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Research article

Second-order general Emden-Fowler differential equations of neutral type: Improved Kamenev-type oscillation criteria

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Abstract: The study of the oscillatory behavior of a general class of neutral Emden-Fowler differential equations is the focus of this work. The main motivations for studying the oscillatory behavior of neutral equations are their many applications as well as the richness of these equations with exciting analytical issues. We obtained novel oscillation conditions in Kamenev-type criteria for the considered equation in the canonical case. We improve the monotonic and asymptotic characteristics of the non-oscillatory solutions to the considered equation and then utilize these characteristics to refine the oscillation conditions. We present, through examples and discussions, what demonstrates the novelty and efficiency of the results compared to previous relevant findings in the literature. In addition, we numerically represent the solutions of some special cases to support the theoretical results.

Keywords: differential equations; Kamenev-type criteria; oscillation theory; canonical case; neutral argument

1. Introduction

Studying the properties of solutions to differential equations (DEs) is the main aim of qualitative theory. This theory is concerned with investigating features such as stability, periodicity, bifurcation, oscillation, synchronization, etc. This theory emerged from researchers' attempts to obtain sufficient information about the nonlinear models that appear when modeling biological, physical, and other phenomena; see [1–4]. The qualitative theory was also extended to include functional, fractional, and partial differential equations.

A functional differential equation (FDE) is a DE with a deviating argument. That is, the FDE contains a dependent variable and some of its derivatives for various argument values. These equations are distinguished by the fact that they take into account the previous and subsequent times when studying any phenomenon or system. In fact, the majority of studies on FDEs until the time of Volterra [5] focused only on some properties of some very special equations. A few books on FDEs were published. In the latter portion of the 1940s and the initial phase of the 1950s. Mishkis [6] laid the groundwork for a broad theory of linear systems in his book by introducing a general class of delayed-argument equations. In 1954, Bellman and Danskin [7] noted the many uses of equations containing past information in disciplines such as economics and biology. Bellman and Cook [8] provide a more comprehensive treatment of the theoretical framework concerning linear equations and the fundamentals of stability theory.

Oscillation theory, as part of qualitative theory, revolves around creating the criteria for the presence of oscillatory and non-oscillatory solutions to FDEs, investigating zero-order distribution laws, estimating the number of zeros in a certain period of time and the distance between neighboring zeros, and other basics. Oscillation theory has now become an important mathematical instrument for numerous advanced fields and technological applications. Obtaining oscillation conditions for specific FDEs has been a widely studied area in the last few decades; see [9–13].

1.1. FDEs with a neutral delay argument

Delay DEs that have the derivative of the solutions of the highest order with and without delay are known as neutral differential equations. These equations are apparent in the investigation of oscillatory masses and the modeling of electrical circuitry with ideal transmission lines (see [14]). Understanding the qualitative characteristics of delay differential equations is becoming more and more important as new models and developments in biology, economics, physics, and engineering keep appearing.

In the canonical scenario, we find new conditions to test the oscillation of nonlinear neutral FDEs of second order. Namely, we consider the FDE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a\,\psi\,(x)\left[\frac{\mathrm{d}}{\mathrm{d}t}z\right]^{\alpha}\right)(t)+q\,(t)\,F\,(x\,(g\,(t)))=0, \tag{1.1}$$

where $t\in\mathbb{I}:=[t_0,\infty)$, $\alpha\in\mathbb{Q}^{+}$ is a ratio of odd numbers, and

$$z:=x+p\,x\,(h)\,.$$

In our study, we consider the following hypotheses:

A1: $a\in\mathbf{C}^1(\mathbb{I},\mathbb{R}^+)$, $q\in\mathbf{C}(\mathbb{I},\mathbb{R}^+)$, and $p\in\mathbf{C}(\mathbb{I},[0,p_0])$, where $p_0<1$.
A2: $h,g\in\mathbf{C}(\mathbb{I},\mathbb{R})$, $h\le t$, $g\le t$, $g'\ge 0$, $\lim_{t\to\infty}h(t)=\infty$, and $\lim_{t\to\infty}g(t)=\infty$.
A3: $\psi\in\mathbf{C}^1(\mathbb{R},[m,M])$, where $0<m\le M$, and $\kappa=m^{-1/\alpha}M^{1/\alpha}$.
A4: $F\in\mathbf{C}^1(\mathbb{R},\mathbb{R})$, $\varkappa F(\varkappa)>0$ for $\varkappa\neq 0$, $F'(\varkappa)\ge 0$, and $-F(-\varkappa w)\ge F(\varkappa w)\ge F(\varkappa)F(w)$ for $\varkappa w>0$.

For a solution of Eq (1.1), we define a function $x\in\mathbf{C}^1([t_x,\infty),\mathbb{R})$, $t_x\in\mathbb{I}$, which has the properties: $a\cdot\psi\cdot(z')^{\alpha}\in\mathbf{C}^1([t_x,\infty),\mathbb{R})$, $\sup\{|x(t)|:t\ge t_*\}>0$ for all $t_*\ge t_x$, and satisfies Eq (1.1) on $[t_x,\infty)$. If a solution x of FDE (1.1) has arbitrarily large zeros, it is referred to as oscillatory; if not, it is referred to as non-oscillatory. The FDE (1.1) is said to be in the canonical case if

$$\int_{t_0}^{\infty}a^{-1/\alpha}(s)\,\mathrm{d}s=\infty. \tag{1.2}$$

1.2. Literature review

We can divide previous related works in the literature into two main parts. The first section consists of studies that are concerned with using different techniques to study oscillation, and the second section consists of studies that focus on improving the relationships and inequalities used in studying oscillation.

Oscillation criteria are often only sufficient conditions to test the oscillation and are not necessary. Therefore, finding different techniques and methods for studying oscillation is inevitable and necessary for the purpose of application to the largest area of special cases and also to get rid of some of the restrictions that may be imposed by some techniques.

In 2000, Džurina and Mihalíková [15] used Riccati substitution to provide some oscillatory conditions for the solutions of the equation

$$(az')'(t) + q(t)(x \circ h)(t) = 0 \tag{1.3}$$

with $h(t) = t - k$, $k > 0$ and $p(t) = -p_0 < 1$. Han et al. [16] acquired some Kamenev-type oscillation conditions for (1.3) when $h'(t) = h_0 > 0$. By comparison method, Džurina [17] studied the oscillation of (1.3) with an advanced neutral term, i.e., when $h(t) \geq t$.

