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Research article

Global existence of weak solutions to a class of higher-order nonlinear evolution equations

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Abstract: This paper deals with the initial boundary value problem for a class of n-dimensional higher-order nonlinear evolution equations that come from the viscoelastic mechanics and have no positive definite energy. Through the analysis of functionals containing higher-order energy of motion, a modified potential well with positive depth is constructed. Then, using the potential well method, and Galerkin method, it has been shown that when the initial data starts from the stable set, there exists a global weak solution to such an evolution problem.

Keywords: potential well method; higher-order n-dimensional nonlinear evolution equations; initial boundary value problem; global weak solution; Galerkin method

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1. Introduction

In this paper, we study the following initial boundary value problem for n-dimensional higher-order nonlinear wave equations with dispersive and dissipative terms:

$$\begin{aligned} u_{tt}(x,t) + u_t(x,t) + (-1)^K\Delta^K u(x,t) + (-1)^K\Delta^K u_t(x,t) + (-1)^K\Delta^K u_{tt}(x,t) \\ = f\left(u(x,t)\right), \qquad (x,t) \in U \times [0,T), \end{aligned} \tag{1.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \qquad x \in U, \tag{1.2}$$

$$D^\alpha u(x,t) = 0 \text{ for any } 0 \le |\alpha| \le K-1, \qquad (x,t) \in \partial U \times [0,T), \tag{1.3}$$

where $U \subset R^n$ is a bounded domain with sufficiently smooth boundary ∂U, $K = 1,2,3,\cdots$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$ means multi-index derivative operator, $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_n)$ is multi-index of nonnegative integers $\alpha_i (i = 1, 2, \cdots, n)$, $\left|\alpha\right| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $u_0(x) \in H_0^K(U)$ and $u_1(x) \in H_0^K(U)$. Moreover,

$$f(u) = |u|^{p-1}u \tag{1.4}$$

with $p > 1$ satisfying:

$$1 < p < +\infty \text{ when } K \geq \frac{n}{2}; \ \ 1 < p \leq \frac{n}{n-2K} \text{ when } K < \frac{n}{2}. \tag{1.5}$$

Problems (1.1)–(1.3) come from viscoelastic mechanics. As $K = 1$, the nonlinear evolution equation

$$u_{tt} - u_{xx} - u_{xxtt} = f(u)$$

describes the propagation of longitudinal strain waves in a slender elastic rod [1, 2]. Similar equations containing a strong damping term u_{xxt} appear in the framework of the Mooney–Rivlin viscoelastic solids of second grade (see [3]). Concerning the higher-dimensional equation

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u), \ \ x \in U, \ \ t > 0, \tag{1.6}$$

a unique existence result of a global strong solution for the initial boundary problem of Eq (1.6) was proved in [4] under some assumptions on $f(u)$ for the positive definite energy. Xu et al. [5] also proved that the global strong solution of Eq (1.6) decays to zero exponentially as the time approaches infinity by using the multiplier method for the positive definite energy.

In [6], Gazzola and Squassina considered the initial boundary value problem of the following equation with both strong and weak damping terms

$$u_{tt} - \Delta u - \omega\Delta u_t + \mu u_t = |u|^{p-2}u \ \text{ in } \ U \times (0, T), \tag{1.7}$$

where $T > 0, \omega \geq 0$ and $\mu > -\omega\lambda_1$ (λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary condition). They got the global existence of solutions with initial data in the potential well, which was first introduced by Sattinger (see [7]). Moreover, they proved the finite time blow up for solutions starting in the unstable set and constructed the high energy initial data for which the solution blows up. In [8], Lian and Xu also obtained the global well-posedness of equation $u_{tt} - \Delta u - \omega\Delta u_t + \mu u_t = u \ln |u|$.

In [9], Xu and Yang studied the following nonlinear wave equation with dispersive–dissipative terms and weak damping

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} + u_t = |u|^{p-1}u. \tag{1.8}$$

Using the technique of [6] and the concavity method, Xu and Yang derived a sufficient condition on the initial data with arbitrarily positive initial energy such that the corresponding local solution of Eq (1.8) blows up in a finite time. However, the global existence of weak solutions and strong solutions for Eq (1.8) is still open.

As $K = 2$, Eq (1.1) represents the elastic plate equation with dispersive and dissipative effects [10, 11]. In [12], Xu et al. studied the global well-posedness of the initial boundary value problem for a class of fourth-order wave equations with a nonlinear damping term and a nonlinear source term, which was introduced to describe the dynamics of a suspension bridge. By the potential well method, in [13] Lin et al. derived global weak solutions and global strong solutions of the initial boundary value problem for a class of damped nonlinear evolutional equations

$$u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t = f(u),\ \ x \in U, t > 0.$$

Up to now, there is no result on the existence of global solutions to the initial boundary value problem for the nonlinear wave equation, including dispersive term $\triangle u_{tt}$, dissipative term $\triangle u_t$ and u_t.

Fourth-order equation models with the main part $u_{tt} + \Delta^2 u + \cdots$ containing weak and strong damping terms such as $u_t, f(u_t), \Delta u_t$ and nonlinear strain $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})$ also attract a lot of attention (see [14–18]). A recent work by Lian et al. (see [19]) considered the solutions of the following equation

$$u_{tt} + \Delta^2 u - \Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) - \Delta u_t + |u_t|^{r-1}u_t = f(u),\ \ (x,t) \in U \times (0,\infty). \tag{1.9}$$

The global existence, asymptotic behavior, and blow-up of solutions for subcritical initial energy and critical initial energy of Eq (1.9) were obtained, and the blow-up of solutions in finite time for the positive initial energy case was also proved.

As $K > 2$, Problems (1.1)–(1.3) also appear in physics. For example, when $n = 2$ and $K = 4$, Eq (1.1) can represent the model of two-dimensional quasicrystal elasticity with dispersive and dissipative effect; when $n = 3$ and $K = 6$, Eq (1.1) represents the model of three-dimensional quasicrystal elasticity with dispersive and dissipative effect (see [20]). As mentioned above, Eq (1.1) has an important physical background, while the mathematical achievements for the arbitrary higher order wave equation with both dispersive and dissipative terms (for any positive integer $K \geq 1$) are scarce. So the aim of the present paper is to establish a global existence result of weak solutions to such an evolution problem.

