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Research article

Elliptic equations in $\mathbb{R}^2$ involving supercritical exponential growth

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Abstract: In this work, we investigated the existence of nontrivial weak solutions for the equation

$$-\mathrm{div}(w(x)\nabla u) = f(x,u), \qquad x \in \mathbb{R}^2,$$

where $w(x)$ is a positive radial weight, the nonlinearity $f(x,s)$ possesses growth at infinity of the type $\exp((\alpha_0 + h(|x|))|s|^{2/(1-\beta)})$, with $\alpha_0 > 0$, $0 < \beta < 1$ and h is a continuous radial function that may be unbounded at infinity. To show the existence of weak solutions, we used variational methods and a new type of the Trudinger-Moser inequality defined on the whole two-dimensional space.

Keywords: Trudinger-Moser inequality; supercritical exponential growth; mountain pass theorem; elliptic equation; variational method

1. Introduction

We begin recalling the following stationary Schrödinger equation:

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

To treat the Eq (1.1) variationally, the Sobolev embedding theorems restrict the nonlinearity f to be of the type $|f(x,u)| \leq c(1 + |u|^{q-1})$, with $1 < q \leq 2^* = \frac{2N}{N-2}$ and $N \geq 3$. Some pioneering results considering the above nonlinearity in a bounded domain $\Omega \subset \mathbb{R}^N$ were treated by Brézis [1], Brézis-Nirenberg [2], Bartsch-Willem [3], and Capozzi-Fortunato-Palmieri [4]. A natural extension of the equation defined on the whole space $\mathbb{R}^N$, considering the nonlinearity $|f(x,u)| \leq c(|u| + |u|^{q-1})$, with $1 < q \leq 2^* = \frac{2N}{N-2}$ in $N \geq 3$, was studied by Kryszewski and Szulkin [5], and Ding and Ni [6], among others. For this case, the Eq (1.1) needs to be rewritten as $-\Delta u + V(x)u = f(x,u)$ for $x \in \mathbb{R}^N$, where $V(x)$ is used to address the compactness properties. Extensions of Eq (1.1) include the p-Laplacian operator, where Δu is replaced by $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. For instance, equations with nonlinearities exhibiting critical Sobolev exponent growth are addressed in [7] for bounded domains in $\mathbb{R}^N$, with similar considerations in the whole space discussed in [8, 9]. Critical exponential growth is considered in [10] for bounded domains and in [11] for the whole space. Additionally, equations involving the (p, q)-Laplacian operator, which address critical Sobolev exponents and related nonlinear growth, can be found in [12, 13]. Another type of equation involves a weight operator $\operatorname{div}(w(x)\nabla u)$, as seen in [14, 15], with Hamiltonian systems using this operator discussed in [16, 17].

In dimension $N = 2$, Sobolev embedding asserts that $H_0^1(\Omega) \subset L^q(\Omega)$ for $q \geq 1$. Therefore, there is no restriction on (1.1) for the values $q > 1$ in $|f(x, u)| \leq c(1+|u|^{q-1})$. Additionally, some examples show that $H_0^1(\Omega) \not\subset L^\infty(\Omega)$. For this case, the maximal growth of the nonlinearity f is of the exponential type (see Pohozaev [18], Trudinger [19], and Yudovich [20]). More precisely, it has been proven that

$$e^{\alpha|u|^2} \in L^1(\Omega), \quad \text{for all} \quad u \in H_0^1(\Omega) \;\text{ and }\; \alpha > 0. \tag{1.2}$$

Furthermore, Moser [21] showed that there exists a positive constant $C = C(\alpha, \Omega)$ such that

$$\sup_{\substack{u\in H_0^1(\Omega)\\ \|\nabla u\|_2\leq 1}} \int_\Omega e^{\alpha|u|^2}\, dx \begin{cases} \leq C, & \alpha \leq 4\pi, \\ +\infty, & \alpha > 4\pi. \end{cases} \tag{1.3}$$

Equation (1.1) with nonlinearities involving exponential growth have been studied by Adimurthi [10], Adimurthi-Yadava [22], and de Figueiredo, Miyagaki, and Ruf [23], among others. Inequality (1.3) is called the Trudinger-Moser inequality. These types of results have been extensively investigated by various authors: in Sobolev spaces over the whole space $\mathbb{R}^2$ [24] and in Sobolev spaces for singular versions [25]; in Lorentz-Sobolev spaces within bounded domains [26,27], in Lorentz-Sobolev spaces over the whole space $\mathbb{R}^N$ [28], and for singular versions in Lorentz-Sobolev spaces [29]; and in weighted Sobolev spaces [14, 30]. Additionally, supercritical versions are discussed in [31].

Now, we introduce a supercritical version of the Trudinger-Moser inequality. Let Ω be a smooth domain in $\mathbb{R}^2$ and w be a weight defined on Ω. We shall denote by $H^1_{0,\mathrm{rad}}(\Omega, w)$ the radial Sobolev weighted space obtained as the closure of all the radially symmetric functions in $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\Omega,w} := \|u\|_{H^1_{0,\mathrm{rad}}(\Omega,w)} = \left(\int_\Omega w(x)|\nabla u|^2\, dx\right)^{\frac{1}{2}}.$$

In particular, if Ω is the whole space $\mathbb{R}^2$, we denote the above Sobolev space as $H^1_{\mathrm{rad}}(\mathbb{R}^2, w)$. Trudinger-Moser inequalities for radial Sobolev spaces with logarithmic weights defined on the unit ball B_1 in $\mathbb{R}^2$ were treated by Calanchi and Ruf [14]. Considering $w(x) = (\log 1/|x|)^\beta$ and $0 \leq \beta < 1$, the mentioned authors proved that

$$\int_{B_1} e^{\alpha|u|^{\frac{2}{1-\beta}}}\, dx < +\infty, \quad \text{for all } \; u \in H^1_{0,\mathrm{rad}}(B_1, w) \;\text{ and for all }\; \alpha > 0. \tag{1.4}$$