Şahiner [18] presented Philos-type conditions for oscillation of (1.1) when $\alpha = 1$, $F'(\upsilon) \geq K > 0$, and $\pm F(\pm\upsilon w) \geq LF(\upsilon)F(w)$, for all $\upsilon w > 0$ and for some $L > 0$.

Theorem 1. *([18], Theorem 2.1) Assume that*

$$D_0 = \{(\varkappa, \varsigma) : \varkappa > \varsigma > \varkappa_0\} \text{ and } D = \{(\varkappa, \varsigma) : \varkappa \geq \varsigma \geq t_0\}.$$

By $P \in \Im$, we mean that the continuous real-valued function P with domain D belongs to class $\Im$ and satisfies

(*i*) $P(\varkappa, \varsigma) = 0$ *for* $\varkappa \geq t_0$;

(*ii*) $P(\varkappa, \varsigma) > 0$ *on the domain* D_0;

(*iii*) $P(\varkappa, \varsigma)$ *possesses a continuous and non-positive partial derivative* $\partial P/\partial\varsigma$ *on the domain* D_0 *where*

$$\frac{\partial P(\varkappa, \varsigma)}{\partial \varsigma} = -k(\varkappa, \varsigma)\sqrt{P(\varkappa, \varsigma)} \text{ for all } (\varkappa, \varsigma) \in D_0$$

holds for $k \in C(D, \mathbb{R})$.

If there is a function $\rho \in \mathbf{C}(\mathbb{I}, \mathbb{R}^+)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{P(t, t_0)} \int_{t_0}^{t} \left(P(t, s)\rho(s)Q(s) - \frac{M\rho(s)a(g(s))}{4LKg'(s)} G^2(t, s) \right) \mathrm{d}s = \infty,$$

then all solutions of (1.1) are oscillatory, where $Q := q[1 - p(g)]$, *and*

$$G(\varkappa, \varsigma) = k(\varkappa, \varsigma) - \frac{\rho'(\varsigma)}{\rho(\varsigma)}\sqrt{P(\varkappa, \varsigma)}.$$

Putting $P(\varkappa,\varsigma) = (\varkappa - \varsigma)^n$, $n > 1$, in the results of Theorem 1, we obtain a Kamenev-type criterion of Eq (1.1).

On the other hand, in the most recent period, there was a significant surge in research activity concentrated on improving the inequalities that are used in the study of oscillation. One of the influential relationships in investigating the oscillation of neutral equations involves examining the interplay between x and its z, as well as the relationship between the corresponding function and its derivatives.

The classical substitution $x > (1-p)z$ is often used in studying the oscillation of neutral equations. Moaaz et al. [19] devised improved conditions for the oscillation of

$$\left(a\left[z'\right]^{\alpha}\right)'(t) + \sum_{i=1}^{L} q_i(t)\, x^{\gamma}(g_i(t)) = 0, \tag{1.4}$$

They provided the following relationship as an improvement on the classical relationship:

$$x > z\sum_{j=1}^{\nu/2} \frac{1}{p_0^{2j-1}}\left(1 - \frac{1}{p_0}\frac{\eta_{t_1}\left(h^{[-2j]}\right)}{\eta_{t_1}\left(h^{[-(2j-1)]}\right)}\right),$$

for $p_0 > 1$ and ν is an even natural number, and

$$x > z(1-p_0)\sum_{j=0}^{(\nu-1)/2} p_0^{2j}\frac{\eta_{t_1}\left(h^{[2j+1]}\right)}{\eta_{t_1}},$$

for $p < 1$, and ν is an odd natural number, where $h^{[\pm l]} = h^{\pm 1}\left(h^{[\pm(l-1)]}\right)$, for $l = 1, 2, ...$, and

$$\eta_{t_1}(t) := \int_{t_1}^{t} a^{-1/\alpha}(s)\,\mathrm{d}s.$$

For the non-canonical situation, Hassan et al. [20] used the improved relationship

$$x > z\sum_{r=0}^{(n-1)/2} p_0^{2r}\left(1 - p_0\frac{\mu_0\left(h^{[2r+1]}\right)}{\mu_0\left(h^{[2r]}(t)\right)}\right),$$

when $z' < 0$.

A research study for the fourth-order DE

$$\left(az'''\right)'(t) + q(t)\,x(g(t)) = 0,$$

Moaaz et al. [21] devised improved substitution for x by z in all cases of positive solutions.

Lemma 1. *([21], Lemma 1) Let x be a solution of (1.1) and $x > 0$ eventually. Then,*

$$x(t) > \sum_{r=0}^{m}\left(\prod_{l=0}^{2r} p\left(h^{[l]}(t)\right)\right)\left[\frac{z\left(h^{[2r]}(t)\right)}{p\left(h^{[2r]}(t)\right)} - z\left(h^{[2r+1]}(t)\right)\right],$$

for any integer $m \geq 0$, where

$$h^{[0]}(t) := t,\ h^{[l]}(t) = \left(h \circ h^{[l-1]}\right)(t),\ \textit{for}\ l = 1, 2, \ldots .$$

See [22–24] for other intriguing findings on the oscillatory nature of solutions to third- and fourth-order DEs that have been published more recently.

The oscillation criteria of neutral DEs depend on several factors:

– The substitution x by z;
– The monotonic and asymptotic features of non-oscillatory solutions;
– The technique used to obtain the oscillation criteria.

Improving any of these factors directly affects the oscillation criteria. In our paper, we test the extent to which the oscillation criteria are affected by improving the relationship between x and z. We used a well-known technique that produces criteria known as Kamenev criteria; but the improvement lies in the new relationships used in the study.

1.3. The function class $\mathbb{Y}$

Here, we define a class of functions to obtain the oscillation condition of the Kamenev-type criteria. Suppose that

$$\mathbb{K} := \left\{(u, v, w) \in \mathbb{I}^3 : w \leq v \leq u\right\}.$$

A $\varphi \in \mathbf{C}(\mathbb{K}, \mathbb{R})$ belongs to the class $\mathbb{Y}$ ($\varphi \in \mathbb{Y}$), if

(i) $\varphi(u, u, w) = 0$, $\varphi(u, w, w) = 0$, and $\varphi(u, s, w) \neq 0$ for $w < s < u$.
(ii) φ has the partial derivative $\partial\varphi/\partial v$ on $\mathbb{K}$ and $\partial\varphi/\partial v$ is locally integrable with respect to v in $\mathbb{K}$.

