simultaneously estimate the energy of both velocity and temperature, and we need to make very detailed estimates. Meanwhile, we make full use of the bootstrap argument.

(ii) Strong nonlinearity due to temperature-dependent heat conductivity. For the case where the heat conductivity is a constant, the key is to obtain a consistent estimate of temperature. If the heat conductivity depends on temperature, the estimation of temperature is a huge challenge. In the process of controlling the additional terms generated by nonlinear terms in the energy equation, we do time-weighted energy estimates. We first obtain the upper bound of $t^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}$, and then combine it with the local existence theorem to obtain a consistent estimate of $\|\nabla^2\theta\|_{L^2}$.

(iii) Degradation caused by vacuum. Vacuum leads to the degeneracy of time evolution in momentum equations and temperature equations. When it comes to the initial vacuum, it is usually natural to introduce the compatibility condition.

These obstacles require us to make full use of its structural characteristics and make precise estimates starting from the equations. The main objective of this article is to investigate the global well-posedness and large time behavior of strong solutions to the (1.1)–(1.4) under the assumption of small initial data.

Without loss of generality, we abbreviate

$$\int f dx \triangleq \int_\Omega f dx. \tag{1.5}$$

For $1 \le r \le \infty, k \ge 1$, the standard Sobolev spaces are abbreviated as follows:

$$\begin{aligned}
&L^r = L^r(\Omega),\ \ W^{k,r} = W^{k,r}(\Omega) = \{f \in L^r | \nabla^a f \in L^r,\ \ \forall |a| \le k\},\ \ H^k = H^k(\Omega) = W^{k,2},\\
&H^1_{0,\sigma} = H^1_{0,\sigma}(\Omega) = \{f \in H^1 |\, \mathrm{div}\, f = 0,\ \ f = 0\ \text{ on }\ \partial\Omega\},\\
&H^2_{\mathbf{n}} = H^2_{\mathbf{n}}(\Omega) = \{f \in H^2 | \nabla f \cdot \mathbf{n} = 0\ \text{ on }\ \partial\Omega\}.
\end{aligned}$$

The definition of strong solutions to be established in this article is as follows:

Definition 1.1. *Given $T \in (0,\infty)$ and $q \in (3,6]$, (ρ,u,θ,P) is called a strong solution to the initial-boundary value problem (1.1)–(1.4) on $[0,T]\times\Omega$, if*

$$\begin{cases}
\rho \in C([0,T]; W^{1,q}),\ \rho_t \in C([0,T]; L^q),\\
u \in C([0,T]; H^1_{0,\sigma} \cap H^2) \cap L^2([0,T]; W^{2,q}),\\
\theta \in C([0,T]; H^2_{\mathbf{n}}) \cap L^2([0,T]; W^{2,q}),\\
P \in C([0,T]; H^1) \cap L^2([0,T]; W^{1,q}),\\
(\sqrt{\rho}u_t,\ \sqrt{\rho}\theta_t) \in L^\infty([0,T]; L^2),\ (u_t,\theta_t) \in L^2([0,T]; H^1),
\end{cases} \tag{1.6}$$

and (ρ,u,θ,P) satisfies the system (1.1)–(1.4) a.e. on $[0,T]\times\Omega$.

We prove the global existence of a strong solution, which allows for an initial density containing vacuum. The specific conclusion is as follows:

Theorem 1.1. *For some positive constants* $\underline{\theta}$ *and* $q \in (3,6]$, *suppose that the initial data* (ρ_0, u_0, θ_0) *satisfies*

$$0 \le \rho_0 \in W^{1,q}, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \quad \underline{\theta} \le \theta_0 \in H^2_{\mathbf{n}}, \tag{1.7}$$

and the compatibility conditions

$$\begin{cases} -\operatorname{div}(2\mu(\theta_0)D(u_0)) + \nabla P_0 = \sqrt{\rho_0}g_1, \\ -\operatorname{div}(\kappa(\theta_0)\nabla\theta_0) - 2\mu(\theta_0)|D(u_0)|^2 = \sqrt{\rho_0}g_2, \end{cases} \tag{1.8}$$

for some $P_0 \in H^1$ *and* $(g_1, g_2) \in L^2$. *Then there exist two positive constants* ϵ_0 *and* ϵ_1 *depending on* Ω, $\underline{\theta}$, α, β *and the initial data, such that if*

$$|V| \le \epsilon_0, \quad \|\sqrt{\rho_0}u_0\|_{L^2} + \beta\|\sqrt{\rho_0}\theta_0\|_{L^2} \le \epsilon_1, \tag{1.9}$$

where $V = \{x \in \Omega | \rho_0(x) = 0\}$, *the initial-boundary value problem* (1.1)–(1.4) *admits a unique global strong solution* (ρ, u, θ, P), *and the following large-time behavior holds:*

$$\|u\|^2_{H^2} + \|P\|^2_{H^1} + \|\sqrt{\rho}u_t\|^2_{L^2} \le Ce^{-C^{-1}t}, \tag{1.10}$$

$$\|\theta - \frac{1}{\overline{\rho_0}|\Omega|}E_0\|^2_{H^2} + \|\sqrt{\rho}\theta_t\|^2_{L^2} \le Ce^{-C^{-1}t}, \tag{1.11}$$

where

$$E_0 = \int \rho_0(\theta_0 + \frac{1}{2}|u_0|^2)dx, \qquad \overline{\rho_0} = \frac{1}{|\Omega|}\int \rho_0 dx.$$

Remark 1.1. *The authors in [34] studied the well-posedness of strong solutions to the system* (1.1)–(1.4) *in 2D and also proposed the condition* $(1.9)_1$. *In addition, for the initial-boundary value problem of temperature-dependent non-isentropic compressible Navier-Stokes equations with vacuum, the unique local strong solution was established by Cao and his collaborators [35, 36] under the assumption that the size of the initial vacuum domain is sufficiently small and compatibility conditions. From this perspective, the conditions* (1.8) *and* $(1.9)_1$ *we proposed are also reasonable. Establishing strong solutions without compatibility conditions as in [37] is meaningful and is left for further research.*

Remark 1.2. *When the initial density is far away from vacuum and* $\|\sqrt{\rho_0}u_0\|_{L^2} + \|\sqrt{\rho_0}\theta_0\|_{L^2}$ *is sufficiently small, the authors in [33] established the global existence of a strong solution to the system* (1.1)–(1.4) *and obtained the exponential decay-in-time results. Compared to their results, our results do not require the initial density to be far away from the vacuum, which is also the highlight of this article.*

Here, we would like to express some opinions on the analysis of this article. The local existence and uniqueness of strong solutions are guaranteed by Lemma 2.1. Our main idea is to reasonably combine the bootstrap argument with time-weighted estimates. We are committed to establishing consistent a prior estimates that are independent of time. As pointed out in [33], the core is to obtain the bound of θ. But the method they use is aimed at the initial density with a positive lower bound and is not suitable for situations with vacuum. In order to overcome the difficulties caused by vacuum, we borrow the technique from [36] and divide the domain Ω into two parts: $\{x \in \Omega | \rho(x) \le c_0\}$ and $\{x \in \Omega | \rho(x) > c_0\}$. Further utilize the smallness of the initial data and measure of the initial vacuum region to obtain consistent estimates. With global a priori estimates, we can successfully extend the local solution to the global one. Finally, we also obtain exponential decay.

The remaining content of this article is arranged as follows: in the second section, we reviewed the existence of local strong solutions, basic inequalities and lemmas. In the third section, we mainly establish a priori estimates and prove Theorem 1.1.

2. Preliminaries

In this section, we review the known results and basic inequalities, which are crucial for our subsequent calculations.

We first elaborate on the conclusion of the local existence of a strong solution, which is guaranteed by [9].

Lemma 2.1. *Assume that the initial data* (ρ_0, u_0, θ_0) *satisfy* (1.7) *and* (1.8). *Then there exists a small time* $T_0 > 0$ *and a unique strong solution* (ρ, u, θ, P) *to the system* (1.1)–(1.4) *on* $\Omega \times (0, T_0)$.

The following Poincaré-type inequality is extremely helpful for studying problems involving vacuum (see [24]).

Lemma 2.2. *Let* $f \in H^1$, $0 \le g \le C_1$, *and* $\int g dx \ge (C_1)^{-1}$. *Then for* $p \in [1, 6]$, *one has*

$$\|f\|_{L^p} \le C\|gf\|_{L^1} + C\|\nabla f\|_{L^2},$$

where C *depends only on* p, C_1, Ω.

The direct application of Lemma 2.2 is to control term $\|\theta_t\|_{L^6}$. Specifically, we can take $g = \rho$ and combine $\int \rho dx = \int \rho_0 dx$ to easily derive

$$\|f\|_{L^p} \le C\|\rho f\|_{L^1} + C\|\nabla f\|_{L^2},$$

where C only depends on $p, \|\rho_0\|_{L^1}, \Omega$. This inequality is frequently used in the a priori estimates in Section 3.1.

The following estimation of high-order derivatives is based on the regularity theory of the Stokes equations with variable viscosity coefficient; please refer to [38] for details.