Motivated by previous papers [6,8,13], we shall use the potential well theory to establish conditions under which the initial boundary value problems (1.1)–(1.3) have global weak solutions. This method proposed by Sattinger (see [7,21]) and its improvements (see [22–25]) allow us to consider the hyperbolic equations without positive definite energy. For example, concerning about Eq (1.7) in [6], in the framework of the potential well method, a Nehari manifold N, a stable set $\mathbf{W}$ (potential well) and an unstable set $\mathbf{V}$ (outside the potential well), should be introduced, and the mountain pass energy level d (also known as potential well depth) can be characterized as

$$d = \inf_{u \in N}\left(\frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right).$$

However, when dealing with the present models with higher-order dispersive term $\Delta^K u_{tt}$, higher-order energy of motion $\|\nabla^K u_t\|_{L^2(U)}$ should be contained in energy functionals. This work brings complicated construction of potential well $\mathbf{W}$, and consequently a detailed computational formula of the modified potential well depth d is needed.

Therefore, in Section 2, we present some notations and definitions for the energy functionals, modified potential well $\mathbf{W}$, and modified potential well depth d. Then we concentrate on the detailed equivalent definition of d and prove that $d > 0$.

In Section 3, using the potential well method and Galerkin method, we construct a global weak solution to the evolution Problems (1.1)–(1.3) when the initial data stars from stable set $\mathbf{W}$.

There are also interesting problems for further studies.

1) As $K = 1$, the blow-up property of corresponding local solution, has been derived in [12], whereas the uniqueness of solution, vacuum isolation of solutions and decay or blow-up properties of solutions are still open for dispersive–dissipative models with any arbitrary higher-order K.

2) The Cauchy problem of such kinds of higher-order evolution equations has not been of concerned so far.

3) It is well worth considering some important physical properties and physical structures in numerical analysis, such as positivity preservation, maximum principle [26], long-term behavior [27], and singular solutions.

2. Notations and preliminaries

Let us give some explanations for constraint (1.5) of exponent p. Note that $2^* = \frac{2n}{n-2K}$ is the critical Sobolev exponent for q in the embedding $H_0^K(U) \hookrightarrow L^q(U)$ (see [28]); it follows from (1.5) that $H_0^K(U) \hookrightarrow L^{p+1}(U)$, hence the following functionals $I(u), J(u)$, and $E(t)$ introduced in (2.1), (2.2), and (2.4) should be well defined. Furthermore, by assumption (1.5), we can control the L^2 norm of the nonlinear term (1.4) by using Sobolev embedding $H_0^K(U) \hookrightarrow L^{2p}(U)$. It will lead to global existence results for the nonlinear ordinary differential systems (3.7) and (3.8) associated to Problems (1.1)–(1.3) when the Galerkin method is applied.

We denote by $\|\cdot\|_q$ the $L^q(U)$ norm for $1 \le q \le \infty$, by $\|\cdot\|$ the $L^2(U)$ norm, and by $\|\cdot\|_{k,p}$ the $W^{k,p}(U)$ norm. Let

$$Z = \left\{u(x,t) \text{ in } L^\infty(0,T; H_0^K(U)) \text{ and } u_t(x,t) \text{ in } L^\infty(0,T; H_0^K(U))\right\}.$$

For any $0 < t < T$, we define functionals $I, J : Z \to \mathbb{R}$ by

$$I(u) = \left\|\nabla^K u\right\|^2 + \left\|\nabla^K u_t\right\|^2 - \|u\|_{p+1}^{p+1}, \quad p > 1 \tag{2.1}$$

and

$$J(u) = \frac{1}{2}\left\|\nabla^K u\right\|^2 + \frac{1}{2}\left\|\nabla^K u_t\right\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \quad p > 1. \tag{2.2}$$

We define the potential well depth (also the mountain pass value of J) as

$$d = \inf_{\substack{0<t<T, u\in Z,\\ \|\nabla^K u\|^2+\|\nabla^K u_t\|^2\neq 0}} \left(\sup_{a\ge 0} J(au)\right). \tag{2.3}$$

The energy functional $E : Z \to \mathbb{R}$ is defined by

$$E(u) = \frac{1}{2}\left\|u_t\right\|^2 + \frac{1}{2}\left\|\nabla^K u\right\|^2 + \frac{1}{2}\left\|\nabla^K u_t(x,t)\right\|^2 - \int_\Omega F(u(x,t))dx, \tag{2.4}$$

where $F(u) = \int_0^u f(s)ds$.

We introduce the modified Nehari manifold (for Nehari manifold we refer to [29] and [30]) as

$$N = \{u \in Z \big| I(u) = 0 \text{ and } \|\nabla^K u\|^2 + \|\nabla^K u_t\|^2 \neq 0\}.$$

Finally, for any $0 < t < T$ the modified potential well is defined as

$$\mathbf{W} = \left\{u \in Z \big| \, I(u) > 0, J(u) < d\right\} \cup \{0\}. \tag{2.5}$$

Theorem 2.1. The depth of the potential well (denoted by d in (2.3)) can also be characterized as

$$d = \inf_{\substack{0<t<T,\\ u\in N}} J(u).$$

In order to prove Theorem 2.1, we introduce the following two lemmas.

Lemma 2.1. If $0 < t < T, u \in Z$ and $\|\nabla^K u\|^2 + \|\nabla^K u_t\|^2 \neq 0$, we have
(i) $\operatorname{Lim}_{a\to+\infty} J(au) = -\infty$, $\operatorname{Lim}_{a\to 0} J(au) = 0$.
(ii) There exists a unique positive number $\tilde{a} = \tilde{a}(u)$ such that $\frac{dJ(au)}{da}\Big|_{a=\tilde{a}} = 0$.
(iii) When $a = \tilde{a}$, $\frac{d^2J(au)}{da^2} < 0$.
(iv) $J(au)$ increases with a as $0 \le a \le \tilde{a}$; $J(au)$ decreases with a as $\tilde{a} \le a < +\infty$.

Proof. (i) is true because

$$J(au) = \frac{a^2}{2}\left\|\nabla^K u_t\right\|^2 + \frac{a^2}{2}\left\|\nabla^K u\right\|^2 - \frac{a^{p+1}}{p+1}\left\|u\right\|_{p+1}^{p+1}, \quad p > 1. \tag{2.6}$$

Calculate

$$\frac{dJ(au)}{da} = a\left\|\nabla^K u_t\right\|^2 + a\left\|\nabla^K u\right\|^2 - a^p\left\|u\right\|_{p+1}^{p+1}. \tag{2.7}$$

Solving $\frac{dJ(au)}{da} = 0$, there is an unique solution

$$\tilde{a} = \frac{\left(\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2\right)^{\frac{1}{p-1}}}{\|u\|_{p+1}^{\frac{p+1}{p-1}}}. \tag{2.8}$$

(ii) is true as $\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2 \neq 0$.
In order to obtain (iii), substitute (2.8) into the expression

$$\frac{d^2J(au)}{da^2} = \left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2 - a^{p-1}p\,\|u\|_{p+1}^{p+1}$$

gives

$$\frac{d^2J(au)}{da^2}\Big|_{a=\tilde{a}} = \left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right)\tilde{a}^{p-1}(1-p) < 0.$$

At last, from (2.7) and (2.8), we have

$$\frac{dJ(au)}{da} = \|u\|_{p+1}^{p+1}\, a(\tilde{a}^{p-1} - a^{p-1}).$$

Hence $\frac{dJ(au)}{da} > 0$ as $0 < a < \tilde{a}$ and $\frac{dJ(au)}{da} < 0$ as $\tilde{a} < a < +\infty$. So (iv) is true.