For any function $f \in \mathbf{C}^1(\mathbb{I}, \mathbb{R})$,we define the following operator:

$$\mathcal{T}[f; u, w] := \int_w^u f(v)\, \varphi(u, v, w)\, \mathrm{d}v,$$

for $w \leq v \leq u$. Moreover, we define the function $\mu(u, v, w)$ by

$$\mu(u, v, w) := \frac{1}{\varphi(u, v, w)} \frac{\partial\varphi(u, v, w)}{\partial s},$$

for all $w < v < u$. We notice that $\mathcal{T}[\cdot\,; u, w]$ is linear. Using integration by parts, we find that

$$\mathcal{T}[f'; u, w] = -\mathcal{T}[(f \cdot \mu); u, w]. \tag{1.5}$$

2. Main results

Here, we investigate the improved monotonic features of non-oscillatory solutions and then establish new oscillation conditions for (1.1). For simplicity, the class of all positive non-oscillatory solutions of (1.1) is denoted by the symbol $\mathbb{S}$. Also, we assume that $\gamma = \frac{1}{\alpha}(\alpha/(\alpha+1))^{\alpha+1}$,

$$A_{t_*}(t) := \int_{t_*}^t \frac{1}{[a(s)]^{1/\alpha}} \mathrm{d}s$$

for $t \geq t_*$, and

$$G_q(t; m) := q(t)\, F\left(\sum_{r=0}^{m} \left(\prod_{l=0}^{2r} p\left(h^{[l]}(g(t))\right)\right)\left[\frac{1}{p(h^{[2r]}(g(t)))} - 1\right]\left[\frac{A_{t_1}\left(h^{[2r]}(g(t))\right)}{A_{t_1}(g(t))}\right]^{\kappa}\right).$$

2.1. Monotonic properties and preliminary results

First, we infer the monotonic behavior of positive solutions in the next lemma.

Lemma 2. *If $x \in \mathbb{S}$, then the corresponding function z of x conforms to $z > 0$, $z' > 0$, and $(a\psi(x)[z']^{\alpha})' < 0$, eventually.*

Proof. Suppose that $x \in \mathbb{S}$. From (A2), there exists a $t_1 \in \mathbb{I}$ whereby $(x \circ h)(t)$ and $(x \circ g)(t)$ are positive for $t \geq t_1$. Therefore, $z(t) > 0$. Using (A4), we have that $F(x(g(t))) > 0$. Hence, (1.1) becomes

$$(a\psi(x)[z']^{\alpha})'(t) = -q(t) F(x(g(t))) < 0.$$

Thus, we find that $a\psi(x)[z']^{\alpha}$ has a constant sign. This is the same as stating that $z' > 0$ or $z' < 0$ for $t \geq t_2$, where t_2 is large enough. But, when $z'(t) < 0$, this case contradicts (1.2), as displayed next:

Let $z'(t) < 0$ for $t \geq t_2$. Then

$$(a\psi(x)[z']^{\alpha})(t) \leq (a\psi(x)[z']^{\alpha})(t_2) := -L < 0.$$

Hence,

$$z' \leq -\frac{L^{1/\alpha}}{a^{1/\alpha}(t)[\psi(x)]^{1/\alpha}},$$

which with (A3) gives

$$z' \leq -\frac{L^{1/\alpha}}{M^{1/\alpha}} a^{-1/\alpha}.$$

Thus,

$$z(t) \leq z(t_2) - \frac{L^{1/\alpha}}{M^{1/\alpha}} \int_{t_2}^{t} a^{-1/\alpha}(s)\, \mathrm{d}s.$$

But condition (1.2) results in $z(t) \to -\infty$ as $t \to \infty$, a contradiction.

Consequently, the proof ends. □

Lemma 3. *Assume that $x \in \mathbb{S}$. Then*

$$z(t) \geq M^{-1/\alpha} [a(t)\psi(x(t))]^{1/\alpha} z'(t) A_{t_1}(t).$$

Moreover,

$$\frac{z(t)}{[A_{t_1}(t)]^{\kappa}}$$

is nonincreasing, for $t \geq t_1$.

Proof. Let $x \in \mathbb{S}$. From (A2), there is a $t_1 \in \mathbb{I}$ such that $(x \circ h)(t) > 0$ and $(x \circ g)(t) > 0$ for $t \geq t_1$. Using Lemma 2, we obtain

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^{t} \frac{1}{[a(\xi)\psi(x(\xi))]^{1/\alpha}} [a(\xi)\psi(x(\xi))]^{1/\alpha} z'(\xi)\, \mathrm{d}\xi \\ &\geq \int_{t_1}^{t} \frac{1}{[Ma(\xi)]^{1/\alpha}} [a(\xi)\psi(x(\xi))]^{1/\alpha} z'(\xi)\, \mathrm{d}\xi \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{M^{1/\alpha}}\left[a(t)\psi(x(t))\right]^{1/\alpha} z'(t)\int_{t_1}^{t}\frac{1}{[a(\xi)]^{1/\alpha}}\mathrm{d}\xi \\ &= \frac{1}{M^{1/\alpha}}\left[a(t)\psi(x(t))\right]^{1/\alpha} z'(t) A_{t_1}(t). \end{aligned}$$

Hence, for any $t \geq t_1$,

$$\begin{aligned} 0 &\geq z' - \frac{M^{1/\alpha}}{\left[a(t)\psi(x)\right]^{1/\alpha} A_{t_1}} z \\ &\geq z' - \frac{M^{1/\alpha}}{m^{1/\alpha}}\frac{[a]^{-1/\alpha}}{A_{t_1}} z. \end{aligned} \tag{2.1}$$

Since $A'_{t_1}(t) \geq 0$, $A_{t_1}(t_1) = 0$ and $A_{t_1}(\infty) = \infty$, there is a $t_2 \geq t_1$ such that $A_{t_1}(t_2) = 1$. Now, from (2.1), we obtain

$$\begin{aligned} 0 &\geq \frac{\mathrm{d}}{\mathrm{d}t}\left(z(t)\exp\left[-\kappa\int_{t_2}^{t}\frac{[a(s)]^{-1/\alpha}}{A_{t_1}(s)}\mathrm{d}s\right]\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(z(t)\exp\left[-\kappa\ln A_{t_1}(t)\right]\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z(t)}{[A_{t_1}(t)]^{\kappa}}\right). \end{aligned}$$

Consequently, this proof ends. □

Next, we improve the oscillation results by deriving a new substitution for x by z.

