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Research article

L^1 local stability to a nonlinear shallow water wave model

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Abstract: A nonlinear shallow water wave equation containing the famous Degasperis−Procesi and Fornberg−Whitham models is investigated. The novel derivation is that we establish the L^2 bounds of solutions from the equation if its initial value belongs to space $L^2(\mathbb{R})$. The L^∞ bound of the solution is derived. The techniques of doubling the space variable are employed to set up the L^1 local stability of short time solutions.

Keywords: local strong solutions; bounds of solution; shallow water wave model; L^1 local stability

1. Introduction

Consider the equation

$$u_t - u_{txx} + \beta u_x + muu_x = 3\alpha u_x u_{xx} + \alpha uu_{xxx}, \tag{1.1}$$

in which constants $m > 0$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Equation (1.1) characterizes the hydrodynamical dynamics of shallow water waves and is a special model derived in Constantin and Lannes [1]. In fact, the nonlinear shallow water wave model holds great significance for the scientific community due to its application in tsunami modeling and forecasting, a critical scientific problem with global implications for coastal communities. The investigation of shallow water wave equations may aid scientists in comprehending and predicting the behavior of tsunamis.

If $m = \frac{3}{2}, \beta = -1$, and $\alpha = \frac{3}{2}$, Eq (1.1) reduces to the Fornberg−Whitham (FW) model [2, 3]

$$u_t - u_{txx} + \frac{3}{2}uu_x = u_x + \frac{9}{2}u_x u_{xx} + \frac{3}{2}uu_{xxx}. \tag{1.2}$$

Many works have been carried out to discuss various dynamical behaviors of the FW equation. Sufficient and necessary conditions, guaranteeing that the wave breaking of Eq (1.2) happens, are found out in Haziot [4]. The sufficient conditions of wave breaking and discontinuous traveling wave solutions to the FW model are considered in H$\ddot{o}$rmann [5, 6]. The continuity solutions of Eq (1.2) in Besov space are explored in Holmes and Thompson [7]. The H*ölder* continuous solutions to the FW model are in detail investigated in Holmes [8]. Ma et al. [9] provide sufficient conditions to ensure the occurrence of wave breaking for a range of nonlocal Whitham type equations. On the basis of $L^2(\mathbb{R})$ conservation law, Wu and Zhang [10] investigate the wave breaking of the Fornberg−Whitham equation. Comparing to the previous wave breaking results for the FW model, Wei [11] gives a novel sufficient condition to guarantee that the wave breaking for Eq (1.2) happens.

Suppose that $m = 4, \beta = 0$, and $\alpha = 1$, Eq (1.1) becomes the well-known Degasperis−Procesi (DP) equation [12]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

Many works have been carried out to study the dynamical characteristics of Eq (1.3). For instances, the integrability of the DP equation is derived in Degasperis and Procesi [12] and Degasperis et al. [13]. Escher et al. [14] investigate the existence of global weak solutions for the DP model. Liu et al. [15] prove the well-posedness of global strong solutions and blow-up phenomena for Eq (1.3) under certain conditions. Yin [16] considers the Cauchy problem for a periodic generalized Degasperis−Procesi model. The large-time asymptotic behavior of the periodic entropy solutions for the DP equation is discussed in Conclite and Karlsen [17]. Various kinds of traveling wave solutions for Eq (1.3) are presented in [18–20]. In the Sobolev space $H^s(\mathbb{R})$ with $s > \frac{3}{2}$, Lai and Wu [21] discuss the local existence for a partial differential equation involving the DP and Camassa−Holm(CH) models. The investigation of wave speed for the DP model is carried out in Henry [22]. The dynamical properties of CH equations are presented in [23–26]. For dynamical features of other nonlinear models, which are closely relevant to the DP and FW models, we refer the reader to [27–30].

As we know, the L^2 conservation law derived from the DP or FW equation takes an essential role in investigating the dynamical features of the DP and FW models. We derive that Eq (1.1) possesses the following L^2 conservation law:

$$\int_{\mathbb{R}} \frac{1+\xi^2}{\frac{m}{\alpha}+\xi^2} |\widehat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{1+\xi^2}{\frac{m}{\alpha}+\xi^2} |\widehat{u_0}(\xi)|^2 d\xi \sim \| u_0 \|^2_{L^2(\mathbb{R})}, \tag{1.4}$$

where $u(0, x) = u_0 \in H^s(\mathbb{R})$ endowed with the index $s > \frac{3}{2}$ is the initial value of u.

A natural question is that as the shallow water wave model (1.1) generalizes the famous Fornberg−Whitham equation (1.2) and Degasperis−Procesi model (1.3), what kinds of dynamical characteristics of DP and FW models still hold for Eq (1.1). For this purpose, the key element of this work is that we derive $L^2(\mathbb{R})$ conservation law for (1.1). Using (1.4) and the technique of transport equation, we establish the boundedness of the solutions for Eq (1.1). Employing the approach called doubling the space variable in Kružkov [31], we investigate the $L^1(\mathbb{R})$ stability of short-time strong solutions provided that $u_0(x)$ belongs to the space $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s > \frac{3}{2}$. To our knowledge, this $L^1(\mathbb{R})$ stability of Eq (1.1) has never been established in literatures.