Lemma 2.3. *Assume that* $\mu(\theta)$ *satisfies*

$$\mu(\theta) \in C^1[0, \infty), \;\; 0 < \underline{\mu} \le \mu(\theta) \le \overline{\mu} < \infty, \;\; \nabla\mu(\theta) \in L^q \; (3 < q \le 6).$$

Let $(u, P) \in H^1_{0,\sigma} \times L^2$ *be the unique weak solution to the following system*

$$\begin{cases} -\operatorname{div}(2\mu(\theta)D(u)) + \nabla P = F, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \displaystyle\int \frac{P}{\mu(\theta)} dx = 0. & \end{cases} \tag{2.1}$$

There are the following conclusions:

(1) If $F \in L^r$ with $2 \le r < q$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and

$$\|u\|_{W^{2,r}} + \|\frac{P}{\mu(\theta)}\|_{W^{1,r}} \le C(1 + \|\nabla\mu(\theta)\|_{L^q}^{\frac{q}{q-3}\times\frac{5r-6}{2r}})\|F\|_{L^r}. \tag{2.2}$$

(2)If $F \in H^1$, $\nabla^2\mu(\theta) \in L^2$, then $(u, P) \in H^3 \times H^2$ and

$$\|u\|_{H^3} + \|\frac{P}{\mu(\theta)}\|_{H^2} \le C(1 + \|\nabla\mu(\theta)\|_{H^1}^{4+\frac{q}{q-3}})\|F\|_{H^1}. \tag{2.3}$$

Here C depends only on $\underline{\mu}, \overline{\mu}, q, r, \Omega$.

Proof. The proof of (2.2) has been given in [38], but for simplicity, we have omitted it here. We now provide a detailed proof process for (2.3). We rewrite (2.1) as

$$-\Delta u + \nabla(\frac{P}{\mu(\theta)}) = \frac{F}{\mu(\theta)} + \frac{2\nabla\mu(\theta)\cdot D(u)}{\mu(\theta)} - \frac{P\nabla\mu(\theta)}{\mu(\theta)^2}.$$

According to the regularity theory of the Stokes system, Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&\|u\|_{H^3} + \|\frac{P}{\mu(\theta)}\|_{H^2} \\
&\le C\|\frac{F}{\mu(\theta)}\|_{H^1} + C\|\frac{\nabla\mu(\theta)\cdot D(u)}{\mu(\theta)}\|_{H^1} + C\|\frac{P\nabla\mu(\theta)}{\mu(\theta)^2}\|_{H^1} \\
&\le C\|F\|_{L^2} + C\|\nabla F\|_{L^2} + C\|F\nabla\mu(\theta)\|_{L^2} + C\|\nabla\mu(\theta)\cdot D(u)\|_{L^2} + C\|\nabla\mu(\theta)\frac{P}{\mu(\theta)}\|_{L^2} \\
&\quad + C\||\nabla^2\mu(\theta)||D(u)|\|_{L^2} + C\||\nabla^2\mu(\theta)||\frac{P}{\mu(\theta)}|\|_{L^2} + C\||\nabla\mu(\theta)||\nabla^2 u|\|_{L^2} \\
&\quad + C\||\nabla\mu(\theta)||\nabla(\frac{P}{\mu(\theta)})|\|_{L^2} + C\||\nabla\mu(\theta)|^2|D(u)|\|_{L^2} + C\||\nabla\mu(\theta)|^2|\frac{P}{\mu(\theta)}|\|_{L^2} \\
&\le C\|F\|_{H^1} + C\|F\|_{L^4}\|\nabla\mu(\theta)\|_{L^4} + C\|\nabla\mu(\theta)\|_{L^q}\|(\nabla u, \frac{P}{\mu(\theta)})\|_{L^{\frac{2q}{q-2}}} \\
&\quad + C\|\nabla^2\mu(\theta)\|_{L^2}\|(\nabla u, \frac{P}{\mu(\theta)})\|_{L^\infty} + C\|\nabla\mu(\theta)\|_{L^4}\|(\nabla^2 u, \nabla(\frac{P}{\mu(\theta)}))\|_{L^4} \\
&\quad + C\|\nabla\mu(\theta)\|_{L^6}^2\|(\nabla u, \frac{P}{\mu(\theta)})\|_{L^6} \\
&\le C(1 + \|\nabla\mu(\theta)\|_{H^1})\|F\|_{H^1} + C\|\nabla\mu(\theta)\|_{H^1}\|(\nabla u, \frac{P}{\mu(\theta)})\|_{H^1} \\
&\quad + C\|\nabla\mu(\theta)\|_{H^1}(\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla^3 u\|_{L^2}^{\frac{3}{4}} + \|\frac{P}{\mu(\theta)}\|_{L^2}^{\frac{1}{4}}\|\nabla^2(\frac{P}{\mu(\theta)})\|_{L^2}^{\frac{3}{4}} + \|(\nabla u, \frac{P}{\mu(\theta)})\|_{L^2}) \\
&\quad + C\|\nabla\mu(\theta)\|_{H^1}(\|\nabla^2 u\|_{L^2}^{\frac{1}{4}}\|\nabla^3 u\|_{L^2}^{\frac{3}{4}} + \|\nabla(\frac{P}{\mu(\theta)})\|_{L^2}^{\frac{1}{4}}\|\nabla^2(\frac{P}{\mu(\theta)})\|_{L^2}^{\frac{3}{4}} + \|(\nabla^2 u, \nabla(\frac{P}{\mu(\theta)}))\|_{L^2}) \\
&\quad + C\|\nabla\mu(\theta)\|_{H^1}^2\|(\nabla u, \frac{P}{\mu(\theta)})\|_{H^1} \\
&\le \frac{1}{2}\|(\nabla^3 u, \nabla^2(\frac{P}{\mu(\theta)}))\|_{L^2} + C(1 + \|\nabla\mu(\theta)\|_{H^1})\|F\|_{H^1}
\end{aligned}$$

$$+ C(\|\nabla\mu(\theta)\|_{H^1} + \|\nabla\mu(\theta)\|_{H^1}^2 + \|\nabla\mu(\theta)\|_{H^1}^4)\|(\nabla u, \frac{P}{\mu(\theta)})\|_{H^1},$$

where we have used the following inequality

$$\|\nabla^2 u\|_{L^4} \le C\|\nabla^2 u\|_{L^2}^{\frac{1}{4}}\|\nabla^3 u\|_{L^2}^{\frac{3}{4}} + C\|\nabla^2 u\|_{L^2}.$$

By virtue of (2.2),

$$\|u\|_{H^2} + \|\frac{P}{\mu(\theta)}\|_{H^1} \le C(1 + \|\nabla\mu(\theta)\|_{L^q}^{\frac{q}{q-3}})\|F\|_{L^2}.$$

Hence,

$$\|u\|_{H^3} + \|\frac{P}{\mu(\theta)}\|_{H^2} \le C(1 + \|\nabla\mu(\theta)\|_{H^1}^{4+\frac{q}{q-3}})\|F\|_{H^1},$$

which implies that (2.3) is valid. We finish the proof of Lemma 2.3. □

3. Global well-posedness

This section is divided into two subsections. The first subsection is to establish time-weighted a priori estimates. The second subsection extends the unique local strong solution to the global strong solution using the bootstrap argument based on the obtained estimates. It also proves the conclusion of exponential decay.

3.1. A priori estimates

In this subsection, we will establish the necessary a priori estimates, which can extend the local solution to the global solution. In order to get consistent estimates that do not depend on time, we do some time-weighted estimates. Let (ρ, u, θ, P) be the strong solution of Navier-Stokes equations (1.1)–(1.4) with initial data (ρ_0, u_0, θ_0) on $\Omega \times (0, T]$, and the initial data satisfies conditions (1.7)–(1.9). For simplicity, let us take $c_v = 1$. For a general positive constant c_v, the calculation is the same. The letter C represents a generic positive constant that depends on $\tilde{\rho} \triangleq \|\rho_0\|_{L^\infty}, \underline{\theta}$, and Ω but does not depend on T.

Our main idea is to apply the bootstrap argument. To this end, we first propose the following a priori assumptions:

$$\sup_{0\le t\le T}(\|\nabla u\|_{H^1}^2 + \|\theta\|_{H^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2) + \int_0^T \|(\sqrt{\rho}\theta_t, \nabla\theta_t, \nabla^2 u, \nabla^2\theta)\|_{L^2}^2 dt \le M. \tag{3.1}$$

Here M is a positive constant that depends only on the initial data.

We first use the standard method to find the upper and lower bounds of density and the lower bound of temperature.