Lemma 4. *Assume that $x \in \mathbb{S}$. Then (1.1) can be articulated as*

$$\left(a(t)\psi(x(t))\left[z'(t)\right]^{\alpha}\right)' + G_q(t;m)F(z(g(t))) \leq 0, \tag{2.2}$$

for $t > t_1$ and any integer $m \geq 0$.

Proof. Let $x \in \mathbb{S}$. From (A2), there is a $t_1 \in \mathbb{I}$ such that $(x \circ h)(t) > 0$ and $(x \circ g)(t) > 0$ for $t \geq t_1$. Using Lemma 1, we obtain

$$x > \sum_{r=0}^{m}\left(\prod_{l=0}^{2r} p\left(h^{[l]}\right)\right)\left[\frac{z\left(h^{[2r]}\right)}{p\left(h^{[2r]}\right)} - z\left(h^{[2r+1]}\right)\right] \tag{2.3}$$

From Lemmas 2–3, we find that

$$z' > 0 \text{ and } \left(\frac{z}{[A_{t_1}]^{\kappa}}\right)' \leq 0.$$

Then,

$$z\left(h^{[2r]}\right) \geq z\left(h^{[2r+1]}\right)$$

and

$$z\left(h^{[2r]}\right) \geq \left[\frac{A_{t_1}\left(h^{[2r]}\right)}{A_{t_1}}\right]^{\kappa} z.$$

Using previous relationships in (2.3), we arrive at

$$x > z \sum_{r=0}^{m} \left(\prod_{l=0}^{2r} p\left(h^{[l]}\right) \right) \left[\frac{1}{p\left(h^{[2r]}\right)} - 1 \right] \left[\frac{A_{t_1}\left(h^{[2r]}\right)}{A_{t_1}} \right]^{\kappa},$$

which with (1.1) gives

$$\begin{aligned} 0 &= \left(a\psi(x)\left[z'\right]^{\alpha}\right)' \\ &\quad + qF\left(z(g) \sum_{r=0}^{m} \left(\prod_{l=0}^{2r} p\left(h^{[l]}(g)\right) \right) \left[\frac{1}{p\left(h^{[2r]}(g)\right)} - 1 \right] \left[\frac{A_{t_1}\left(h^{[2r]}(g)\right)}{A_{t_1}(g)} \right]^{\kappa} \right) \\ &\geq \left(a\psi(x)\left[z'\right]^{\alpha}\right)' \\ &\quad + qF(z(g))\, F\left(\sum_{r=0}^{m} \left(\prod_{l=0}^{2r} p\left(h^{[l]}(g)\right) \right) \left[\frac{1}{p\left(h^{[2r]}(g)\right)} - 1 \right] \left[\frac{A_{t_1}\left(h^{[2r]}(g)\right)}{A_{t_1}(g)} \right]^{\kappa} \right) \\ &= \left(a\psi(x)\left[z'\right]^{\alpha}\right)' + G_q(t;m)\, F(z(g)), \end{aligned}$$

where assumption (A4) was used.

Therefore, the proof ends. □

The following theorem transforms the studied equation into the form of a Riccati inequality, or what is known as the Riccati technique.

Theorem 2. *Assume that $x \in \mathbb{S}$, and there is a constant $L > 0$ such that*

$$F'(\upsilon) \geq L\left[F(\upsilon)\right]^{1-1/\alpha}, \tag{2.4}$$

for $u \neq 0$. If we define the function ω as

$$\omega := \upsilon \frac{a\psi(x)\left[z'\right]^{\alpha}}{F(z(g))}, \tag{2.5}$$

then ω satisfies

$$\omega'(t) \leq \frac{\upsilon'(t)}{\upsilon(t)}\omega(t) - \upsilon(t)\, G_q(t;m) - \frac{L\, g'(t)}{\left(Ma(g(t))\,\upsilon(t)\right)^{1/\alpha}}\left[\omega(t)\right]^{1+1/\alpha}, \tag{2.6}$$

where $\upsilon \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$.

Proof. Assume that $x \in \mathbb{S}$. By differentiating ω, we find

$$\omega' = \frac{\upsilon'}{\upsilon} \cdot \omega + \upsilon \cdot \left[\frac{\left(a \cdot (\psi \circ x) \cdot \left[z'\right]^{\alpha}\right)'}{(F \circ z \circ g)} - \frac{a \cdot (\psi \circ x) \cdot \left[z'\right]^{\alpha}}{(F \circ z \circ g)^2} (F \circ z \circ g)' \right],$$

which with (2.2) yields

$$\omega' \leq \frac{\upsilon'}{\upsilon} \cdot \omega + \upsilon \cdot \left[-G_q - \frac{a \cdot (\psi \circ x) \cdot \left[z'\right]^{\alpha}}{(F \circ z \circ g)^2} \cdot \left(F' \circ z \circ g\right) \cdot \left(z' \circ g\right) \cdot g' \right]. \tag{2.7}$$

Since $(a(t)\psi(x(t))[z'(t)]^{\alpha})' < 0$ and $g(t) \le t$, we have

$$\begin{aligned} a(t)\psi(x(t))[z'(t)]^{\alpha} &\le a(g(t))\psi(x(g(t)))[z'(g(t))]^{\alpha} \\ &\le Ma(g(t))[z'(g(t))]^{\alpha}, \end{aligned}$$

and then

$$z'(g(t)) \ge \frac{(a(t)\psi(x(t))[z'(t)]^{\alpha})^{1/\alpha}}{M^{1/\alpha}[a(g(t))]^{1/\alpha}}. \tag{2.8}$$