Furthermore, setting $\alpha^*_\beta = 2\left[2\pi(1-\beta)\right]^{\frac{1}{1-\beta}}$, there exists $C = C(\alpha,\beta) > 0$ such that

$$\sup_{\substack{u\in H^1_{0,\mathrm{rad}}(B_1,w)\\ \|u\|_{B_1,w}\leq 1}} \int_{B_1} e^{\alpha|u|^{\frac{2}{1-\beta}}}\, dx \begin{cases} \leq C, & \alpha \leq \alpha^*_\beta, \\ +\infty, & \alpha > \alpha^*_\beta. \end{cases} \tag{1.5}$$

A supercritical version of the Trudinger-Moser inequality defined on $H^1_{0,\mathrm{rad}}(B_1) := H^1_{0,\mathrm{rad}}(B_1, I)$, where the weight is the identity function on B_1, was proved by Ngô and Nguyen [31]. The mentioned authors considered the following assumptions:

(h_1) $h : [0,1) \to \mathbb{R}$ is a radial function, $h(0) = 0$ and $h(r) > 0$ for $r \in (0,1)$.
(h_2) There exists some $c > 0$ such that

$$h(r) \le \frac{c}{-\ln r}, \quad \text{near to } 0.$$

(h_3') There exists $\gamma \in (0,1)$ such that

$$h(r) \le \frac{2\gamma\pi \ln(1-r)}{\ln r}, \quad \text{near to } 1.$$

In [31], it was shown that

$$\int_{B_1} \exp\Big((\alpha + h(|x|))|u|^2\Big)\, dx < +\infty, \quad \text{for all } u \in H^1_{0,\mathrm{rad}}(B_1) \text{ and for all } \alpha > 0. \tag{1.6}$$

Furthermore, there exists $C = C(\alpha, h) > 0$ such that

$$\sup_{\substack{u \in H^1_{0,\mathrm{rad}}(B_1) \\ \|u\|_{B_1,I} \le 1}} \int_{B_1} \exp\Big((\alpha + h(|x|))|u|^2\Big)\, dx \begin{cases} \le C, & \alpha \le 4\pi, \\ = +\infty, & \alpha > 4\pi. \end{cases} \tag{1.7}$$

Let us consider

(h_3'') There exist $\gamma \in (0,1)$ such that

$$h(r) \le \frac{\gamma\alpha^*_\beta \ln(1-r)}{\ln r}, \quad \text{near to } 1.$$

The next proposition combines the above results.

Proposition 1.1 (See [30]). *Assume that h satisfies (h_1), (h_2), and (h_3''), and that w is the weight defined by $w(x) = (\log 1/|x|)^\beta$ for $0 < |x| < 1$, where $\beta \in [0,1)$. Then,*

$$\int_{B_1} \exp\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big)\, dx < +\infty, \quad \textit{for all } u \in H^1_{0,\mathrm{rad}}(B_1, w) \textit{ and for all } \alpha > 0.$$

Furthermore, there exists $C = C(\alpha, h) > 0$ such that

$$\sup_{\substack{u \in H^1_{0,\mathrm{rad}}(B_1,w) \\ \|u\|_{B_1,w} \le 1}} \int_{B_1} \exp\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big)\, dx \begin{cases} \le C, & \alpha < \alpha^*_\beta, \\ +\infty, & \alpha > \alpha^*_\beta. \end{cases}$$

We point out that conditions (h_3') or (h_3'') allow the function $h(r) \to +\infty$ as $r \to 1^-$, and this motivates us to say that a function f possesses supercritical exponential growth if there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(x,s)}{\exp\Big((\alpha + h(|x|))|s|^{2/(1-\beta)}\Big)} = \begin{cases} +\infty, & \alpha < \alpha_0, \\ 0, & \alpha > \alpha_0, \end{cases}$$

uniformly on $x \in \mathbb{R}^2$. The above limit implies that $f(x,s) = g(x,s)\exp\Big((\alpha_0 + h(|x|))|s|^{2/(1-\beta)}\Big)$, where

$$\lim_{s\to+\infty} \frac{g(x,s)}{\exp\Big((\alpha + h(|x|))|s|^{2/(1-\beta)}\Big)} = 0, \qquad \text{uniformly on } \ x \in \mathbb{R}^2, \ \text{for all } \ \alpha > 0.$$

Our first objective in this work is to extend Proposition 1.1, in the sense of obtaining a Trudinger-Moser inequality on the whole space $\mathbb{R}^2$. Following [32], we consider the weight

$$w(x) = \begin{cases} \Big[\ln\Big(\frac{1}{|x|}\Big)\Big]^{\beta}, & 0 < |x| < 1 \\ |x|^a, & |x| \geq 1, \end{cases} \tag{1.8}$$

where $0 \leq \beta < 1$ and $a > 2$. On h, we assume that

(h_3) $h(r) > 0$ for $r \in [1, +\infty)$. Moreover, there exist $c > 0$ and $\xi < a/(1-\beta) - 2$ such that

$$h(r) \leq cr^{\xi}, \quad \text{for } \ r \text{ sufficiently large},$$

where the constants a and β are given by (1.8).

In particular, (h_3) allows us to consider the case where $h(r) \to +\infty$ as $r \to +\infty$. Next, we present our adaptation of the Trudinger-Moser inequality which will be utilized in our proof of the existence result.

