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*Research article*

# Representation rings of extensions of Hopf algebra of Kac-Paljutkin type

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**Abstract:** In this paper, we focus on studying two classes of finite dimensional Δ-associative algebras, which are extensions of a family of $2n^2$-dimensional Kac-Paljutkin type semi-simple Hopf algebras $H_{2n^2}$. All their indecomposable modules are classified. Furthermore, their representation rings are described by generators with suitable relations.

**Keywords:** Δ-associative algebra; tensor product; representation ring

## 1. Introduction

As an invariant of a finite tensor category, the representation ring plays a very important role, having gained great attention in recent years. For example, Chen et al. [1] constructed the representation rings of Taft algebras $H_n(q)$. Wang et al. studied the Green rings of finite-dimensional pointed rank-one Hopf algebras of nilpotent and non-nilpotent types in [2, 3]. It should be noted that the small quantum group [4, 5] and the Drinfeld double of a Taft algebra [6, 7] are not of finite representation type. Their representation rings are not finitely generated, as described by [8] and [9, 10], respectively. In [11, 12], the authors considered the Grothendieck rings of the quotient algebras of Wu-Liu-Ding algebras (see [13, 14] for definitions), and provided their Casimir numbers, which are a class of non-pointed semi-simple Hopf algebras. The representations and Grothendieck rings of the Hopf algebra $H_{2n^2}$ are described in [15]. Furthermore, Guo and Yang [16] explicitly described the Grothendieck rings of the category of Yetter-Drinfeld modules over $H_{2n^2}$ by generators and relations. Sun et al. [17] described the structure of the representation ring of the small quasi-quantum group. For more results on representations and representation rings, one can refer to [18–25].

In order to understand and extend the concept of representation rings, we can weaken the definition of weak Hopf algebras to more general cases. The most interesting one is the so-called Δ-associative algebra, which was introduced in [26], where the representation ring of Δ-associative algebra was defined and described for Kac-Paljutkin Hopf algebra $K_8$(i.e., $H_8$). One consideration is that the representation rings of Δ-associative algebras may not be commutative and may not even contain the identity (see [26]). In this paper, we provide two classes of Δ-associative algebras to understand them. Roughly speaking, based on Kac-Paljutkin-type Hopf algebras $H_{2n^2}$ (see [27] or [15]), which are generalizations of the 8-dimensional Kac-Paljutkin Hopf algebra, we weaken the definition of Hopf algebra $H_{2n^2}$ to obtain two classes of Δ-associative algebras, $\overline{H}_{2n^2}$ and $\widehat{H}_{2n^2}$. One can see that $\overline{H}_{2n^2}$ is still a weak Hopf algebra but $\widehat{H}_{2n^2}$ is a non-trivial Δ-associative algebra. Following that, we describe their representations as well as their representation rings. We show that the representation ring $r(\overline{H}_{2n^2})$ is a commutative ring, but the representation ring $r(\widehat{H}_{2n^2})$ is noncommutative with no identity. However, we can embed the ring $r(\widehat{H}_{2n^2})$ into a ring $r^*(\widehat{H}_{2n^2})$ with an identity in the natural way. The rings $r(\overline{H}_{2n^2})$ and $r^*(\widehat{H}_{2n^2})$ are described by generators with suitable relations.

The paper is organized as follows. In Section 2, we recall the definition of Δ-associative algebra, and review the definition and representations of the semi-simple Hopf algebra $H_{2n^2}$. In Section 3, by weakening the definition of Hopf algebra $H_{2n^2}$, we obtain two classes of Δ-associative algebras $\overline{H}_{2n^2}$ and $\widehat{H}_{2n^2}$, where $\overline{H}_{2n^2}$ is a weak Hopf algebra, and $\widehat{H}_{2n^2}$ is just a non-trivial Δ-associative algebra. Some properties of $\overline{H}_{2n^2}$ and $\widehat{H}_{2n^2}$ are discussed. In Section 4, all irreducible modules of $\overline{H}_{2n^2}$ are listed, and there are $(n^2+7n+2)/2$ non-isomorphic finite dimensional irreducible modules for $\overline{H}_{2n^2}$. The decomposition formulas of the tensor product of two arbitrary irreducible $\overline{H}_{2n^2}$-modules are also established. The representation ring $r(\overline{H}_{2n^2})$, described by generators and relations, shows that $r(\overline{H}_{2n^2})$ is a commutative ring. In Section 5, the representations and representation ring of $\widehat{H}_{2n^2}$ are studied. We found that the representation ring $r(\widehat{H}_{2n^2})$ is a noncommutative ring with no identity, which can be embedded into a ring $r^*(\widehat{H}_{2n^2})$ with an identity.

## 2. Preliminaries

Throughout this research, we work over a fixed algebraically closed field $\Bbbk$ of characteristic 0, unless otherwise stated. All algebras, Hopf algebras, and modules are defined over $\Bbbk$; all modules are left modules and finite dimensional; all maps are $\Bbbk$-linearity; dim, $\otimes$, and hom stand for $\dim_{\Bbbk}$, $\otimes_{\Bbbk}$, and $\hom_{\Bbbk}$, respectively.

The definition of weak Hopf algebra was introduced by Li (see [28]). We recall that a $\Bbbk$-bialgebra $(H,\mu,\eta,\Delta,\varepsilon)$ is called a weak Hopf algebra if there exists a map $T\in\hom(H,H)$ such that $T*\mathrm{id}*T=T$ and $\mathrm{id}*T*\mathrm{id}=\mathrm{id}$, where $*$ is the convolution map in $\hom(H,H)$. By weakening the definition of weak Hopf algebras, the following definition is established.

**Definition 2.1.** *An associative $\Bbbk$-algebra $A$ with an identity is called a* **Δ-associative algebra** *if there exists an algebra homomorphism $\Delta: A\to A\otimes A$ such that $(\Delta\otimes\mathrm{id})\Delta=(\mathrm{id}\otimes\Delta)\Delta$.*

All Hopf algebras, bialgebras, and weak Hopf algebras are Δ-associative algebras. If the Δ-associative algebra $A$ is not a coalgebra, $A$ is said to be non-trivial.

In the sequel, we always assume that $A$ is a Δ-associative algebra, and the Sweedler's notations [29] are used. For example, for $a\in A$, we denote

$$\Delta(a)=\sum_{(a)}a_{(1)}\otimes a_{(2)}.$$

Now, suppose that $M$ and $N$ are two $A$-modules, then $M\otimes N$ is also an $A$-module defined as follows:

$$a\cdot(m\otimes n)=\sum_{(a)}a_{(1)}m\otimes a_{(2)}n,$$

for all $m\in M, n\in N$.