Combining (2.7) and (2.8), we arrive at

$$\begin{aligned} \omega' &\le \frac{\upsilon'}{\upsilon}\cdot\omega + \upsilon\cdot\left[-G_q - \frac{[a\cdot(\psi\circ x)\cdot[z']^{\alpha}]^{1+1/\alpha}}{(F\circ z\circ g)^2}\cdot\frac{(F'\circ z\circ g)\cdot g'}{M^{1/\alpha}(a\circ g)^{1/\alpha}}\right] \\ &= \frac{\upsilon'}{\upsilon}\cdot\omega + \upsilon\cdot\left[-G_q - \frac{\omega^{1+1/\alpha}}{\upsilon^{1+1/\alpha}}\cdot\frac{(F'\circ z\circ g)}{(F\circ z\circ g)^{1-1/\alpha}}\cdot\frac{g'}{M^{1/\alpha}(a\circ g)^{1/\alpha}}\right]. \end{aligned} \tag{2.9}$$

Using (2.4), we find

$$\omega' \le \frac{\upsilon'}{\upsilon}\cdot\omega - \upsilon\cdot G_q - \frac{Lg'}{M^{1/\alpha}(a\circ g)^{1/\alpha}\cdot\upsilon^{1/\alpha}}\omega^{1+1/\alpha}.$$

Therefore, the proof ends. □

2.2. *Kamenev-type oscillation criteria*

Here, we employ the previous results to derive a novel criterion that tests whether all solutions are oscillatory.

Theorem 3. *Assume that $\varphi \in \mathbb{Y}$, $\upsilon \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$, and there is a constant $L > 0$ such that (2.4) holds. If*

$$\limsup_{t\to\infty}\mathcal{T}\left[\upsilon\cdot G_q - \frac{\gamma M}{L^{\alpha}}\left(\frac{\upsilon'}{\upsilon}+\mu\right)^{\alpha+1}\frac{(a\circ g)\cdot\upsilon}{(g')^{\alpha}}; t, l\right] > 0, \tag{2.10}$$

then (1.1) oscillates.

Proof. Assuming the opposite of what is required means that there is a non-oscillatory solution to the studied equation, and this necessarily leads to guaranteeing the existence of a solution that eventually becomes positive for this equation. Let it be x. From Theorem 2, if we define the function ω as in (2.5), then ω satisfies (2.6).

Next, applying the operator $\mathcal{T}[\cdot\,; t, l]$ to (2.6), we obtain

$$\mathcal{T}[\omega'; t, l] \le \mathcal{T}\left[\left(\frac{\upsilon'}{\upsilon}\cdot\omega - \frac{Lg'}{(M(a\circ g)\cdot\upsilon)^{1/\alpha}}\omega^{1+1/\alpha}\right); t, l\right] - \mathcal{T}\left[\upsilon\cdot G_q; t, l\right].$$

Using the property (1.5), we obtain

$$\mathcal{T}\left[\upsilon\cdot G_q; t, l\right] \le \mathcal{T}\left[\left(\frac{\upsilon'}{\upsilon}+\mu\right)\cdot\omega - \frac{Lg'}{(M(a\circ g)\cdot\upsilon)^{1/\alpha}}\omega^{1+1/\alpha}; t, l\right]. \tag{2.11}$$

By simple calculation, we find that the function $H(\omega) = c_1\omega - c_2\omega^{1+1/\alpha}$, where $c_1, c_2 > 0$, has the maximum

$$H(\omega) \leq H(\omega_{\max}) = \gamma c_1^{\alpha+1} c_2^{-\alpha}, \tag{2.12}$$

at $\omega_{\max} = (\alpha c_1 / ((\alpha+1) c_2))^{\alpha}$. Using (2.12), (2.11) becomes

$$\mathcal{T}\left[\upsilon \cdot G_q; t, l\right] \leq \mathcal{T}\left[\frac{\gamma M}{L^\alpha}\left(\frac{\upsilon'}{\upsilon} + \mu\right)^{\alpha+1} \frac{(a \circ g) \cdot \upsilon}{(g')^\alpha}; t, l\right]. \tag{2.13}$$

Taking the super limit for (2.13), we have

$$\limsup_{t\to\infty} \mathcal{T}\left[\upsilon \cdot G_q - \frac{\gamma M}{L^\alpha}\left(\frac{\upsilon'}{\upsilon} + \mu\right)^{\alpha+1} \frac{(a \circ g) \cdot \upsilon}{(g')^\alpha}; t, l\right] \leq 0.$$

This contradicts assumption (2.10).

Therefore, the proof ends. □

Corollary 1. *Assume that there is a constant $L > 0$ such that (2.4) holds. If there are $\upsilon, \eta \in \mathbf{C}^1(\mathbb{I}, \mathbb{R}^+)$ and $\varrho, k > \max\{1/2, \alpha\}$ such that*

$$\limsup_{t\to\infty} \int_l^t \eta(s)(t-s)^\varrho (s-l)^k \left[\upsilon(s) G_q(s; m) - \frac{\gamma M}{L^\alpha} H(t, s, l)\right] \mathrm{d}s > 0, \tag{2.14}$$

for $l \geq t_0$, then all solutions of (1.1) are oscillatory, where

$$H(t, s, l) := \left[\frac{\upsilon'(s)}{\upsilon(s)} + \frac{\eta'(s)}{\eta(s)} + \frac{kt - (\varrho + k)s + \varrho l}{(t-s)(s-l)}\right]^{\alpha+1} \frac{(a(g(s)))\upsilon(s)}{(g'(s))^\alpha}. \tag{2.15}$$

Proof. By choosing

$$\varphi(u, v, w) = \eta(v)(u-v)^\varrho (v-w)^k,$$

we find

$$\mu(u, v, w) = \frac{\eta'(v)}{\eta(v)} + \frac{ku - (\varrho + k)v + \varrho w}{(u-v)(v-w)}.$$

Using Theorem 3, condition (2.10) reduces to (2.14).