Theorem 1.2. *Suppose that h satisfies* (h_1) − (h_3) *and that w is the weight defined by* (1.8)*. Then,*

$$\int_{\mathbb{R}^2} \exp\Big[\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\,dx < +\infty, \quad \textit{for all } \ u \in H^1_{\mathrm{rad}}(\mathbb{R}^2, w) \textit{ and } \alpha > 0. \tag{1.9}$$

Moreover, if $\alpha < \alpha^*_\beta$*, there exists* $C > 0$ *satisfying*

$$\sup_{\|u\|_{\mathbb{R}^2,w}\leq 1} \int_{\mathbb{R}^2} \exp\Big[\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\,dx \leq C. \tag{1.10}$$

If $\alpha > \alpha^*_\beta$*, it holds that*

$$\sup_{\|u\|_{\mathbb{R}^2,w}\leq 1} \int_{\mathbb{R}^2} \exp\Big[\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\,dx = +\infty. \tag{1.11}$$

In the subsequent section, we will outline the proof of Theorem 1.2. The aim of this study is to find a nontrivial weak solution to the following stationary Schrödinger equation:

$$-\operatorname{div}(w(x)\nabla u) \ = \ f(x,u), \qquad x \in \mathbb{R}^2. \tag{1.12}$$

Here, w represents the weight defined on (1.8) which allows that f possesses the maximal growth established in Theorem 1.2. More precisely, we assume the following hypotheses:

(H_1) $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is continuous and possesses radial symmetry in its first variable, namely $f(x,s) = f(y,s)$ whenever $|x| = |y|$. Additionally, $f(x,s) = 0$ for all $x \in \mathbb{R}^2$ and $s \leq 0$.

(H_2) The following limit holds:

$$\lim_{s\to 0}\frac{f(x,s)}{s}=0,\quad \text{uniformly on } x\in\mathbb{R}^2.$$

(H_3) There exists a constant $\mu>2$ such that

$$0<\mu F(x,s):=\mu\int_0^s f(x,t)\le sf(x,s),\quad \text{for all}\quad x\in\mathbb{R}^2 \text{ and for all } s>0.$$

(H_4) There exists a constant $\alpha_0>0$ such that

$$\lim_{s\to+\infty}\frac{f(x,s)}{\exp\Big((\alpha+h(|x|))|s|^{2/(1-\beta)}\Big)}=\begin{cases}+\infty, & \alpha<\alpha_0,\\ 0, & \alpha>\alpha_0,\end{cases}$$

uniformly on $x\in\mathbb{R}^2$, where h satisfies $(h_1)-(h_3)$.

(H_5) There exist constants $p>2$ and $C_p>0$ such that

$$f(x,s)\ge C_p s^{p-1},\quad \text{for all}\quad s\ge 0 \text{ and for all } x\in\mathbb{R}^2,$$

where

$$C_p>\frac{S_p^p\Big(\frac{\alpha_0}{\alpha_\beta^*}\Big)^{(1-\beta)(p-2)/2}\Big(\frac{1}{2}-\frac{1}{p}\Big)^{(p-2)/2}}{\Big(\frac{1}{2}-\frac{1}{\mu}\Big)^{(p-2)/2}}$$

and

$$S_p:=\inf_{0\neq u\in H^1_{\mathrm{rad}}(\mathbb{R}^2,w)}\frac{\Big(\int_{\mathbb{R}^2}w(x)|\nabla u|^2\,dx\Big)^{1/2}}{\Big(\int_{\mathbb{R}^2}|u|^p\,dx\Big)^{1/p}}.$$

In the forthcoming text, we shall denote the Hilbert space $E:=H^1_{\mathrm{rad}}(\mathbb{R}^2,w)$ equipped with the inner product defined as

$$\langle u,v\rangle_E=\int_{\mathbb{R}^2}w(x)\nabla u\nabla v\,dx,\quad \text{for all } u,\ v\in E,$$

which induces the norm

$$\|u\|:=\|u\|_E=\Big(\int_{\mathbb{R}^2}w(x)|\nabla u|^2\,dx\Big)^{1/2}.$$

Additionally, E^* denotes the dual space of E equipped with its standard norm. We define $u\in E$ to be a weak solution of (1.12) if

$$\int_{\mathbb{R}^2}w(x)\nabla u\nabla\phi\,dx=\int_{\mathbb{R}^2}f(x,u)\phi\,dx,\quad \text{for all } \phi\in E. \tag{1.13}$$

To find weak solutions of our problem (1.12), we will employ variational methods. For this purpose, let us consider the functional $J:E\to\mathbb{R}$ defined as:

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^2}w(x)|\nabla u|^2\,dx-\int_{\mathbb{R}^2}F(x,u)\,dx.$$

Moreover, based on established arguments (see [33]), it follows that J belongs to $C^1(E,\mathbb{R})$ and

$$J'(u)\phi = \int_{\mathbb{R}^2} w(x)\nabla u\nabla\phi\, dx - \int_{\mathbb{R}^2} f(x,u)\phi\, dx, \quad \text{for all } u,\ \phi \in E.$$

The main result of this article is presented as follows:

Theorem 1.3. *Suppose that f satisfies $(H_1)-(H_5)$ and h satisfies $(h_1)-(h_3)$. Then, problem* (1.12) *possesses a nontrivial weak solution.*