The organization of this job is that Section 2 prepares several Lemmas. The $L^1(\mathbb{R})$ stability of short time solution to Eq (1.1) is established in Section 3.

2. Lemmas

For the nonlinear shallow water wave equation (1.1), we write out its initial problem

$$\begin{cases} u_t - u_{txx} + \beta u_x + muu_x = 3\alpha u_x u_{xx} + \alpha uu_{xxx}, \\ u(0,x) = u_0(x). \end{cases} \tag{2.1}$$

Utilizing inverse operator $\mathbb{A}^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$, we obtain the equivalent form of (2.1), which reads as

$$\begin{cases} u_t + \alpha uu_x = -\beta\mathbb{A}^{-2}u_x + \frac{\alpha-m}{2}\mathbb{A}^{-2}(u^2)_x, \\ u(0,x) = u_0(x). \end{cases} \tag{2.2}$$

In fact, for any function $D(x) \in L^r(\mathbb{R})$ with $1 \le r \le \infty$, we have

$$\mathbb{A}^{-2}D(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}D(z)dz.$$

Writing $Q_u = \beta\mathbb{A}^{-2}u + \frac{m-\alpha}{2}\mathbb{A}^{-2}(u^2)$ and $J_u = \beta\mathbb{A}^{-2}\partial_x u + \frac{m-\alpha}{2}\partial_x\mathbb{A}^{-2}(u^2)$ yields

$$u_t + \frac{\alpha}{2}(u^2)_x + J_u = 0. \tag{2.3}$$

We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\| h \|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in\mathbb{R}\setminus e} |h(t,x)|$. For any real number s, we let $H^s = H^s(\mathbb{R})$ denote the Sobolev space with the norm defined by

$$\| h \|_{H^s} = \left(\int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\hat{h}(t,\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi) = \int_{-\infty}^{\infty} e^{-ix\xi}h(t,x)dx$. For $T > 0$ and nonnegative number s, let $C([0,T); H^s(\mathbb{R})$ denote the Frechet space of all continuous H^s-valued functions on $[0,T)$.

Lemma 2.1. *([21]) Provided that $s > \frac{3}{2}$ and initial value $u_0(x) \in H^s(\mathbb{R})$, then there has a unique solution u which belongs to the space $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$, in which T represents maximal existence time for solution u^*.*

Lemma 2.2. *Suppose that $m > 0$, $\alpha > 0$, $u_0 \in H^s(\mathbb{R})$, and $s > \frac{3}{2}$. Let u be the solution of (2.1). Set $y = u - \frac{\partial^2 u}{\partial x^2}$ and $Y = (\frac{m}{\alpha} - \frac{\partial^2}{\partial x^2})^{-1}u$. Then*

$$\int_{\mathbb{R}} yYdx = \int_{\mathbb{R}} \frac{1+\xi^2}{\frac{m}{\alpha}+\xi^2}|\widehat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{1+\xi^2}{\frac{m}{\alpha}+\xi^2}|\widehat{u_0}(\xi)|^2 d\xi \sim \| u_0 \|^2_{L^2(\mathbb{R})}. \tag{2.4}$$

*In the sense of Lemma 2.1, for $s > \frac{3}{2}$, the maximal existence time T means $\lim_{t\to T} \| u(t,\cdot) \|_{H^s(\mathbb{R})} = \infty$.

Moreover,

$$\begin{cases} \| u \|_{L^2} \leq \sqrt{\frac{\alpha}{m}} \| u_0 \|_{L^2}, & if \quad \frac{m}{\alpha} \leq 1, \\ \| u \|_{L^2} \leq \sqrt{\frac{m}{\alpha}} \| u_0 \|_{L^2}, & if \quad \frac{m}{\alpha} \geq 1. \end{cases} \tag{2.5}$$

Proof. We have $u = \frac{m}{\alpha} Y - \partial^2_{xx} Y$ and $\partial^2_{xx} Y = \frac{m}{\alpha} Y - u$. Utilizing integration by parts and Eq (1.1) yields

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} yY dx &= \int_{\mathbb{R}} y_t Y dx + \int_{\mathbb{R}} y Y_t dx = 2 \int_{\mathbb{R}} Y y_t dx \\ &= 2 \int_{\mathbb{R}} \Big[(-\frac{m}{2} u^2)_x - \beta u_x + \frac{\alpha}{2} \partial^3_{xxx} (u^2) \Big] Y dx \\ &= 2 \int_{\mathbb{R}} \Big[(-\frac{m}{2} u^2)_x Y - \beta u_x Y + \frac{\alpha}{2} (u^2)_x \partial^2_{xx} Y \Big] dx \\ &= \int_{\mathbb{R}} \Big[(-mu^2)_x Y - 2\beta u_x Y + \alpha (u^2)_x (\frac{m}{\alpha} Y - u) \Big] dx \\ &= \int_{\mathbb{R}} \Big(-2\beta u_x Y - \alpha (u^2)_x u \Big) dx \\ &= 2\beta \int_{\mathbb{R}} u Y_x dx \\ &= 2\beta \int_{\mathbb{R}} (\frac{m}{\alpha} Y - \partial^2_{xx} Y) Y_x dx \\ &= 0. \end{aligned}$$

Utilizing the above identity and the Parserval identity gives rise to (2.4). Inequality (2.5) is derived directly from (2.4).