Lemma 2.2. If $0 < t < T, u \in Z$ and $\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2 \neq 0$, $J(\alpha u) = \sup_{a\ge 0} J(au)$ is equivalent to $I(\alpha u) = 0$.

Proof. Since

$$I(\alpha u) = \alpha^2 \left\|\nabla^K u_t\right\|^2 + \alpha^2 \left\|\nabla^K u\right\|^2 - \alpha^{p+1} \|u\|_{p+1}^{p+1},$$

$\tilde{a}$ in (2.8) coincides with the solution to equation $I(\alpha u) = 0$. From Lemma 2.1 (iv), $J(\alpha u) = \sup_{a\geq 0} J(au)$.

Conversely, if $J(\alpha u) = \sup_{a\geq 0} J(au)$, Lemma 2.1 gives $\alpha = \tilde{a}$ in (2.8), then

$$I(\alpha u) = \tilde{a}^2 \left\|\nabla^K u_t\right\|^2 + \tilde{a}^2 \left\|\nabla^K u\right\|^2 - \tilde{a}^{p+1} \|u\|_{p+1}^{p+1} = 0.$$

Proof of Theorem 2.1: By Lemma 2.1, the depth of potential well (see (2.3)) should be

$$\begin{aligned} d &= \inf_{\substack{0<t<T, u\in Z,\\ \|\nabla^K u\|^2+\|\nabla^K u_t\|^2\neq 0}} \left(\sup_{a\geq 0} J(au)\right) \\ &= \inf_{\substack{0<t<T, u\in Z,\\ \|\nabla^K u\|^2+\|\nabla^K u_t\|^2\neq 0}} J(\tilde{a}u). \end{aligned} \tag{2.9}$$

Let $w = \tilde{a}u$ then from Lemma 2.2 we have $d = \inf J(w)$, where the infimum is taken for all $t \in (0, T)$ and all functions $w \in Z$ satisfying that $I(u)$ attains 0 on $(0, T)$ with $\|\nabla^K u\|^2 + \|\nabla^K u_t\|^2 \neq 0$, which means $w \in N$.

The proof of Theorem 2.1 is completed.

Lemma 2.3. As $p > 1$ satisfies (1.5), a computational formula for the potential well depth is

$$d = \frac{1}{\kappa\Lambda^\kappa}. \tag{2.10}$$

Here

$$\kappa = \frac{2(p+1)}{p-1} \tag{2.11}$$

and

$$\Lambda^2 = \sup_{\substack{t\in(0,T), u\in Z\\ \|\nabla^K u_t\|^2+\|\nabla^K u\|^2\neq 0}} \frac{\|u\|_{p+1}^2}{\left\|\nabla^K u_t\right\|^2 + \|\nabla^K u\|^2}. \tag{2.12}$$

Moreover,

$$d \geq \frac{1}{\kappa S_{p+1}^\kappa} > 0,$$

where S_{p+1} is the best Sobolev constant for the embedding $H_0^K(U) \hookrightarrow L^{p+1}(U)$, i.e.,

$$S_{p+1} = \sup_{u\in H_0^K(U)\setminus\{0\}} \frac{\|u\|_{p+1}}{\|\nabla^K u\|}.$$

Proof. As the process of proof in Theorem 2.1, substituting (2.8) into the computation of $J(au)$ (see (2.6)) we obtain

$$\begin{aligned}
J(\tilde{a}u) &= \frac{\tilde{a}^2}{2}\left(\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2\right) - \frac{\tilde{a}^{p+1}}{p+1}\|u\|_{p+1}^{p+1} \\
&= \frac{\tilde{a}^2}{2}\left(\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2\right) - \frac{\tilde{a}^2}{p+1}\cdot \tilde{a}^{p-1}\|u\|_{p+1}^{p+1} \\
&= \frac{\tilde{a}^2}{2}\left(\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2\right) - \frac{\tilde{a}^2}{p+1}\cdot \frac{\left(\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2\right)}{\|u\|_{p+1}^{p+1}}\|u\|_{p+1}^{p+1} \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\cdot \tilde{a}^2\left(\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2\right) \\
&= \frac{p-1}{2(p+1)}\cdot\left(\frac{\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2}{\|u\|_{p+1}^2}\right)^{\frac{p+1}{p-1}}.
\end{aligned}$$

Value of κ and Λ in (2.11), (2.12) gives

$$\sup_{\substack{0<t<T, u\in Z,\\ \|\nabla^K u\|^2+\|\nabla^K u_t\|^2\neq 0}}\left(\frac{\|u\|_{p+1}^2}{\|\nabla^K u_t\|^2 + \|\nabla^K u\|^2}\right)^{\frac{p+1}{p-1}} = \Lambda^{\frac{2(p+1)}{p-1}} = \Lambda^{\kappa}.$$

Therefore

$$\begin{aligned}
d &= \inf_{\substack{0<t<T, u\in Z,\\ \|\nabla^K u\|^2+\|\nabla^K u_t\|^2\neq 0}} J(\tilde{a}u) \\
&= \frac{1}{\sup_{\substack{0<t<T, u\in Z,\\ \|\nabla^K u\|^2+\|\nabla^K u_t\|^2\neq 0}} \frac{1}{J(\tilde{a}u)}} \\
&= \frac{1}{\kappa\Lambda^{\kappa}}.
\end{aligned}$$

Furthermore,

$$\Lambda^2 \leq \sup_{\substack{0<t<T, u\in Z,\\ \|\nabla^K u\|^2+\|\nabla^K u_t\|^2\neq 0}} \frac{\|u\|_{p+1}^2}{\left\|\nabla^K u\right\|^2} \leq S_{p+1}^2.$$

It follows that

$$d \geq \frac{1}{\kappa S_{p+1}^{\kappa}} > 0.$$

Lemma 2.4. If $J(u) \leq d$, then $I(u) > 0$ is equivalent to

$$0 < \left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2 < \Lambda^{-\kappa}.$$

Proof. From (2.1) and (2.2), the following equality holds:

$$J(u) = \frac{1}{p+1}I(u) + \frac{1}{\kappa}\left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right). \tag{2.13}$$

Since $d = \frac{1}{\kappa\Lambda^{\kappa}}$ where $\kappa = \frac{2(p+1)}{p-1}$, for $I(u) > 0$ and $J(u) \le d$ we have

$$0 < \left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2 < \Lambda^{-\kappa}.$$

