Research article

Exponential synchronization of neural networks with mixed delays under impulsive control

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Abstract: In this paper, the exponential synchronization problem of a class of neural networks with mixed delays under impulsive control is studied. Combining the impulsive comparison principle and the concept of an average impulsive interval, two impulsive differential inequalities with mixed delays are discussed, and the sufficient conditions for the existence of exponential decay are obtained. Based on two different impulsive control strategies, and then by means of the Lyapunov function, the inequality technique, and these two new inequalities, a set of sufficient conditions are derived to ensure the synchronization of the drive and response systems. In order to prove the effectiveness of the proposed control scheme, two numerical examples are given to prove its practicability and effectiveness.

Keywords: neural networks; impulsive differential inequalities; exponential synchronization; delay; impulsive control

1. Introduction

Originating from the groundbreaking M-P model devised by McCulloch and Pitts [1], neural networks (NNs) have evolved into powerful machine learning algorithms [2]. Their ubiquitous application across diverse fields stems from their remarkable aptitude for learning and discerning patterns within data. Whether in associative memory tasks that need to process a large amount of information [3], or in automatic control systems that require precise control, NNs have shown their unique advantages. Concurrently, NNs occupy an indispensable position in realms such as pattern recognition, signal processing [4–6], secure communication, and tackling intricate optimization challenges [7, 8]. Because of its characteristics, it has attracted the attention of scholars.

In the specific implementation, a time delay is widespread and inevitable, which is due to the limited transmission of information between systems [9,10]. On the other hand, NNs usually have spatial char-

acteristics because numerous parallel pathways exist. The dissimilar axis-cylinder size and length,as well as the propagation velocity distribution along these pathways, brings about a propagation distribution delay [11, 12]. Numerous scholars engaged in profound discourses, thereby embracing the perspective that the presence of a time delay inherently influenced the dynamic behavior exhibited by NNs, such as oscillation, divergence, or instability, and other complex dynamic behaviors [13,14]. The above phenomena is harmful to NNs and practical applications. Therefore, it is a useful theoretical research and practical application to introduce a time delay into the modeling of neural networks.

Synchronization is a dynamic behavior of NNs, whereby two or more systems have the same behavior under coupling or external control. Synchronization is widely used in secure communication [15, 16], image encryption [17, 18], and image protection [19]. Many scholars studied various types of synchronizations according to various situations, see [20–23].Therefore, the study of NN synchronization has both theoretical and practical values.

Designing a suitable controller to achieve synchronization is a challenge. Based on the time interval operation, the intermittent controller automatically switches the working / resting state of the equipment, optimizes energy consumption, and reduces costs. Because of its energy saving effect and practicability, it has attracted wide attention and application, see [24–28]. Impulse control turns a continuous system into a discontinuous one at the impulse moment. Both pulse control and intermittent control exert infleunces at the corresponding time. However, compared with intermittent control, pulse control is a time rather than a time period, which helps reduce the cost of control. Therefore, impulsive control is widely used in synchronization problems. In [29], the exponential synchronization problem of delayed coupled NNs was studied by means of a Razumikhin-type inequality. In [30], the exponential synchronization problem of stochastic coupled NNs was studied by using the stochastic impulsive differential inequality. In [31], the quasi-synchronization of NNs was studied using the Lyapunov function and the impulsive comparison principle. In [32], the quasi-synchronization problem of directed coupled heterogeneous NNs was studied using the extended pulse comparison principle and the saturated nonlinear dead-zone function. In [33], the μ-synchronization of coupled NNs was studied by constructing a new impulsive differential inequality. In [34], the quasi-synchronization problem of the quasi-synchronization of multi-layer delayed NNs was studied using the Lyapunov function and the pulse comparison principle. In [35], the synchronization problem of nonlinear, delayed, semi-Markov, jump NNs was studied using the stochastic Lyapunov function and the Razumikhin technique. In [36], the synchronization problem of coupled delayed inertial NNs was studied using the improved pulse comparison principle. In [37], the synchronization problem of hybrid coupled NNs was studied using delay impulsive differential inequalities. For example, in the above cited literature, the main proof of [33] and [36] is to transform the original problem into an impulsive differential inequality by means of the Lyapunov function. The forms of the two are similar. [33] proved the impulsive differential inequality first, while [36] was based on the improved impulsive comparison principle. Inspired by this, we can first discuss the transformed impulsive differential inequalities. This is the source of the idea of the impulsive inequality in this paper.

The main results of this paper focus on the following aspects:

(i) Two delay impulsive differential inequalities are given; and

(ii) The exponential synchronization problem of two different impulsive control strategies is considered.

The subsequent sections of this paper are organized in the following manner: Section 2 presents

an overview of the model and introduces the prerequisite knowledge; Section 3 delves into the presentation of sufficient conditions to achieve exponential synchronization in impulsive neural networks that incorporate mixed delays; in Section 4, two numerical examples are provided to demonstrate the practical efficacy of the proposed approach; and finally, Section 5 concludes the paper with a summary of the key findings.

2. Model description and preliminaries

Hereof, unless otherwise elucidated, we will use the following notation. We define R as the set of real numbers, N as the set of positive integers, and R^n as the set of n-dimensional real vectors equipped with the Euclidean norm $\|\cdot\|$. Additionally, we let $R^{m \times n}$ represent the set of $m \times n$ real matrices. The $n \times n$ unit matrix is represented by E. When A and B are regular intersection matrices, the notation $A > B$ (respectively, $A < B$) indicates that $A - B$ is a positive definite (respectively, a negative definite). The transpose and the inverse of matrix A are denoted by A^{T} and A^{-1}, respectively. The maximal and minimal eigenvalues of symmetric A are $\lambda_{max}(A)$ and $\lambda_{min}(A)$. The function $\sigma(\cdot)$ denotes the spectral radius of a matrix. For any interval $J \subset R$, $\mathcal{P}C = \{\psi : J \to R^n$ that are continuous everywhere except at a finite number of points t, where $\psi(t^+), \psi(t^-)$ exist and $\psi(t^+) = \psi(t)\}$. For $\varsigma(t) \in \mathcal{P}C([-\tau, 0], R^n)$, define $\|\varsigma(t)\| = \sqrt{\varsigma^T(t)\varsigma(t)}$.

The driving system considered in this paper is a NN with mixed delays, which is defined as follows:

$$\begin{cases} \frac{dx(t)}{dt} = -\tilde{D}x(t) + \tilde{A}f(x(t)) + \tilde{B}f(x(t - \tau(t))) + \tilde{C}\int_{t-h}^{t} f(x(s))ds + \tilde{I}(t), \\ x(t) = \tilde{\phi}(t), \ -\varrho < t < 0, \end{cases} \tag{2.1}$$

where the neuron state vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and matrices $\tilde{D} > 0$, $\tilde{A}$, $\tilde{B}, \tilde{C}$ represent the diagonal matrix, connection weight matrix, delayed weight matrix, and distributively delayed connection weight matrix of the neural network, respectively. The activation function is $f(\cdot) = (f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot))^{\mathrm{T}}$. Time-varying delays $\tau(t)$ satisfy $0 \leq \tau(t) \leq \tau$. The distributed-delay is h, and $\varrho = max\{\tau, h\}$. $\tilde{\phi}(t) \in \mathcal{P}C([-\varrho, 0], R^n)$. with components $\tilde{\phi}_i(t), i = 1, 2, \ldots, n$.