For each time $t \in [0, T)$, we write the transport system

$$\begin{cases} q_t = \alpha u(t, q), \\ q(0, x) = x. \end{cases} \tag{2.6}$$

The next lemma demonstrates that $q(t, x)$ possesses the feature of increasing diffeomorphism.

Lemma 2.3. *Provided that T is defined as in Lemma 2.1 and $u_0 \in H^s(\mathbb{R})$ endowed with $s \geq 3$, then system (2.6) possesses a unique q belonging to $C^1([0, T) \times \mathbb{R})$. In addition, $q_x(t, x) > 0$ in the region $[0, T) \times \mathbb{R}$.*

Proof. Employing Lemma 2.1 derives that $u_x \in C^2(\mathbb{R})$ and $u_t \in C^1[0, T)$ if $(t, x) \in [0, T) \times \mathbb{R}$. Subsequently, it is concluded that solution $u(t, x)$ and its slope $u_x(t, x)$ possess boundness and are Lipschitz continuous in the region $[0, T) \times \mathbb{R}$. Using the theorem of existence and uniqueness for ODE guarantees that system (2.6) possesses a unique solution $q \in C^1([0, T) \times \mathbb{R})$.

Making use of system (2.6) gives rise to $\frac{d}{dt} q_x = \alpha u_x(t, q) q_x$ and $q_x(0, x) = 1$. Thus, we have

$$q_x(t, x) = e^{\int_0^t \alpha u_x(\tau, q(\tau, x)) d\tau}.$$

If $T' < T$, we acquire

$$\sup_{(t,x) \in [0, T') \times R} |u_x(t, x)| < \infty,$$

implying that it must have a constant $C_0 > 0$ to ensure $q_x(t, x) \geq e^{-C_0 t}$. The proof is finished.

For writing concisely in the following discussions, we utilize notations $L^\infty = L^\infty(\mathbb{R})$, $L^1 = L^1(\mathbb{R})$, and $L^2 = L^2(\mathbb{R})$.

Lemma 2.4. *Assume $t \in [0, T]$, $s > \frac{3}{2}$, and $u_0 \in H^s(\mathbb{R})$. Then*

$$\| u(t,x) \|_{L^\infty} \leq \| u_0 \|_{L^\infty} + \Big(\frac{|\beta| c_0}{2} \| u_0 \|_{L^2} + \frac{|\alpha - m| c_0^2}{4} \| u_0 \|_{L^2}^2 \Big) t, \tag{2.7}$$

in which $c_0 = \max\Big(\sqrt{\frac{\alpha}{m}}, \sqrt{\frac{m}{\alpha}}\Big)$.

Proof. Set $\eta(x) = \frac{1}{2} e^{-|x|}$. Utilizing the density arguments utilized in [15], we only need to deal with the case $s = 3$ to verify Lemma 2.4. For $u_0 \in H^3(\mathbb{R})$, using Lemma 2.1 ensures the existence of u belonging to $H^3(\mathbb{R})$. Applying system (2.2) arises

$$u_t + \alpha u u_x = (\alpha - m) \eta \star (u u_x) - \beta \eta \star u_x, \tag{2.8}$$

where $\star$ stands for the convolution. Using $\int_{\mathbb{R}} e^{2|x-z|} dz = 1$, we acquire

$$\begin{aligned} |\eta(x) \star u_x| &= \frac{1}{2} | - \int_{-\infty}^{x} e^{-x+z} u(t,z) dz + \int_{x}^{\infty} e^{x-z} u(t,z) dz | \\ &\leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-z|} |u(t,z)| dz \\ &\leq \frac{1}{2} \Big(\int_{\mathbb{R}} e^{-2|x-z|} dz \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} u^2(t,z) dz \Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \| u \|_{L^2} \leq \frac{c_0}{2} \| u_0 \|_{L^2} . \end{aligned} \tag{2.9}$$

We have

$$\begin{aligned} |\eta \star (u u_x)| &= | \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-z|} u u_z dz | \\ &= \frac{1}{2} | \int_{-\infty}^{x} e^{-x+z} u u_z dz + \frac{1}{2} \int_{x}^{+\infty} e^{x-z} u u_z dz | \\ &= | - \frac{1}{4} \int_{-\infty}^{x} e^{-|x-z|} u^2 dz + \frac{1}{4} \int_{x}^{\infty} e^{-|x-z|} u^2 dz | \\ &\leq \frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-z|} u^2 dz \leq \frac{1}{4} c_0^2 \| u_0 \|_{L^2}^2 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \frac{du(t, q(t,x))}{dt} &= u_t(t, q(t,x)) + u_x(t, q(t,x)) \frac{dq(t,x)}{dt} \\ &= u_t(t, q(t,x)) + \alpha u u_x(t, q(t,x)). \end{aligned} \tag{2.11}$$