Lemma 3.1. *It holds that* $\forall\,(x, t) \in \Omega \times [0, T]$,

$$0 \le \rho(x, t) \le \tilde{\rho}, \quad \underline{\theta} \le \theta(x, t), \tag{3.2}$$

$$\sup_{0\le t\le T}(t^i\|\sqrt{\rho}u\|_{L^2}^2) + \int_0^T t^i\|\nabla u\|_{L^2}^2 dt \le C\|\sqrt{\rho_0}u_0\|_{L^2}^2, \quad i = 0, ..., 8. \tag{3.3}$$

Proof. The proof of $(3.2)_1$ is provided by [7, 24, 33]. Then, applying the standard maximum principle (see [36] or [24, p.43]) to $(1.1)_3$ along with $\theta_0 \geq \underline{\theta}$ shows $(3.2)_2$. Finally, multiplying $(1.1)_2$ by u and then integrating over Ω, using integration by parts, one has

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2dx + 2\int \theta^\alpha|D(u)|^2dx = 0. \tag{3.4}$$

By integrating the above equation on $[0,t]$ and with the help of $\theta \geq \underline{\theta}$ and $2\|D(u)\|_{L^2}^2 = \|\nabla u\|_{L^2}^2$, it can be inferred that

$$\sup_{0\leq t\leq T}\|\sqrt{\rho}u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2dt \leq C\|\sqrt{\rho_0}u_0\|_{L^2}^2. \tag{3.5}$$

Furthermore, multiplying (3.4) by t, integrating with respect to t, and taking advantage of Poincaré inequality and (3.5) yield

$$\sup_{0\leq t\leq T}(t\|\sqrt{\rho}u\|_{L^2}^2) + \int_0^T t\|\nabla u\|_{L^2}^2dt \leq C\int_0^T \|\sqrt{\rho}u\|_{L^2}^2dt \leq C\int_0^T \|\nabla u\|_{L^2}^2dt \leq C\|\sqrt{\rho_0}u_0\|_{L^2}^2. \tag{3.6}$$

Similarly, for $i = 2, ..., 8$, the following formula can also be obtained

$$\sup_{0\leq t\leq T}(t^i\|\sqrt{\rho}u\|_{L^2}^2) + \int_0^T t^i\|\nabla u\|_{L^2}^2dt \leq C\|\sqrt{\rho_0}u_0\|_{L^2}^2, \tag{3.7}$$

which, combined with (3.5) and (3.6), can prove that (3.3) holds. □

Lemma 3.2. *Suppose (ρ, u, θ, P) satisfies the assumptions* (3.1)*. There exists a small positive constant ϵ_1 depending on Ω, $\underline{\theta}$, α, β and the initial data, such that for any $i = 1, 2, 3, 4$,*

$$\sup_{0\leq t\leq T}\|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2dt \leq C, \tag{3.8}$$

$$\sup_{0\leq t\leq T}(t^i\|\nabla u\|_{L^2}^2) + \int_0^T t^i\|\sqrt{\rho}u_t\|_{L^2}^2dt \leq \epsilon_1^{\frac{1}{3}}. \tag{3.9}$$

Proof. Notice that Lemma 2.2 can directly provide

$$\|\theta_t\|_{L^p} \leq C\|\rho\theta_t\|_{L^1} + \|\nabla\theta_t\|_{L^2}, \quad \forall\, p \in [1,6]. \tag{3.10}$$

Multiplying $(1.1)_2$ by u_t and integrating it over Ω, according to (3.10) and the Gagliardo-Nirenberg inequality, we can easily show

$$\begin{aligned}
&\frac{d}{dt}\int \theta^\alpha|D(u)|^2dx + \|\sqrt{\rho}u_t\|_{L^2}^2 \\
&= \int \alpha\theta^{\alpha-1}\theta_t|D(u)|^2dx - \int \rho(u\cdot\nabla u)\cdot u_tdx \\
&\leq C(M)\|\theta_t\|_{L^6}\|\nabla u\|_{L^{\frac{12}{5}}}^2 + C\|\sqrt{\rho}u\|_{L^3}\|\nabla u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2} \\
&\leq C(M)\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}} + \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}^2.
\end{aligned} \tag{3.11}$$

Then, integrating the last identity over $[0,t]$, by virtue of Holder's inequality, (3.1) and Lemma 3.1, we can establish

$$\begin{aligned}
&\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2+\int_0^T\|\sqrt{\rho}u_t\|_{L^2}^2dt\\
&\le C\|\nabla u_0\|_{L^2}^2+C(M)\sup_{0\le t\le T}\|\nabla u\|_{L^2}\Big(\int_0^T\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2}^2dt\Big)^{\frac{1}{2}}\Big(\int_0^T\|\nabla u\|_{L^2}^2dt\Big)^{\frac{1}{4}}\Big(\int_0^T\|\nabla u\|_{H^1}^2dt\Big)^{\frac{1}{4}}\\
&\quad+C\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_{L^2}\|\nabla u\|_{L^2})\int_0^T\|\nabla u\|_{H^1}^2dt\\
&\le C+C(M)\epsilon_1^{\frac{1}{2}}+C(M)\epsilon_1\\
&\le C,
\end{aligned}$$

provided that $\epsilon_1\ll 1$.

Similarly, multiplying (3.11) by $t^i(i\ge 1)$, then integrating it over $[0,t]$, one has from (3.1) and Lemma 3.1 that

$$\begin{aligned}
&\sup_{0\le t\le T}(t^i\|\nabla u\|_{L^2}^2)+\int_0^T t^i\|\sqrt{\rho}u_t\|_{L^2}^2dt\\
&\le C\int_0^T t^{i-1}\|\nabla u\|_{L^2}^2dt+C(M)\Big(\int_0^T t^{4i}\|\nabla u\|_{L^2}^2dt\Big)^{\frac{1}{4}}+C(M)\sup_{0\le t\le T}(t^i\|\sqrt{\rho}u\|_{L^2})\\
&\le C\epsilon_1^2+C(M)\epsilon_1^{\frac{1}{2}}+C(M)\epsilon_1\\
&\le \epsilon_1^{\frac{1}{3}}.
\end{aligned}$$

The proof of Lemma 3.2 is completed. □

Lemma 3.3. *Suppose* (ρ,u,θ,P) *satisfies the assumptions* (3.1). *There exist two small positive constants* ϵ_0 *and* ϵ_1 *depending on* Ω, $\underline{\theta}$, α, β *and the initial data, such that*

$$\sup_{0\le t\le T}((1+t)\|\sqrt{\rho}u_t\|_{L^2}^2)+\int_0^T(1+t)\|\nabla u_t\|_{L^2}^2dt\le C, \tag{3.12}$$

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C. \tag{3.13}$$

Proof. Differentiating $(1.1)_2$ with respect to t, multiplying the result identity by u_t, then integrating over Ω yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2+2\int\theta^\alpha|D(u_t)|^2dx\\
&=\int\operatorname{div}(\rho u)|u_t|^2dx+\int\operatorname{div}(\rho u)u\cdot\nabla u\cdot u_tdx-\int\rho u_t\cdot\nabla u\cdot u_tdx-2\alpha\int\theta^{\alpha-1}\theta_tD(u):\nabla u_tdx\\
&\le 2\int\rho|u||u_t||\nabla u_t|dx+\int\rho|u||\nabla(u\cdot\nabla u\cdot u_t)|dx+\int\rho|u_t|^2|\nabla u|dx+C(M)\int|\theta_t||\nabla u||\nabla u_t|dx.
\end{aligned}\tag{3.14}$$

It follows from Hölder's inequality, the Gagliardo-Nirenberg inequality, Lemmas 3.1–3.2, and (3.1) that

$$\int\rho|u||\nabla(u\cdot\nabla u\cdot u_t)|dx$$

$$
\begin{aligned}
&\le C\|u\|_{L^6}(\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\|u_t\|_{L^6} + \|u\|_{L^6}\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6} + \|u\|_{L^6}\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2})\\
&\le C\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}\\
&\le \frac{1}{4}\int \theta^\alpha |D(u_t)|^2 dx + C\|\nabla u\|_{L^2}^4\|\nabla u\|_{H^1}^2,
\end{aligned}
$$

$$
\begin{aligned}
&2\int \rho|u||u_t||\nabla u_t|dx + \int \rho|u_t|^2|\nabla u|dx\\
&\le C\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2} + (\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{4}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{3}{4}})^2\|\nabla u\|_{L^2}\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}\\
&\le \frac{1}{4}\int \theta^\alpha |D(u_t)|^2 dx + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4.
\end{aligned}
$$

The estimate of the last term in inequality (3.14) is more subtle. In view of $\rho_0 \in W^{1,q} \hookrightarrow C$, we know that there exists a positive constant c_0 such that

$$
|V_{c_0}| \le 2|V|,
$$

where $V_{c_0} = \{x \in \Omega | \rho_0(x) \le c_0\}$ and $V = \{x \in \Omega | \rho_0(x) = 0\}$. Then

$$
\begin{aligned}
&C(M)\int |\theta_t||\nabla u||\nabla u_t|dx\\
&= C(M)\int_{\rho\le c_0} |\theta_t||\nabla u||\nabla u_t|dx + C(M)\int_{\rho>c_0} |\theta_t||\nabla u||\nabla u_t|dx\\
&\le C(M)\|1\|_{L^{12}(\rho\le c_0)}\|\theta_t\|_{L^6}\|\nabla u\|_{L^4}\|\nabla u_t\|_{L^2} + C(M)c_0^{-\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla u\|_{L^\infty}\|\nabla u_t\|_{L^2}\\
&\le C(M)|V_{c_0}|^{\frac{1}{12}}\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{H^1}^{\frac{3}{4}}\|\nabla u_t\|_{L^2} + C(M)c_0^{-\frac{1}{2}}\|\nabla u\|_{L^\infty}\|\nabla u_t\|_{L^2}\\
&\le \frac{1}{4}\int \theta^\alpha |D(u_t)|^2 dx + C(M)|V|^{\frac{1}{6}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^2 + C(M)c_0^{-1}\|\nabla u\|_{L^\infty}^2 \qquad (3.15)
\end{aligned}
$$

owing to $|\{x \in \Omega | \rho(x,t) \le c_0\}| = |V_{c_0}|$ (see [7, Theorem 2.1]), (3.1), (3.10) and the Gagliardo-Nirenberg inequality. Moreover, since

$$
\begin{aligned}
\|\nabla^2 u\|_{L^6} &\le C(M)\|\rho u_t + \rho u\cdot\nabla u\|_{L^6}\\
&\le C(M)\|u_t\|_{L^6} + C(M)\|u\|_{L^\infty}\|\nabla u\|_{L^6}\\
&\le C(M)\|\nabla u_t\|_{L^2} + C(M)\|\nabla u\|_{H^1}^2,
\end{aligned}
$$

then according to the Gagliardo-Nirenberg inequality, we find

$$
\begin{aligned}
\|\nabla u\|_{L^\infty} &\le C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla^2 u\|_{L^6}^{\frac{3}{4}} + C\|\nabla u\|_{L^2}\\
&\le C(M)\|\nabla u\|_{L^2}^{\frac{1}{4}}(\|\nabla u_t\|_{L^2}^{\frac{3}{4}} + \|\nabla u\|_{H^1}^{\frac{3}{2}}) + C\|\nabla u\|_{L^2}. \qquad (3.16)
\end{aligned}
$$