We denote by $[M]$ the isomorphism class of an $A$-module $M$ and

$$\mathscr{P}=\{[M]|M\in \text{mod-}A\}.$$

Let $R(A)$ be a free abelian group spanned by $\mathscr{P}$. For all $[M],[N]\in\mathscr{P}$, we define $[M][N]=[M\otimes N]$, and it is easy to see that $R(A)$ is a ring. Let

$$r(A)=\frac{R(A)}{\langle [M\oplus N]-[M]-[N]\rangle},$$

where $\langle [M\oplus N]-[M]-[N]\rangle$ is the ideal of $R(A)$ generated by $[M\oplus N]-[M]-[N]$ for all $[M],[N]\in\mathscr{P}$, then $r(A)$ is also a ring and is called the representation ring or the Green ring of $A$. If $A$ is a non-trivial $\Delta$-associative algebra, then $r(A)$ may have no identity $[\Bbbk_{\varepsilon}]$, the trivial 1-dimensional $A$-module. In this case, let

$$r^*(A)=\{(k,\alpha)|k\in\Bbbk,\alpha\in r(A)\}.$$

For $(k,\alpha)$, $(k',\alpha')\in r^*(A)$, we define the addition and multiplication in $r^*(A)$ as the following

$$(k,\alpha)+(k',\alpha')=(k+k',\alpha+\alpha'),$$

$$(k,\alpha)\cdot(k',\alpha')=(kk',k'\alpha+k\alpha'+\alpha\alpha').$$

Then, $r^*(A)$ is a ring with the identity $(1,0)$, and $r(A)$ is embedded into $r^*(A)$ naturally.

In the sequel, we always assume that $n>1$ and $q$ is a primitive $n$-th root of unity.

The Hopf algebra $H_{2n^2}$ can be found in [15, 27], which is a generalization of Kac-Paljutkin Hopf algebra.

**Definition 2.2.** *The* **Hopf algebra** $H_{2n^2}$ *is an associative algebra generated by* $x, y$ *and* $z$*, with the following relations*

$$x^n=1,\ y^n=1,\ xy=yx,\ zx=yz,\ zy=xz,$$
$$z^2=\frac{1}{n}\sum_{i,j=0}^{n-1}q^{-ij}x^iy^j.$$

*The comultiplication, counit, and antipode are as follows*

$$\begin{aligned}
&\Delta(x)=x\otimes x, && \varepsilon(x)=1,\quad S(x)=x^{-1},\\
&\Delta(y)=y\otimes y, && \varepsilon(y)=1,\quad S(y)=y^{-1},\\
&\Delta(z)=\tfrac{1}{n}\sum_{i,j=0}^{n-1}q^{-ij}x^iz\otimes y^jz, && \varepsilon(z)=1,\quad S(z)=z.
\end{aligned}$$

The Hopf algebra $H_{2n^2}$ is a $2n^2$-dimensional semi-simple Hopf algebra. It is of a basis $\{x^iy^j,\ x^iy^jz|0\leq i,j\leq n-1\}$. Indeed,

$$\ell=\left(\sum_{i=0}^{n-1}x^i\right)\left(\sum_{j=0}^{n-1}y^j\right)(1+z)$$

is the left and right integral of $H_{2n^2}$ and $\varepsilon(\ell)=2n^2\neq 0$. It is also a quasi-triangular Hopf algebra with a universal $R$-matrix

$$R=\frac{1}{n}\sum_{i,j=0}^{n-1}q^{-ij}x^j\otimes y^{n-i}.$$

Therefore, the representation ring of $H_{2n^2}$ is semi-simple and commutative. The representations and representation ring of the Hopf algebra $H_{2n^2}$ are described in [15]. We list them as follows, where $q$ is any primitive $n$-th root of unity, but $q=e^{2p\pi i/n}$ with an even number $p\in\mathbb{Z}$ if $n$ is odd for convenience.

Set

$$\sigma(m)=\begin{cases}1, & 0\leq m\leq n-1,\\ -1, & n\leq m\leq 2n-1.\end{cases}$$

(a) 1-dimensional irreducible $H_{2n^2}$-module $S_m, m\in\mathbb{Z}_{2n}$: it is of basis $v^m$, and the actions of $H_{2n^2}$ on $S_m$ are

$$x.v^m=q^mv^m,\quad y.v^m=q^mv^m,\quad z.v^m=\sigma(m)q^{\frac{m^2}{2}}v^m;$$

(b) 2-dimensional irreducible $H_{2n^2}$-module $S_{i,j}, 0\leq i<j\leq n-1$: it is of basis $v_1^{ij}$ and $v_2^{ij}$, and the actions of $H_{2n^2}$ on $S_{i,j}$ are

$$\begin{array}{lll}x.v_1^{ij}=q^iv_1^{ij}, & y.v_1^{ij}=q^jv_1^{ij}, & z.v_1^{ij}=v_2^{ij},\\ x.v_2^{ij}=q^jv_2^{ij}, & y.v_2^{ij}=q^iv_2^{ij}, & z.v_2^{ij}=q^{ij}v_1^{ij}.\end{array}$$

The set

$$\mathscr{S}=\{S_m,S_{i,j}\mid m\in\mathbb{Z}_{2n},0\leq i<j\leq n-1\}$$

forms a complete list of non-isomorphic irreducible $H_{2n^2}$-modules.

**Lemma 2.3.** [15, Theorem 1] *The decomposition formulas of tensor product of two $H_{2n^2}$-modules in $\mathscr{S}$ are as follows.*

*(1) (a) For all $0\leq m,m'\leq n-1$, we have*

$$S_m\otimes S_{m'}\cong S_{m+m'(\text{mod } n)};$$

*(b) for all $0\leq m\leq n-1, n\leq m'\leq 2n-1$ and $n\leq m+m'\leq 2n-1$, or $0\leq m'\leq n-1, n\leq m\leq 2n-1$ and $n\leq m'+m\leq 2n-1$, we have*

$$S_m\otimes S_{m'}\cong S_{m+m'};$$

*(c) for all $0\leq m\leq n-1, n\leq m'\leq 2n-1$ and $2n\leq m+m'\leq 3n-1$, or $0\leq m'\leq n-1, n\leq m\leq 2n-1$ and $2n\leq m'+m\leq 3n-1$, we have*

$$S_m\otimes S_{m'}\cong S_{m+m'-n};$$

(d) *for all* $n \le m, m' \le 2n-1$ *and* $2n \le m+m' \le 3n-1$*, we have*

$$S_m \otimes S_{m'} \cong S_{m+m'(\text{mod } 2n)};$$

(e) *for all* $n \le m, m' \le 2n-1$ *and* $3n \le m+m' \le 4n-1$*, we have*

$$S_m \otimes S_{m'} \cong S_{m+m'(\text{mod } 3n)}.$$

(2) *For all* $m \in \mathbb{Z}_{2n}, 0 \le i < j \le n-1$*, we have*

$$S_m \otimes S_{i,j} \cong S_{m+i(\text{mod } n), j+m(\text{mod } n)} \cong S_{i,j} \otimes S_m.$$