Conversely, if

$$\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2 < \Lambda^{-\kappa}, \tag{2.14}$$

then

$$\Lambda^{\kappa} \cdot \left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right) < 1.$$

By value of Λ in (2.12),

$$\begin{aligned}
1 &> \Lambda^{\frac{\kappa(p-1)}{2}} \cdot \left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right)^{\frac{p-1}{2}} \\
&= \Lambda^{p+1} \cdot \left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right)^{\frac{p-1}{2}} \\
&\ge \|u\|_{p+1}^{p+1} \cdot \left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right)^{-\frac{p+1}{2}} \cdot \left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right)^{\frac{p-1}{2}} \\
&= \|u\|_{p+1}^{p+1} \cdot \left(\left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2\right)^{-1}.
\end{aligned}$$

Hence

$$\|u\|_{p+1}^{p+1} < \left\|\nabla^K u_t\right\|^2 + \left\|\nabla^K u\right\|^2.$$

Thus $I(u) = \left\|\nabla^K u\right\|^2 + \left\|\nabla^K u_t\right\|^2 - \left\|u\right\|_{p+1}^{p+1} > 0$.

3. Global weak solution

We denote the inner product in $L^2(U)$ by

$$(u, v) = \int_U u(x)v(x)dx. \tag{3.1}$$

A continuous linear functional defined on the locally convex linear topological space $\mathfrak{D}(0, T)$ is called the "distribution" or the "generalized function" (see [31], Chapter 8). We denote the space of generalized functions on $(0, T)$ by $\mathfrak{D}'(0, T)$.

Definition 3.1. For $T > 0$, if the function $u(x, t) \in Z$ satisfies:
1) for any $v(x) \in H_0^K(U)$ and for almost $t \in [0, T)$,

$$\begin{aligned}
&(u_t, v) + \int_0^t (\nabla^K u, \nabla^K v)d\tau + (\nabla^K u, \nabla^K v) + (\nabla^K u_t, \nabla^K v) + (u, v) \\
&\qquad = \int_0^t (f(u), v)d\tau + (u_1, v) + (\nabla^K u_0, \nabla^K v) + (\nabla^K u_1, \nabla^K v) + (u_0, v).
\end{aligned} \tag{3.2}$$

2) $u(x, 0) = u_0(x)$ in $H_0^K(U)$ and $u_t(x, 0) = u_1(x)$ in $L^2(U)$.
Then we call $u = u(x, t)$ a global weak solution to Problems (1.1)–(1.3).

For $u_0 \in H_0^K(U)$ and $u_1 \in H_0^K(U)$, we introduce the following initial functionals:

$$E(0) = \frac{1}{2}\left\|u_1(x)\right\|^2 + \frac{1}{2}\left\|\nabla^K u_0(x)\right\|^2 + \frac{1}{2}\left\|\nabla^K u_1(x)\right\|^2 - \int_U F(u_0(x))dx, \tag{3.3}$$

$$J(0) = \frac{1}{2}\left\|\nabla^K u_0(x)\right\|^2 + \frac{1}{2}\left\|\nabla^K u_1(x)\right\|^2 - \frac{1}{p+1}\left\|u_0(x)\right\|_{p+1}^{p+1}, \tag{3.4}$$

$$I(0) = \|\nabla^K u_1(x)\|^2 + \|\nabla^K u_0(x)\|^2 - \|u_0(x)\|_{p+1}^{p+1}. \tag{3.5}$$

Theorem 3.1. If $T > 0$, $f(s) = |s|^{p-1}s$ where p satisfies (1.5) and $E(0) < d$, there exists a global weak solution to Problems (1.1)–(1.3) as long as $I(0) > 0$, $J(0) < d$ for $u_0 \in H_0^K(\Omega)$ and $u_1 \in H_0^K(\Omega)$. Moreover, for any $0 \leq t < T$, $u \in \mathbf{W}$.

3.1. Step 1: Galerkin method.

Let $\{\omega_k(x)\}(k = 1, 2, 3, \cdots)$ be a complete orthogonal basis in $H^{2K}(\Omega) \cap H_0^K(\Omega)$, which solves the following eigenvalue system

$$(-1)^K \Delta^K \omega_k = \lambda_k \omega_k,\ D^\alpha \omega_k\big|_{\partial U} = 0,\ 0 \leq |\alpha| \leq K - 1.$$

It is also a complete orthonormal basis for $L^2(U)$ and a complete orthogonal basis for $H_0^K(U)$. (see [32, 33]).

Based on the Galerkin method, an approximate solution to Problems (1.1)–(1.3) can be constructed by

$$u_m(x,t) = \sum_{k=1}^{m} g_{km}(t)\omega_k(x),\ m = 1, 2, 3, \cdots, \tag{3.6}$$

where $u_m(x,t)$ satisfies a system of nonlinear ordinary differential equations

$$\begin{aligned} &(\omega_k, u_{mtt}) + (\omega_k, (-1)^K\Delta^K u_m) + (\omega_k, (-1)^K\Delta^K u_{mt}) + (\omega_k, (-1)^K\Delta^K u_{mtt}) + (\omega_k, u_{mt}) \\ &= (\omega_k(x), f(u_m(x,t))) \end{aligned} \tag{3.7}$$

with initial values

$$g_{km}(0) = a_{km} \text{ and } g'_{km}(0) = b_{km} \tag{3.8}$$

for $k = 1, 2, \cdots, m$.

Since $u_0(x) \in H_0^K(U)$ and $u_1(x) \in H_0^K(U)$, when $m \to +\infty$ there exist a_{km} and b_{km} $(k = 1, 2, \cdots, m)$ such that

$$u_m(x,0) = \sum_{k=1}^{m} a_{km}\omega_k(x) \longrightarrow u_0(x) \text{ in } H_0^K(U), \tag{3.9}$$

$$u_{mt}(x,0) = \sum_{k=1}^{m} b_{km}\omega_k(x) \longrightarrow u_1(x) \text{ in } H_0^K(U). \tag{3.10}$$

Notice that $f(s) = |s|^{p-1}s$ $(p > 1)$ is locally Lipschitz continuous with respect to s. According to classical existence theory for nonlinear ordinary differential equations (see [34], corollary 1.1.1), systems (3.7) and (3.8) with initial data satisfying (3.9) and (3.10) have a local solution $u_m(x,t)$ for each m. In order to extend it to a global solution on $[0,T)$, we will make priori estimates of $u_m(x,t)$ $(m=1,2,\cdots)$.