Combining with (2.8)–(2.11) and Lemma 2.2 gives rise to

$$
\begin{aligned}
\left| \frac{du(t,q(t,x))}{dt} \right| &\le \frac{|m-\alpha|}{4}\int_{-\infty}^{\infty} e^{-|q(t,x)-z|}u^2dz + |\beta\eta \star u_x| \\
&\le \frac{|m-\alpha|}{4}\int_{-\infty}^{\infty} u^2dz + \frac{|\beta|}{2}\left|\int_{-\infty}^{\infty} e^{-|q(t,x)-z|}u_zdz\right| \\
&\le \frac{|m-\alpha|}{4}\| u \|_{L^2}^2 + \frac{|\beta|}{2}\| u \|_{L^2} \\
&\le \frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2 .
\end{aligned} \tag{2.12}
$$

From (2.12), we have

$$
\begin{cases}
\frac{du(t,q(t,x))}{dt} \le \frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2, \\
\frac{du(t,q(t,x))}{dt} \ge -\Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big).
\end{cases} \tag{2.13}
$$

Integrating (2.13) on the interval $[0, t]$ yields

$$
\begin{cases}
u(t,q(t,x)) - u_0 \le \Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big)t, \\
u(t,q(t,x)) - u_0 \ge -\Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big)t.
\end{cases} \tag{2.14}
$$

From the first inequality in (2.14), we have

$$
\| u(t,q(t,x)) \|_{L^\infty} \le \Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big)t + \| u_0 \|_{L^\infty} . \tag{2.15}
$$

Using the second inequality in (2.14) gives rise to

$$
\begin{aligned}
|u(t,q(t,x))| &\ge \Big|u_0 - \Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big)t.\Big| \\
&\ge -\Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big)t - |u_0|,
\end{aligned}
$$

from which we have

$$
\| u(t,q(t,x)) \|_{L^\infty} \ge -\Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big)t - \| u \|_{L^\infty} . \tag{2.16}
$$

Utilizing (2.15) and (2.16), we obtain

$$
\| u(t,q(t,x)) \|_{L^\infty} \le \| u_0 \|_{L^\infty} + \Big(\frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2\Big)t. \tag{2.17}
$$

Utilizing Lemma 2.3 and (2.17) yields (2.7).

Lemma 2.5. *If* $u_0 \in L^2(\mathbb{R})$, *then*

$$
\begin{cases}
\| Q_u(t,\cdot) \|_{L^\infty(\mathbb{R})} \le \frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2, \\
\\
\| J_u(t,\cdot) \|_{L^\infty(\mathbb{R})} \le \frac{|\beta|c_0}{2}\| u_0 \|_{L^2} + \frac{|\alpha-m|c_0^2}{4}\| u_0 \|_{L^2}^2,
\end{cases} \tag{2.18}
$$

in which $c_0 = \max\Big(\sqrt{\frac{\alpha}{m}}, \sqrt{\frac{m}{\alpha}}\Big)$.

Proof. From (2.3), we have

$$Q_u = \frac{m-\alpha}{4}\int_{\mathbb{R}} e^{-|x-z|}u^2(t,z)dz + \frac{\beta}{2}\int_{\mathbb{R}} e^{-|x-z|}u(t,z)dz, \tag{2.19}$$

$$J_u = \frac{m-\alpha}{4}\int_{\mathbb{R}} e^{-|x-z|}sgn(z-x)u^2(t,z)dz + \frac{\beta}{2}\int_{\mathbb{R}} e^{-|x-z|}sgn(z-x)u(t,z)dz. \tag{2.20}$$

Utilizing (2.9), (2.19), (2.20), Lemma 2.2, and the Schwartz inequality, we obtain (2.18).

Lemma 2.6. *Let $u_0, v_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Provided that functions u and v satisfy system (2.2), for any $g(t,x) \in C_0^\infty([0,\infty)\times(-\infty,\infty))$, then*

$$\int_{-\infty}^{\infty}\Big|J_u(t,x)-J_v(t,x)\Big||g(t,x)|dx \le c(1+t)\int_{-\infty}^{\infty}|u(t,x)-v(t,x)|dx, \tag{2.21}$$

in which $c > 0$ depends on $m, \alpha, \beta, g, \| u_0 \|_{L^2}$ and $\| v_0 \|_{L^2}$.