(3) *Set* $I_1 = \{0 \le i < j, k < l < n \mid i+k \equiv j+l (\text{mod } n)\}$ *and* $I_2 = \{0 \le i < j, k < l < n \mid i+l \equiv j+k(\text{mod } n)\}$*, then*

(a) *if* $i,j,k,l \notin I_1 \cup I_2$*, we have*

$$S_{i,j} \otimes S_{k,l} \cong S_{i+k(\text{mod } n), j+l(\text{mod } n)} \oplus S_{i+l(\text{mod } n), j+k(\text{mod } n)};$$

(b) *if* $i,j,k,l \in I_1 \backslash I_2$*, we have*

$$S_{i,j} \otimes S_{k,l} \cong S_{i+k} \oplus S_{j+l} \oplus S_{i+l(\text{mod } n), j+k(\text{mod } n)};$$

(c) *if* $i,j,k,l \in I_2 \backslash I_1$*, we have*

$$S_{i,j} \otimes S_{k,l} \cong S_{i+k(\text{mod } n), j+l(\text{mod } n)} \oplus S_{i+l(\text{mod } n)} \oplus S_{j+k(\text{mod } n)+n};$$

(d) *if* $n$ *is even, and* $i,j,k,l \in I_1 \cap I_2$*, we have*

$$S_{i,j} \otimes S_{k,l} \cong S_{i+k} \oplus S_{j+l} \oplus S_{i+l(\text{mod } n)} \oplus S_{j+k(\text{mod } n)+n}.$$

Let

$$A_l(y,z,u) = \begin{bmatrix} z & uy & 0 & \cdots & 0 & 0 \\ 1 & z & y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & y \\ 0 & 0 & 0 & \cdots & 1 & z \end{bmatrix}_{l \times l}$$

and

$$B_l(y,z) = \det\left(A_l(y,z,1+y^n)\right), \;\; D_l(y,z,x) = \det\left(A_l(y,z,1+y^{2m-1}x)\right).$$

Then, we have

$$B_0(y,z) = 1+y^n, \quad B_1(y,z) = z, \quad B_2(y,z) = z^2 - y - y^{n+1};$$

$$D_0(y,z,x) = 1+y^{2m-1}x, \quad D_1(y,z,x) = z, \quad D_2(y,z,x) = z^2 - y - y^{2m}x.$$

[15, Theorem 2, Theorem 3] can be stated as follows.

**Lemma 2.4.** *Assume that* $n \ge 2$ *is an integer, we have*

*(1) if $n \geq 3$ is odd and $m = \frac{n-1}{2}$, then*

$$r(H_{2n^2}) \cong \mathbb{Z}[y,z]/\langle y^{2n}-1, \quad zy^n - z, \quad B_{m+1}(y,z) - y^{m+1}B_m(y,z)\rangle;$$

*(2) if $n = 2$, then*

$$r(H_{2n^2}) \cong \mathbb{Z}[x,y,z]/\langle\ y^2-1, \quad x^2-y^2, \quad zx-zy, \quad z-zy, \quad z^2-x-y-xy-1\ \rangle;$$

*(3) if $n > 2$ is even with $m = \frac{n}{2}$, then*

$$r(H_{2n^2}) \cong \mathbb{Z}[x,y,z] \Big/ \left\langle \begin{array}{ll} x^n-1, & D_{m+1}(y,z,x) - y^{m+1}D_{m-1}(y,z,x), \\ x^2-y^2, & zx-zy, \quad D_m(y,z,x) - y^m D_m(y,z,x) \end{array} \right\rangle.$$

## 3. Extensions of Hopf algebra of Kac-Paljutkin type

In this section, we establish two classes of extensions of Hopf algebra $H_{2n^2}$ of Kac-Paljutkin type, which are denoted by $\overline{H}_{2n^2}$ and $\widehat{H}_{2n^2}$, where $\overline{H}_{2n^2}$ is a weak Hopf algebra in the sense of Li in [28], but $\widehat{H}_{2n^2}$ is a non-trivial $\Delta$-associative algebra.

**Definition 3.1.** *Let $\overline{H}_{2n^2}$ be an associative algebra generated by $x, y$, and $z$, with the following relations*

$$\begin{aligned} &x^{n+1} = x,\ y^{n+1} = y,\ xy = yx, \\ &z = yzx^{n-1},\ z = xzy^{n-1}, \\ &z^2 = \frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^i y^j. \end{aligned}$$

For convenience, we set

$$e_i = \frac{1}{n}\sum_{l=1}^{n} q^{-il}x^l, \quad f_i = \frac{1}{n}\sum_{k=1}^{n} q^{-ik}y^k,$$

for $i = 1, 2, \cdots, n$, then

$$x^s e_i = q^{is}e_i,\ y^t e_i = e_i y^t,\ x^s f_i = f_i x^s,\ y^t f_i = q^{it} f_i,$$

for $s, t = 1, 2, \cdots, n$, and

$$z^2 = \sum_{j=1}^{n} e_j y^j = \sum_{i=1}^{n} x^i f_i.$$

We also have

$$\Delta(z) = \left(\sum_{j=1}^{n} e_j \otimes y^j\right)(z \otimes z) = \left(\sum_{i=1}^{n} x^i \otimes f_i\right)(z \otimes z).$$

Let $J_1 = x^n, J_2 = y^n$ and $J = J_1 J_2$.

The following lemma helps us to understand the deep relations between the algebra $\overline{H}_{2n^2}$ and the algebra $H_{2n^2}$.

**Lemma 3.2.** $z = yzx^{n-1},\ z = xzy^{n-1}$ *if and only if* $zJ = Jz = z,\ zx = yz,\ zy = xz.$

*Proof.* Indeed, assume that $z = yzx^{n-1}$, $z = xzy^{n-1}$, then

$$z = y^n z x^{n(n-1)} = J_2 z J_1 \quad \text{and} \quad z = x^n z y^{n(n-1)} = J_1 z J_2.$$

It follows that

$$z = J_1 z J_2 = J_1 J_2 z J_1 J_2 = JzJ$$

and $zJ = Jz = z$. Also,

$$zx = yzx^n = yzJ_1 = y(J_2 z J_1) = yz, \; zy = xzy^n = xzJ_2 = x(J_1 z J_2) = xz.$$

Conversely, if $zJ = Jz = z$ and $xz = zy, \; yz = zx$, then

$$zJ_i = zJJ_i = zJ = z = Jz = J_i Jz = J_i z$$

for $i = 1, 2$. So we have

$$z = zJ_2 = zy^n = xzy^{n-1}, \; z = zJ_1 = zx^n = yzx^{n-1}.$$

The result follows. □

It is easy to see that

$$\{J, J_1 - J, J_2 - J, 1 + J - J_1 - J_2\}$$

is a set of orthogonal central idempotents of $\overline{H}_{2n^2}$.