Therefore, the proof ends. □

Corollary 2. *Assume that there is a constant $L > 0$ such that (2.4) holds. If $g(t) = \lambda t$, $\lambda \in (0, 1]$, $\alpha \in \mathbb{Z}^+$, $a(t) \geq 1$, and there is $\varrho, k > \alpha$ such that*

$$\limsup_{t\to\infty} \frac{\mathcal{T}\left[G_q; t, l\right]}{\left[A_{t_1}(t) - A_{t_1}(l)\right]^{\varrho+k-\alpha}} > \frac{1}{\lambda^\alpha} \frac{\gamma M}{L^\alpha} \sum_{i=0}^{\alpha+1} \binom{\alpha+1}{i} (-\varrho)^i k^{\alpha+1-i} \beta(k + i - \alpha, \varrho - i + 1), \tag{2.16}$$

for $l \geq t_0$, then all solutions of (1.1) are oscillatory, where $\beta(\cdot, \cdot)$ is the beta function.

Proof. Assume that

$$\upsilon(t) \equiv 1 \text{ and } \varphi(u, v, w) = \left(A_{t_1}(u) - A_{t_1}(v)\right)^\varrho \left(A_{t_1}(v) - A_{t_1}(w)\right)^k. \tag{2.17}$$

Then, we find that

$$\begin{aligned}
&\mathcal{T}\left[(\mu)^{\alpha+1}\frac{(a\circ g)}{(g')^{\alpha}};t,l\right]\\
=\ &\frac{1}{\lambda^{\alpha}}\int_{l}^{t}\frac{(A_{t_1}(t)-A_{t_1}(s))^{\varrho}}{(A_{t_1}(s)-A_{t_1}(l))^{-k}}\left[\frac{\varrho A_{t_1}(t)-(\varrho+k)A_{t_1}(s)+kA_{t_1}(l)}{a(s)(A_{t_1}(t)-A_{t_1}(s))(A_{t_1}(s)-A_{t_1}(l))}\right]^{\alpha+1}(a(\lambda s))\,\mathrm{d}s\\
=\ &\frac{1}{\lambda^{\alpha}}\int_{l}^{t}\frac{(A_{t_1}(t)-A_{t_1}(s))^{\varrho-\alpha-1}}{(A_{t_1}(s)-A_{t_1}(l))^{-k+\alpha+1}}\frac{[\varrho A_{t_1}(t)-(\varrho+k)A_{t_1}(s)+kA_{t_1}(l)]^{\alpha+1}}{[a(s)]^{\alpha+1}}(a(\lambda s))\,\mathrm{d}s.
\end{aligned}$$

Since $a'(t)\geq 0$, we have that $a(\lambda s)/a(s)\leq 1$. Hence,

$$\begin{aligned}
&\mathcal{T}\left[(\mu)^{\alpha+1}\frac{(a\circ g)}{(g')^{\alpha}};t,l\right]\\
\leq\ &\frac{1}{\lambda^{\alpha}}\int_{l}^{t}\frac{(A_{t_1}(t)-A_{t_1}(s))^{\varrho-\alpha-1}}{(A_{t_1}(s)-A_{t_1}(l))^{-k+\alpha+1}}\frac{[\varrho A_{t_1}(t)-(\varrho+k)A_{t_1}(s)+kA_{t_1}(l)]^{\alpha+1}}{[a(s)]^{\alpha}}\mathrm{d}s\\
\leq\ &\frac{1}{\lambda^{\alpha}}\int_{l}^{t}\frac{(A_{t_1}(t)-A_{t_1}(s))^{\varrho-\alpha-1}}{(A_{t_1}(s)-A_{t_1}(l))^{-k+\alpha+1}}\frac{[\varrho A_{t_1}(t)-(\varrho+k)A_{t_1}(s)+kA_{t_1}(l)]^{\alpha+1}}{[a(s)]^{1/\alpha}}\mathrm{d}s\\
=\ &\frac{1}{\lambda^{\alpha}}\int_{l}^{t}\frac{(A_{t_1}(t)-A_{t_1}(s))^{\varrho-\alpha-1}}{(A_{t_1}(s)-A_{t_1}(l))^{-k+\alpha+1}}[\varrho A_{t_1}(t)-(\varrho+k)A_{t_1}(s)+kA_{t_1}(l)]^{\alpha+1}\,\mathrm{d}A_{t_1}(s).
\end{aligned}$$

Let $w:=A_{t_1}(s)-A_{t_1}(l)$. Thus,

$$\begin{aligned}
&\mathcal{T}\left[(\mu)^{\alpha+1}\frac{(a\circ g)}{(g')^{\alpha}};t,l\right]\\
\leq\ &\frac{1}{\lambda^{\alpha}}\int_{0}^{\vartheta}(\vartheta-w)^{\varrho-\alpha-1}w^{k-\alpha-1}[k(\vartheta-w)-\varrho w]^{\alpha+1}\,\mathrm{d}w\\
=\ &\frac{1}{\lambda^{\alpha}}\int_{0}^{\vartheta}\left[(\vartheta-w)^{\varrho-\alpha-1}w^{k-\alpha-1}\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-\varrho)^{i}k^{\alpha+1-i}(\vartheta-w)^{\alpha+1-i}w^{i}\right]\mathrm{d}w\\
=\ &\frac{1}{\lambda^{\alpha}}\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-\varrho)^{i}k^{\alpha+1-i}\int_{0}^{\vartheta}(\vartheta-w)^{\varrho-i}w^{k+i-\alpha-1}\mathrm{d}w\\
=\ &\frac{1}{\lambda^{\alpha}}\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-\varrho)^{i}k^{\alpha+1-i}\vartheta^{\varrho+k-\alpha}\int_{0}^{\vartheta}\left(1-\frac{w}{\vartheta}\right)^{\varrho-i}\left(\frac{w}{\vartheta}\right)^{k+i-\alpha-1}\frac{1}{\vartheta}\mathrm{d}w\\
=\ &\frac{1}{\lambda^{\alpha}}\vartheta^{\varrho+k-\alpha}\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-\varrho)^{i}k^{\alpha+1-i}\int_{0}^{1}(1-y)^{\varrho-i}(y)^{k+i-\alpha-1}\,\mathrm{d}y\\
=\ &\frac{1}{\lambda^{\alpha}}\vartheta^{\varrho+k-\alpha}\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-\varrho)^{i}k^{\alpha+1-i}\beta(k+i-\alpha,\varrho-i+1).
\end{aligned}\tag{2.18}$$