The response system is as follows:

$$\begin{cases} \frac{dy(t)}{dt} = -\tilde{D}y(t) + \tilde{A}f(y(t)) + \tilde{B}f(y(t - \tau(t))) + \tilde{C}\int_{t-h}^{t} f(y(s))ds \\ \qquad + \tilde{I}(t) + \tilde{u}(t), \\ y(t) = \tilde{\varphi}(t), \ -\varrho < t < 0, \end{cases} \tag{2.2}$$

where $\tilde{\varphi}(t) = (\tilde{\varphi}_1(t), \tilde{\varphi}_2(t), \ldots, \tilde{\varphi}_n(t))^T$, has components $\tilde{\varphi}_i(t)$ that belong to $\mathcal{P}C([-\varrho, 0], R^n)$.

To achieve the article's objective, the error $e(t) = y(t) - x(t)$ is defined.

Combining networks (2.1) and (2.2), the error system is formulated as follows:

$$\begin{cases} \frac{de(t)}{dt} = -\tilde{D}e(t) + \tilde{A}g(e(t)) + \tilde{B}g(e(t - \tau(t))) + \tilde{C}\int_{t-h}^{t} g(e(s))ds + \tilde{u}(t), \\ e(t) = \tilde{\psi}(t), \ -\varrho < t < 0, \end{cases} \tag{2.3}$$

where $g(e(t)) = f(y(t)) - f(x(t))$, $\tilde{\psi}(t) = \tilde{\varphi}(t) - \tilde{\phi}(t)$.

The impulsive control protocol has been designed in the following manner:

$$\tilde{u}(t) = \tilde{K}_1 e(t) + \tilde{K}_2 g(e(t)) - \mu \sum_{k=1}^{\infty} \tilde{R}_k e(t)\delta(t - t_k), \tag{2.4}$$

where the gain matrices are represented by $\tilde{K}_1$ and $\tilde{K}_2$ in this context. $\tilde{R}_k > 0$ is the control gain. The fixed impulsive moments t_k (for $k \in N$) form a strictly increasing sequence that satisfy $0 = t_0 < t_1 < t_2 < \ldots, t_k < \ldots$ with the property that the sequence diverges to infinity, (i.e., $\lim_{t \to \infty} t_k = +\infty$). The Dirac impulsive function is denoted by $\delta(\cdot)$.

Remark 2.1. *In [38], the controller only considered the feedback of the activation function in t and $t - \tau(t)$. In [34], the controller was composed of the linear term of the error function and the dirac impulsive function. In [35], the controller was only composed of the dirac impulsive function. The controller in this paper considers the non-delay term of the error function and the dirac impulsive function.*

By combining Eqs (2.3) and (2.4), we obtain a rephrased version of the error system:

$$
\begin{cases}
\frac{de(t)}{dt} = -(\tilde{D} - \tilde{K}_1)e(t) + (\tilde{A} + \tilde{K}_2)g(e(t)) + \tilde{B}g(e(t - \tau(t))) \\
\qquad + \tilde{C} \int_{t-h}^{t} g(e(s))ds, \ t \neq t_k, \\
\qquad := -\hat{D}e(t) + \hat{A}g(e(t)) + \tilde{B}g(e(t - \tau(t))) + \tilde{C} \int_{t-h}^{t} g(e(s))ds, \ t \neq t_k, \\
\Delta e(t_k) = e(t_k^+) - e(t_k^-) = -\mu \tilde{R}_k e(t_k^-), \ t = t_k, \\
e(t) = \tilde{\psi}(t), \ -\varrho < t < 0.
\end{cases}
\tag{2.5}
$$

Definition 2.1. *[38] The systems (2.1) and (2.2) were deemed to be exponentially synchronized if there existed positive constants λ and $M \geq 1$, such that for every $t > 0$, the norm of the error function $\|e(t)\|$ between the two systems satisfied the inequality $\|e(t)\| \leq \sup_{-\tau \leq s \leq 0} \|\varphi(s) - \phi(s)\| M e^{-\lambda t}$. Here, λ serves as the convergence rate (or degree), which characterizes the exponential decline in the synchronization error as time progresses.*

Definition 2.2. *[31] The average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \ldots, t_k, \ldots\}$ was equal to T_a, if there existed the positive integer N_0 and the positive scalar T_a such that for all $0 \leq t \leq T$, the number of impulsive times $N_\zeta(T, t)$ of the sequence ζ within the time interval (t, T) satisfied the following inequality:*

$$
\frac{T - t}{T_a} - N_0 \leq N_\zeta(T, t) \leq \frac{T - t}{T_a} + N_0, \ 0 \leq t \leq T.
$$

Here, $N_\zeta(T, t)$ represents the count of impulsive instants which belong to the sequence ζ that strictly occur between the times t and T.

Lemma 2.1. *[31] Let $\tau_1(t)$ and $\tau_2(t)$ be functions which satisfy $0 \leq \tau_1(t), \tau_2(t) \leq \tau$ for all t. Consider the function $G(t, u, \bar{u}_1) : R^+ \times R \times R \to R$, which is nondecreasing in its third argument $\bar{u}_1$ for fixed values of t and u. Additionally, let $I_k(u) : R \to R$ be a function that is nondecreasing in u. If there exists a positive constant $\alpha > 0$ such that*

$$
\begin{cases}
D^+ u(t) \leq G(t, u(t), u(t - \tau_1(t))) + \alpha \int_{t-\tau_2(t)}^{t} u(s)ds, \ t \neq t_k, \\
u(t_k) \leq I_k(u(t_k^-)), \ k \in N,
\end{cases}
\tag{2.6}
$$

$$
\begin{cases}
D^+ v(t) > G(t, v(t), v(t - \tau_1(t))) + \alpha \int_{t-\tau_2(t)}^{t} v(s)ds, \ t \neq t_k, \\
v(t_k) \geq I_k(v(t_k^-)), \ k \in N,
\end{cases}
\tag{2.7}
$$

then $u(t) \leq v(t)$ holds for $-\tau \leq t \leq 0$, and under the assumption that certain properties related to the right upper derivative $D^+ u(t)$ hold, it can be shown that $u(t) \leq v(t)$ also holds for all $t > 0$.