We point out that equations or systems with nonlinearities involving the classical Trudinger-Moser inequalities imply that the growth of f is of type $\exp(|s|^2)$ as s tends to infinity (see [23–25, 34–36], among others). Equations considering Trudinger-Moser inequalities on Lorentz-Sobolev spaces allow us to consider f of the type $\exp(|s|^p)$ with $p > 1$ as s tends to infinity (see [1, 37–39]). Equations with logarithmic weight defined on the unit ball in $\mathbb{R}^2$ may have nonlinearities of the form $\exp(|s|^{2/(1-\beta)})$ for $0 \le \beta < 1$ (see [14, 16]), $\exp((\alpha + h(|x|))|s|^2)$ (see [31, 40]), or $\exp\left(\alpha + h(|x|)|s|^{2/(1-\beta)}\right)$ (see [16, 30, 41]). Furthermore, our existence theorem complements the work in [30] since we consider the whole space $\mathbb{R}^2$. Our main contribution is given by the assumption (H_4), which allows us to consider the behavior of $f(x,s)$ as $\exp\left(\alpha + h(|x|)|s|^{2/(1-\beta)}\right)$ for some $\alpha > 0$, as s tends to infinity, where the radial function h may be unbounded at infinity. Finally, note that the class of functions which satisfy conditions $(H_1)-(H_5)$ is not empty, for instance, consider the following function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,s) = \begin{cases} As^{p-1} + p(1+|x|^{\xi})s^{p-1}\exp\left((1+|x|^{\xi})s^p\right), & s \ge 0 \\ 0, & s < 0, \end{cases}$$

for some positive constants $a > 2$, $0 < \beta < 1$, $0 < \xi < a/(1-\beta) - 2$, $p = 2/(1-\beta)$, and A sufficiently large.

2. Preliminaries

We begin this section by presenting a version of the Strauss result [42], which follows from [14, 32] and plays an important role to prove our version of the supercritical Trudinger-Moser inequality.

Lemma 2.1 (See [14, 32])**.** *Let u be a function in E. Then,*

$$|u(x)| \le \begin{cases} \dfrac{(-\ln|x|)^{\frac{1-\beta}{2}}}{\sqrt{2\pi(1-\beta)}}\|u\|, & \text{if} \quad 0 < |x| < 1, \\ \dfrac{1}{\sqrt{2\pi a}|x|^{a/2}}\|u\|, & \text{if} \quad |x| \ge 1. \end{cases}$$

The next lemma is related to the embeddings of the space E into Lebesgue spaces.

Lemma 2.2 (See [32])**.** *The space E is continuously and compactly embedded in $L^p(\mathbb{R}^2)$ for $p > 4/a$.*

2.1. Proof of Theorem 1.2

Proof. Let us consider $u \in E$ with $\|u\| \leq 1$ and $\alpha < \alpha_\beta^*$. By Lemma 2.1, we have

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_1}\Big[\exp\Big((\alpha+h(|x|))|u|^{2/(1-\beta)}\Big)-1\Big]\,dx &= \sum_{k=1}^{+\infty}\frac{1}{k!}\int_{\mathbb{R}^2\setminus B_1}[\alpha+h(|x|)]^k|u|^{2k/(1-\beta)}\,dx\\ &\leq \sum_{k=1}^{+\infty}\frac{1}{k!}\int_{\mathbb{R}^2\setminus B_1}\frac{[\alpha+h(|x|)]^k}{|x|^{\frac{ak}{1-\beta}}}\,dx\\ &\leq \sum_{k=1}^{+\infty}\frac{2^k\alpha^k}{k!}\int_{\mathbb{R}^2\setminus B_1}\frac{1}{|x|^{\frac{ak}{1-\beta}}}\,dx+\sum_{k=1}^{+\infty}\frac{2^k}{k!}\int_{\mathbb{R}^2\setminus B_1}\frac{h^k(|x|)}{|x|^{\frac{ak}{1-\beta}}}\,dx. \qquad (2.1)\end{aligned}$$

Since $a > 2(1-\beta)$, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^2\setminus B_1}\frac{1}{|x|^{\frac{ak}{1-\beta}}}\,dx \leq \int_{\mathbb{R}^2\setminus B_1}\frac{1}{|x|^{\frac{a}{1-\beta}}}\,dx = C_1, \quad \text{for all} \quad k \geq 1. \qquad (2.2)$$

From (h_3), there exist $c_1 > 0$ and $R_0 > 1$ such that

$$h(|x|) \leq c_1|x|^\xi, \quad \text{for all} \quad |x| \geq R_0.$$

Since $a > (2+\xi)(1-\beta)$, we can get $C_2 > 0$ such that

$$\int_{\mathbb{R}^2\setminus B_{R_0}}\frac{h^k(|x|)}{|x|^{\frac{ak}{1-\beta}}}\,dx \leq \int_{\mathbb{R}^2\setminus B_{R_0}}\frac{c_1^k}{|x|^{(\frac{a}{1-\beta}-\xi)k}}\,dx \leq c_1^k\int_{\mathbb{R}^2\setminus B_{R_0}}\frac{1}{|x|^{\frac{a}{1-\beta}-\xi}}\,dx = C_2, \quad \text{for all } k \geq 1. \qquad (2.3)$$

Using the continuity of h, we can find $c_2 > 0$ such that $h(|x|) \leq c_2$ for $1 \leq |x| \leq R_0$. Then, we can get $C_3 > 0$ such that

$$\int_{B_{R_0}\setminus B_1}\frac{h^k(|x|)}{|x|^{\frac{\alpha k}{1-\beta}}}\,dx \leq \int_{B_{R_0}\setminus B_1}\frac{c_2^k}{|x|^{\frac{\alpha k}{1-\beta}}}\,dx \leq c_2^k\int_{B_{R_0}\setminus B_1}\frac{1}{|x|^{\frac{a}{1-\beta}}}\,dx = C_3, \quad \text{for all } k \geq 1. \qquad (2.4)$$