Now, as in the proof of Theorem 3, if we assume the contrary, then we arrive at (2.13). Under assumptions in (2.17), inequality (2.13) reduces to

$$\mathcal{T}\left[G_{q};t,l\right]\leq\frac{\gamma M}{L^{\alpha}}\mathcal{T}\left[(\mu)^{\alpha+1}\frac{(a\circ g)}{(g')^{\alpha}};t,l\right],$$

which with (2.18) gives

$$\mathcal{T}\left[G_q;t,l\right] \leq \frac{1}{\lambda^{\alpha}}\frac{\gamma M}{L^{\alpha}}\vartheta^{\varrho+k-\alpha}\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-\varrho)^i k^{\alpha+1-i}\beta\left(k+i-\alpha,\varrho-i+1\right).$$

Thus, we obtain

$$\limsup_{t\to\infty}\frac{1}{\vartheta^{\varrho+k-\alpha}}\mathcal{T}\left[G_q;t,l\right] \leq \frac{1}{\lambda^{\alpha}}\frac{\gamma M}{L^{\alpha}}\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-\varrho)^i k^{\alpha+1-i}\beta\left(k+i-\alpha,\varrho-i+1\right).$$

This contradicts assumption (2.16).

Therefore, the proof ends. □

In the following corollaries, we present an oscillation criterion for a special case of the studied equation, which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a(t)\,(\psi\circ x)(t)\left[\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right]^{\alpha}\right)+q(t)\left(x^{\alpha}\circ g\right)=0.$$

It is easy to notice that the function $F(u)=u^{\alpha}$ satisfies (A4) and (2.4) with $L=\alpha$.

Corollary 3. *If there are* $\upsilon,\eta\in\mathbf{C}^1(\mathbb{I},\mathbb{R}^+)$ *and* $\varrho,k>\max\{1/2,\alpha\}$ *such that*

$$\limsup_{t\to\infty}\int_l^t \eta(s)(t-s)^{\varrho}(s-l)^k\left[\upsilon(s)\,G_q(s;m)-\frac{\gamma M}{\alpha^{\alpha}}H(t,s,l)\right]\mathrm{d}s>0, \tag{2.19}$$

for $l\geq t_0$*, then all solutions of (1.1) are oscillatory, where* $H(t,s,l)$ *is defined as in (2.15).*

Corollary 4. *If* $g(t)=\lambda t$, $\lambda\in(0,1]$, $\alpha=1$, $a(t)\geq 1$*, and there is* $\varrho>1/2$ *such that*

$$\limsup_{t\to\infty}\frac{1}{\left[A_{t_1}(t)\right]^{2\varrho+1}}\mathcal{T}\left[G_q;t,l\right]>\frac{M}{\lambda}\frac{\varrho}{4\varrho^2-1}, \tag{2.20}$$

for $l\geq t_0$*, then all solutions of (1.1) are oscillatory.*

2.3. *Examples and discussion*

Example 1. *Consider the delay equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1+\sin^2(x(t))}\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)+\frac{1}{2}x(h_0t)\right]\right)+\frac{q_0}{t^2}x(g_0t)=0, \tag{2.21}$$

where h_0, $g_0\in(0,1]$*, and* $q_0>0$*. We note that* $\alpha=1$, $a(t)=1$, $p(t)=1/2$, $h(t)=h_0t$, $g(t)=g_0t$, $q(t)=q_0/t^2$, $F(u)=u$*, and* $\psi(u)=1/\left(1+\sin^2u\right)$*. It is easy to verify that*

$$m=\frac{1}{2}\leq\psi(t)\leq 1=M,$$

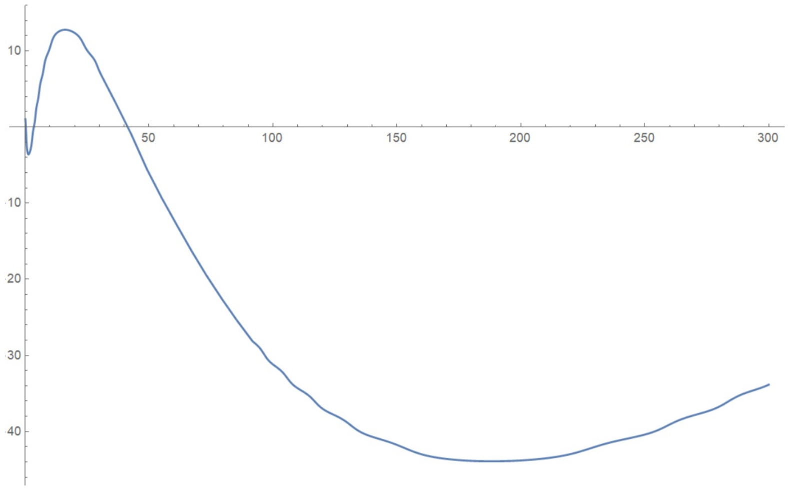

Figure 1. The numerical solution of (2.21) when $h_0 = g_0 = 1$ and $q_0 = 2$.

$\kappa = 2$, $A_{t_0}(\infty) = \infty$, $h^{[r]}(t) = h_0^r t$ for $r = 0, 1, ...$,

$$\begin{aligned} G_q(t;m) &= \frac{q_0}{t^2}\left(\sum_{r=0}^{m} h_0^{4r}\left(\prod_{l=0}^{2r}\left(\frac{1}{2}\right)\right)\right) \\ &= \frac{q_0}{2t^2}\sum_{r=0}^{m}\left(\frac{h_0^2}{2}\right)^{2r}. \end{aligned}$$

Now, we have

$$\begin{aligned} \frac{1}{[A_{t_1}(t)]^{2\varrho+1}}\mathcal{T}\left[G_q;t,l\right] &= \frac{1}{[t-t_0]^{2\varrho+1}}\int_l^t G_q(s;m)\varphi(t,s,l)\,\mathrm{d}s \\ &= \frac{q_0}{2[t-t_0]^{2\varrho+1}}\sum_{r=0}^{m}\left(\frac{h_0^2}{2}\right)^{2r}\int_l^t \frac{1}{s^2}(t-s)^{2\varrho}(s-l)^2\,\mathrm{d}s. \end{aligned}$$

Hence, condition (2.20) becomes

$$\frac{q_0}{2(1+2\varrho)}\sum_{r=0}^{m}\left(\frac{h_0^2}{2}\right)^{2r} > \frac{1}{g_0}\frac{\varrho}{4\varrho^2-1}.$$