Lemma 2.2. *[32] Given any real, symmetric, and positive definite matrix M of an appropriate dimension, a positive scalar $\eta > 0$, and a vector function $\omega(\cdot) : [a, b] \to R^n$ such that the relevant integrals are well-defined, the following inequality holds:*

$$\left[\int_a^b \omega(s)ds\right]^{\mathrm{T}} M \left[\int_a^b \omega(s)ds\right] \le (b-a) \int_a^b \omega^{\mathrm{T}}(s)M\omega(s)ds.$$

In order to study the exponential synchronization of NNs with mixed discrete delays and finite distributed delays, the following impulsive differential inequalities which involve discrete delays and finite distributed delays are first established.

Lemma 2.3. *Let $u(t) \in \mathcal{PC}([-\varrho, +\infty), R)$ with $u(t) \ge 0$; for the impulsive sequence $\zeta = \{t_1, t_2, \ldots, t_k, \ldots\}$, consider the following differential inequalities:*

$$\begin{cases} D^+u(t) \le \alpha u(t) + \theta_1 u(t - \tau(t)) + \theta_2 \int_{t-h}^t v(s)ds, \ t \ne t_k, \\ u(t_k) \le \beta u(t_k^-), \ k \in N, \\ u(s) = \varphi(s) \ge 0, s \in [-\varrho, 0], \end{cases} \tag{2.8}$$

where $\alpha > 0$, $\theta_1 > 0$, $\theta_2 > 0, 0 \le \tau(t) \le \tau$, for $t \in R$, $0 < \beta < 1$ for $k \in N$, $\varrho = max\{\tau, h\}$. If $\alpha + \frac{ln\beta}{T_a} + \frac{\theta_1}{\beta^{N_0}} + \frac{\theta_2 h}{\beta^{N_0}} < 0$, then $\exists \lambda > 0$ with the following property:

$$u(t) \le \beta^{-N_0} \sup_{-\varrho \le s \le 0} \|\varphi(s)\|^2 e^{-\lambda_0 t}, t \ge 0,$$

where λ_0 satisfies $\lambda_0 + \alpha + \frac{ln\beta}{T_a} + \frac{\theta_1 e^{\lambda_0 \tau}}{\beta^{N_0}} + \frac{e^{\lambda_0 h} - 1}{\lambda_0} \frac{\theta_2}{\beta^{N_0}} = 0$.

Proof. The following comparison system is considered:

$$\begin{cases} \dot{\tilde{u}}(t) = \alpha \tilde{u}(t) + \theta_1 \tilde{u}(t - \tau(t)) + \theta_2 \int_{t-h}^t \tilde{u}(s)ds + \varepsilon, \ t \ne t_k, \\ \tilde{u}(t_k) = \beta \tilde{u}(t_k^-), \ k \in N, \\ \tilde{u}(s) = \varphi(s) + \varepsilon, \ -\varrho \le s \le 0. \end{cases} \tag{2.9}$$

For any $\varepsilon > 0$, $\tilde{u}(t)$ denotes the unique solution. According to Lemma 2.1, we have the inequality $\tilde{u}(t) \ge u(t) \ge 0$ holding for $-\varrho \le s \le 0$. Consequently, it follows that $\tilde{u}(t) \ge u(t)$ also holds for $t \ge 0$. Leveraging the formula for the variation of parameters, we can describe $\tilde{u}(t)$ as follows:

$$\tilde{u}(t) = W(t, 0)\tilde{u}(0) + \int_0^t W(t, s)\left[\theta_1 \tilde{u}(s - \tau(s)) + \theta_2 \int_{t-h}^t \tilde{u}(r)dr + \varepsilon\right]ds. \tag{2.10}$$

The function $W(t, s)$ defined for all $t, s \ge 0$, serves as the Cauchy matrix for the subsequent linear impulsive system:

$$\begin{cases} \dot{\tilde{u}}(t) = \alpha \tilde{u}(t), \ t \ne t_k, \\ \tilde{u}(t_k) = \beta \tilde{u}(t_k^-), \ k \in N. \end{cases} \tag{2.11}$$

Definition 2.2 states that it can acquire the following:

$$W(t, s) \le e^{\alpha(t-s)} \Pi_{s < t_k < t} \beta = e^{\alpha(t-s)} \beta^{N_\zeta(s,t)} \le e^{\alpha(t-s)} \beta^{\frac{t-s}{T_a} - N_0}$$

$$= \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t-s)}. \tag{2.12}$$

Let $q = \beta^{-N_0} \sup_{-\varrho \le s \le 0} \|\varphi(s)\|^2$. The result of substituting (2.12) into (2.10) is as follows:

$$\tilde{u}(t) \le q e^{(\alpha + \frac{\ln\beta}{T_a})t} + \int_0^t \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t-s)} \left[\theta_1 \tilde{u}(s - \tau(s)) + \theta_2 \int_{s-h}^s \tilde{u}(r)dr + \varepsilon \right] ds, \tag{2.13}$$

Denote $G(\lambda) = \lambda + \alpha + \frac{\ln\beta}{T_a} + \frac{\theta_1 e^{\lambda\tau}}{\beta^{N_0}} + \frac{e^{\lambda h} - 1}{\lambda} \frac{\theta_2}{\beta^{N_0}}$. Then, we have the following:

$$G(0^+) = \alpha + \frac{\ln\beta}{T_a} + \frac{\theta_1}{\beta^{N_0}} + \lim_{\lambda \to 0^+} \left(\frac{e^{\lambda h} - 1}{\lambda} \frac{\theta_2}{\beta^{N_0}} \right)$$

$$= \alpha + \frac{\ln\beta}{T_a} + \frac{\theta_1}{\beta^{N_0}} + \frac{\theta_2 h}{\beta^{N_0}} < 0.$$

For any $\lambda \in (0, +\infty)$, it is evident that $G(+\infty) > 0$ and $G'(\lambda) > 0$. As a result, the equation $G(\lambda) = 0$ admits a unique positive solution, denoted by λ_0, where $\lambda_0 > 0$.

Since $\varepsilon > 0, \lambda_0 > 0, \alpha + \frac{\ln\beta}{T_a} + \frac{\theta_1}{\beta^{N_0}} + \frac{\theta_2 h}{\beta^{N_0}} < 0$ and $0 < \beta < 1$, we have the following:

$$\tilde{u}(s) < q < q e^{-\lambda_0 s} + \frac{\varepsilon}{\sigma_0}, -\varrho \le s \le 0, \tag{2.14}$$

where $\varpi = -(\alpha + \frac{\ln\beta}{T_a} + \frac{\theta_1}{\beta^{N_0}} + \frac{\theta_2 h}{\beta^{N_0}})\beta^{N_0}$.