Now, we define three maps $\Delta : \overline{H}_{2n^2} \to \overline{H}_{2n^2} \otimes \overline{H}_{2n^2}$, $\varepsilon : \overline{H}_{2n^2} \to \Bbbk$, and $T : \overline{H}_{2n^2} \to \overline{H}_{2n^2}$ as follows

$$\begin{array}{lll} \Delta(x) = x \otimes x, & \varepsilon(x) = 1, & T(x) = x^{n-1}, \\ \Delta(y) = y \otimes y, & \varepsilon(y) = 1, & T(y) = y^{n-1}, \\ \Delta(z) = \frac{1}{n} \sum\limits_{i,j=1}^{n} q^{-ij} x^i z \otimes y^j z, & \varepsilon(z) = 1, & T(z) = z. \end{array}$$

We get the first main result as follows.

**Theorem 3.3.** *$\overline{H}_{2n^2}$ is a weak Hopf algebra with the weak antipode $T$.*

*Proof.* We will prove this theorem in three steps.

a) The $\Bbbk$-map $\Delta$ keeps the defining relations, hence it can be extended into the whole algebra $\overline{H}_{2n^2}$ such that $\Delta$ is an algebra homomorphism. Indeed, we note that

$$\begin{aligned} e_i &= \frac{1}{n} \sum_{l=1}^{n} q^{-il} x^l, \quad f_i = \frac{1}{n} \sum_{k=1}^{n} q^{-jk} y^k, \text{ for } i = 1, 2, \cdots, n, \\ z^2 &= \sum_{j=1}^{n} e_j y^j = \sum_{i=1}^{n} x^i f_i, \\ \Delta(z) &= \frac{1}{n} \sum_{i,j=1}^{n} q^{-ij} x^i z \otimes y^j z = \left( \sum_{j=1}^{n} e_j \otimes y^j \right) (z \otimes z) = \left( \sum_{i=1}^{n} x^i \otimes f_i \right) (z \otimes z). \end{aligned}$$

So it is easy to see that

$$\begin{aligned}
(\Delta(x))^{n+1} &= \Delta(x), \quad (\Delta(y))^{n+1} = \Delta(y), \quad \Delta(x)\Delta(y) = \Delta(y)\Delta(x),\\
\Delta(y)\Delta(z)\Delta(x)^{n-1} &= (y\otimes y)\left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz\right)(x^{n-1}\otimes x^{n-1})\\
&= \frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iyzx^{n-1}\otimes y^jyzx^{n-1}\\
&= \frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz = \Delta(z),\\
\Delta(x)\Delta(z)\Delta(y)^{n-1} &= (x\otimes x)\left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz\right)(y^{n-1}\otimes y^{n-1})\\
&= \frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^ixzy^{n-1}\otimes y^jxzy^{n-1}\\
&= \frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz = \Delta(z),\\
(\Delta(z))^2 &= \left(\sum_{j=1}^{n} e_j\otimes y^j\right)(z\otimes z)\left(\sum_{i=1}^{n} x^i\otimes f_i\right)(z\otimes z)\\
&= \sum_{i,j=1}^{n} e_jy^{i+j}\otimes e_iy^{i+j} = \frac{1}{n^2}\sum_{i,j,l,k=1}^{n} q^{-il-jk}x^ly^{i+j}\otimes x^ky^{i+j}\\
&= \frac{1}{n^2}\sum_{i,j,l,k=1}^{n} q^{-jl+i(l-k)}x^ly^j\otimes x^ky^j = \frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iy^j\otimes x^iy^j\\
&= \frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}\Delta(x)^i\Delta(y)^j.
\end{aligned}$$

Hence, $\Delta$ can define a homomorphism of algebra. Similarly, $\varepsilon$ can also define a homomorphism of algebra.

b) $(\overline{H}_{2n^2}, \Delta, \varepsilon)$ is a coalgebra. Indeed, the $\Bbbk$-map $\Delta$ is coassociative. To see this fact, we have

$$(\Delta\otimes \mathrm{id})\Delta(x) = x\otimes x\otimes x = (\mathrm{id}\otimes\Delta)\Delta(x),$$

$$(\Delta\otimes \mathrm{id})\Delta(y) = y\otimes y\otimes y = (\mathrm{id}\otimes\Delta)\Delta(y),$$

and

$$\begin{aligned}
(\Delta\otimes \mathrm{id})\Delta(z) &= (\Delta\otimes\mathrm{id})\left(\sum_{i=1}^{n} x^iz\otimes f_iz\right)\\
&\overset{\text{definition and (a)}}{=\!=\!=} \sum_{i,j=1}^{n} x^ie_jz\otimes x^iy^jz\otimes f_iz = \sum_{i,j=1}^{n} q^{ij}e_jz\otimes x^iy^jz\otimes f_iz,\\
&= (\mathrm{id}\otimes\Delta)\left(\sum_{j=1}^{n} e_jz\otimes y^jz\right) = (\mathrm{id}\otimes\Delta)\Delta(z).
\end{aligned}$$

Hence,

$$(\Delta\otimes \mathrm{id})\Delta(a)=(\mathrm{id}\otimes\Delta)\Delta(a)$$

for $a\in\{x,y,z\}$. Then, the map $\Delta$ is coassociative in $\overline{H}_{2n^2}$ by the statement a). Furthermore, $\varepsilon$ also satisfies the counit axiom.

By the statements a) and b), we see that $\overline{H}_{2n^2}$ is a bialgebra.

c) The map $T$ can define a weak antipode of $\overline{H}_{2n^2}$ in a natural way. To see this fact, we note that

$$\begin{aligned}
(T(x))^{n+1} &= ((x)^{n-1})^{n+1}=x^{n-1}=T(x);\\
(T(y))^{n+1} &= ((y)^{n-1})^{n+1}=y^{n-1}=T(y);\\
T(y)T(x) &= (y)^{n-1}(x)^{n-1}=(x)^{n-1}(y)^{n-1}=T(x)T(y);\\
(T(x))^{n-1}T(z)T(y) &= ((x)^{n-1})^{n-1}z(y)^{n-1}=J_1z=z=T(z);\\
(T(y))^{n-1}T(z)T(x) &= ((y)^{n-1})^{n-1}z(x)^{n-1}=J_2z=z=T(z);\\
\frac{1}{n}\sum_{i,j=1}^{n}q^{-ij}(T(y))^j(T(x))^i &= \frac{1}{n}\sum_{i,j=1}^{n}q^{-ij}y^{(n-1)j}x^{(n-1)i}=\frac{1}{n}\sum_{i,j=1}^{n}q^{-ij}x^iy^j=z^2=(T(z))^2.
\end{aligned}$$

Thus, the map $T$ can be extended into an anti-algebra homomorphism $T:\overline{H}_{2n^2}\to\overline{H}_{2n^2}$.