Then, by using Corollary 4, all solutions of (2.21) are oscillatory if

$$q_0\sum_{r=0}^{m}\left(\frac{h_0^2}{2}\right)^{2r} > \frac{1}{g_0}. \tag{2.22}$$

Remark 1. *Using Theorem 1 with $H(t,s) = (t-s)^2$ and $\rho(t) = t^2$, we obtain that all solutions of (2.21) are oscillatory if*

$$q_0 > \frac{2}{g_0}. \tag{2.23}$$

Applying to the special case when $g_0 = 0.5$ and $h_0 = 0.9$, we notice that criteria (2.22) and (2.23) lead to $q_0 > 1.672$ and $q_0 > 4$, respectively. Therefore, our results improve results in [18]. For example, we find that our results guarantee that all solutions of equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1+\sin^2(x(t))}\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)+\frac{1}{2}x\left(\frac{t}{2}\right)\right]\right)+\frac{2}{t^2}x\left(\frac{t}{2}\right)=0$$

oscillate while the criteria of [18] do not apply ($2 = q_0 \not> \frac{2}{g_0} = 4$).
Figure 1 shows one of the numerical solutions to Eq (2.21).

Example 2. *Consider the delay equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1+\mathbf{e}^{-x^2(t)}}{\mathbf{e}^t}\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)+p_0x(t-h_0)\right]\right)+q_0\mathbf{e}^{-t}x(t-g_0)=0, \tag{2.24}$$

where h_0, g_0, and q_0 are positive. We note that $\alpha = 1$, $a(t) = \mathbf{e}^{-t}$, $p(t) = p_0$, $h(t) = t - h_0$, $g(t) = t - g_0$, $q(t) = q_0\mathbf{e}^{-t}$, $F(u) = u$, and $\psi(u) = 1 + \mathbf{e}^{-u^2}$. It is easy to verify that $m = 1$, $M = 2$, $\kappa = 2$,

$$G_q(t;m) = q_0\mathbf{e}^{-t}[1-p_0]\sum_{r=0}^{m} p_0^{2r}\mathbf{e}^{-4rh_0}.$$

By choosing $\eta(t) = \upsilon(t) = e^t$ and $\varrho = k = 2$, we obtain

$$H(t,s,l) = \mathbf{e}^{g_0}\left[2+\frac{2t-4s+2l}{(t-s)(s-l)}\right]^2.$$

Thus,

$$\begin{aligned}
&\limsup_{t\to\infty}\int_l^t \mathbf{e}^s(t-s)^2(s-l)^2\\
&\quad\times\left[q_0[1-p_0]\sum_{r=0}^{m}p_0^{2r}\mathbf{e}^{-4rh_0}-\frac{1}{2}\mathbf{e}^{g_0}\left[2+\frac{2t-4s+2l}{(t-s)(s-l)}\right]^2\right]\mathrm{d}s\\
=&\left(q_0[1-p_0]\sum_{r=0}^{m}p_0^{2r}\mathbf{e}^{-4rh_0}-\mathbf{e}^{g_0}\right)(+\infty),
\end{aligned}$$

which is fulfilled if

$$q_0[1-p_0]\sum_{r=0}^{m}p_0^{2r}\mathbf{e}^{-4rh_0} > \mathbf{e}^{g_0}. \tag{2.25}$$

Then, by using Corollary 3, all solutions of (2.24) are oscillatory if (2.25) holds.

Remark 2. *Recent results in papers [26–30] provided many improved criteria that test the oscillatory characteristics of second-order neutral DEs. However, these results fail to apply to Eqs (2.21) and (2.24), because these results only apply in the case of $\psi(u) = u$. Figure 2 shows one of the numerical solutions to Eq (2.24).*

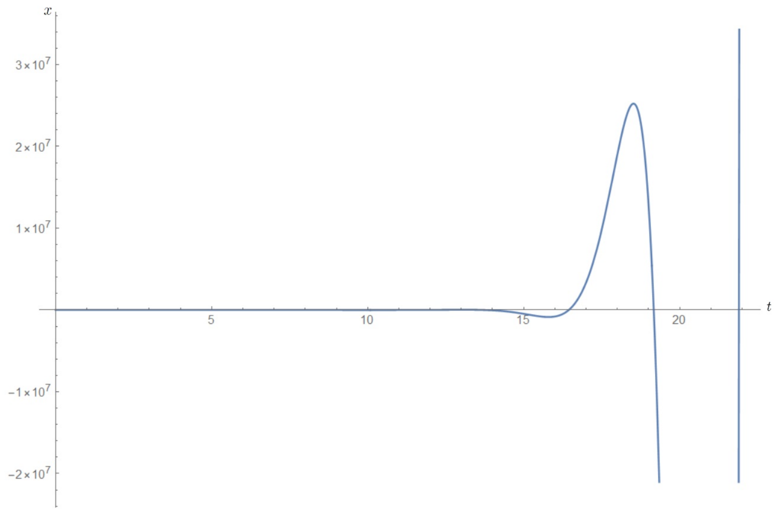

Figure 2. The numerical solution of (2.24) when $h_0 = g_0 = 1$, $p_0 = 1/2$, and $q_0 = e^5$.

3. Conclusions

The investigation into the oscillatory behavior of FDEs is affected by the accuracy of the relationships and inequalities used. In this article, we studied the oscillations of solutions of the class of FDEs of the neutral type (1.1). As an extension of the results in [21], we have derived a novel relation between x and z. We used the Riccati approach to couple the studied equation with an inequality of the Riccati type. Then, we presented Kamenev-type criteria that ensure the oscillation of (1.1).

Our results—as shown in Remark 2—have the advantage of being applied to a more general class of second-order FDEs of the neutral type compared to the results in [26–28]. Our results also presented more sharp conditions in the oscillation test than the results that dealt with the same equation (see Remark 1).

It would be of interest to formally extend our findings to the noncanonical case ($A_{t_*}(\infty) < \infty$). Also, an interesting point, as a future work, is to obtain an improved relation between x and z without the need for the constraint $\psi(u) \geq m > 0$, which excludes a large class of bounded functions such as $\sin^2 u$, $\mathbf{e}^{-u^2}$ and $\frac{1}{1+u^2}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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