Next, we prove that

$$\tilde{u}(t) \le q e^{-\lambda_0 t} + \frac{\varepsilon}{\varpi}, t > 0. \tag{2.15}$$

There exists a $t^* > 0$ such that, if (2.15) is false, then

$$\tilde{u}(t^*) > q e^{-\lambda_0 t^*} + \frac{\varepsilon}{\varpi} \tag{2.16}$$

and

$$\tilde{u}(t) \le q e^{-\lambda_0 t} + \frac{\varepsilon}{\varpi}, t < t^*. \tag{2.17}$$

Combine (2.15) with (2.17), we have the following:

$$\tilde{u}(t^*) \le q e^{(\alpha + \frac{\ln\beta}{T_a})t^*} + \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} \theta_1 (q e^{-\lambda_0(s-\tau(s))} + \frac{\varepsilon}{\varpi}) ds$$

$$+ \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} \theta_2 \int_{s-h}^s (q e^{-\lambda_0 r} + \frac{\varepsilon}{\varpi}) dr ds$$

$$+ \varepsilon \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} ds$$

$$\le q e^{(\alpha + \frac{\ln\beta}{T_a})t^*} + \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} \theta_1 (q e^{-\lambda_0(s-\tau)} + \frac{\varepsilon}{\varpi}) ds$$

$$+ \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} \theta_2 \int_{s-h}^s (q e^{-\lambda_0 r} + \frac{\varepsilon}{\varpi}) dr ds$$

$$+ \varepsilon \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} ds$$

$$\leq q e^{(\alpha + \frac{\ln\beta}{T_a})t^*} + \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})t^*} \int_0^{t^*} e^{-(\lambda_0 + \alpha + \frac{\ln\beta}{T_a})s} q\theta_1 e^{\lambda_0\tau} ds$$

$$+ \frac{\varepsilon}{\varpi} \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} \theta_1 ds$$

$$+ \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})t^*} \int_0^{t^*} e^{-(\lambda_0 + \alpha + \frac{\ln\beta}{T_a})s} q\theta_2 \frac{e^{\lambda_0 h} - 1}{\lambda_0} ds$$

$$+ \frac{\varepsilon}{\varpi} \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} \theta_2 h ds$$

$$+ \varepsilon \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} ds$$

$$\leq q e^{(\alpha + \frac{\ln\beta}{T_a})t^*} + q e^{(\alpha + \frac{\ln\beta}{T_a})t^*} \int_0^{t^*} e^{-(\lambda_0 + \alpha + \frac{\ln\beta}{T_a})s} (-\lambda_0 - \alpha - \frac{\ln\beta}{T_a}) ds$$

$$+ \frac{\varepsilon(\theta_1 + \theta_2 h + \varpi)}{\varpi} \int_0^{t^*} \beta^{-N_0} e^{(\alpha + \frac{\ln\beta}{T_a})(t^*-s)} ds$$

$$\leq q e^{(\alpha + \frac{\ln\beta}{T_a})t^*} + q e^{-\lambda_0 t^*} - q e^{(\alpha + \frac{\ln\beta}{T_a})t^*}$$

$$+ \frac{-\varepsilon(\alpha + \frac{\ln\beta}{T_a})\beta^{N_0}}{\varpi} \frac{\beta^{-N_0}}{-(\alpha + \frac{\ln\beta}{T_a})} (1 - e^{(\alpha + \frac{\ln\beta}{T_a})t^*})$$

$$\leq q e^{-\lambda_0 t^*} + \frac{\varepsilon}{\varpi}, \tag{2.18}$$

which contracts with (2.16). Therefore, (2.15) holds true.

Let $\varepsilon \to 0$; then

$$u(t) \leq \tilde{u}(t) \leq q e^{-\lambda_0 t} = \beta^{-N_0} \sup_{-\varrho \leq s \leq 0} \|\varphi(s)\|^2 e^{-\lambda_0 t}, \ t \geq 0. \tag{2.19}$$

This completes the proof. $\qquad\square$

Lemma 2.4. *Let* $u(t) \in \mathcal{PC}([-\varrho, +\infty), R)$ *with* $u(t) \geq 0$; *for the impulsive sequence* $\zeta = \{t_1, t_2, \ldots, t_k, \ldots\}$, *consider the following differential inequalities:*

$$\begin{cases} D^+ u(t) \leq \alpha u(t) + \theta_1 u(t - \tau(t)) + \theta_2 \int_{t-h}^t v(s) ds, \ t \neq t_k, \\ u(t_k) \leq \beta u(t_k^-), \ k \in N, \\ u(s) = \varphi(s) \geq 0, \ s \in [-\varrho, 0], \end{cases} \tag{2.20}$$

where $\alpha < 0$, $\theta_1 > 0$, $\theta_2 > 0, 0 \leq \tau(t) \leq \tau$, *for* $t \in R$, $\beta \geq 1$ *for* $k \in N$, $\varrho = max\{\tau, h\}$. *If* $\alpha + \frac{\ln\beta}{T_a} + \theta_1 \beta^{N_0} + \theta_2 h \beta^{N_0} < 0$, *then* $\exists \lambda > 0$ *with the following property:*

$$u(t) \leq \beta^{N_0} \sup_{-\varrho \leq s \leq 0} \|\varphi(s)\|^2 e^{-\lambda_0 t}, t \geq 0,$$

where λ_0 *satisfies* $\lambda_0 + \alpha + \frac{\ln\beta}{T_a} + \theta_1 e^{\lambda_0\tau}\beta^{N_0} + \frac{e^{\lambda_0 h}-1}{\lambda_0}\theta_2\beta^{N_0} = 0$.

Proof. The proof process follows a similar approach to that of Lemma 2.4, specifically involving the Cauchy matrix as follows:

$$W(t, s) \leq e^{\alpha(t-s)} \Pi_{s < t_k < t} \beta = e^{\alpha(t-s)} \beta^{N_\zeta(s,t)} \leq e^{\alpha(t-s)} \beta^{\frac{t-s}{T_a} + N_0}$$

$$= \beta^{N_0} e^{(\alpha + \frac{ln\beta}{T_a})(t-s)}. \tag{2.21}$$

This completes the proof. □

Remark 2.2. *In [33], our focus was on the impulsive differential inequality that incorporated multiple discrete delays. Subsequently, in [37], we delved into the delayed impulsive differential inequality characterized by finite distributed delays. Furthermore, in [39], we explored the scenario where the integral term within the impulsive inequality assumed the form of a kernel function. However, this paper presents a comprehensive study of the impulsive differential inequality that encompasses both discrete delays and finite distributed delays, with no imposed constraints on the intensity of the impulse.*

The following presumptions are made throughout this paper:

(A1) For every $u_1, u_2 \in R, u_1 \neq u_2$, where F_j, $j = 1, 2, \ldots, n$ are real positive constants, the activation functions $f_j(\cdot)$, where j ranges from 1 to n, are bounded functions that satisfy a specific Lipschitz condition:

$$|f_j(u_1) - f_j(u_2)| \leq F_j |u_1 - u_2|; \text{ and}$$

(A2) $\beta = \sigma((E - \mu \tilde{R}_k)^{\mathrm{T}}(E - \mu \tilde{R}_k)).$

3. Main results

Within this section, we explore the attainment of exponential synchronization between systems (2.1) and (2.2) through the construction of an appropriate Lyapunov functional.