Replacing (2.2)–(2.4) in (2.1), one has

$$\int_{\mathbb{R}^2\setminus B_1}\Big[\exp\Big((\alpha+h(|x|))|u|^{2/(1-\beta)}\Big)-1\Big]\,dx \leq C_1e^{2\alpha}+(C_2+C_3)e^2. \qquad (2.5)$$

On the other hand, consider $v(x) = u(x) - u(e)$ for $|x| < 1$ and $v(x) = 0$ for $|x| \geq 1$, where e is fixed in $\mathbb{R}^2$ such that $|e| = 1$. Then, $v \in H^1_{0,\mathrm{rad}}(B_1, w)$ for each $u \in E$. Moreover, using the fact that $\|u\| \leq 1$, we have that $\|v\|_{H^1_{0,\mathrm{rad}}(B_1,w)} \leq 1$. Taking $\epsilon > 0$ sufficiently small satisfying $\alpha(1+\epsilon) < \alpha_\beta^*$, we can find $C_\epsilon > 0$ such that

$$|u(x)|^{2/(1-\beta)} \leq (1+\epsilon)|v(x)|^{2/(1-\beta)} + C_\epsilon|u(e)|^{2/(1-\beta)}.$$

Then,

$$\int_{B_1}\Big[\exp\Big((\alpha+h(|x|))|u|^{2/(1-\beta)}\Big)-1\Big]\,dx \leq \int_{B_1}\exp\Big((\alpha+h(|x|))|u|^{2/(1-\beta)}\Big)\,dx$$

$$\leq \int_{B_1} \exp\Big((\alpha + h(|x|))((1+\epsilon)|v(x)|^{2/(1-\beta)} + C_\epsilon |u(e)|^{2/(1-\beta)})\Big)\,dx$$
$$\leq \sup_{|x|\leq 1} \exp\Big((\alpha + h(|x|))C_\epsilon |u(e)|^{2/(1-\beta)}\Big) \int_{B_1} \exp\Big(((1+\epsilon)\alpha + (1+\epsilon)h(|x|))|v(x)|^{2/(1-\beta)}\Big)\,dx.$$

Using the continuity of h and Lemma 2.1, there exists $C_4 > 0$ such that

$$\sup_{|x|\leq 1} \exp\Big((\alpha + h(|x|))C_\epsilon |u(e)|^{2/(1-\beta)}\Big) \leq C_4.$$

Therefore,

$$\int_{B_1} \Big[\exp\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\,dx \leq C_4 \int_{B_1} \exp\Big(\alpha^*_\beta + (1+\epsilon)h(|x|))|v(x)|^{2/(1-\beta)}\Big)\,dx. \tag{2.6}$$

Note that the function $h_\epsilon(r) = (1+\epsilon)h(r)$ defined on $r \in [0,1)$ satisfies the conditions of Proposition 1.1 and using the fact that $v \in H^1_{0,\text{rad}}(B_1, w)$, we can find $C_5 > 0$ such that

$$\int_{B_1} \Big[\exp\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\,dx \leq C_4 \sup_{\substack{v\in H^1_{0,\text{rad}}(B_1,w)\\ \|v\|_{B_1,w}\leq 1}} \int_{B_1} \exp\Big((\alpha^*_\beta + h_\epsilon(|x|))|v(x)|^{2/(1-\beta)}\Big)\,dx \leq C_5. \tag{2.7}$$

Using the above inequality and (2.5), we obtain $C > 0$, independent of the election of $u \in E$, satisfying

$$\int_{\mathbb{R}^2} \Big[\exp\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\,dx \leq C.$$

Therefore, the inequalities (1.9) and (1.10) follow. Moreover, we consider the sequence $(\psi_k) \subset E$ defined as

$$\psi_k(x) = \Big(\frac{1}{\alpha^*_\beta}\Big)^{(1-\beta)/2} \begin{cases} k^{\frac{2}{1-\beta}} \ln\Big(\frac{1}{|x|^2}\Big)^{1-\beta}, & 0 \leq |x| \leq e^{-k/2}, \\ k^{\frac{1-\beta}{2}}, & e^{-k/2} \leq |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Note that $\|\psi_k\| = 1$ for each $k \geq 1$, and for $\alpha > \alpha^*_\beta$, it follows that

$$\int_{\mathbb{R}^2} \exp\Big((\alpha + h(|x|))|\psi_k|^{2/(1-\beta)}\Big)\,dx \geq \int_{B_1} \exp\Big(\alpha|\psi_k|^{2/(1-\beta)}\Big)\,dx \geq 2\pi \int_{e^{-k}/2}^1 \exp\Big(\frac{\alpha}{\alpha^*_\beta}k\Big) r\,dr.$$

Consequently,

$$\int_{\mathbb{R}^2} \exp\Big((\alpha + h(|x|))|\psi_k|^{2/(1-\beta)}\Big)\,dx \geq \pi e^{k\left(\frac{\alpha}{\alpha^*_\beta}-1\right)}\Big(e^k - 1\Big) \to +\infty, \quad \text{as } k \to \infty,$$

and the proof of the last assertion follows.

Remark 2.3.

(a) An example of a function h that satisfies conditions $(h_1) - (h_3)$ is given by $h(r) = r^\xi$ for some $0 < \xi < a/(1-\beta) - 2$ where a and β are given in (1.8).

(b) As it was observed in [31], the assertions of Theorem 1.2 are no longer valid when considering the space of nonradial functions $H^1(\mathbb{R}^2, w)$.

3. Mountain pass structure

We now outline several results necessary for utilizing variational methods.