Now (3.14) implies

$$
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int \theta^\alpha |D(u_t)|^2 dx
$$

$$\leq C(M)(1+c_0^{-4})\|\nabla u\|_{L^2}^2 + C(M)|V|^{\frac{1}{6}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^2$$
$$+ C(M)c_0^{-1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^3 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4. \tag{3.17}$$

Next, multiplying (3.17) by $1+t$ and integrating it over $[0, t]$, we obtain after using Grönwall's inequality, Lemmas 3.1–3.2, and (3.1) that

$$\begin{aligned}
&\sup_{0\leq t\leq T}((1+t)\|\sqrt{\rho}u_t\|_{L^2}^2) + \int_0^T (1+t)\|\nabla u_t\|_{L^2}^2 dt \\
&\leq \exp\{C\sup_{0\leq t\leq T}\|\nabla u\|_{L^2}^2\int_0^T \|\nabla u\|_{L^2}^2 dt\}[C + C\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2 dt + C(M)(1+c_0^{-4})\epsilon_1^2 + C(M)\epsilon_0^{\frac{1}{6}} \\
&\quad + C(M)c_0^{-1}(\int_0^T (1+t)^4\|\nabla u\|_{L^2}^2 dt)^{\frac{1}{4}}(\int_0^T \|\nabla^2 u\|_{L^2}^2 dt)^{\frac{3}{4}}] \\
&\leq C + C(M)c_0^{-1}\epsilon_1^{\frac{1}{2}} \\
&\leq C,
\end{aligned}$$

provided that $\epsilon_0 \ll 1,\ \epsilon_1 \ll 1$.

Furthermore, we calculate from $(1.1)_1$ that

$$\partial_i\rho_t + u\cdot\nabla\partial_i\rho + \partial_i u\cdot\nabla\rho = 0.$$

Then, we multiply the last identity by $q|\nabla\rho|^{q-2}\partial_i\rho$ and integrate it over Ω to obtain

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \leq C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}. \tag{3.18}$$

Combining (3.16) and Lemmas 3.1 and 3.2, we ascertain

$$\begin{aligned}
&\int_0^T \|\nabla u\|_{L^\infty} dt \\
&\leq C(M)(\int_0^T \|\nabla u\|_{L^2}^2 dt)^{\frac{1}{8}}((\int_0^T \|\nabla u_t\|_{L^2}^2 dt)^{\frac{3}{8}}T^{\frac{1}{2}} + (\int_0^T \|\nabla u\|_{H^1}^2 dt)^{\frac{3}{4}}T^{\frac{1}{8}}) \\
&\quad + C(\int_0^T \|\nabla u\|_{L^2}^2 dt)^{\frac{1}{2}}T^{\frac{1}{2}} \\
&\leq C(M)\epsilon_1^{\frac{1}{4}} + C\epsilon_1 \\
&\leq C, \qquad if\ 0\leq T\leq 1,
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\int_0^T \|\nabla u\|_{L^\infty} dt \\
&= \int_0^1 \|\nabla u\|_{L^\infty} dt + \int_1^T \|\nabla u\|_{L^\infty} dt \\
&\leq C(M)\epsilon_1^{\frac{1}{4}} + C(M)(\int_1^T t^8\|\nabla u\|_{L^2}^2 dt)^{\frac{1}{8}}(\int_1^T \|\nabla u_t\|_{L^2}^2 dt)^{\frac{3}{8}}(\int_1^T t^{-2} dt)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+C(M)(\int_1^T t^2\|\nabla u\|_{L^2}^2 dt)^{\frac{1}{8}}(\int_1^T \|\nabla u\|_{H^1}^2 dt)^{\frac{3}{4}}(\int_1^T t^{-2}dt)^{\frac{1}{8}}\\
&+C(\int_1^T t^2\|\nabla u\|_{L^2}^2 dt)^{\frac{1}{2}}(\int_1^T t^{-2}dt)^{\frac{1}{2}}\\
\le\ & C(M)\epsilon_1^{\frac{1}{4}}+C\epsilon_1\\
\le\ & C, \qquad if\ T\ge 1.
\end{aligned}
\tag{3.20}
$$

Now according to (3.18) and Grönwall's inequality, one has

$$
\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le\|\nabla\rho_0\|_{L^q}\exp(C\int_0^T\|\nabla u\|_{L^\infty}dt)\le C.
$$

The proof of this lemma is finished. □

Next, we will make estimates for each order regarding θ. This is the key to all a priori estimates.

Lemma 3.4. *Suppose* (ρ, u, θ, P) *satisfies the assumptions* (3.1). *It holds that*

$$
\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_{L^2}^2+\int_0^T\|\nabla\theta\|_{L^2}^2dt\le C\|\sqrt{\rho_0}\theta_0\|_{L^2}^2+\epsilon_1, \tag{3.21}
$$

$$
\sup_{0\le t\le T}\|\rho\theta^{\beta+2}\|_{L^1}+\int_0^T\|\nabla\theta^{\beta+1}\|_{L^2}^2dt\le C\|\rho_0\theta_0^{\beta+2}\|_{L^1}+\epsilon_1, \tag{3.22}
$$

$$
\sup_{0\le t\le T}\|\nabla\theta^{\beta+1}\|_{L^2}^2+\int_0^T\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2dt\le C, \tag{3.23}
$$

$$
\sup_{0\le t\le T}(t\|\nabla\theta^{\beta+1}\|_{L^2}^2)+\int_0^T t\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2dt\le C\|\rho_0\theta_0^{\beta+2}\|_{L^1}+\epsilon_1^{\frac{1}{2}}, \tag{3.24}
$$

$$
\sup_{0\le t\le T}\|\theta\|_{H^1}\le C. \tag{3.25}
$$

Proof. Firstly, multiplying $(1.1)_3$ by θ and integrating it over Ω yields

$$
\frac{1}{2}\frac{d}{dt}\int\rho|\theta|^2dx+\int\theta^\beta|\nabla\theta|^2dx=2\int\theta^{\alpha+1}|D(u)|^2dx.
$$

By integrating the above equation on $[0,t]$, we obtain from Lemma 3.1 that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_{L^2}^2+\int_0^T\|\nabla\theta\|_{L^2}^2dt\\
\le\ & C\|\sqrt{\rho_0}\theta_0\|_{L^2}^2+C(M)\int_0^T\|\nabla u\|_{L^2}^2dt\\
\le\ & C\|\sqrt{\rho_0}\theta_0\|_{L^2}^2+C(M)\epsilon_1^2\\
\le\ & C\|\sqrt{\rho_0}\theta_0\|_{L^2}^2+\epsilon_1.
\end{aligned}
$$

Similarly, multiplying $(1.1)_3$ by $\theta^{\beta+1}$ and integrating it over $\Omega\times[0,t]$, it can be inferred that

$$
\sup_{0\le t\le T}\|\rho\theta^{\beta+2}\|_{L^1}+\int_0^T\|\nabla\theta^{\beta+1}\|_{L^2}^2dt\le C\|\rho_0\theta_0^{\beta+2}\|_{L^1}+\epsilon_1. \tag{3.26}
$$

Next, taking $(1.1)_1$ and (1.3) into account, multiplying $(1.1)_3$ by $\theta^\beta\theta_t$, and integrating by parts over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\theta^{\beta+1}\|_{L^2}^2+\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2\\
&=-\int(\rho u\cdot\nabla\theta)\theta^\beta\theta_t dx+\int 2\theta^\alpha|D(u)|^2\theta^\beta\theta_t dx\\
&\le\|\sqrt{\rho}u\|_{L^3}\|\nabla\theta\|_{L^6}\|\sqrt{\rho}\theta^\beta\theta_t\|_{L^2}+2\|\theta\|_{L^\infty}^{\alpha+\beta}\|D(u)\|_{L^{\frac{12}{5}}}^2\|\theta_t\|_{L^6}\\
&\le C(M)\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}\\
&\quad+C(M)\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2}\\
&\le\frac{1}{2}\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2+C(M)\|\sqrt{\rho}u\|_{L^2}\|\nabla^2\theta\|_{L^2}^2+C(M)\|\nabla u\|_{L^2}\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2},
\end{aligned}\tag{3.27}$$

by the Gagliardo-Nirenberg inequality and (3.1). On the one hand, integrating (3.27) with respect to t leads to

$$\begin{aligned}
&\sup_{0\le t\le T}\|\nabla\theta^{\beta+1}\|_{L^2}^2+\int_0^T\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2dt\\
&\le\|\nabla\theta_0^{\beta+1}\|_{L^2}^2+C(M)\sup_{0\le t\le T}\|\sqrt{\rho}u\|_{L^2}\int_0^T\|\nabla^2\theta\|_{L^2}^2dt\\
&\quad+C(M)(\int_0^T\|\nabla u\|_{L^2}^2dt)^{\frac{1}{2}}(\int_0^T\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2}^2dt)^{\frac{1}{2}}\\
&\le C+C(M)\epsilon_1\\
&\le C,
\end{aligned}\tag{3.28}$$

where in the next to last inequality one has used Lemma 3.1 and (3.1). On the other hand, it follows from (3.27), (3.1), (3.26), and Lemma 3.1 that

$$\begin{aligned}
&\sup_{0\le t\le T}(t\|\nabla\theta^{\beta+1}\|_{L^2}^2)+\int_0^T t\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2dt\\
&\le\int_0^T\|\nabla\theta^{\beta+1}\|_{L^2}^2dt+C(M)\sup_{0\le t\le T}(t\|\sqrt{\rho}u\|_{L^2})+C(M)(\int_0^T t^2\|\nabla u\|_{L^2}^2dt)^{\frac{1}{2}}\\
&\le C\|\rho_0\theta_0^{\beta+2}\|_{L^1}+\epsilon_1+C(M)\epsilon_1\\
&\le C\|\rho_0\theta_0^{\beta+2}\|_{L^1}+\epsilon_1^{\frac{1}{2}}.
\end{aligned}$$