Proof. Applying the Tonelli Theorem and Lemmas 2.2 and 2.4 gives rise to

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Big|J_u(t,x)-J_v(t,x)\Big||g(t,x)|dx\\
&\le \frac{|\beta|}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|x-z|}|sgn(z-x)||u-v||g(t,x)|dzdx\\
&\qquad + \frac{|m-\alpha|}{2}\int_{-\infty}^{\infty}|\partial_x\mathbb{A}^{-2}(u^2-u^2)||g(t,x)|dx\\
&\le c\int_{-\infty}^{\infty}|u-v|dz\int_{-\infty}^{\infty}e^{-|x-z|}|g(t,x)|dx\\
&\qquad + \frac{|m-\alpha|}{4}\Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-|x-z|}|sgn(z-x)|\Big|u^2-v^2\Big|dz|g(t,x)|dx\Big|\\
&\le c\int_{-\infty}^{\infty}|u(t,z)-v(t,z)|dz\\
&\qquad + \frac{|m-\alpha|}{4}\int_{-\infty}^{\infty}\Big|(u-v)(u+v)\Big|dz\Big|\int_{-\infty}^{\infty}|g(t,x)|dx\Big|\\
&\le c(1+t)\int_{-\infty}^{\infty}|u(t,z)-v(t,z)|dz,
\end{aligned}$$

from which we acquire (2.21).

Suppose that function $\gamma(y)$ is infinitely differentiable on $\mathbb{R}$ such that $\gamma(y) \ge 0$, $\gamma(y) = 0$ when $|y| \ge 1$, and $\int_{-\infty}^{\infty}\gamma(y)dy = 1$. For arbitrary constant $h > 0$, set $\gamma_h(y) = \frac{\gamma(h^{-1}y)}{h} \ge 0$. Thus, $\gamma_h(y)$ belongs to $C^\infty(-\infty,\infty)$ and

$$|\gamma_h(y)| \le \frac{c}{h}, \quad \int_{-\infty}^{\infty}\gamma_h(y)dy = 1; \qquad \gamma_h(y) = 0 \quad \text{if} \quad |y| \ge h.$$

Suppose that $G(x)$ is locally integrable in $\mathbb{R}$. Its mean function is written as

$$G^h(x) = \frac{1}{h}\int_{-\infty}^{\infty} \gamma(\frac{x-y}{h})G(y)dy, \quad h > 0.$$

For the Lebesgue point x_0 of $G(x)$, it has

$$\lim_{h\to 0}\frac{1}{h}\int_{|x-x_0|\le h} |G(x) - G(x_0)|dx = 0. \tag{2.22}$$

If x is an arbitrary Lebesgue point of $G(x)$, it has $\lim_{h\to 0} G^h(x) = G(x)$. Provided that point x is not Lebesque point of $G(x)$, (2.22) always holds. Thus, $G^h(x) \to G(x)$ $(h \to 0)$ is valid almost everywhere.

We illustrate the notation of a characteristic cone. Suppose that $N > \max_{t\in[0,T]} \| W(t,\cdot) \|_{L^\infty} < \infty$, $0 \le t \le T_0 = min(T, R_0N^{-1})$ and $\mho = \{(t, x) : |x| < R_0 - Nt\}$. We write that S_τ represents the cross section of $\mho$ endowed with $t = \tau, \tau \in [0, T_0]$. For $r > 0, \rho > 0$, set $K_r = \Big\{x : |x| \le r\Big\}$. Let $\theta_T = [0, T]\times\mathbb{R}$ and $D_1 = \Big\{(t, x, \tau, y)\Big|\,|\frac{t-\tau}{2}| \le h, \rho \le \frac{t+\tau}{2} \le T - \rho, |\frac{x-y}{2}| \le h, |\frac{x+y}{2}| \le r - \rho\Big\}$.

Lemma 2.7. *[31] If function $Q(t, x)$ is measurable and bounded in $\Omega_T = [0, T] \times K_r$, for $h \in (0, \rho)$, $\rho \in (0, \min[r, T])$, setting*

$$H_h = \frac{1}{h^2}\iiiint\limits_{D_1} |Q(t, x) - Q(\tau, y)|dxdtdyd\tau,$$

then $\lim_{h\to 0} H_h = 0$.

Lemma 2.8. *[31] Provided that $|\frac{\partial M(u)}{\partial u}|$ is bounded and*

$$L(u, v) = sgn(u - v)(M(u) - M(v)),$$

then for any functions u and v, function $L(u, v))$ obeys the Lipschitz condition.

Lemma 2.9. *Suppose that $u_0(x) \in H^s(\mathbb{R})$ endowed with $s > \frac{3}{2}$. Provided that u satisfies (2.2), $g(t, x) \in C_0^\infty(\theta_T)$ and $g(0, x) = 0$, for every constant k, then*

$$\iint\limits_{\theta_T} \Big\{|u - k|g_t + sgn(u - k)\frac{\alpha}{2}[u^2 - k^2]g_x - sgn(u - k)J_u g\Big\}dxdt = 0.$$