Lemma 3.1. *Assume that* $(H_1), (H_2)$*, and* (H_4) *hold. Then, there exist* $\sigma, \rho > 0$*, such that*

$$J(u) \geq \sigma, \quad \text{for all} \quad u \in E \ \text{ with } \ \|u\| = \rho.$$

Proof. Given $q > 4/a$ and $\epsilon > 0$, from (H_1), (H_2), and (H_4), there exists $c > 0$ such that

$$|F(x, s)| \leq \epsilon |s|^2 + c|s|^q \exp\Big[\Big((2\alpha_0 + h(|x|))|s|^{2/(1-\beta)}\Big) - 1\Big], \quad \text{for all} \quad (x, s) \in \mathbb{R}^2 \times \mathbb{R}.$$

By the Cauchy-Schwarz inequality and the inequality $(e^w - 1)^2 \leq e^{2w} - 1$ for all $w \geq 0$, we obtain

$$\int_{\mathbb{R}^2} F(x, u)\, dx \leq \epsilon \|u\|_2^2 + c\|u\|_{2q}^q \left(\int_{\mathbb{R}^2} \Big[\exp\Big((4\alpha_0 + 2h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\, dx \right)^{1/2}. \tag{3.1}$$

Using Lemma 2.1, for u in E with $\|u\| \leq 1$, one has

$$|u(x)| \leq \frac{1}{\sqrt{2\pi a}|x|^{a/2}}, \quad \text{for all } \ |x| \geq 1.$$

By (h_3), there exist $R_0 > 1$ and $c_1 > 0$ such that

$$h(|x|) \leq c_1 |x|^{\xi}, \quad \text{for all } \ |x| \geq R_0.$$

Therefore, we can get $C_1 > 0$ such that

$$(4\alpha_0 + 2h(|x|))|u|^{2/(1-\beta)} \leq \frac{4\alpha_0}{(2\pi a)^{1/(1-\beta)}|x|^{\frac{a}{1-\beta}}} + \frac{2c_1}{(2\pi a)^{1/(1-\beta)}|x|^{\frac{a}{1-\beta}-\xi}} \leq \frac{C_1}{|x|^{\eta}}, \quad \text{for all} \quad |x| \geq R_0,$$

where $\eta = \min\{a/(1-\beta) - \xi, a/(1-\beta)\} > 2$, which implies the existence of $C_2 > 0$ such that

$$\int_{\mathbb{R}^2 \setminus B_{R_0}} \Big[\exp\Big((4\alpha_0 + 2h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\, dx \leq 2\pi \int_{R_0}^{+\infty} r\Big(\exp(C_1 r^{-\eta}) - 1\Big)\, dr = C_2. \tag{3.2}$$

Let $h_0 = \max_{0 \leq r \leq R_0} h(r)$. Using Theorem 1.2, we can get $C_3 > 0$ such that

$$\begin{aligned} \int_{B_{R_0}} \Big[\exp\Big((4\alpha_0 + 2h(|x|))|u|^{2/(1-\beta)}\Big) - 1\Big]\, dr &\leq \int_{B_{R_0}} \Big[\exp\Big((4\alpha_0 + 2h_0)|u|^{2/(1-\beta)}\Big) - 1\Big]\, dx \\ &\leq \int_{B_{R_0}} \Big[\exp\Big((4\alpha_0 + 2h_0)\|u\|^{2/(1-\beta)}\big(\frac{|u|}{\|u\|}\big)^{2/(1-\beta)}\Big) - 1\Big]\, dx \leq C_3, \end{aligned} \tag{3.3}$$

provided that $\|u\| \leq \rho_1$ for some $\rho_1 > 0$ such that $(4\alpha_0 + 2h_0)\rho_1^{2/(1-\beta)} < \alpha_\beta^*$. From (3.1)–(3.3), and Lemma 2.2, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} F(x, u)\, dx \leq \epsilon C\|u\|^2 + C\|u\|^q,$$

provided that $\|u\| \leq \rho_0$ for some $0 < \rho_0 \leq \min\{1, \rho_1\}$. Then,

$$J(u) \geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(x, u)\, dx \geq \Big(\frac{1}{2} - \epsilon C\Big)\|u\|^2 - C\|u\|^q.$$

Note that we can assume that $\epsilon > 0$ satisfies $(1/2 - \epsilon C) \geq 1/4$. Consequently, it is possible to choose $\rho > 0$ and $\sigma > 0$ with $0 < \rho \leq \rho_0$ such that $J(u) \geq \sigma > 0$, for all $u \in E$ with $\|u\| = \rho$.

The next lemma follows the same lines as [30, Lemma 3.3].

Lemma 3.2. *Suppose that* $(H_1) - (H_2)$ *hold. If* $e_0 \neq 0$ *in* E*, then there exists* $t > 0$ *large enough such that* $e = te_0$ *satisfies*

$$J(e) < 0 \quad \text{and} \quad \|e\| > \rho,$$

where $\rho > 0$ *is given by Lemma 3.1.*

4. Palais-Smale sequence

In this section, we show some results related to the Palais-Smale sequences. Let us recall that we say that $(u_n) \subset E$ is a $(PS)_c$ sequence for the functional J if

$$J(u_n) \to c \quad \text{and} \quad \|J'(u_n)\|_{E^*} \to 0. \tag{4.1}$$

Moreover, if (u_n) satisfying (4.1) possesses a convergent subsequence, we say that (u_n) satisfies the Palais-Smale condition at the level c.

The following lemma asserts that each Palais-Smale sequence associated with J is bounded.