$|g_i(e_i(t))| \leq F_i(e_i(t))$, which indicates that $g_i(\cdot)$ in system (2.5) is constrained from (A1). Thus, for any positive diagonal matrices A, the function $g(e(t))$ satisfies the following inequality:

$$g^{\mathrm{T}}(e(t)) A g(e(t)) \leq e^{\mathrm{T}}(t) F^{\mathrm{T}} A F e(t), \tag{3.1}$$

where $F = diag\{F_1, F_2, \ldots, F_n\}$.

Theorem 3.1. *Assume that the assumptions (A1) and (A2) are satisfied. Then, systems (2.1) and (2.2) are exponentially synchronized if there exist real constants $\alpha > 0$, $\theta_1 > 0$, $\theta_2 > 0$, along with an $n \times n$−matrix $P > 0$, four diagonal positive definite $n \times n$−matrix $\Sigma_i > 0, i = 1, 2, 3$, and an M such that the subsequent conditions are fulfilled:*

Case I: The impulsive strength is within the range $0 < \beta < 1$:
(a1)

$$\Xi = \begin{bmatrix} \xi_{11} & 0 & 0 \\ 0 & F^{\mathrm{T}}\Sigma_2 F - \theta_1 P & 0 \\ 0 & 0 & \Sigma_3 - \frac{1}{h}M \end{bmatrix} < 0,$$

where $\xi_{11} = -P\hat{D} - \hat{D}^{\mathrm{T}}P + P\hat{A}\Sigma_1^{-1}\hat{A}^{\mathrm{T}}P + F^{\mathrm{T}}\Sigma_1 F + P\tilde{B}\Sigma_2^{-1}\tilde{B}^{\mathrm{T}}P + P\tilde{C}\Sigma_3^{-1}\tilde{C}^{\mathrm{T}}P - \alpha P.$

(b1)

$$F^{\mathrm{T}}MF - \theta_2 P < 0, \ and$$

(c1)

$$\alpha + \frac{ln\beta}{T_a} + \frac{\theta_1}{\beta^{N_0}} + \frac{\theta_2 h}{\beta^{N_0}} < 0.$$

Case II: The impulsive strength is within the range $\beta \geq 1$:

The three real constants in the theorem satisfy the relation $\alpha < 0$, $\theta_1 > 0$, $\theta_2 > 0$, conditions (a2) and (b2) are identical to conditions (a1) and (b1) respectively, as defined in Case I, and

(c2)

$$\alpha + \frac{ln\beta}{T_a} + (\theta_1 + \theta_2 h)\beta^{N_0} < 0.$$

Proof. We build the subsequent Lyapunov function as follows:

$$V(t) = e^{\mathrm{T}}(t)Pe(t), \tag{3.2}$$

where P is strictly positive definite matrix.

$$
\begin{aligned}
\dot{V}(t) &= 2e^{\mathrm{T}}(t)P\dot{e}(t) \\
&= 2e^{\mathrm{T}}(t)P\{-\hat{D}e(t) + \hat{A}g(e(t)) + \tilde{B}g(e(t-\tau(t))) + \tilde{C}\int_{t-h}^{t} g(e(s))ds\} \\
&= -2e^{\mathrm{T}}(t)P\hat{D}e(t) + 2e^{\mathrm{T}}(t)P\hat{A}g(e(t)) + 2e^{\mathrm{T}}(t)P\tilde{B}g(e(t-\tau(t))) \\
&\quad + 2e^{\mathrm{T}}(t)P\tilde{C}\int_{t-h}^{t} g(e(s))ds \\
&\leq e^{\mathrm{T}}(t)(-P\hat{D} - \hat{D}^{\mathrm{T}}P)e(t) + e^{\mathrm{T}}(t)P\hat{A}\Sigma_1^{-1}\hat{A}^{\mathrm{T}}Pe(t) + g^{\mathrm{T}}(e(t))\Sigma_1 g(e(t)) \\
&\quad + e^{\mathrm{T}}(t)P\tilde{B}\Sigma_2^{-1}\tilde{B}^{\mathrm{T}}Pe(t) + g^{\mathrm{T}}(e(t-\tau(t)))\Sigma_2 g(e(t-\tau(t))) \\
&\quad + e^{\mathrm{T}}(t)P\tilde{C}\Sigma_3^{-1}\tilde{C}^{\mathrm{T}}Pe(t) + [\int_{t-h}^{t} g(e(s))ds]^{\mathrm{T}}\Sigma_3 \int_{t-h}^{t} g(e(s))ds \\
&\leq e^{\mathrm{T}}(t)(-P\hat{D} - \hat{D}^{\mathrm{T}}P)e(t) + e^{\mathrm{T}}(t)P\hat{A}\Sigma_1^{-1}\hat{A}^{\mathrm{T}}Pe(t) + e^{\mathrm{T}}(t)F^{\mathrm{T}}\Sigma_1 Fe(t) \\
&\quad + e^{\mathrm{T}}(t)P\tilde{B}\Sigma_2^{-1}\tilde{B}^{\mathrm{T}}Pe(t) + e^{\mathrm{T}}(t-\tau(t))F^{\mathrm{T}}\Sigma_2 Fe(t-\tau(t)) \\
&\quad + e^{\mathrm{T}}(t)P\tilde{C}\Sigma_3^{-1}\tilde{C}^{\mathrm{T}}Pe(t) + [\int_{t-h}^{t} g(e(s))ds]^{\mathrm{T}}\Sigma_3 \int_{t-h}^{t} g(e(s))ds \\
&= e^{\mathrm{T}}(t)(-P\hat{D} - \hat{D}^{\mathrm{T}}P + P\hat{A}\Sigma_1^{-1}\hat{A}^{\mathrm{T}}P + F^{\mathrm{T}}\Sigma_1 F + P\tilde{B}\Sigma_2^{-1}\tilde{B}^{\mathrm{T}}P \\
&\quad + P\tilde{C}\Sigma_3^{-1}\tilde{C}^{\mathrm{T}}P)e(t) + e^{\mathrm{T}}(t-\tau(t))F^{\mathrm{T}}\Sigma_2 Fe(t-\tau(t)) \\
&\quad + [\int_{t-h}^{t} g(e(s))ds]^{\mathrm{T}}\Sigma_3 \int_{t-h}^{t} g(e(s))ds. \tag{3.3}
\end{aligned}
$$

According to Eqs (a1)–(c1), the following inequality is valid for all $t \neq t_k$, where $k \in N$.

$$\dot{V}(t) \leq \alpha e^{\mathrm{T}}(t)Pe(t) + \theta_1 e^{\mathrm{T}}(t-\tau(t))Pe(t-\tau(t)) + \theta_2 \int_{t-h}^{t} e^{\mathrm{T}}(s)Pe(t)ds$$

$$\leq \alpha V(t) + \theta_1 V(t - \tau(t)) + \theta_2 \int_{t-h}^{t} V(s)ds. \tag{3.4}$$

Otherwise,

$$
\begin{aligned}
V(t_k) &= e^{\mathrm{T}}(t_k)Pe(t_k) \\
&= e^{\mathrm{T}}(t_k^-)(E - \mu \tilde{R}_k)^{\mathrm{T}} P(E - \mu \tilde{R}_k)e(t_k^-) \\
&\leq \sigma((E - \mu \tilde{R}_k)^{\mathrm{T}}(E - \mu \tilde{R}_k))e^{\mathrm{T}}(t_k^-)Pe(t_k^-) \\
&\leq \beta V(t_k^-)
\end{aligned} \tag{3.5}
$$

holds for $k \in N$.