Furthermore, it is easy to see that

$$\begin{aligned}
&(\mathrm{id}*T*\mathrm{id})(x)=xT(x)x=x^{n+1}=x=\mathrm{id}(x),\\
&(T*\mathrm{id}*T)(x)=T(x)xT(x)=x^{n-1}=T(x),\\
&(\mathrm{id}*T*\mathrm{id})(y)=yT(y)y=y^{n+1}=y=\mathrm{id}(y),\\
&(T*\mathrm{id}*T)(y)=T(y)yT(y)=y^{n-1}=T(y),\\
&(\mathrm{id}*T*\mathrm{id})(z)=\sum_{i,j=1}^{n}q^{ij}e_jzT(x^iy^jz)f_iz=\sum_{i,j=1}^{n}q^{ij}e_jz^2y^{(n-1)j}x^{(n-1)i}f_iz\\
&\qquad=\sum_{i,j=1}^{n}q^{-ij}e_j(\sum_{k=1}^{n}e_ky^k)f_iz=\sum_{i,j=1}^{n}e_jf_iz=Jz=z=\mathrm{id}(z),\\
&(T*\mathrm{id}*T)(z)=\sum_{i,j=1}^{n}q^{ij}T(e_jz)x^iy^jzT(f_iz)=\sum_{i,j=1}^{n}q^{-ij}ze_{(n-1)j}z^2f_{(n-1)i}\\
&\qquad=\sum_{i,j=1}^{n}q^{-ij}ze_{(n-1)j}(\sum_{k=1}^{n}e_ky^k)f_{(n-1)i}=z\sum_{i,j=1}^{n}e_jf_i=Jz=z=T(z).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
&\sum_{(x)}T(x_{(1)})x_{(2)}=J_1=\sum_{(x)}x_{(1)}T(x_{(2)}),\\
&\sum_{(y)}T(y_{(1)})y_{(2)}=J_2=\sum_{(y)}y_{(1)}T(y_{(2)}),\\
&\sum_{(z)}T(z_{(1)})z_{(2)}=\sum_{i=1}^{n}T(x^iz)f_iz=\sum_{i=1}^{n}y^{n-i}e_iz^2=\sum_{i=1}^{n}y^ne_i=J_1J_2=J,
\end{aligned}$$

$$\sum_{(z)} z_{(1)}T(z_{(2)}) = \sum_{j=1}^{n} e_j z T(y^j z) = \sum_{j=1}^{n} e_j z^2 y^{n-j} = \sum_{i=1}^{n} e_j y^n = J_1 J_2 = J.$$

The elements $J_1, J_2$ and $J$ are all the center of $\overline{H}_{2n^2}$ by Lemma 3.2. Hence,

$$\sum_{(a)} a_{(1)}T(a_{(2)}), \sum_{(a)} T(a_{(1)})a_{(2)} \in C(\overline{H}_{2n^2}),$$

the center of $\overline{H}_{2n^2}$, for all $a \in \overline{H}_{2n^2}$.

The above facts show that

$$\sum_{(a)} a_{(1)}T(a_{(2)})a_{(3)} = a = \mathrm{id}(a), \quad \sum_{(a)} T(a_{(1)})a_{(2)}T(a_{(3)}) = T(a)$$

for all $a \in \overline{H}_{2n^2}$ by induction.

This means that $T$ can indeed define a weak antipode of $\overline{H}_{2n^2}$, and hence $\overline{H}_{2n^2}$ is a weak Hopf algebra. Here, we point that $\overline{H}_{2n^2}$ is a noncommutative and noncocommutative weak Hopf algebra. □

Now, we assume that

$$J\overline{H}_{2n^2} = \overline{A}_0,\ (J_1 - J)\overline{H}_{2n^2} = \overline{A}_1,\ (J_2 - J)\overline{H}_{2n^2} = \overline{A}_2,\ (1 + J - J_1 - J_2)\overline{H}_{2n^2} = \overline{A}_3.$$

One can get the following result.

**Proposition 3.4.** *As algebras, we have*

*(1)* $\overline{H}_{2n^2} = \overline{A}_0 \oplus \overline{A}_1 \oplus \overline{A}_2 \oplus \overline{A}_3$;
*(2)* $\overline{A}_0 \cong H_{2n^2}$;
*(3)* $\overline{A}_1 \cong \overline{A}_2 \cong \Bbbk[g]/(g^n - 1)$;
*(4)* $\overline{A}_3 \cong \Bbbk$.

*Proof.* (1) The first statement follows from the fact that $\{J, J_1 - J, J_2 - J, 1 + J - J_1 - J_2\}$ is a set of orthogonal central idempotents.

(2) For the statement, we note that $\overline{A}_0$ can be viewed as a subalgebra of $\overline{H}_{2n^2}$ with the identity element $J$, generated by $Jx$, $Jy$, and $Jz$ with the relations

$$\begin{aligned}
&(Jx)^n = J,\ (Jy)^n = J,\\
&(Jx)(Jy) = (Jy)(Jx),\ (Jz)(Jx) = (Jy)(Jz),\ (Jz)(Jy) = (Jx)(Jz),\\
&(Jz)^2 = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij}(Jx)^i(Jy)^j.
\end{aligned}$$

Let $\rho_0 : H_{2n^2} \to \overline{A}_0$ be the map defined by

$$\rho_0(1) = J, \quad \rho_0(x) = Jx, \quad \rho_0(y) = Jy \quad \rho_0(z) = Jz,$$

it is straightforward to see that $\rho_0$ is a well-defined surjective algebra homomorphism. Let $\phi : \overline{H}_{2n^2} \to H_{2n^2}$ be the map given by

$$\phi(1) = 1, \quad \phi(x) = x, \quad \phi(y) = y, \quad \phi(z) = z.$$

The map $\phi$ is also a well-defined algebra epimorphism. Considering the restriction $\phi|_{\overline{A}_0}$ of $\phi$ on $\overline{A}_0$, we have $\phi|_{\overline{A}_0} \circ \rho_0 = \mathrm{id}_{H_{2n^2}}$. Hence, $\rho_0$ is injective and $\overline{A}_0 \cong H_{2n^2}$ as algebras.