Proof. Assume that $\Psi(u)$ is a convex downward and twice smooth function for $-\infty < u < \infty$. Let $g(t, x) \in C_0^\infty(\theta_T)$. Using $\Psi'(u)g(t, x)$ to multiply Eq (2.3), integrating over the domain θ_T, we transfer the derivatives to g and acquire

$$\iint\limits_{\theta_T} \Big\{\Psi(u)g_t + \alpha\Big[\int_k^u \Psi'(y)ydy\Big]g_x - \Psi'(u)J_u(t, x)g\Big\}dtdx = 0, \tag{2.23}$$

in which for any constant k, the identity $\int_{-\infty}^{\infty}\Big[\int_k^u \Psi'(y)ydy\Big]g_x dx = -\int_{-\infty}^{\infty}\Big[g\Psi'(u)uu_x\Big]dx$ is utilized. We have the expression

$$\int_{-\infty}^{\infty}\Big[\int_k^u \Psi'(y)ydy\Big]g_x dx = \int_{-\infty}^{\infty}\Big[\frac{1}{2}\Psi'(u)u^2-\frac{1}{2}\Psi'(k)k^2 -\frac{1}{2}\int_k^u y^2\Psi''(y)dy\Big]g_x dx. \tag{2.24}$$

Let $\Psi^h(u)$ be the mean function of $|u-k|$ and set $\Psi(u)=\Psi^h(u)$. Letting $h\to 0$ and employing the features of $sgn(u-k)$, (2.23), and (2.24) complete the proof.

Actually, the derivation of Lemma 2.9 can also be found in [31].

3. L^1 local stability

Utilizing the bounded property of solution $u(t,x)$ for system (2.2), we investigate the $L^1(\mathbb{R})$ local stability of $u(t,x)$, which is written in the following theorem.

Theorem 3.1. *Suppose that u and v satisfy Eq (1.1) endowed with initial values $u_0, v_0 \in H^s(\mathbb{R})\cap L^1(\mathbb{R})$ $(s>\frac{3}{2})$, respectively. Let $t\in[0,T]$. Then there is a C_T depending on $\| u_0 \|_{L^2(\mathbb{R})}, \| v_0 \|_{L^2(\mathbb{R})}, T, \alpha, \beta$ and m, to satisfy*

$$\| u(t,\cdot)-v(t,\cdot) \|_{L^1(\mathbb{R})} \le C_T \| u_0 - v_0 \|_{L^1(\mathbb{R})}\ . \tag{3.1}$$

Proof. Utilizing Lemmas 2.1 and 2.4 deduces that u and v remain bounded and continuous in $[0,T]\times\mathbb{R}$. Set $\uplus = \{(t,x)\} = [\rho, T-2\rho]\times K_{r-2\rho}$, where $0<2\rho\le \min(T,r)$, and $\theta_T=[0,T]\times\mathbb{R}$. Assume $b(t,x)\in C_0^\infty([0,\infty)\times\mathbb{R})$ associated with $b(t,x)=0$ outside $\uplus$.

For $h\le\rho$, we construct the function

$$g=b(\frac{t+\tau}{2},\frac{x+y}{2})\gamma_h(\frac{t-\tau}{2})\gamma_h(\frac{x-y}{2}) = b(...)\lambda_h(*),$$

in which $(...)=(\frac{t+\tau}{2},\frac{x+y}{2})$ and $(*)=(\frac{t-\tau}{2},\frac{x-y}{2})$. By the definition of function $\gamma(y)$, we have

$$g_t+g_\tau = b_t(...)\lambda_h(*),\quad g_x+g_y=b_x(...)\lambda_h(*).$$

Choosing $k=v(\tau,y)$ in Lemma 2.9 and applying the methods called doubling the space variables in [31] yield

$$\iiiint_{\theta_T\times\theta_T}\Big\{|u(t,x)-v(\tau,y)|g_t +sgn(u(t,x)-v(\tau,y))\frac{\alpha}{2}\Big(u^2(t,x)-v^2(\tau,y)\Big)g_x -sgn(u(t,x)-v(\tau,y))J_u(t,x)g\Big\}dtdxd\tau dy=0. \tag{3.2}$$

Taking $k=u(t,x)$ in Lemma 2.9 gives rise to

$$\iiiint\limits_{\theta_T\times\theta_T} \Big\{|v(\tau,y)-u(t,x)|g_\tau$$
$$+sgn(v(\tau,y)-u(t,x))\frac{\alpha}{2}\Big(u^2(t,x)-v^2(\tau,y)\Big)g_y$$
$$-sgn(v(\tau,y)-u(t,x))J_v(\tau,y)g\Big\}d\tau dydtdx=0. \tag{3.3}$$

Using (3.2) and (3.3) yields

$$0\le \iiiint\limits_{\theta_T\times\theta_T} \Big\{|u(t,x)-v(\tau,y)|(g_t+g_\tau)$$
$$+sgn(u(t,x)-v(\tau,y))\frac{\alpha}{2}\Big(u^2(t,x)-v^2(\tau,y)\Big)(g_x+g_y)\Big\}dxdtdyd\tau$$
$$+\Big|\iiiint\limits_{\theta_T\times\theta_T} sgn(u(t,x)-v(t,x))(J_u(t,x)-J_v(\tau,y))gdxdtdyd\tau\Big|.$$
$$=P_1+P_2+\Big|\iiiint\limits_{\theta_T\times\theta_T} P_3dxdtdyd\tau\Big|. \tag{3.4}$$