Multiplying by $g_{km}^{'}(t)$ on both sides of (3.7) and summing up from $k=1$ to $k=m$, we obtain

$$(\sum_{j=1}^{m}\omega_k(x)g_{km}^{'}(t),u_{mtt})+(\sum_{k=1}^{m}\omega_k(x)g_{km}^{'}(t),(-1)^K\Delta^K u_m)+(\sum_{k=1}^{m}\omega_k(x)g_{km}^{'}(t),(-1)^K\Delta^K u_{mt})$$
$$+(\sum_{k=1}^{m}\omega_k(x)g_{km}^{'}(t),(-1)^K\Delta^K u_{mtt})+(\sum_{k=1}^{m}\omega_k(x)g_{km}^{'}(t),u_{mt})=(\sum_{k=1}^{m}\omega_k(x)g_{km}^{'}(t),f(u_m)).$$

That is,

$$(u_{mt},u_{mtt})+(u_{mt},(-1)^K\Delta^K u_m)+(u_{mt},(-1)^K\Delta^K u_{mt})$$
$$+(u_{mt},(-1)^K\Delta^K u_{mtt})+(u_{mt},u_{mt})=(u_{mt},f(u_m)).$$

Integrating the above equality by parts with respect to x,

$$\frac{1}{2}\frac{d}{dt}\left\|u_{mt}\right\|^2+\frac{1}{2}\frac{d}{dt}\left\|\nabla^K u_m\right\|^2+\left\|\nabla^K u_{mt}\right\|^2+\frac{1}{2}\frac{d}{dt}\left\|\nabla^K u_{mt}\right\|^2+\|u_{mt}\|^2-(f(u_m),u_{mt})=0.$$

Let $F(u_m)=\int_0^{u_m}f(s)ds$, calculation

$$\frac{d}{dt}\int_U F(u_m)dx=\int_U\frac{d}{dt}F(u_m)dx=\int_U f(u_m)\cdot u_{mt}dx=(f(u_m),u_{mt})$$

gives that

$$\frac{d}{dt}(\frac{1}{2}\|u_{mt}\|^2+\frac{1}{2}\left\|\nabla^K u_m\right\|^2+\frac{1}{2}\left\|\nabla^K u_{mt}\right\|^2-\int_U F(u_m)dx)+\left\|\nabla^K u_{mt}\right\|^2+\|u_{mt}\|^2=0. \tag{3.11}$$

Let $E_m(t)=E(u_m)$, from (2.4) we have

$$E_m(t)=\frac{1}{2}\left\|u_{mt}\right\|^2+\frac{1}{2}\left\|\nabla^K u_m\right\|^2+\frac{1}{2}\left\|\nabla^K u_{mt}\right\|^2-\int_U F(u_m)dx \tag{3.12}$$

and

$$E_m(0)=\frac{1}{2}\left\|u_{mt}(x,0)\right\|^2+\frac{1}{2}\left\|\nabla^K u_m(x,0)\right\|^2+\frac{1}{2}\left\|\nabla^K u_{mt}(x,0)\right\|^2-\int_U F(u_m(x,0))dx. \tag{3.13}$$

Integrating (3.11) with respect to t on $(0,t)$ for $0\le t<T$ gives

$$E_m(t)+\int_0^t\left\|\nabla^K u_{m\tau}(x,\tau)\right\|^2d\tau+\int_0^t\left\|u_{m\tau}(x,\tau)\right\|^2d\tau=E_m(0),\quad 0\le t<T. \tag{3.14}$$

It concludes that

$$E_m(t)\le E_m(0),\qquad 0\le t<T. \tag{3.15}$$

3.2. Step 2: proving $u_m(x,t) \in \mathbf{W}$ for sufficiently large m and $0 < t < T$.

First, we claim that there exists $N_1 > 0$ such that

$$I(u_m)(0) > 0 \text{ and } J(u_m)(0) < d \quad \text{for all } m > N_1. \tag{3.16}$$

From (3.9), (3.10), and (1.5), using Sobolev imbedding theorem we find $\left\|\nabla^K u_m(x,0)\right\|$ converges to $\left\|\nabla^K u_0(x)\right\|$, $\left\|u_m(x,0)\right\|_{p+1}^{p+1}$ converges to $\left\|u_0(x)\right\|_{p+1}^{p+1}$ and $\left\|\nabla^K u_{mt}(x,0)\right\|$ converges to $\left\|\nabla^K u_1(x)\right\|$ as $m \to +\infty$. Therefore, when m tends to $+\infty$,

$$I(u_m)(0) = \left\|\nabla^K u_m(x,0)\right\|^2 + \left\|\nabla^K u_{mt}(x,0)\right\|^2 - \left\|u_m(x,0)\right\|_{p+1}^{p+1}$$

converges to $I(0)$ and

$$J(u_m)(0) = \frac{1}{2}\left\|\nabla^M u_m(x,0)\right\|^2 + \frac{1}{2}\left\|\nabla^K u_{mt}(x,0)\right\|^2 - \frac{1}{p+1}\left\|u_m(x,0)\right\|_{p+1}^{p+1}$$

converges to $J(0)$.

Since $I(0) > 0$ and $J(0) < d$, we conclude that $I(u_m)(0) > 0$ and $J(u_m)(0) < d$ for sufficiently large integer m, which implies (3.16).

Next we prove that $\int_U F(u_m(x,0))dx$ converges to $\int_U F(u_0(x))dx$ when m increases to $+\infty$.

By mean value theorem of integral, there exists $\xi^{(m)}$ between $u_0(x)$ and $u_m(x,0)$ such that

$$F(u_m(x,0)) - F(u_0(x)) = \int_{u_0(x)}^{u_m(x,0)} f(s)ds = f(\xi^{(m)})(u_m(x,0) - u_0(x)),$$

hence

$$\left|\int_U \Big(F(u_m(x,0)) - F(u_0(x))\Big)dx\right| \le \left\||\xi^{(m)}|^p\right\|_{\frac{p+1}{p}} \|u_m(x,0) - u_0(x)\|_{p+1}.$$

Under condition (1.5) of p, $H_0^K(U)$ is embedded in $L^{p+1}(U)$. Since $u_m(x,0)$ converges to $u_0(x)$ in $H_0^K(U)$ as m increases to $+\infty$ (see (3.9)), $\left\|u_m(x,0) - u_0(x)\right\|_{p+1} \to 0$ as $m \to +\infty$ and $\left\||\xi^{(m)}|^p\right\|_{\frac{p+1}{p}}$ is uniformly bounded for $m = 1, 2, \cdots$. So we arrive at

$$\int_U F(u_m(x,0))dx \longrightarrow \int_U F(u_0(x))dx \quad (m \to +\infty). \tag{3.17}$$

By (3.9), (3.10), and (3.17), $E_m(0)$ in (3.13) converges to

$$\frac{1}{2}\left(\left\|u_1\right\|^2 + \left\|\nabla^K u_0\right\|^2 + \left\|\nabla^K u_1\right\|^2\right) - \int_U F(u_0)dx,$$

that is,

$$E_m(0) \longrightarrow E(0), \qquad m \to +\infty. \tag{3.18}$$

Since $E(0) < d$, there exists $N_2 > 0$ satisfying $E_m(0) < d$ for all $m > N_2$.