(3) We note that $\overline{A}_1$ and $\overline{A}_2$ are the subalgebras of $\overline{H}_{2n^2}$ with the unit $J_1 - J$ and $J_2 - J$, respectively. In $\overline{A}_1$, we have

$$(J_1 - J)y = J_1 y - J_1 y = 0,\ (J_1 - J)z = z - z = 0.$$

It follows that $\overline{A}_1$ is generated by $(J_1 - J)x$, with the relation

$$[(J_1 - J)x]^n = J_1 - J.$$

Similarly, the subalgebra $\overline{A}_2$ is generated by $(J_2 - J)y$, with the relations

$$[(J_2 - J)y]^n = J_2 - J,$$

and $(J_2 - J)x = J_2 x - J_2 x = 0,\ (J_2 - J)z = z - z = 0$. Finally, it is easy to see that

$$\overline{A}_1 \cong \overline{A}_2 \cong \Bbbk[g]/(g^n - 1).$$

(4) It is obvious that $\overline{A}_3$ is a one-dimensional subalgebra of $\overline{H}_{2n^2}$ with the unit $1 + J - J_1 - J_2$. Indeed,

$$(1 + J - J_1 - J_2)x = 0,\ (1 + J - J_1 - J_2)y = 0,\ (1 + J - J_1 - J_2)z = 0.$$

So, $\overline{A}_3 = \Bbbk(1 + J - J_1 - J_2)$.

The proof is completed. □

By Proposition 3.4, $\overline{H}_{2n^2}$ is a semi-simple algebra of dimension $2n^2 + 2n + 1$ with a $\Bbbk$-basis:

$$\left\{ (Jx)^i (Jy)^j,\ (Jx)^i (Jy)^j z,\ [(J_1 - J)x]^i,\ [(J_2 - J)y]^j, 1 + J - J_1 - J_2 \middle| 1 \le i, j \le n \right\}.$$

Now, we consider the second class of extensions of $H_{2n^2}$. The definition is given as follows.

**Definition 3.5.** *The associative algebra $\widehat{H}_{2n^2}$ is generated by $x, y$, and $z$, with the relations*

$$\begin{aligned} &x^{n+1} = x,\ y^{n+1} = y,\ xy = yx, \\ &zx = yz,\ zy = xz, \\ &z^2 = \frac{1}{n} \sum_{i,j=1}^{n} q^{-ij} x^i y^j. \end{aligned}$$

We define $\Delta : \widehat{H}_{2n^2} \to \widehat{H}_{2n^2} \otimes \widehat{H}_{2n^2}$, as follows

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{n} \sum_{i,j=1}^{n} q^{-ij} x^i z \otimes y^j z$$

and extend it to $\widehat{H}_{2n^2}$ in the natural way. As previously, let $J_1 = x^n, J_2 = y^n$ and $J = J_1 J_2$.

**Theorem 3.6.** *$\widehat{H}_{2n^2}$ is a $\Delta$-associative algebra.*

*Proof.* We should show that the $\Delta$ is an algebra homomorphism of $\widehat{H}_{2n^2}$. In fact,

$$\begin{aligned}(\Delta(x))^{n+1} &= \Delta(x), \quad (\Delta(y))^{n+1}=\Delta(y), \quad \Delta(x)\Delta(y)=\Delta(y)\Delta(x),\\ \Delta(y)\Delta(z) &= (y\otimes y)\left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz\right)\\ &= \left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz\right)(x\otimes x)=\Delta(z)\Delta(x),\\ \Delta(x)\Delta(z) &= (x\otimes x)\left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz\right)\\ &= \left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^iz\otimes y^jz\right)(y\otimes y)=\Delta(z)\Delta(y).\end{aligned}$$

Furthermore, we have that

$$(\Delta(z))^2=\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}\Delta(x)^i\Delta(y)^j$$

by the proof of Theorem 3.3 a). Hence, the map $\Delta$ can be extended to a homomorphism of algebra in $\widehat{H}_{2n^2}$. Also, $\Delta$ satisfies the coassociative axiom by the proof of Theorem 3.3 b). Hence, $\widehat{H}_{2n^2}$ is a $\Delta$-associative algebra. □

**Remark 3.7.** *The $\Delta$-associative algebra $\widehat{H}_{2n^2}$ may be non-trivial. Indeed, if there exists a $\Bbbk$-map $\varepsilon:\widehat{H}_{2n^2}\to\Bbbk$ :*

$$\varepsilon(x)=a,\ \varepsilon(y)=b,\ \varepsilon(z)=c,\quad a,b,c\in\Bbbk,$$

*such that $\varepsilon$ is a counit of $\widehat{H}_{2n^2}$. Then, we have*

$$x\varepsilon(x)=\varepsilon(x)x=x,\quad y\varepsilon(y)=\varepsilon(y)y=y,$$

*we get that $a=b=1$ and*

$$m\circ(\varepsilon\otimes\mathrm{id})\Delta(z)=c\left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}y^j\right)z=cJ_2z=cy^nz,$$

$$m\circ(\mathrm{id}\otimes\varepsilon)\Delta(z)=c\left(\frac{1}{n}\sum_{i,j=1}^{n} q^{-ij}x^i\right)z=cJ_1z=cx^nz,$$

*where $m$ is the multiplication of $\widehat{H}_{2n^2}$.*

*If there exists a $c\in\Bbbk$ such that $cJ_1z=cJ_2z=z$, then $\widehat{H}_{2n^2}$ is trivial. Otherwise, $\widehat{H}_{2n^2}$ is non-trivial.*

*Here, we always assume that $x$, $y$, and $z$ are freely without any additional conditions in the definition of $\widehat{H}_{2n^2}$. Consequently, in this case, $\widehat{H}_{2n^2}$ is a non-trivial $\Delta$-associative algebra.*

Recall that in $\widehat{H}_{2n^2}$, the set

$$\{J,\ J_1 - J + J_2 - J, 1 + J - J_1 - J_2\}$$

is a set of orthogonal central idempotents. Set

$$\widehat{A}_0 := J\widehat{H}_{2n^2},\ \widehat{A}_1 := (J_1 - J + J_2 - J)\widehat{H}_{2n^2},\ \widehat{A}_2 := (1 + J - J_1 - J_2)\widehat{H}_{2n^2}.$$

**Proposition 3.8.** *As an algebra, we have*

*(1)* $\widehat{H}_{2n^2} = \widehat{A}_0 \oplus \widehat{A}_1 \oplus \widehat{A}_2$.
*(2)* $\widehat{A}_0 \cong H_{2n^2}$.
*(3)* $\widehat{A}_1 \cong \underbrace{H_2 \oplus H_2 \oplus \cdots \oplus H_2}_{n \text{ copies}}$, *where* $H_2$ *is the Sweedler's algebra.*
*(4)* $\widehat{A}_2 \cong \Bbbk[h]/(h^2)$.

*Proof.* (1) It is obvious.

(2) The proof is similar to Proposition 3.4 (2).