Lemma 4.1. *Assume* $(H_1) - (H_4)$*. Then any Palais-Smale sequence for the functional* J *is bounded in* E*.*

Proof. Using (H_3), we obtain

$$J(u_n) - \frac{1}{\mu}J'(u_n)u_n = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 - \frac{1}{\mu}\int_{\mathbb{R}^2} (\mu F(x, u_n) - f(x, u_n)u_n)\, dx \geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2.$$

Using (4.1), we have

$$J(u_n) = c + o_n(1) \quad \text{and} \quad \|J'(u_n)\|_{E^*} = o_n(1).$$

Therefore, for n sufficiently large, we obtain

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 \leq c + o_n(1) + o_n(1)\|u_n\|.$$

Consequently, the sequence (u_n) is bounded in E.

Lemma 4.2. *Assume that* $(H_1) - (H_4)$ *are satisfied . Then,* J *satisfies the Palais-Smale condition at the level* c*, where*

$$c < \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(\frac{\alpha^*_\beta}{\alpha_0}\right)^{1-\beta}.$$

Proof. Take a Palais-Smale sequence $(u_n) \subset E$ for J at the level c of J. Using Lemma 4.1, we can find $u \in E$, up to a subsequence, such that $u_n \rightharpoonup u$ weakly in E. Setting $v_n := u_n - u$, we have that $v_n \rightharpoonup 0$ weakly in E. Then,

$$\int_{\mathbb{R}^2} w(x)\nabla u_n \nabla v_n\, dx - \int_{\mathbb{R}^2} f(x, u_n)v_n\, dx = J'(u_n)v_n = o_n(1)$$

and

$$\int_{\mathbb{R}^2} w(x)\nabla u_n \nabla v_n\, dx = \|u_n\|^2 - \|u\|^2 + o_n(1).$$

Therefore,

$$\|u_n\|^2 - \|u\|^2 = \int_{\mathbb{R}^2} f(x, u_n)v_n \, dx + o_n(1). \tag{4.2}$$

It remains to show that, up to a subsequence, the integral in (4.2) tends to zero as $n \to +\infty$. From Lemma 4.1 and the assumption on c, we obtain

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 = c + o_n(1) < \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(\frac{\alpha_\beta^*}{\alpha_0}\right)^{1-\beta} + o_n(1).$$

Thus, without loss of generality, we can find $\delta > 0$ such that

$$\|u_n\|^{2/(1-\beta)} \leq \frac{\alpha_\beta^*}{\alpha_0} - \delta, \quad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

Now, take $m > 1$ and $\epsilon > 0$ such that

$$m(\alpha_0 + 2\epsilon)\Big(\frac{\alpha_\beta^*}{\alpha_0} - \delta\Big) < \alpha_\beta^*. \tag{4.4}$$

From assumptions on f, there exists $C_\epsilon > 0$ such that

$$|f(x, s)| \leq \epsilon|s| + C_\epsilon\Big[\exp\Big((\alpha_0 + \epsilon + h(|x|))|s|^{2/(1-\beta)}\Big) - 1\Big], \quad \text{for all} \quad (x, s) \in \mathbb{R}^2 \times \mathbb{R}.$$

By the Hölder inequality with $1/m + 1/m' = 1$ and the identity $(e^r - 1)^m \leq e^{rm} - 1$ for all $r \geq 0$, we obtain

$$\int_{\mathbb{R}^2} |f(x, u_n)v_n| \, dx \leq \epsilon\|u_n\|_2\|v_n\|_2 + C_\epsilon\|v_n\|_{m'}\left(\int_{\mathbb{R}^2}\Big[\exp\Big(m(\alpha_0 + \epsilon + h(|x|))|u_n|^{2/(1-\beta)}\Big) - 1\Big]\,dx\right)^{1/m}. \tag{4.5}$$

Using the continuity of h and $h(0) = 0$, there exists $0 < r_1 < 1$ such that

$$h(|x|) < \epsilon, \quad \text{for all} \quad |x| \leq r_1.$$

Thus,

$$\int_{B_{r_1}}\Big[\exp\Big(m(\alpha_0 + \epsilon + h(|x|))|u_n|^{2/(1-\beta)}\Big) - 1\Big]\,dx \leq \int_{B_{r_1}}\Big[\exp\Big(m(\alpha_0 + 2\epsilon)\|u_n\|^{2/(1-\beta)}\Big(\frac{|u_n|}{\|u_n\|}\Big)^{2/(1-\beta)}\Big) - 1\Big]\,dx.$$

Using (4.3), (4.4), and Theorem 1.2, we can get $C_1 > 0$ such that

$$\int_{B_{r_1}}\Big[\exp\Big(m(\alpha_0 + \epsilon + h(|x|))|u_n|^{2/(1-\beta)}\Big) - 1\Big]\,dx \leq \int_{B_{r_1}}\Big[\exp\Big(\alpha_\beta^*\Big(\frac{|u_n|}{\|u_n\|}\Big)^{2/(1-\beta)}\Big) - 1\Big]\,dx \leq C_1. \tag{4.6}$$

By (h_3), there exist $c > 0$ and $r_2 > 1$ such that

$$h(r) \leq c|x|^\xi, \quad \text{for all} \quad |x| \geq r_2.$$

Using the above inequality, the boundedness of the sequence $(\|u_n\|)$, and Lemma 2.1, there exists $C_2 > 0$ such that

$$m(\alpha_0 + \epsilon + h(|x|))|u_n(x)|^{2/(1-\beta)} \leq \frac{C_2}{|x|^\eta}, \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad |x| \geq r_2,$$

where $\eta = \min\{a/(1-\beta) - \xi, a/(1-\beta)\} > 2$, which implies the existence of $C_3 > 0$ such that

$$\int_{\mathbb{R}^2 \setminus B_{r_2}} \left[\exp\Big(m(\alpha_0 + \epsilon + h(|x|))|u_n|^{2/(1-\beta)}\Big) - 1\right] dx \le 2\pi \int_{r_2}^{+\infty} \left[\exp\Big(C_2|x|^{-\eta}\Big) - 1\right] dr = C_3. \tag{4.7}$$