By applying Lemma 2.2, (3.26) and (3.28), one can derive

$$\sup_{0\le t\le T}\|\theta\|_{H^1}\le C\sup_{0\le t\le T}(\|\rho\theta\|_{L^1}+\|\nabla\theta\|_{L^2})\le C.$$

Therefore, we finish the proof of Lemma 3.4. □

Lemma 3.5. *Suppose* (ρ,u,θ,P) *satisfies the assumptions* (3.1). *It has*

$$\sup_{0\le t\le T}\|(\sqrt{\rho}\theta_t,\nabla^2\theta)\|_{L^2}^2+\int_0^T\|(\nabla^2\theta,\theta^{\frac{\beta}{2}}\nabla\theta_t)\|_{L^2}^2dt\le C.\tag{3.29}$$

Proof. Differentiating $(1.1)_3$ with respect to t yields

$$\begin{aligned}\rho\theta_{tt}+\rho u\cdot\nabla\theta_t-\operatorname{div}(\theta^{\beta}\nabla\theta_t)=&-\rho_t\theta_t-\rho_t u\cdot\nabla\theta-\rho u_t\cdot\nabla\theta+2\alpha\theta^{\alpha-1}\theta_t|D(u)|^2\\&+2\theta^{\alpha}\partial_t(|D(u)|^2)+\operatorname{div}(\beta\theta^{\beta-1}\theta_t\nabla\theta).\end{aligned}\tag{3.30}$$

Multiplying (3.30) by θ_t, then integrating it over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\theta^{\frac{\beta}{2}}\nabla\theta_t\|_{L^2}^2\\&=-\int\rho_t\theta_t^2dx-\int\rho_t u\cdot\nabla\theta\theta_tdx-\int\rho u_t\cdot\nabla\theta\theta_tdx+2\int\theta^{\alpha}(|D(u)|^2)_t\theta_tdx\\&\quad+2\alpha\int\theta^{\alpha-1}\theta_t|D(u)|^2\theta_tdx-\beta\int\theta^{\beta-1}\theta_t\nabla\theta\cdot\nabla\theta_tdx\\&\triangleq\sum_{i=1}^{6}J_i.\end{aligned}\tag{3.31}$$

Now, we will use (1.1), the Gagliardo-Nirenberg inequality, (3.1), (3.10), Lemmas 3.1–3.4, and (3.15) to estimate each term on the right hand of (3.31) as follows:

$$\begin{aligned}J_1&\le\int\rho|u||\theta_t||\nabla\theta_t|dx\le\|\rho u\|_{L^3}\|\theta_t\|_{L^6}\|\nabla\theta_t\|_{L^2}\\&\le C\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2}\|\nabla\theta_t\|_{L^2}\\&\le C(M)\epsilon_1^{\frac{1}{2}}\|(\sqrt{\rho}\theta_t,\theta^{\frac{\beta}{2}}\nabla\theta_t)\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}J_2&=\int u\cdot\nabla\rho u\cdot\nabla\theta\theta_tdx\le\|u\|_{L^{\frac{6q}{q-3}}}^2\|\nabla\rho\|_{L^q}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}\\&\le C\|u\|_{L^2}^{\frac{2q-3}{2q}}\|\nabla^2u\|_{L^2}^{\frac{2q+3}{2q}}\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2}\\&\le\frac{1}{8}\|(\sqrt{\rho}\theta_t,\theta^{\frac{\beta}{2}}\nabla\theta_t)\|_{L^2}^2+C\|u\|_{L^2}^{\frac{2q-3}{q}}\|\nabla^2u\|_{L^2}^{\frac{2q+3}{q}},\end{aligned}$$

$$\begin{aligned}J_3&\le\|\sqrt{\rho}u_t\|_{L^6}\|\nabla\theta\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^3}\\&\le C\|\nabla u_t\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2}^{\frac{1}{2}}\\&\le\frac{1}{8}\|\theta^{\frac{\beta}{2}}\nabla\theta_t\|_{L^2}^2+C\|(\nabla u_t,\ \sqrt{\rho}\theta_t)\|_{L^2}^2,\end{aligned}$$

$$J_4\le C\|\nabla u_t\|_{L^2}^2+C(M)\epsilon_0^{\frac{1}{6}}\|(\sqrt{\rho}\theta_t,\theta^{\frac{\beta}{2}}\nabla\theta_t)\|_{L^2}^2+C(M)c_0^{-1}\|\nabla u\|_{L^\infty}^2,$$

$$\begin{aligned}J_5&\le C(M)\int_{\rho\le c_0}|\nabla u|^2\theta_t^2dx+C(M)\int_{\rho>c_0}|\nabla u|^2\theta_t^2dx\\&\le C(M)\|1\|_{L^3(\rho\le c_0)}\|\nabla u\|_{L^6}^2\|\theta_t\|_{L^6}^2+C(M)c_0^{-1}\int_{\rho>c_0}|\nabla u|^2\rho\theta_t^2dx\end{aligned}$$

$$\leq C(M)\epsilon_0^{\frac{1}{3}}\|(\sqrt{\rho}\theta_t,\theta^{\frac{\beta}{2}}\nabla\theta_t)\|_{L^2}^2+C(M)c_0^{-1}\|\nabla u\|_{L^\infty}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2,$$

and

$$\begin{aligned}
J_6 &\leq C(M)\beta\int_{\rho\leq c_0}|\theta_t||\nabla\theta\cdot\nabla\theta_t|dx+C(M)c_0^{-\frac{1}{2}}\beta\int_{\rho>c_0}\sqrt{\rho}|\theta_t||\nabla\theta\cdot\nabla\theta_t|dx\\
&\leq C(M)\|1\|_{L^6(\rho\leq c_0)}\|\theta_t\|_{L^6}\|\nabla\theta\|_{L^6}\|\nabla\theta_t\|_{L^2}+C(M)c_0^{-\frac{1}{2}}\beta\|\sqrt{\rho}\theta_t\|_{L^4}\|\nabla\theta\|_{L^4}\|\nabla\theta_t\|_{L^2}\\
&\leq C(M)|V|^{\frac{1}{6}}\|(\sqrt{\rho}\theta_t,\theta^{\frac{\beta}{2}}\nabla\theta_t)\|_{L^2}^2\\
&\quad+C(M)c_0^{-\frac{1}{2}}\beta\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{4}}\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|_{L^2}^{\frac{3}{4}}\|\nabla\theta\|_{L^2}^{\frac{1}{4}}\|\nabla^2\theta\|_{L^2}^{\frac{3}{4}}\|\nabla\theta_t\|_{L^2}\\
&\leq (C(M)\epsilon_0^{\frac{1}{6}}+\frac{1}{8})\|(\sqrt{\rho}\theta_t,\theta^{\frac{\beta}{2}}\nabla\theta_t)\|_{L^2}^2+C(M)c_0^{-4}\beta^8\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla\theta\|_{L^2}^2.
\end{aligned}$$

Substituting all the estimates of $J_i(i=1,...,6)$ into (3.31), then taking ϵ_0 and ϵ_1 satisfy

$$\frac{3}{8}+C(M)\epsilon_1^{\frac{1}{2}}+C(M)\epsilon_0^{\frac{1}{6}}+C(M)\epsilon_0^{\frac{1}{3}}\leq\frac{1}{2},$$

we deduce

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\theta^{\frac{\beta}{2}}\nabla\theta_t\|_{L^2}^2\\
&\leq C\|(\nabla u_t,\ \sqrt{\rho}\theta_t)\|_{L^2}^2+C\|u\|_{L^2}^{\frac{2q-3}{q}}\|\nabla^2u\|_{L^2}^{\frac{2q+3}{q}}+C(M)c_0^{-1}\|\nabla u\|_{L^\infty}^2\\
&\quad+C(M)c_0^{-1}(\|\nabla u\|_{L^\infty}^2+c_0^{-3}\beta^8\|\nabla\theta\|_{L^2}^2)\|\sqrt{\rho}\theta_t\|_{L^2}^2.
\end{aligned}\tag{3.32}$$

In view of Poincaré's inequality, Lemma 3.1 and (3.1), one has

$$\begin{aligned}
&C\int_0^T t^i\|u\|_{L^2}^{\frac{2q-3}{q}}\|\nabla^2u\|_{L^2}^{\frac{2q+3}{q}}dt\\
&\leq C\sup_{0\leq t\leq T}\|\nabla^2u\|_{L^2}^2\Big(\int_0^T t^{i\frac{2q}{2q-3}}\|\nabla u\|_{L^2}^2dt\Big)^{\frac{2q-3}{2q}}\Big(\int_0^T\|\nabla^2u\|_{L^2}^2dt\Big)^{\frac{3}{2q}}\\
&\leq C(M)\epsilon_1^{\frac{2q-3}{q}}\\
&\leq C,\qquad i=0,1.
\end{aligned}\tag{3.33}$$