(3) Let

$$x_1 = (J_1 + J_2 - 2J)x,\ y_1 = (J_1 + J_2 - 2J)y,\ z_1 = (J_1 + J_2 - 2J)z.$$

Note that $\widehat{A}_1$ is a subalgebra of $\widehat{H}_{2n^2}$ with the identity $J_1 + J_2 - 2J$, which is generated by $x_1$, $y_1$, and $z_1$ with the relations

$$\begin{array}{lll} x_1^n = J_1 - J, & y_1^n = J_2 - J, & x_1y_1 = y_1x_1 = 0, \\ z_1^2 = 0, & z_1x_1 = y_1z_1, & x_1z_1 = z_1y_1. \end{array}$$

Let

$$\lambda_s = \frac{1}{n}\sum_{t=1}^{n} q^{-st}x_1^t,\ \mu_s = \frac{1}{n}\sum_{t=1}^{n} q^{-st}y_1^t,$$

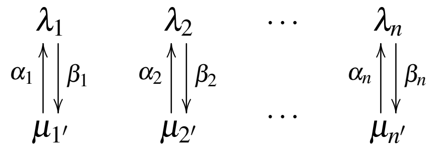
for $s = 1, 2, \cdots, n$. One can see that

$$\{\lambda_1, \lambda_2, \cdots, \lambda_n, \mu_1, \mu_2, \cdots, \mu_n\}$$

is a complete set of primitive orthogonal idempotents of $\widehat{A}_1$, and

$$\begin{array}{lll} x_1\lambda_s = \lambda_s x_1 = q^s\lambda_s, & y_1\lambda_s = \lambda_s y_1 = 0, & z_1\lambda_s = \mu_s z_1, \\ x_1\mu_s = \mu_s x_1 = 0, & y_1\mu_s = \mu_s y_1 = q^s\mu_s, & z_1\mu_s = \lambda_s z_1. \end{array}$$

Thus, the bounded quiver of $\widehat{A}_1$ is as follows



with the admissible ideal

$$\alpha_i\beta_i = \beta_i\alpha_i = 0, (i = 1, 2, \cdots, n).$$

Therefore, the set

$$\{\lambda_1+\mu_1, \lambda_2+\mu_2, \cdots, \lambda_n+\mu_n\}$$

forms a complete set of orthogonal central idempotents of $\widehat{A}_1$ and

$$\widehat{A}_1 = \bigoplus_{l=1}^{n} (\lambda_l+\mu_l)\widehat{A}_1.$$

In addition, each ideal $(\lambda_l+\mu_l)\widehat{A}_1$ of $\widehat{A}_1$ is a 4-dimensional subalgebra of $\widehat{A}_1$ with the identity $\lambda_l+\mu_l$. Also, it has the $\Bbbk$-basis $\{\lambda_l, \mu_l, \lambda_l z, \mu_l z\}$ for $l \in \{1,2,\cdots,n\}$.

Let $H_2$ be the 4-dimensional Sweedler's algebra generated by $\eta, \chi$ subjecting to the relations

$$\eta^2 = 1,\ \chi^2 = 0,\ \eta\chi = -\chi\eta.$$

For $l = 1,2,\cdots,n$, let $\rho_l : (\lambda_l+\mu_l)\widehat{A}_1 \to H_2$ be the map given by

$$\rho_l(\lambda_l+\mu_l) = 1,\ \rho_l(\lambda_l) = \frac{1+\eta}{2},$$

$$\rho_l(\mu_l) = \frac{1-\eta}{2},\ \rho_l(\lambda_l z) = \frac{1+\eta}{2}\chi,\ \rho_l(\mu_l z) = \frac{1-\eta}{2}\chi.$$

Obviously, $\rho_l$ is an algebra isomorphism.

(4) The ideal $\widehat{A}_2$ is a subalgebra of $\widehat{H}_{2n^2}$ with the identity $1+J-J_1-J_2$. Since

$$(1+J-J_1-J_2)x = 0,\ (1+J-J_1-J_2)y = 0,\ (1+J-J_1-J_2)z \neq 0,$$

it can be viewed as an algebra generated by $(1+J-J_1-J_2)z$ with the relation

$$[(1+J-J_1-J_2)z]^2 = 0.$$

This shows that $\widehat{A}_2$ is isomorphic to $\Bbbk[h]/(h^2)$.

The proof is completed. □

By Proposition 3.8, we see that $\widehat{H}_{2n^2}$ is not a semi-simple algebra with the dimension $2n^2+4n+2$. Its radical has a $\Bbbk$-basis as follows

$$\{\lambda_l z, \mu_l z, (1+J-J_1-J_2)z | l = 1,2,\cdots,n\}.$$

## 4. The representations of $\overline{H}_{2n^2}$

In this section, the aim is to classify all indecomposable representations of the weak Hopf algebra $\overline{H}_{2n^2}$, and then to characterize its representation ring using generators with generating relations.

The representation theory of semi-simple algebra $H_{2n^2}$ and $\Bbbk[g]/(g^n-1)$ was studied by many authors; see, for example, [15, 30, 31]. Let ind-$R$ denote the set of all indecomposable representations of some ring $R$. By Proposition 3.4, we have

$$\overline{H}_{2n^2} = \overline{A}_0 \oplus \overline{A}_1 \oplus \overline{A}_2 \oplus \overline{A}_3$$

as algebras, therefore,

$$\text{ind-}\overline{H}_{2n^2} = \text{ind-}\overline{A}_0 \cup \text{ind-}\overline{A}_1 \cup \text{ind-}\overline{A}_2 \cup \text{ind-}\overline{A}_3.$$

where ind-$\overline{A}_i$ can be viewed as the subset of ind-$\overline{H}_{2n^2}$ in the natural way. We note that $\overline{A}_0 \cong H_{2n^2}$, $\overline{A}_1 \cong \overline{A}_2 \cong \Bbbk[g]/(g^n - 1)$ and $\overline{A}_3 \cong \Bbbk$,

Now, we list all irreducible representations of $\overline{A}_i$, $i = 1,2,3$.

(1) Let $M_i, i \in \mathbb{Z}_n$. The 1-dimensional irreducible $\overline{H}_{2n^2}$-module with basis $v^{(i)}$, the actions of $\overline{H}_{2n^2}$ on $M_i$ are

$$x.v^{(i)} = q^i v^{(i)}, \quad y.v^{(i)} = 0, \quad z.v^{(i)} = 0.$$

In fact,

$$\text{ind-}\overline{A}_1 = \{M_i | i \in \mathbb{Z}_n\}.$$

(2) Let $N_i, i \in \mathbb{Z}_n$. The 1-dimensional irreducible $\overline{H}_{2n^2}$-module with basis $w^{(i)}$, the actions of $\overline{H}_{2n^2}$ on $N_i$ are

$$x.w^{(i)} = 0, \quad y.w^{(i)} = q^i w^{(i)}, \quad z.w^{(i)} = 0.$$

In fact,

$$\text{ind-}\overline{A}_2 = \{N_i | i \in \mathbb{Z}_n\}.$$

(3) Let $L$ be a 1-dimensional irreducible $\overline{H}_{2n^2}$-module with basis $u$, the actions of $\overline{H}_{2n^2}$ on $L$ are

$$x.u = 0, \quad y.u = 0, \quad z.u = 0.$$

In fact,

$$\text{ind-}\overline{A}_3 = \{L\}.$$

Therefore, we have the following results.