By integrating the conditions (a1)–(c1) with Lemma 2.3, we derive the subsequent conclusion:

$$\lambda_{min}(P)\|e^{\mathrm{T}}(t)e(t)\| \leq V(t) \leq \beta^{N_0} \sup_{-\varrho \leq s \leq 0}\|\tilde{\varphi}(s) - \tilde{\phi}(s)\|^2 e^{-\lambda_0 t}, t \geq 0, \tag{3.6}$$

quasi,

$$\|e(t)\| \leq \sqrt{\frac{\beta^{N_0}}{\lambda_{min}(P)}} \sup_{-\varrho \leq s \leq 0}\|\tilde{\varphi}(s) - \tilde{\phi}(s)\| e^{-\frac{\lambda_0}{2}t}, t \geq 0. \tag{3.7}$$

Systems (2.1) and (2.2) exhibit an exponential synchronization, which conforms to the criteria specified in Definition 2.1.

In a similar fashion, utilizing the constraints (a2)–(c2) in conjunction with Lemma 2.4, we can ascertain that systems (2.1) and (2.2) exhibit an exponential synchronization.

The proof is completed. □

Replace the impulse control mentioned in Eq (2.4) with the following impulsive control

$$\tilde{u}(t) = -\tilde{L}e(t) - \mu \sum_{k=1}^{\infty} \tilde{R}_k e(t)\delta(t - t_k), \tag{3.8}$$

where the notation $L = diag(l_1, l_2, \ldots, l_n)$, $l_i > 0$ designates the feedback control gain matrix. The impulsive gain is μ. The control gain is $R_k > 0$.

Remark 3.1. *This kind of controller is widely used, which can be seen in [34], and the secondary controller can be used to expand the existing conclusions.*

Consider the error function $e(t) = y(t) - x(t)$, which quantifies the deviation between the outputs of networks (2.1) and (2.2). When these networks are interconnected, and the error dynamics are governed by Eq (3.8), the resulting error system is derived as outlined below:

$$
\begin{cases}
\frac{de(t)}{dt} = -\tilde{D}e(t) + \tilde{A}g(e(t)) + \tilde{B}g(e(t - \tau(t))) + \tilde{C}\int_{t-h}^{t} g(e(s))ds \\
\quad -\tilde{L}e(t), \ t \neq t_k, \\
\quad := -\hat{D}e(t) + \tilde{A}g(e(t)) + \tilde{B}g(e(t - \tau(t))) + \tilde{C}\int_{t-h}^{t} g(e(s))ds, \ t \neq t_k, \\
\Delta e(t_k) = e(t_k^+) - e(t_k^-) = -\mu \tilde{R}_k e(t_k^-), \ t = t_k, \\
e(t) = \tilde{\psi}(t), \ -\varrho < t < 0.
\end{cases} \tag{3.9}
$$

By repeating the previous proof process, we can obtain the following theorem:

Theorem 3.2. *Assume that the assumptions (A1) and (A2) are satisfied. Then, systems (2.1) and (2.2) are exponentially synchronized if exist real constants $\alpha > 0$, $\theta_1 > 0$, $\theta_2 > 0$, an $n \times n-$matrix $P > 0$, three diagonal positive definite $n \times n-$matrix $\Sigma_i > 0$, $i = 1, 2, 3$, and an M such that the subsequent conditions are fulfilled:*

Case I: The impulsive strength is within the range $0 < \beta < 1$:
(a1)

$$\hat{\Xi} = \begin{bmatrix} \hat{\xi}_{11} & 0 & 0 \\ 0 & F^{\mathrm{T}}\Sigma_2 F - \theta_1 P & 0 \\ 0 & 0 & \Sigma_3 - \frac{1}{h}M \end{bmatrix} < 0,$$

where $\hat{\xi}_{11} = -P\hat{D} - \hat{D}^{\mathrm{T}}P + P\tilde{A}\Sigma_1^{-1}\tilde{A}^{\mathrm{T}}P + F^{\mathrm{T}}\Sigma_1 F + P\tilde{B}\Sigma_2^{-1}\tilde{B}^{\mathrm{T}}P + P\tilde{C}\Sigma_3^{-1}\tilde{C}^{\mathrm{T}}P - \alpha P$.
(b1)

$$F^{\mathrm{T}}MF - \theta_2 P < 0, \ and$$

(c1)

$$\alpha + \frac{ln\beta}{T_a} + \frac{\theta_1 + \theta_2 h}{\beta^{N_0}} < 0.$$

Case II: The impulsive strength is within the range $\beta \geq 1$:
The three real constants in the theorem satisfy the relation $\alpha < 0$, $\theta_1 > 0$, $\theta_2 > 0$, conditions (a2) and (b2) are same as the conditions (a1) and (b1) in Case I, and
(c2)

$$\alpha + \frac{ln\beta}{T_a} + (\theta_1 + \theta_2 h)\beta^{N_0} < 0.$$

Proof. If we assume that $V(t) = e^{\mathrm{T}}(t)Pe(t)$, and P is a positive integer martix, then we may show a method analogous to Theorem 3.1. The proof is completed. □