Next, similar to (3.19), we infer that

$$\begin{aligned}
&C(M)c_0^{-1}\int_0^T t^i\|\nabla u\|_{L^\infty}^2dt\\
&\leq C(M)c_0^{-1}\Big(\int_0^T t^{4i}\|\nabla u\|_{L^2}^2dt\Big)^{\frac{1}{4}}\Big(\Big(\int_0^T\|\nabla u_t\|_{L^2}^2dt\Big)^{\frac{3}{4}}+\Big(\int_0^T\|\nabla u\|_{H^1}^4dt\Big)^{\frac{3}{4}}\Big)\\
&\quad+C(M)c_0^{-1}\int_0^T t^i\|\nabla u\|_{L^2}^2dt\\
&\leq C(M)c_0^{-1}\epsilon_1^{\frac{1}{2}}+C(M)c_0^{-1}\epsilon_1^2\\
&\leq C,\qquad i=0,1.
\end{aligned}\tag{3.34}$$

Furthermore, we employ Lemma 3.4 to obtain

$$\begin{aligned} C(M)c_0^{-4}\beta^8 \int_0^T \|\nabla\theta\|_{L^2}^2 dt &\le C(M)c_0^{-4}\beta^8 (C\|\sqrt{\rho_0}\theta_0\|_{L^2}^2 + \epsilon_1) \\ &\le C(M)c_0^{-4}(C\epsilon_1^2 + \epsilon_1) \\ &\le C. \end{aligned} \tag{3.35}$$

Combining (3.32)–(3.35), Lemmas 3.3, 3.4, and Grönwall's inequality gives

$$\sup_{0\le t\le T} ((1+t)\|\sqrt{\rho}\theta_t\|_{L^2}^2) + \int_0^T (1+t)\|\theta^{\frac{\beta}{2}}\nabla\theta_t\|_{L^2}^2 dt \le C. \tag{3.36}$$

We then rewrite $(1.1)_3$ as

$$\begin{cases} -\Delta\theta^{\beta+1} = (\beta+1)(2\theta^\alpha|D(u)|^2 - \rho\theta_t - \rho u\cdot\nabla\theta), & \text{in } \Omega, \\ \nabla\theta\cdot\mathbf{n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.37}$$

It follows from the standard L^2-theory of elliptic equations that

$$\begin{aligned} \|\nabla^2\theta^{\beta+1}\|_{L^2} &\le C\|(\Delta\theta^{\beta+1}, \nabla\theta^{\beta+1})\|_{L^2} \\ &\le C\|\theta\|_{L^\infty}^\alpha \|\nabla u\|_{L^4}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2} + C\|u\|_{L^\infty}\|\nabla\theta\|_{L^2} + C\|\nabla\theta^{\beta+1}\|_{L^2} \\ &\le C(M)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{3}{2}} + C\|\sqrt{\rho}\theta_t\|_{L^2} + C\|\nabla\theta^{\beta+1}\|_{L^2}, \end{aligned} \tag{3.38}$$

where in the last inequality we have used (3.1). Note that

$$\nabla^2\theta^{\beta+1} = \beta(\beta+1)\theta^{\beta-1}\nabla\theta\otimes\nabla\theta + (\beta+1)\theta^\beta\nabla^2\theta,$$

thus

$$\begin{aligned} \|\nabla^2\theta\|_{L^2} &\le C\|\theta^\beta\nabla^2\theta\|_{L^2} \\ &\le C\|\nabla^2\theta^{\beta+1}\|_{L^2} + C\|\nabla\theta^{\beta+1}\|_{L^6}\|\nabla\theta\|_{L^3} \\ &\le C\|\nabla^2\theta^{\beta+1}\|_{L^2} + C\|\nabla\theta^{\beta+1}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta^{\beta+1}\|_{L^2}^{\frac{3}{2}} \\ &\le C(M)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{3}{2}} + C\|\sqrt{\rho}\theta_t\|_{L^2} + C\|\nabla\theta^{\beta+1}\|_{L^2} \\ &\quad + C\|\nabla\theta^{\beta+1}\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{3}{2}} + C\|\nabla\theta^{\beta+1}\|_{L^2}^2. \end{aligned} \tag{3.39}$$

On the one hand, according to (3.38), (3.1), Lemma 3.2, (3.36), and Lemma 3.4, we compute

$$\begin{aligned} t^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2} &\le C(M)t^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + Ct^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2} + Ct^{\frac{1}{2}}\|\nabla\theta^{\beta+1}\|_{L^2} \\ &\quad + Ct^{\frac{1}{2}}\|\nabla\theta^{\beta+1}\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{3}{2}} + Ct^{\frac{1}{2}}\|\nabla\theta^{\beta+1}\|_{L^2}^2 \\ &\le C(M)\epsilon_1^{\frac{1}{12}} + C \\ &\le C. \end{aligned} \tag{3.40}$$

According to the local existence theorem of the solution (see Lemma 2.1), there exists a small positive time T_0 such that

$$\sup_{0\le t\le T_0} \|\nabla^2\theta\|_{L^2} \le C.$$

This, together with (3.40), yields

$$\sup_{0\le t\le T} \|\nabla^2\theta\|_{L^2} \le \sup_{0\le t\le T_0} \|\nabla^2\theta\|_{L^2} + T_0^{-\frac{1}{2}} \sup_{T_0\le t\le T} (t^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}) \le C. \tag{3.41}$$

Moreover, Lemma 3.4, the Gagliardo-Nirenberg inequality, and (3.41) imply

$$\sup_{0\le t\le T} \|\theta\|_{L^\infty} \le C \sup_{0\le t\le T} \|\theta\|_{H^2} \le C, \tag{3.42}$$

$$\sup_{0\le t\le T} \|\nabla^2\theta^{\beta+1}\|_{L^2} \le C \sup_{0\le t\le T} (\|\nabla\theta\|_{L^4}^2 + \|\nabla^2\theta\|_{L^2}) \le C. \tag{3.43}$$

On the other hand, by (3.1), (3.38), and (3.39), we discover

$$\begin{aligned}
&\int_0^T \|\nabla^2\theta\|_{L^2}^2 dt \\
&\le C(M) \sup_{0\le t\le T} \|\nabla u\|_{H^1}^2 (\int_0^T \|\nabla u\|_{L^2}^2 dt)^{\frac{1}{2}} (\int_0^T \|\nabla u\|_{H^1}^2 dt)^{\frac{1}{2}} + C \int_0^T \|(\sqrt{\rho}\theta_t, \nabla\theta^{\beta+1})\|_{L^2}^2 dt \\
&\quad + C \sup_{0\le t\le T} (\|\nabla\theta^{\beta+1}\|_{L^2} \|\sqrt{\rho}\theta_t\|_{L^2}) \int_0^T \|\sqrt{\rho}\theta_t\|_{L^2}^2 dt + C \sup_{0\le t\le T} \|\nabla\theta^{\beta+1}\|_{L^2}^2 \int_0^T \|\nabla\theta^{\beta+1}\|_{L^2}^2 dt \\
&\le C(M)\epsilon_1 + C \\
&\le C.
\end{aligned} \tag{3.44}$$

Consequently, combining (3.36), (3.41) with (3.44), we complete the proof of this lemma. □

By synthesizing Lemmas 3.1–3.5, the following corollary can be obtained.

Corollary 3.1. *There exist two small positive constants ϵ_0 and ϵ_1 as described in Theorem 1.1, such that if*

$$|V| \le \epsilon_0, \quad \|\sqrt{\rho_0}u_0\|_{L^2} + \beta\|\sqrt{\rho_0}\theta_0\|_{L^2} \le \epsilon_1,$$

then it holds for any $(x,t) \in \Omega \times [0,T]$ that

$$\rho(x,t) \ge 0, \qquad \theta(x,t) \ge \underline{\theta},$$

$$\begin{aligned}
&\sup_{0\le t\le T} (\|\rho\|_{W^{1,q}} + \|\rho_t\|_{L^q} + \|(u,\theta)\|_{H^2}^2 + \|\frac{P}{\theta^\alpha}\|_{H^1} + \|(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t)\|_{L^2}^2) \\
&+ \int_0^T (\|(u,\theta)\|_{H^3}^2 + \|\frac{P}{\theta^\alpha}\|_{H^2}^2 + \|(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t, \nabla u_t, \nabla\theta_t)\|_{L^2}^2) dt \le C.
\end{aligned}$$

Proof. Now, we consider the following Stokes equations:

$$\begin{cases} -\operatorname{div}(2\theta^{\alpha}D(u)) + \nabla P = -\rho u_t - \rho u\cdot\nabla u, & \text{in } \Omega,\\ \operatorname{div} u = 0, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.45}$$