Recalling $\left|f(u)\right| = \left|u\right|^p$, a control of $F(u) = \int_0^u f(s)ds$ is

$$0 \le F(u) \le \frac{1}{p+1}\left|u\right|^{p+1},$$

then

$$-\frac{1}{p+1}\int_U |u|^{p+1}dx \leq -\int_U F(u)dx. \tag{3.19}$$

Combining with (3.19), it follows from (2.2), (3.12) that

$$E_m(t) \geq J(u_m) + \frac{1}{2}\left\|u_{mt}\right\|^2, \quad 0 \leq t < T.$$

Therefore, from (3.14) we have

$$\begin{aligned} E_m(0) &\geq \int_0^t \left\|\nabla^K u_{m\tau}(x,\tau)\right\|^2 d\tau + E_m(t) \\ &\geq \int_0^t \left\|\nabla^K u_{m\tau}(x,\tau)\right\|^2 d\tau + J(u_m) + \frac{1}{2}\left\|u_{mt}\right\|^2, \quad 0 \leq t < T. \end{aligned} \tag{3.20}$$

Hence

$$J(u_m(x,t)) < d \text{ for all } t \in (0,T) \text{ and } m > N_2. \tag{3.21}$$

In what follows, we prove that $u_m(x,t) \in W$ for sufficiently large integer m. Set $m > \max\{N_1, N_2\}$ and $T > 0$, if there exists $t_0 = t_0(m) \in (0,T)$ such that $u_m(x,t)$ attains $\partial\mathbf{W}$ at $t = t_0$, then $I(u_m)(t_0) = 0$ with $\left\|\nabla^K u_m(x,t_0)\right\| + \left\|\nabla^K u_{mt}(x,t_0)\right\| \neq 0$ or $J(u_m)(t_0) = d$.

Inequality (3.21) means $J(u_m)(t_0) = d$ is impossible; on the other hand, by Theorem 2.1 we find $J(u_m)(t_0) \geq d$, which also contradicts (3.21). Therefore, when m is large enough and $0 < t < T$, $u_m(x,t)$ always stays in $\mathbf{W}$. That is, $I(u_m) > 0$ and $J(u_m) < d$.

3.3. Step 3: existence of a global solution for nonlinear ordinary differential systems (3.7) *and* (3.8).

Substituting

$$J(u_m(x,t)) = \frac{p-1}{2(p+1)}\left(\left\|\nabla^K u_m\right\|^2 + \left\|\nabla^K u_{mt}\right\|^2\right) + \frac{1}{p+1}I(u_m(x,t)) \tag{3.22}$$

into (3.20), we obtain

$$\frac{p-1}{2(p+1)}\left(\left\|\nabla^K u_m\right\|^2 + \left\|\nabla^K u_{mt}\right\|^2\right) + \frac{1}{p+1}I(u_m(x,t)) + \frac{1}{2}\left\|u_{mt}\right\|^2 < d$$

and

$$\int_0^t \left\|\nabla^K u_{m\tau}(x,\tau)\right\|^2 d\tau < d, \quad 0 < t < T. \tag{3.23}$$

Since $I(u_m) > 0$, when sufficiently large m we have the estimates

$$||\nabla^K u_m||^2 + ||\nabla^K u_{mt}||^2 < \kappa d \tag{3.24}$$

and

$$||u_{mt}||^2 < 2d, \quad 0 < t < T. \tag{3.25}$$

Inequality (3.24) shows $\left\|u_m\right\|_{H_0^K(U)}$ and $\left\|u_{mt}\right\|_{H_0^K(U)}$ are uniformly bounded for $m = 1, 2, \cdots$. Consequently $\left\|u_m\right\|$ and $\left\|\nabla u_m\right\|$ are also uniformly bounded for $m = 1, 2, \cdots$.

From (1.5), the Sobolev space $H_0^K(U)$ is embedded in $L^{p+1}(U)$ and $L^{2p}(U)$. Thus $\|u_m\|_{p+1}$ and $\|f(u_m)\|^2 = \|u_m\|_{2p}^{2p}$ are also uniformly bounded for $m = 1, 2, \cdots$.

When $0 < t < T$,

$$\left|(\omega_k(x), f(u_m))\right| \leq \|\omega_k(x)\|\|f(u_m)\| = \|f(u_m)\|$$

should be uniformly bounded for $m = 1, 2, \cdots$, where $\{\omega_k(x)\}_{k=1}^{+\infty}$ is a complete orthonormal basis in $L^2(\Omega)$ with $\|\omega_i(x)\| = 1$ $(i = 1, 2, \cdots)$.

Now we conclude that there exist global solutions

$$g_{km}(t), \quad k = 1, 2, 3, \cdots, m$$

to problems (3.7) and (3.8) on $[0, T)$, according to classical theory of nonlinear ordinary differential system (see [34]).

3.4. Step 4: deriving global weak solutions that satisfy (3.2).

Let $q = \frac{p+1}{p}$ and $Q_T = U \times [0, T)$. It follows from

$$|f(u)|^q = |u|^{pq} = |u|^{p+1}$$

that

$$\{f(u_m)\}_{m=1}^{+\infty} \text{ is uniformly bounded in } L^\infty(0, T; L^q(U)). \tag{3.26}$$

Furthermore,

$$\{f(u_m)\}_{m=1}^{+\infty} \text{ is uniformly bounded in } L^q(Q_T). \tag{3.27}$$

By (3.24) there exists a subsequence of $\{u_m(x,t)\}_{m=1}^{+\infty}$ (still denoted by $\{u_m(x,t)\}_{m=1}^{+\infty}$), a function $u(x,t)$ satisfying the following two:

$$u_m(x,t) \text{ converges to } u(x,t) \text{ in } L^\infty(0, T; H_0^K(U)) \text{ weakly-star as } m \text{ increases to } +\infty, \tag{3.28}$$

$$u_{mt}(x,t) \text{ converges to } u_t(x,t) \text{ in } L^\infty(0, T; H_0^K(U)) \text{ weakly-star as } m \text{ increases to } +\infty. \tag{3.29}$$

By (3.26), there exists another subsequence of $\{u_m(x,t)\}_{m=1}^{+\infty}$ (still denoted by $\{u_m(x,t)\}_{m=1}^{+\infty}$ again), a function $X(x,t)$ satisfy that

$$f(u_m(x,t)) \text{ converges to } X(x,t) \text{ in } L^\infty(0, T; L^q(U)) \text{ weakly-star as } m \text{ increases to } +\infty. \tag{3.30}$$

By (3.25), $\{u_m(x,t)\}_{m=1}^{+\infty}$ is uniformly bounded in $H^1(Q_T)$. Since $H^1(Q_T)$ is compactly imbedded into $L^2(Q_T)$, there exists a subsequence of $\{u_m(x,t)\}_{m=1}^{+\infty}$ (still denoted by $\{u_m(x,t)\}_{m=1}^{+\infty}$) such that

$$u_m(x,t) \text{ converges to } u(x,t) \text{ in } L^2(Q_T) \text{ as } m \text{ increases to } +\infty,$$

and then

$$u_m(x,t) \text{ converges to } u(x,t) \text{ in } Q_T \text{ almost everywhere as } m \text{ increases to } +\infty.$$

Moreover,

$$f(u_m(x,t)) \text{ converges to } f(u(x,t)) \text{ in } Q_T \text{ almost everywhere as } m \text{ increases to } +\infty,$$

because $f(s) = |s|^{p-1}s$ is continuous.