4. Numerical examples

Example 4.1. *Consider the following model:*

$$\begin{cases} \frac{de_i(t)}{dt} = -d_i e_i(t) + \sum_{j=1}^{2} a_{ij}g_j(e_j(t)) + \sum_{j=1}^{2} b_{ij}g_j(e_j(t - \tau(t))) \\ \qquad + \sum_{j=1}^{2} c_{ij} \int_{t-0.9}^{t} f_j(e_j(s))ds + k_{1i} * e_i(t) \\ \qquad + k_{2i} * g_i(e_i(t), t \neq t_k, t \geq 0 \\ \triangle e_i(t_k) = e_i(t_k^+) - e_i(t_k^-) = -\mu R_k e(t_k^-), t = t_k, \end{cases} \qquad (4.1)$$
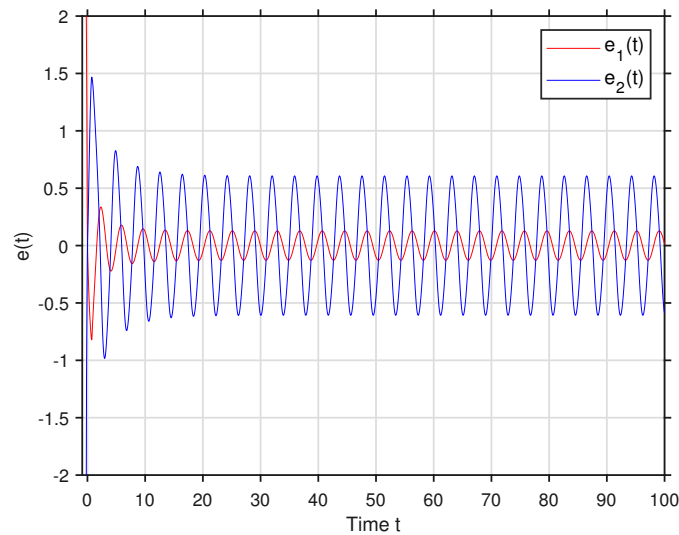
where the coefficients and functions are taken as follows:

$$D = \begin{bmatrix} 7.5 & 0 \\ 0 & 6 \end{bmatrix}, \ A = \begin{bmatrix} 6 & 0.15 \\ -3.2 & 6 \end{bmatrix}, \ B = \begin{bmatrix} -2.4 & 0.12 \\ -0.26 & -2.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 2.5 & 0 \\ 0 & 2 \end{bmatrix}, \ K_1 = \begin{bmatrix} -1.5 & 0 \\ 0 & -3.3 \end{bmatrix}, \ K_2 = \begin{bmatrix} -2.1 & 0 \\ 0 & -2.2 \end{bmatrix}.$$

$$g_1(x) = g_2(x) = tanh(x), \ \tau(t) = 0.8.$$

*In order to see the effect of control more clearly, we first consider the situation without control, see Figure 1.*

Figure 1. The state trajectory of the system (4.1) without control.

Case(1). Let $\mu = 0.9$, $P = E$, $R_k = E$, $t_k = t_{k-1} + 0.02k, k \in N$, and $\beta = 0.01$. It is easy to calculate that $\varrho = 0.9$, $F_i = 1, i = 1, \ldots, n$, $T_a = 0.02, N_0 = 1$. By employing the MATLAB toolbox, we are capable of obtaining the subsequent details:

$$\alpha = 150, \ \theta_1 = \theta_2 = 0.1,$$

$$\Sigma_1 = \begin{bmatrix} 500 & 0 \\ 0 & 300 \end{bmatrix}, \ \Sigma_2 = \Sigma_3 = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix}.$$
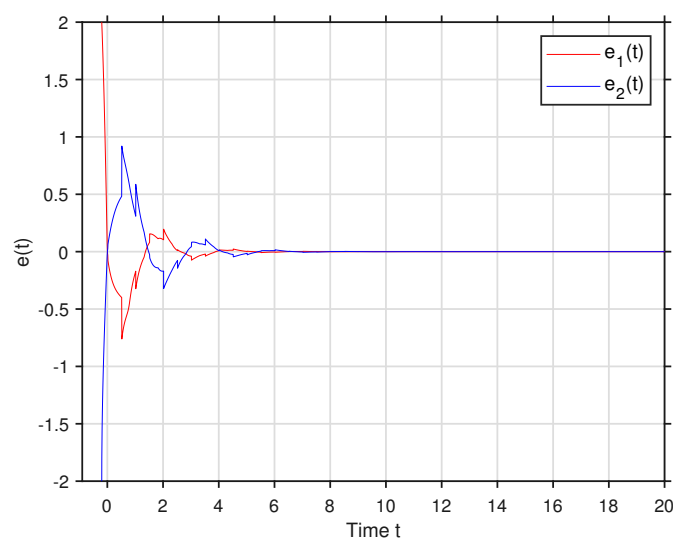
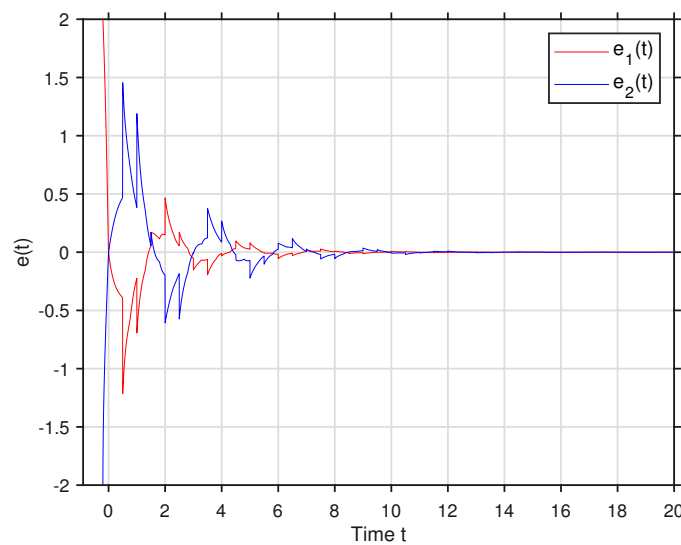**Figure 2.** The state trajectory of the system (4.1) with control ($0.9E$).

Then, the system after the controller is made by MATLAB (Figure 2).

Case(2). Let $\mu = 2.1$, $P = E$, $R_k = E$, $t_k = t_{k-1} + 0.5k$, $k \in N$, and $\beta = 1.21$. It is easy to calculate that $\varrho = 0.9$, $F_i = 1$, $i = 1, \ldots, n$, $T_a = 0.5$, $N_0 = 1$. By employing the MATLAB toolbox, we are capable of obtaining the subsequent details:

$$\alpha = -6, \ \theta_1 = \theta_2 = 2,$$

$$\Sigma_1 = \begin{bmatrix} 50 & 0 \\ 0 & 200 \end{bmatrix}, \ \Sigma_2 = \Sigma_3 = \begin{bmatrix} 1.9 & 0 \\ 0 & 1.9 \end{bmatrix}$$

.

Then, the system after the controller is made by MATLAB (Figure 3).

Figure 3. The state trajectory of the system (4.1) with control (2.1E).