On the basis of the approaches in [31], we aim to verify the inequality

$$0\le \iint\limits_{\theta_T} \Big\{|u(t,x)-v(t,x)|b_t+sgn(u(t,x)-v(t,x))\frac{\alpha}{2}\Big(u^2(t,x)-v^2(t,x)\Big)b_x\Big\}dxdt$$
$$+\Big|\iint\limits_{\theta_T} sgn(u(t,x)-v(t,x))[J_u(t,x)-J_v(t,x)]bdxdt\Big|. \tag{3.5}$$

We write the integrands of P_1 and P_2 in (3.4) as

$$Y_h=Y(t,x,\tau,y,u(t,x),v(\tau,y))\lambda_h(*).$$

Using Lemma 2.4, we obtain $\parallel u\parallel_{L^\infty}< C_T$ and $\parallel v\parallel_{L^\infty}< C_T$. From Lemmas 2.7 and 2.8, for both functions u and v, it is deduced that Y_h obeys the Lipschitz condition. Combining function g, we find $Y_h=0$ outside region $\uplus$ and

$$\iiiint\limits_{\theta_T\times\theta_T} Y_h dxdtdyd\tau=\iiiint\limits_{\theta_T\times\theta_T}\Big[Y(t,x,\tau,y,u(t,x),v(\tau,y))$$
$$-Y(t,x,t,x,u(t,x),v(t,x))\Big]\lambda_h(*)dxdtdyd\tau$$
$$+\iiiint\limits_{\theta_T\times\theta_T} Y(t,x,t,x,u(t,x),v(t,x))\lambda_h(*)dxdtdyd\tau=G_{11}(h)+G_{12}. \tag{3.6}$$

Utilizing $|\lambda(*)|\le\frac{c}{h^2}$ yields

$$|G_{11}(h)|\le c\Big[h+\frac{1}{h^2}\iiiint\limits_{D_1}|u(t,x)-v(\tau,y)|dxdtdyd\tau\Big], \tag{3.7}$$

in which c does not rely on h. Employing Lemma 2.9 deduces that $G_{11}(h) \to 0$ when $h \to 0$. Now we consider G_{12}. Substituting $\frac{t-\tau}{2} = \delta, \frac{x-y}{2} = \omega$, we have

$$\int_{-h}^{h} \int_{-\infty}^{\infty} \lambda_h(\delta, \omega) d\delta d\omega = 1 \tag{3.8}$$

and

$$\begin{aligned} G_{12} &= 2^2 \iint\limits_{\theta_T} Y(t, x, t, x, u(t, x), v(t, x)) \Big\{ \int_{-h}^{h} \int_{-\infty}^{\infty} \lambda_h(\delta, \omega) d\delta d\omega \Big\} dxdt \\ &= 4 \iint\limits_{\theta_T} Y(t, x, t, x, u(t, x), v(t, x)) dxdt. \end{aligned} \tag{3.9}$$

From (3.6)–(3.9), we obtain

$$\lim_{h\to 0} \iiiint\limits_{\theta_T \times \theta_T} Y_h dxdtdyd\tau = 4 \iint\limits_{\theta_T} Y(t, x, t, x, u(t, x), v(t, x)) dxdt. \tag{3.10}$$

Note that

$$\begin{aligned} P_3 &= sgn(u(t, x) - v(\tau, y))(J_u(t, x) - J_v(\tau, y)) b(...) \lambda_h(*) \\ &= \overline{P_3}(t.x, \tau, y) \lambda_h(*) \end{aligned}$$

and

$$\begin{aligned} \iiiint\limits_{\theta_T \times \theta_T} P_3 dxdtdyd\tau &= \iiiint\limits_{\theta_T \times \theta_T} \Big[\overline{P_3}(t.x, \tau, y) - \overline{P_3}(t.x, t, x) \Big] \lambda_h(*) dxdtdyd\tau \\ &+ \iiiint\limits_{\theta_T \times \theta_T} \overline{P_3}(t.x, t, x) \lambda_h(*) dxdtdyd\tau = G_{21}(h) + G_{22}. \end{aligned} \tag{3.11}$$

We obtain

$$|G_{21}(h)| \le c\Big(h + \frac{1}{h^2} \times \iiiint\limits_{D_1} |J_u(t, x) - J_v(\tau, y)| dxdtdyd\tau\Big).$$

Using Lemmas 2.5 and 2.7 derives $G_{21}(h) \to 0$ when $h \to 0$. Applying (3.8) gives rise to

$$\begin{aligned} G_{22} &= 2^2 \iint\limits_{\theta_T} \overline{P_3}(t, x, t, x) \Big\{ \int_{-h}^{h} \int_{-\infty}^{\infty} \lambda_h(\delta, \omega) d\delta d\omega \Big\} dxdt \\ &= 4 \iint\limits_{\theta_T} \overline{P_3}(t, x, t, x) dxdt \\ &= 4 \iint\limits_{\theta_T} sgn(u - v)(J_u - J_v) b(t, x) dxdt. \end{aligned} \tag{3.12}$$

Employing (3.6), (3.10)–(3.12), we obtain inequality (3.5).