On the other hand, $f(u_m(x,t))$ is bounded in $L^q(Q_T)$ from (3.27), according to J. L. Lions' Lemma ([35], Lemma 1.3) we find

$$f(u_m(x,t)) \text{ weakly converges to } f(u(x,t)) \text{ in } L^q(Q_T) \text{ as } m \text{ increases to } +\infty. \tag{3.31}$$

For $0 \le t < T$, integrating by parts with respect to x and integrating with respect to t from 0 to t on both sides of (3.7), we obtain

$$\begin{aligned}
&(\omega_k, u_{mt}) + \int_0^t (\nabla^K \omega_k, \nabla^K u_m)dt + (\nabla^K \omega_k, \nabla^K u_m) + (\nabla^K \omega_k, \nabla^K u_{mt},) + (\omega_k, u_m)\\
&= \int_0^t (\omega_k, f(u_m),)dt + (\omega_k, u_{mt}(x,0),) + (\nabla^K \omega_k(x), \nabla^K u_m(x,0),)\\
&\qquad + (\nabla^K \omega_k(x), \nabla^K u_{mt}(x,0)) + (\omega_k(x), u_m(x,0)), \quad k = 1,2,3,\cdots.
\end{aligned}$$

Let $m \to +\infty$, we obtain

$$\begin{aligned}
&(\omega_k, u_t) + \int_0^t (\nabla^K \omega_k, \nabla^K u)dt + (\nabla^K \omega_k, \nabla^K u) + (\nabla^K \omega_k, \nabla^K u_t) + (\omega_k, u)\\
&= \int_0^t (\omega_k, f(u))dt + (\omega_k, u_1) + (\nabla^K \omega_k, \nabla^K u_0) + (\nabla^K \omega_k, \nabla^K u_1) + (\omega_k, u_0),\\
&\qquad k = 1,2,3,\cdots.
\end{aligned}$$

Since $\{\omega_k(x)\}_{k=1}^{+\infty}$ is a complete orthogonal basis in $H_0^K(U)$, the above equality still holds if we replace ω_k by arbitrary $v \in H_0^K(U)$.

3.5. Step 5: verifying $u(x,0) = u_0(x)$ in $H_0^K(U)$ and $u_t(x,0) = u_1(x)$ in $L^2(U)$.

According to Lemma 1.2 in [35], it can be deduced from (3.28) and (3.29) that $u_m(x,t) \in C(0,T;H_0^M(U))$, $u(x,t) \in C(0,T;H_0^K(U))$. Then

$$u_m(x,0) \to u(x,0) \text{ in } H_0^K(U) \text{ weakly-star as } m \to +\infty.$$

On the other hand, from (3.9) we see that $u_m(x,0)$ strongly converges to $u_0(x)$ in $H_0^K(U)$ as m increases to $+\infty$, so

$$u(x,0) = u_0(x) \text{ in } H_0^K(U).$$

Next, we will verify $u_t(x,0) = u_1(x)$ in $L^2(U)$. Integrating by parts with respect to x on both sides of (3.7), we get

$$\begin{aligned}
&(u_{mtt}, \omega_k) + (\nabla^K u_{mtt}, \nabla^K \omega_k)\\
&= (f(u_m), \omega_k) - (\nabla^K u_m, \nabla^K \omega_k) - (\nabla^K u_{mt}, \nabla^K \omega_k) - (u_{mt}, \omega_k), \ k = 1,2,3,\cdots,m.
\end{aligned} \tag{3.32}$$

By (3.28)–(3.30), when $m \to +\infty$, for $k = 1, 2, 3, \cdots$,

$$(f(u_m), \omega_k) \longrightarrow (X, \omega_k) \text{ in } L^\infty(0, T) \text{ weakly-star,} \tag{3.33}$$

$$(\nabla^K u_m, \nabla^K \omega_k) \longrightarrow (\nabla^K u, \nabla^K \omega_k) \text{ in } L^\infty(0, T) \text{ weakly-star,} \tag{3.34}$$

$$(\nabla^K u_{mt}, \nabla^K \omega_k) \longrightarrow (\nabla^K u_t, \nabla^K \omega_k) \text{ in } L^\infty(0, T) \text{ weakly-star,} \tag{3.35}$$

$$(u_{mt}, \omega_k) \longrightarrow (u_t, \omega_k) \text{ in } L^\infty(0, T) \text{ weakly-star.} \tag{3.36}$$

So the right side in (3.32) converges to $(X, \omega_k) - (\nabla^K u, \nabla^K \omega_k) - (\nabla^K u_t, \nabla^K \omega_k) - (u_t, \omega_k)$ in $L^\infty(0, T)$ weakly-star as $m \to +\infty$, which means that the left side of (3.32) is also convergent in $L^\infty(0, T)$ weakly-star.

Moreover, by (3.36) and (3.35) when $m \to +\infty$, for $k = 1, 2, 3, \cdots$

$$(u_{mt}, \omega_k) \longrightarrow (u_t, \omega_k) \text{ in } \mathfrak{D}'(0, T)$$

and

$$(\nabla^K u_{mt}, \nabla^K \omega_k) \longrightarrow (\nabla^M u_t, \nabla^M \omega_k) \text{ in } \mathfrak{D}'(0, T).$$

Furthermore, when $m \to +\infty$, for $k = 1, 2, 3, \cdots$,

$$(u_{mtt}, \omega_k) \longrightarrow (u_{tt}, \omega_k) \text{ in } \mathfrak{D}'(0, T)$$

and

$$(\nabla^K u_{mtt}, \nabla^K \omega_k) \longrightarrow (\nabla^K u_{tt}, \nabla^K \omega_k) \text{ in } \mathfrak{D}'(0, T).$$

Hence the left side in (3.32) converges to $(u_{tt}, \omega_k) + (\nabla^K u_{tt}, \nabla^K \omega_k)$ in $\mathfrak{D}'(0, T)$, and then it converges to the same limit in $L^\infty(0, T)$ weakly-star by the uniqueness of limit.