Since the sequence (u_n) is bounded in E, by Lemma 2.1, one has

$$|u_n(x)| \le M_0, \quad \text{for all} \quad r_1 \le |x| \le r_2 \quad \text{and for all} \quad n \ge 1.$$

Additionally, since h is continuous, there exists $C_3 > 0$ such that

$$\int_{B_{r_2} \setminus B_{r_1}} \left[\exp\Big(m(\alpha_0 + \epsilon + h(|x|))|u_n|^{2/(1-\beta)}\Big) - 1\right] dx \le C_3. \tag{4.8}$$

Using (4.6)–(4.8), the integral on the right-hand side of (4.5) is bounded. Moreover, by the compact embeddings $E \hookrightarrow L^2(\mathbb{R}^2)$ and $E \hookrightarrow L^{m'}(\mathbb{R}^2)$, and the weakly convergence $v_n \rightharpoonup 0$ in E, up to a subsequence, we obtain

$$\int_{\mathbb{R}^2} |f(x, u_n)v_n| \, dx \le \epsilon \|u\|_2 \|v_n\|_2 + C\|v_n\|_{m'} \to 0, \quad \text{as} \quad n \to +\infty,$$

and the lemma follows.

5. Proof of Theorem 1.3

First, we will show that S_p is attained in a function in E. Consider a sequence $(u_k) \subset E$ such that

$$\int_{\mathbb{R}^2} |u_k|^p \, dx = 1 \qquad \text{and} \qquad \left(\int_{\mathbb{R}^2} w(x)|\nabla u_k|^2 \, dx\right)^{1/2} \to S_p.$$

Therefore, (u_k) is bounded in E. Thus, we can assume that there exists some $u_p \in E$ such that $u_k \rightharpoonup u_p$ weakly in E, $u_k \to u_p$ strongly in $L^p(\mathbb{R}^2)$, and $u_k(x) \to u_p(x)$ almost everywhere in $\mathbb{R}^2$. Hence, $\|u_p\|_p = 1$ and $\|u_p\| \le \liminf_{k \to +\infty} \|u_k\| = S_p$. Noticing that $S_p \le \|u_p\|$, and taking the absolute value of the functions, we can guarantee that $u_p \ge 0$. Thus there exists $u_p \in E$ such that $u(x) \ge 0$ in $\mathbb{R}^2$ with $\|u_p\|_p = 1$ satisfying

$$S_p = \inf_{0 \ne u \in H^1_{\text{rad}}(\mathbb{R}^2, w)} \frac{\Big(\int_{\mathbb{R}^2} w(x)|\nabla u|^2 \, dx\Big)^{1/2}}{\Big(\int_{\mathbb{R}^2} |u|^p \, dx\Big)^{1/p}} = \|u_p\|.$$

This will be the element e_0 considered in Lemma 3.2. From Lemmas 3.1 and 3.2, based on the well-known pass mountain theorem by Ambrosetti-Rabinowitz [43,44]), we obtain a Palais-Smale $(u_n) \subset E$ at the level $d \ge \sigma$, where σ is given by Lemma 3.1, and $d > 0$ is given by

$$d = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

From (H_5), we get

$$J(tu_p) = \frac{t^2}{2}\|u_p\|^2 - \int_{\mathbb{R}^2} F(x, tu_p)\, dx \leq \frac{t^2}{2}\|u_p\|^2 - \frac{C_p t^p}{p}\int_{\mathbb{R}^2} |u_p|^p\, dx.$$

By the assumption on C_p, we obtain

$$\sup_{t\geq 0} J(tu_p) \leq \max_{t\geq 0}\left\{\frac{t^2 S_p^2}{2} - \frac{C_p t^p}{p}\right\} = \frac{(p-2)S_p^{2p/(p-2)}}{2pC_p^{2/(p-2)}} < \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\Big(\frac{\alpha_\beta^*}{\alpha_0}\Big)^{1-\beta}. \tag{5.1}$$

Note that $e = t_0 u_p$ with $t_0 > 0$ is given by Lemma 3.2. Consider $\gamma_0 \in \Gamma$ defined by $\gamma_0(t) = tt_0u_p$. By (5.1), we get

$$d = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \max_{t\in[0,1]} J(\gamma_0(t)) \leq \max_{t\in[0,1]} J(tt_0u_p) \leq \max_{t\geq 0} J(tu_p) < \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\Big(\frac{\alpha_\beta^*}{\alpha_0}\Big)^{1-\beta}.$$

Using Lemma 4.2, the sequence (u_n), up to a sequence, is convergent, that is, we can get $u \in E$ such that $u_n \to u$ in E. By the continuity of J and J', we have that $J(u) = d$ and $J'(u) = 0$. Therefore, u is a solution of the problem (1.12). Moreover, using the fact that $J(u) = d \geq \sigma$, we conclude that u is nontrivial.

6. Conclusions

In this paper, we presented a new type of Trudinger-Moser inequality defined on a radial weighted Sobolev space. Additionally, as an application of the above result, by applying the mountain pass theorem, we found nontrivial weak solutions for a nonlinear equation. Our main contribution is to extend previous results by establishing equations defined on $\mathbb{R}^2$, involving a nonlinear equation with supercritical exponential growth.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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