According to Lemma 2.3, we have the following regularity results:

$$\begin{aligned} \|(\nabla u, \frac{P}{\theta^{\alpha}})\|_{H^1} &\le C\|\rho u_t + \rho u\cdot\nabla u\|_{L^2}\\ &\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3}\\ &\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}\\ &\le \frac{1}{2}\|\nabla u\|_{H^1} + C\|\sqrt{\rho}u_t\|_{L^2} + C\|\nabla u\|_{L^2}^{3}, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} \|(\nabla u, \frac{P}{\theta^{\alpha}})\|_{H^2} &\le C\|\rho u_t + \rho u\cdot\nabla u\|_{H^1}\\ &\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\nabla\rho\|_{L^q}\|u_t\|_{L^{\frac{q}{q-2}}} + C\|\nabla u_t\|_{L^2}\\ &\quad + \|\nabla\rho\|_{L^q}\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{q}{q-2}}} + C\|\nabla u\|_{L^4}^2 + C\|u\|_{L^\infty}\|\nabla^2 u\|_{L^2}\\ &\le C\|(\sqrt{\rho}u_t, \nabla u_t)\|_{L^2} + C\|\nabla u\|_{H^1}^2, \end{aligned} \tag{3.47}$$

which indicates that

$$\|(\nabla u, P)\|_{H^1} \le C(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^3). \tag{3.48}$$

Using Lemmas 3.1–3.3, it can be inferred that

$$\sup_{0\le t\le T}\|(\nabla u, P)\|_{H^1} \le C\sup_{0\le t\le T}(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^3) \le C, \tag{3.49}$$

$$\int_0^T \|(\nabla u, P)\|_{H^2}^2 dt \le C\int_0^T (\|(\sqrt{\rho}u_t, \nabla u_t)\|_{L^2}^2 + \|\nabla u\|_{H^1}^4)dt \le C. \tag{3.50}$$

Hence,

$$\|\rho_t\|_{L^q} = \|u\cdot\nabla\rho\|_{L^q} \le \|u\|_{L^\infty}\|\nabla\rho\|_{L^q} \le C\|\nabla u\|_{H^1}\|\nabla\rho\|_{L^q} \le C. \tag{3.51}$$

On the other hand, by virtue of (3.37) and the regularity theory to the elliptic equation, one has

$$\begin{aligned} \|\nabla^2\theta\|_{H^1} &\le C\|\theta^{-1}\nabla\theta\cdot\nabla\theta + \theta^{-\beta}(2\theta^{\alpha}|D(u)|^2 - \rho\theta_t - \rho u\cdot\nabla\theta)\|_{H^1} + \|\theta\|_{H^1}\\ &\le C\|\nabla\theta\|_{L^4}^2 + C\|\nabla u\|_{L^4}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2} + C\|u\|_{L^\infty}\|\nabla\theta\|_{L^2} + C\|\nabla\theta\|_{L^6}^3\\ &\quad + C\|\nabla^2\theta\|_{L^4}\|\nabla\theta\|_{L^4} + C\|\nabla\theta\|_{L^6}\|\nabla u\|_{L^6}^2 + C\|\nabla^2 u\|_{L^4}\|\nabla u\|_{L^4} + C\|\nabla\theta\|_{L^4}\|\theta_t\|_{L^4}\\ &\quad + C\|\nabla\rho\|_{L^q}\|\theta_t\|_{L^{\frac{2q}{q-2}}} + C\|\nabla\theta_t\|_{L^2} + C\|\nabla\theta\|_{L^4}^2\|u\|_{L^\infty} + C\|\nabla\rho\|_{L^q}\|u\|_{L^\infty}\|\nabla\theta\|_{L^{\frac{2q}{q-2}}}\\ &\quad + C\|\nabla u\|_{L^4}\|\nabla\theta\|_{L^4} + C\|u\|_{L^\infty}\|\nabla^2\theta\|_{L^2} + \|\rho\theta\|_{L^1} + \|\nabla\theta\|_{L^2} \end{aligned}$$

$$\leq \frac{1}{2}\|\nabla^3\theta\|_{L^2} + C\|\nabla^3 u\|_{L^2} + C\|(\nabla u, \nabla\theta)\|_{H^1} + C\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2} + \|\rho\theta\|_{L^1}. \tag{3.52}$$

Combining (3.47), (3.52) and estimates we have obtained, it can be inferred that

$$\int_0^T \|\nabla^3\theta\|_{L^2}^2 dt \leq C\int_0^T (\|(\nabla u, \nabla\theta)\|_{H^1}^2 + \|(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t, \nabla u_t, \nabla\theta_t)\|_{L^2}^2)dt \leq C. \tag{3.53}$$

Considering the estimates obtained in Lemmas 3.1–3.5, combined with (3.49)–(3.51) and (3.53), we have completed the proof of the Corollary 3.1. In this way, we close the a priori assumptions (3.1). □

3.2. *Proof of Theorem 1.1*

With all the a priori estimates established in Corollary 3.1 at hand, the global well-posedness part of Theorem 1.1 then follows by standard procedures.

From Lemma 2.1, we know that there exists a T_0 such that the system (1.1)–(1.4) has a unique local strong solution (ρ, u, θ, P) on $\Omega \times (0, T_0]$. We now extend this local solution to the global solution by contradiction. Therefore, from now on, we assume that $|V| \leq \epsilon_0$, $\|\sqrt{\rho_0}u_0\|_{L^2} + \beta\|\sqrt{\rho_0}\theta_0\|_{L^2} \leq \epsilon_1$ holds, where ϵ_0 and ϵ_1 are the same as in Corollary 3.1. Lemma 2.1 and (1.7) indicate that there is a $T_1 \in (0, T_0)$ such that (3.1) holds at $T = T_1$. Denote

$$T^* \triangleq \sup\{T|(\rho, u, \theta, P) \text{ is a strong solution on } \Omega \times (0, T] \text{ and } (3.1) \text{ holds}\}. \tag{3.54}$$

Obviously, $T^* \geq T_1 > 0$. With the help of Corollary 3.1 and the standard embedding, one can deduce that for any $0 < T \leq T^*$,

$$\rho \in C([0, T]; W^{1,q}), \quad u \in C([0, T]; H^1_{0,\sigma} \cap H^2), \quad \theta \in C([0, T]; H^2_{\mathbf{n}}). \tag{3.55}$$

Now we prove

$$T^* = \infty. \tag{3.56}$$

Otherwise, we assume that $T^* < \infty$. Since $|V| \leq \epsilon_0$, $\|\sqrt{\rho_0}u_0\|_{L^2} + \beta\|\sqrt{\rho_0}\theta_0\|_{L^2} \leq \epsilon_1$, Lemmas 3.1–3.5 and the continuity argument show that the global estimates stated in Corollary 3.1 hold on $[0, T^*]$. Let $\dot{v} \triangleq v_t + u \cdot \nabla v$, it can be inferred from (3.55) that

$$\begin{aligned}(\rho^*, u^*, \theta^*)(x) &\triangleq (\rho, u, \theta)(x, T^*) = \lim_{t\to T^*}(\rho, u, \theta)(x, t),\\ (\rho^*\dot{u}^*, \rho^*\dot{\theta}^*)(x) &\triangleq (\rho\dot{u}, \rho\dot{\theta})(x, T^*) = \lim_{t\to T^*}(\rho\dot{u}, \rho\dot{\theta})(x, t)\end{aligned} \tag{3.57}$$

satisfies regularity condition

$$0 \leq \rho^* \in W^{1,q}, \quad u^* \in H^1_{0,\sigma} \cap H^2, \quad \underline{\theta} \leq \theta^* \in H^2_{\mathbf{n}}.$$

Thus

$$\begin{cases} -\operatorname{div}(2\mu(\theta^*)D(u^*)) + \nabla P^* = \sqrt{\rho^*}g_1, \\ -\operatorname{div}(\kappa(\theta^*)\nabla\theta^*) - 2\mu(\theta^*)|D(u^*)|^2 = \sqrt{\rho^*}g_2, \end{cases}$$

with

$$g_1 = \begin{cases} 0, & x \in \{x|\rho^*(x) = 0\}, \\ \frac{1}{\sqrt{\rho^*}}\rho^*\dot{u^*}, & x \in \{x|\rho^*(x) > 0\}, \end{cases}$$

$$g_2 = \begin{cases} 0, & x \in \{x|\rho^*(x) = 0\}, \\ \frac{1}{\sqrt{\rho^*}}\rho^*\dot{\theta^*}, & x \in \{x|\rho^*(x) > 0\}, \end{cases}$$

satisfying g_1, $g_2 \in L^2$, which can be guaranteed by Corollary 3.1. Therefore, taking (ρ^*, u^*, θ^*) as the initial data, according to Lemma 2.1, the local strong solution can be extended beyond T^*. This contradicts the definition of T^*. So, (3.54) has been proven.

To complete the entire proof of Theorem 1.1, we only need to prove (1.10) and (1.11), i.e., the following proposition.

Proposition 3.1. *It holds that*

$$\sup_{0\le t\le T}(e^{C^{-1}t}(\|u\|_{H^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|P\|_{H^1}^2)) + \int_0^T e^{C^{-1}t}\|(\nabla u, u_t)\|_{H^1}^2 dt \le C, \tag{3.58}$$

$$\sup_{0\le t\le T}(e^{C^{-1}t}(\|\theta - \frac{1}{\overline{\rho_0}|\Omega|}E_0\|_{H^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2)) + \int_0^T e^{C^{-1}t}\|(\nabla\theta, \theta_t)\|_{H^1}^2 dt \le C, \tag{3.59}$$

where the positive constants $\overline{\rho_0}$ and E_0 are defined in Theorem 1.1.

Proof. The proof is divided into the following two steps.
Step 1. The proof of (3.58).