From the above discussions, for $k = 1, 2, \cdots$

$$(u_t, \omega_k) + (\nabla^K u_t, \nabla^K \omega_k) \in L^\infty(0, T)$$

and

$$(u_{tt}, \omega_k) + (\nabla^K u_{tt}, \nabla^K \omega_k) \in L^\infty(0, T).$$

Again, using Lemma 1.2 in [35], for $k = 1, 2, \cdots$ we have

$$(u_{mt}, \omega_k) + (\nabla^K u_{mt}, \nabla^K \omega_k) \in C(0, T; \mathbb{R})$$

and

$$(u_t, \omega_k) + (\nabla^K u_t, \nabla^K \omega_k) \in C(0, T; \mathbb{R}).$$

Therefore, when $m \to +\infty$, for $k = 1, 2, \cdots$

$$(u_{mt}(x, 0), \omega_k(x)) + (\nabla^K u_{mt}(x, 0), \nabla^K \omega_k(x)) \longrightarrow (u_t(x, 0), \omega_k(x)) + (\nabla^K u_t(x, 0), \nabla^K \omega_k(x)).$$

On the other hand, from (3.10) when $m \to +\infty$, for $k = 1, 2, \cdots$

$$(u_{mt}(x,0), \omega_k(x)) + (\nabla^K u_{mt}(x,0), \nabla^K \omega_k(x)) \longrightarrow (u_1, \omega_k) + (\nabla^K u_1, \nabla^K \omega_k).$$

By uniqueness of limit, for $k = 1, 2, \cdots$

$$(u_t(x,0), \omega_k(x)) + (\nabla^K u_t(x,0), \nabla^K \omega_k(x)) = (u_1, \omega_k) + (\nabla^K u_1, \nabla^K \omega_k).$$

Integrating by parts with respect to x,

$$(u_t(x,0), \omega_k(x)) + (u_t(x,0), (-1)^K \Delta^K \omega_k(x)) = (u_1, \omega_k) + (u_1, (-1)^K \Delta^K \omega_k), \quad \text{for } k = 1, 2, \cdots .$$

It follows that

$$(u_t(x,0), \omega_k(x)) + (u_t(x,0), \lambda_k \omega_k(x)) = (u_1, \omega_k) + (u_1, \lambda_k \omega_k) \quad \text{for } k = 1, 2, \cdots .$$

Equivalently,

$$(1 + \lambda_k)(u_t(x,0) - u_1(x), \omega_k(x)) = 0 \quad \text{for } k = 1, 2, \cdots .$$

Since all eigenvalues $\lambda_k > 0$ $(k = 1, 2, \cdots)$ (see Theorem 7.23 in [32]), there should be

$$(u_t(x,0) - u_1(x), \omega_k(x)) = 0 \quad \text{for } k = 1, 2, \cdots .$$

Thus, we have $u_t(x,0) - u_1(x) = 0$ in $L^2(U)$, that is, $u_t(x,0) = u_1(x)$ in $L^2(U)$.

We finally have a global weak solution $u(x,t) \in L^\infty(0,T; H_0^K(U))$ with $u_t(x,t) \in L^\infty(0,T; H_0^K(U))$ to Problems (1.1)–(1.3).

3.6. *Step 6: proving $u(x,t) \in \mathbf{W}$ for $0 \le t < T$.*

The process will be an analogue as in Step 2.

We denote the inner product in $L^\infty(0,T; L^2(U))$ by

$$[u(\cdot,t), v(\cdot,t)] = \int_0^t (u(\cdot,\tau), v(\cdot,\tau)) d\tau, \quad 0 \le t < T.$$

Making an inner product in $L^2(U)$ by u_t on both sides of (1.1), we obtain

$$\begin{aligned}&(u_t, u_{tt}) + (u_t, (-1)^K \Delta^K u) + (u_t, (-1)^K \Delta^K u_t) + (u_t, (-1)^K \Delta^K u_{tt}) + (u_t, u_t)\\ &= (u_t, f(u)) \quad \text{in } \mathfrak{D}'(0,T).\end{aligned}$$

Integrating by parts with respect to x, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|u_t\right\|^2 + \|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla^K u\right\|^2 + \left\|\nabla^K u_t\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla^K u_t\right\|^2\\ &= \frac{d}{dt}\int_U F(u) dx.\end{aligned}$$

Integrating with respect to t from 0 to t $(0 < t < T)$, we have

$$\frac{1}{2}\left\|u_t\right\|^2 + [u_t, u_t] + \frac{1}{2}\left\|\nabla^K u\right\|^2 + \frac{1}{2}\left\|\nabla^K u_t\right\|^2 + [\nabla^K u_t, \nabla^K u_t] - \int_U F(u)dx$$
$$= \frac{1}{2}\left\|u_1(x)\right\|^2 + \frac{1}{2}\left\|\nabla^K u_0(x)\right\|^2 + \frac{1}{2}\left\|\nabla^K u_1(x)\right\|^2 - \int_U F(u_0)dx.$$

That is,

$$E(t) + [\nabla^K u_t, \nabla^K u_t] + [u_t, u_t] = E(0).$$

Hence

$$E(t) \leq E(0) \ \text{ for } 0 \leq t < T.$$

Similar to (3.20), by some computations we obtain

$$\frac{1}{2}\left\|u_t\right\|^2 + J(u) \leq E(t) \leq E(0) \ \text{ for } 0 \leq t < T.$$

Since $E(0) < d$,

$$J(u) < d \ \text{ for } 0 \leq t < T. \tag{3.37}$$

If there exists $t_0 \in (0, T)$ such that $u \in \mathbf{W}$ for $0 \leq t < t_0$ and u attains $\partial\mathbf{W}$ at $t = t_0$, then the nontrivial solution $u \in Z$ satisfies that $J(u)(t_0) = d$ or $I(u)(t_0) = 0$ with $\left\|\nabla^K u_t(x, t_0)\right\| + \left\|\nabla^K u(x, t_0)\right\| \neq 0$.

Inequality (3.37) implies that $J(u)(t_0) = d$ is impossible. On the other hand, if $u(x, t) \in Z$ satisfies that $I(u)(t_0) = 0$ and $\left\|\nabla^K u_t(x, t_0)\right\| + \left\|\nabla^K u(x, t_0)\right\| \neq 0$, that is $u(x, t_0) \in N$, from Theorem 2.1 we should have that $J(u)(t_0) \geq d$, which is also in contradiction with (3.37). So we conclude that $u(x, t) \in \mathbf{W}$ for $0 \leq t < T$.

The proof of Theorem 3.1 is completed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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