Set

$$F(t) = \int_{-\infty}^{\infty} |u - v| dx.$$

In order to prove the inequality (3.1), we define

$$A_h(z) = \int_{-\infty}^{z} \gamma_h(z) dz \qquad \Big(A_h'(z) = \gamma_h(z) \geq 0\Big).$$

In (3.5), provided that two numbers $\rho < \tau_1$, $\tau_1, \rho \in (0, T_0)$, and $h < min(\rho, T_0 - \tau_1)$, we set

$$b(t, x) = [A_h(t - \rho) - A_h(t - \tau_1)]B(t, x),$$

where

$$B(t, x) = B_\varepsilon(t, x) = 1 - A_\varepsilon\Big(|x| + Nt - R_0 + \varepsilon\Big), \quad \varepsilon > 0.$$

Provided that (t, x) does not belong to $\uplus$, then $b(t, x) = 0$. If (t, x) does not belong to $\mho$, we have $B(t, x) = 0$. It arises for $(t, x) \in \mho$ that

$$0 = B_t + N|B_x| \geq B_t + NB_x.$$

Using the above analysis and (3.5) yields

$$\begin{aligned} 0 \leq & \int_0^{T_0} \int_{-\infty}^{\infty} \Big\{[\gamma_h(t - \rho) - \gamma_h(t - \tau_1)]B_\varepsilon|u - v|\Big\} dxdt \\ & + \int_0^{T_0} \int_{-\infty}^{\infty} [A_h(t - \rho) - A_h(t - \tau_1)]|[J_u - J_v]b(t, x)| dxdt, \end{aligned}$$

which together with Lemma 2.6 (when $\varepsilon \to \infty$ and $R_0 \to \infty$) gives rise to

$$\begin{aligned} 0 \leq & \int_0^{T_0} \Big\{[\gamma_h(t - \rho) - \gamma_h(t - \tau_1)] \int_{-\infty}^{\infty} |u - v| dx\Big\} dt \\ & + c(1 + T_0) \int_0^{T_0} [A_h(t - \rho) - A_h(t - \tau_1)] \int_{-\infty}^{\infty} |u - v| dxdt. \end{aligned} \tag{3.13}$$

The property of $\gamma_h(z)$ for $h \leq \min(\rho, T_0 - \rho)$ derives that

$$\begin{aligned} \Big| \int_0^{T_0} \gamma_h(t - \rho) F(t) dt - F(\rho) \Big| &= \Big| \int_0^{T_0} \gamma_h(t - \rho) \Big(F(t) - F(\rho)\Big) dt \Big| \\ &\leq c \frac{1}{h} \int_{\rho - h}^{\rho + h} |F(t) - F(\rho)| dt \to 0, \quad \text{when} \quad h \to 0, \end{aligned}$$

in which $c > 0$ is independent of h.

Setting

$$Z(\rho) = \int_0^{T_0} A_h(t - \rho) F(t) dt = \int_0^{T_0} \int_{-\infty}^{t - \rho} \gamma_h(z) F(t) dzdt,$$

we derive that

$$Z'(\rho) = -\int_0^{T_0} \gamma_h(t-\rho)F(t)dt \to -F(\rho), \quad \text{when} \quad h \to 0.$$

Thus, we acquire

$$Z(\rho) \to Z(0) - \int_0^{\rho} F(z)dz, \quad \text{when} \quad h \to 0. \tag{3.14}$$

and

$$Z(\tau_1) \to Z(0) - \int_0^{\tau_1} F(z)dz, \quad \text{when} \quad h \to 0. \tag{3.15}$$

Using (3.14) and (3.15) directly deduces that

$$Z(\rho) - Z(\tau_1) \to \int_{\rho}^{\tau_1} F(z)dz, \quad \text{when} \quad h \to 0. \tag{3.16}$$

Sending $\tau_1 \to t, \rho \to 0$, from (3.13) and (3.16), we have

$$\int_{-\infty}^{\infty} |u-v|dx \le \int_{-\infty}^{\infty} |u_0 - v_0|dx + c(1+T_0)\int_0^t \int_{-\infty}^{\infty} |u-v|dxdt. \tag{3.17}$$

Utilizing (3.17) and the Gronwall inequality leads to the inequality (3.1).

Remark: We establish the L^1 local stability of strong solutions for the nonlinear shallow water wave equation (1.1) provided that its initial value belongs to the space $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s > \frac{3}{2}$. The asymptotic or uniform stability of strong solutions for Eq (1.1) deserves to be investigated. To study the asymptotic stability, we need to find certain restrictions on the initial data, which may be our future works.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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