It follows from the Poincaré inequality that

$$\int \rho|u|^2 dx \le C\int |\nabla u|^2 dx \le C\int \theta^\alpha |D(u)|^2 dx. \tag{3.60}$$

Substituting (3.60) into (3.4) yields

$$\frac{d}{dt}\int \rho|u|^2 dx + C^{-1}\int \rho|u|^2 dx + \int \theta^\alpha|D(u)|^2 dx \le 0. \tag{3.61}$$

Then multiplying (3.61) by $e^{C^{-1}t}$ and integrating it over $[0, t]$, we arrive at

$$\sup_{0\le t\le T}(e^{C^{-1}t}\|\sqrt{\rho}u\|_{L^2}^2) + \int_0^T e^{C^{-1}t}\|\nabla u\|_{L^2}^2 dt \le C. \tag{3.62}$$

Next, substitute (3.46) into (3.11), then (3.11) becomes

$$\begin{aligned}
&\frac{d}{dt}\int \theta^\alpha|D(u)|^2 dx + \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 \\
&\le C\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}} + C\|\sqrt{\rho}u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}^2 \\
&\le C\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}\|\nabla u\|_{L^2}^{\frac{3}{2}}(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}} + \|\nabla u\|_{L^2}^{\frac{3}{2}}) + C\|\sqrt{\rho}u\|_{L^2}\|\nabla u\|_{L^2}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^6)
\end{aligned}$$

$$\leq (\frac{1}{8} + C\epsilon_1)\|(\sqrt{\rho}u_t, \nabla u)\|_{L^2}^2 + C\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^{\frac{4}{3}}\|\nabla u\|_{L^2}^2,$$

owing to Young's inequality, Lemma 3.1 and Corollary 3.1. Multiply the last inequality by $e^{C^{-1}t}$, using Grönwall's inequality, Lemma 3.4, (3.36), (3.62) and

$$\int_0^T \|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^{\frac{4}{3}}dt \leq (\int_0^T (1+t)\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^2 dt)^{\frac{2}{3}}(\int_0^T (1+t)^{-2}dt)^{\frac{1}{3}} \leq C,$$

imply

$$\sup_{0\leq t\leq T}(e^{C^{-1}t}\|\nabla u\|_{L^2}^2) + \int_0^T e^{C^{-1}t}\|\sqrt{\rho}u_t\|_{L^2}^2 dt \leq Ce^{C\int_0^T \|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^{\frac{4}{3}}dt} \leq C. \tag{3.63}$$

Similarly, in view of the proof of Lemma 3.3, (3.48) and

$$\begin{aligned}\int |\theta_t||\nabla u||\nabla u_t|dx &\leq \|\theta_t\|_{L^6}\|\nabla u\|_{L^3}\|\nabla u_t\|_{L^2}\\ &\leq C\|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}\|(\sqrt{\rho}u_t, \nabla u)\|_{L^2}\|\nabla u_t\|_{L^2},\end{aligned} \tag{3.64}$$

we thereby obtain

$$\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int \theta^\alpha |D(u_t)|^2 dx \leq C(1 + \|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^2)\|(\sqrt{\rho}u_t, \nabla u)\|_{L^2}^2.$$

Multiplying the last inequality by $e^{C^{-1}t}$, applying (3.62), (3.63), Grönwall's inequality, and Corollary 3.1, we deduce

$$\sup_{0\leq t\leq T}(e^{C^{-1}t}\|\sqrt{\rho}u_t\|_{L^2}^2) + \int_0^T e^{C^{-1}t}\|\nabla u_t\|_{L^2}^2 dt \leq Ce^{C\int_0^T \|(\sqrt{\rho}\theta_t, \nabla\theta_t)\|_{L^2}^2 dt} \leq C. \tag{3.65}$$

Thus, collecting (3.62)–(3.65), we get (3.58) immediately.

Step 2. The proof of (3.59).

The method in [34] is used in this process. Integrating $(1.1)_3$ over Ω, we arrive at

$$\frac{d}{dt}\int \rho\theta dx - 2\int \theta^\alpha |D(u)|^2 dx = 0. \tag{3.66}$$

Combining (3.4) and (3.66), it can be inferred that

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2 dx + \frac{d}{dt}\int \rho\theta dx = 0$$

Hence

$$\frac{1}{2}\int \rho|u|^2 dx + \int \rho\theta dx = \frac{1}{2}\int \rho_0|u_0|^2 dx + \int \rho_0\theta_0 dx \triangleq E_0. \tag{3.67}$$

Multiplying $(1.1)_3$ by $\theta - \frac{1}{\overline{\rho_0}|\Omega|}E_0$ and integrating it over Ω, one obtains

$$\frac{1}{2}\frac{d}{dt}\int \rho(\theta - \frac{1}{\overline{\rho_0}|\Omega|}E_0)^2 dx + \int \theta^\beta |\nabla\theta|^2 dx = 2\int (\theta - \frac{1}{\overline{\rho_0}|\Omega|}E_0)\theta^\alpha |D(u)|^2 dx \leq C\int |\nabla u|^2 dx.$$

It follows from Poincaré's inequality that

$$
\begin{aligned}
&\int \rho(\theta-\frac{1}{\overline{\rho_0}|\Omega|}E_0)^2dx\\
&\le C\int \Big(|\theta-\bar\theta|^2+|\bar\theta-\frac{1}{\bar{\rho_0}|\Omega|}\int\rho\theta dx|^2+|\frac{1}{\bar{\rho_0}|\Omega|}\int\rho\theta dx-\frac{1}{\bar{\rho_0}|\Omega|}E_0|^2\Big)dx\\
&\le C\Big(\int|\nabla\theta|^2dx+\int|\int\rho(\theta-\bar\theta)dx|^2dx+\int|\int\rho|u|^2dx|^2dx\Big)\\
&\le C\Big(\int|\nabla\theta|^2dx+(\int|\nabla u|^2dx)^2\Big)\\
&\le C\Big(\int\theta^\beta|\nabla\theta|^2dx+\int|\nabla u|^2dx\Big),
\end{aligned}
$$

where $\bar\theta=\frac{1}{|\Omega|}\int\theta dx$. Thus, we have

$$
\frac{d}{dt}\int\rho(\theta-\frac{1}{\overline{\rho_0}|\Omega|}E_0)^2dx+C^{-1}\int\rho(\theta-\frac{1}{\overline{\rho_0}|\Omega|}E_0)^2dx+\int\theta^\beta|\nabla\theta|^2dx\le C\int|\nabla u|^2dx,
$$

and this combined with (3.62) gives

$$
\sup_{0\le t\le T}(e^{C^{-1}t}\|\sqrt{\rho}(\theta-\frac{1}{\overline{\rho_0}|\Omega|}E_0)\|_{L^2}^2)+\int_0^T e^{C^{-1}t}\|\nabla\theta\|_{L^2}^2dt\le C. \tag{3.68}
$$

Now, we re-estimate (3.27) as follows:

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\theta^{\beta+1}\|_{L^2}^2+\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2\\
&\le C\|\nabla u\|_{L^2}^2\|\nabla^2\theta\|_{L^2}^2+\frac{4}{\alpha+\beta+1}\frac{d}{dt}\int\theta^{\alpha+\beta+1}|D(u)|^2dx+C\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}.
\end{aligned} \tag{3.69}
$$

Multiplying it by $e^{C^{-1}t}$, according to (3.58), and Corollary 3.1, we have

$$
\begin{aligned}
&\frac{d}{dt}(e^{C^{-1}t}\|\nabla\theta^{\beta+1}\|_{L^2}^2)+e^{C^{-1}t}\|\sqrt{\rho}\theta^{\frac{\beta}{2}}\theta_t\|_{L^2}^2\\
&\le Ce^{C^{-1}t}\|(\nabla\theta,\nabla u,\nabla u_t)\|_{L^2}^2+C\|\nabla^2\theta\|_{L^2}^2+\frac{4}{\alpha+\beta+1}\frac{d}{dt}(e^{C^{-1}t}\int\theta^{\alpha+\beta+1}|D(u)|^2dx).
\end{aligned}
$$

Consequently,

$$
\sup_{0\le t\le T}(e^{C^{-1}t}\|\nabla\theta\|_{L^2}^2)+\int_0^T e^{C^{-1}t}\|\sqrt{\rho}\theta_t\|_{L^2}^2dt\le C. \tag{3.70}
$$

Recalling (3.31), (3.64) and Corollary 3.1 leads to

$$
\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\theta^{\frac{\beta}{2}}\nabla\theta_t\|_{L^2}^2\le C\|(\sqrt{\rho}\theta_t,\nabla u_t)\|_{L^2}^2+C\|u\|_{H^2}^4+C(\|\nabla u\|_{L^\infty}^2+\|\nabla\theta\|_{L^2}^2)\|\sqrt{\rho}\theta_t\|_{L^2}^2.
$$

Then, multiplying the last inequality by $e^{C^{-1}t}$, combining the result with Grönwall's inequality, (3.58), (3.70) and Corollary 3.1 gives

$$\sup_{0\le t\le T}(e^{C^{-1}t}\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\int_0^T e^{C^{-1}t}\|\nabla\theta_t\|_{L^2}^2dt\le C. \tag{3.71}$$

Therefore, we complete the proof of (3.59) by (3.68), (3.70), (3.71), and the following estimate

$$\|\nabla^2\theta\|_{L^2}\le C\|\nabla^2\theta^{\beta+1}\|_{L^2}\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{3}{2}}+C\|(\sqrt{\rho}\theta_t,\nabla\theta)\|_{L^2}.$$

The proof of Proposition 3.1 is finished. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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