Example 4.2. *Consider the following model:*

$$\begin{cases} \frac{\mathrm{d}e_i(t)}{\mathrm{d}t} = -d_i e_i(t) + \sum_{j=1}^{2} a_{ij} g_j(e_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(e_j(t - \tau(t))) \\ \qquad + \sum_{j=1}^{2} c_{ij} \int_{t-0.9}^{t} f_j(e_j(s)) ds - l_i * e_i(t), t \neq t_k, t \geq 0 \\ \triangle e_i(t_k) = e_i(t_k^+) - e_i(t_k^-) = -\mu R_k e(t_k^-), t = t_k, k \in N, \end{cases} \qquad (4.2)$$
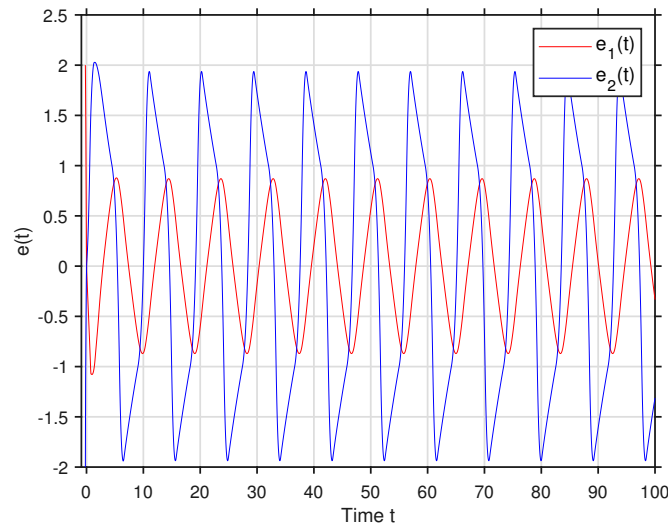
where the coefficients and functions are defined as follows:

$$D = \begin{bmatrix} 2.5 & 0 \\ 0 & 2 \end{bmatrix}, \ A = \begin{bmatrix} 2 & 0.15 \\ -1.2 & 3.5 \end{bmatrix}, \ B = \begin{bmatrix} -1.7 & 0.12 \\ -0.26 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 2.5 & 0 \\ 0 & 2 \end{bmatrix}, \ L = \begin{bmatrix} 4.5 & 0 \\ 0 & 5 \end{bmatrix},$$

$$g_1(x) = g_2(x) = \frac{1}{2}(|x + 1| - |x - 1|), \ \tau(t) = 0.8.$$

In order to see the effect of control more clearly, we first consider the situation without control, see Figure 4.
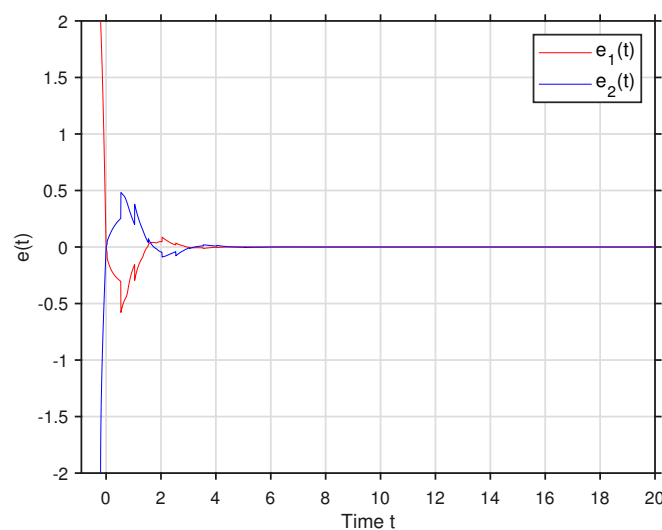

Figure 4. The state trajectory of the system (4.2) without control.

Case(1). Let $\mu = 0.9$, $P = E$, $R_k = E$, $t_k = t_{k-1} + 0.04k, k \in N$, and $\beta = 0.01$. It is easy to calculate that $\varrho = 0.9$, $F_i = 1, i = 1, \ldots, n$, $T_a = 0.04, N_0 = 1$. By employing the MATLAB toolbox, we are capable of obtaining the subsequent details:

$$\alpha = 95, \ \theta_1 = \theta_2 = 0.1,$$

$$\Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \ \Sigma_2 = \Sigma_3 = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix}.$$

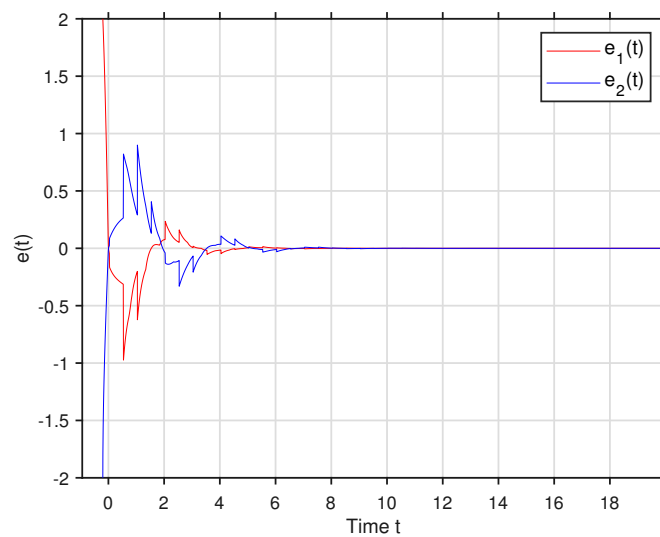**Figure 5.** The state trajectory of the system (5.2) with control (0.9E).

Then, the system after the controller is made by MATLAB (Figure 5).

Case(2). Let $\mu = 2.1$, $P = E$, $R_k = E$, $t_k = t_{k-1} + 0.5k$, $k \in N$, and $\beta = 1.21$. It is easy to calculate that $\varrho = 0.9$, $F_i = 1$, $i = 1, \ldots, n$, $T_a = 0.5$, $N_0 = 1$. By employing the MATLAB toolbox, we are capable of obtaining the subsequent details:

$$\alpha = -6, \; \theta_1 = \theta_2 = 2,$$

$$\Sigma_1 = \begin{bmatrix} 50 & 0 \\ 0 & 200 \end{bmatrix}, \; \Sigma_2 = \Sigma_3 = \begin{bmatrix} 1.9 & 0 \\ 0 & 1.9 \end{bmatrix}.$$

Then, the system after the controller is made by MATLAB (Figure 6).

Figure 6. The state trajectory of the system (4.2) with control (2.1E).

5. Conclusions

This paper delved into the exponential synchronization challenge posed by NNs that incorporated time-varying delays alongside finite distributed delays. To tackle this, a hybrid impulsive controller was devised. Initially, two impulsive differential inequalities were rigorously proven. Subsequently, in the course of proving the main theorem, the original synchronization problem was strategically reframed into a category of impulsive differential inequalities through the application of an apt Lyapunov function. This transformation facilitated the derivation of sufficient corresponding conditions that leveraged lemmas. Ultimately, the validity and efficacy of the obtained conclusions were showcased through two illustrative numerical examples which focused on error analysis.

The attained conclusions affirmed the system's attainment of objectives, yet post-achievement, the control process persisted in the resource exhaustion. Consequently, contemplating a novel impulsive control strategy is advisable to halt the control upon the goal fulfillment. Furthermore, the model overlooked network couplings, which hindered its applicability to broader contexts.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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