**Proposition 4.1.** *The set*

$$\mathscr{R} = \{S_m,\ S_{i,j},\ M_s,\ N_s,\ L \mid m \in \mathbb{Z}_{2n}, 0 \le i < j \le n-1, s \in \mathbb{Z}_n\}$$

*forms a complete list of non-isomorphic irreducible representations of* $\overline{H}_{2n^2}$.

For the decomposition formulas of the tensor product of two irreducible representations of $\overline{H}_{2n^2}$, we have the following several lemmas.

**Lemma 4.2.** *For all* $i \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$ *and* $0 \le s < t \le n-1$, *we have*

*(1)* $M_i \otimes S_m \cong S_m \otimes M_i \cong M_{m+i(\text{mod } n)}$;
*(2)* $M_i \otimes S_{s,t} \cong S_{s,t} \otimes M_i \cong M_{i+s(\text{mod } n)} \oplus M_{i+t(\text{mod } n)}$;
*(3)* $N_i \otimes S_m \cong S_m \otimes N_i \cong N_{m+i(\text{mod } n)}$;
*(4)* $N_i \otimes S_{s,t} \cong S_{s,t} \otimes N_i \cong N_{i+s(\text{mod } n)} \oplus N_{i+t(\text{mod } n)}$;
*(5)* $L \otimes S_m \cong S_m \otimes L \cong L$;
*(6)* $L \otimes S_{s,t} \cong S_{s,t} \otimes L \cong L \oplus L$.

*Proof.* Suppose that $S_m$ and $S_{s,t}$ are irreducible $\overline{H}_{2n^2}$-modules with the basis $\{v^m\}$ and $\{v_1^{st}, v_2^{st}\}$, respectively, for all $m \in \mathbb{Z}_{2n}$ and $0 \leq s < t \leq n-1$.

(1) Considering the decomposition formulas of tensor product $M_i \otimes S_m$ for $i \in \mathbb{Z}_n$. We have

$$x.(v^{(i)} \otimes v^m) = q^{i+m} v^{(i)} \otimes v^m, \quad y.(v^{(i)} \otimes v^m) = 0, \quad z.(v^{(i)} \otimes v^m) = 0.$$

Hence, $M_i \otimes S_m \cong M_{i+m(\text{mod } n)}$. Similarly, one can show that $S_m \otimes M_i \cong M_{i+m(\text{mod } n)}$.

(2) Considering the decomposition formulas of tensor product $M_i \otimes S_{s,t}$ for all $i \in \mathbb{Z}_n$. We have

$$\begin{array}{lll} x.(v^{(i)} \otimes v_1^{st}) = q^{i+s} v^{(i)} \otimes v_1^{st}, & y.(v^{(i)} \otimes v_1^{st}) = 0, & z.(v^{(i)} \otimes v_1^{st}) = 0, \\ x.(v^{(i)} \otimes v_2^{st}) = q^{i+t} v^{(i)} \otimes v_2^{st}, & y.(v^{(i)} \otimes v_2^{st}) = 0, & z.(v^{(i)} \otimes v_2^{st}) = 0. \end{array}$$

So, $M_i \otimes S_{s,t} \cong M_{i+s(\text{mod } n)} \oplus M_{i+t(\text{mod } n)}$. Similarly, $S_{s,t} \otimes M_i \cong M_{i+s(\text{mod } n)} \oplus M_{i+t(\text{mod } n)}$ can be proven.

The remaining cases can be proven in a similar way. □

**Lemma 4.3.** *For all $i, i' \in \mathbb{Z}_n$, we have*

*(1)* $M_i \otimes M_{i'} \cong M_{i+i'(\text{mod } n)}$;
*(2)* $M_i \otimes N_{i'} \cong N_{i'} \otimes M_i \cong L$;
*(3)* $M_i \otimes L \cong L \otimes M_i \cong L$;
*(4)* $N_i \otimes N_{i'} \cong N_{i+i'(\text{mod } n)}$;
*(5)* $N_i \otimes L \cong L \otimes N_i \cong L$;
*(6)* $L \otimes L \cong L$.

*Proof.* It is similar to the proof of Lemma 4.2. □

Let $a = [S_1]$, $b = [S_{n+1}]$, $c = [S_{0,1}]$, $d = [M_0]$, $e = [N_0]$ in $r(\overline{H}_{2n^2})$, we have the following lemma.

**Lemma 4.4.** *For all $i, i' \in \mathbb{Z}_n$, the following statements hold in $r(\overline{H}_{2n^2})$.*

*(1) If $n$ is odd, then*
- *(a)* $[M_i] = b^i d$, *and* $b^n d = d$;
- *(b)* $[N_i] = b^i e$, *and* $b^n e = e$;
- *(c)* $[L] = ed$;
- *(d)* $cd = d + bd$, $ce = e + be$, $d^2 = d$ *and* $e^2 = e$.

*(2) If $n$ is even, then*
- *(a)* $[M_i] = a^i d$, *and* $a^n d = d$;
- *(b)* $[N_i] = a^i e$, *and* $a^n e = e$;
- *(c)* $[L] = ed$;
- *(d)* $bd = ad$, $be = ae$, $cd = d + ad$, $ce = e + ae$, $d^2 = d$ *and* $e^2 = e$.

*Proof.* (1) If $n$ is odd, it can be concluded that $b^i = S_{i+\frac{(-1)^{i+1}+1}{2}n}$, for $i \in \mathbb{Z}_n$ and $b^n = S_n$ by [15, Lemma 3]. Hence, we get that

$$\begin{array}{l} M_0 \otimes S_{i+\frac{(-1)^{i+1}+1}{2}n} \cong S_{i+\frac{(-1)^{i+1}+1}{2}n} \otimes M_0 \cong M_i, \\ M_0 \otimes S_n \cong S_n \otimes M_0 \cong M_0, \\ N_0 \otimes S_{i+\frac{(-1)^{i+1}+1}{2}n} \cong S_{i+\frac{(-1)^{i+1}+1}{2}n} \otimes N_0 \cong N_i, \\ N_0 \otimes S_n \cong S_n \otimes N_0 \cong N_0, \\ M_0 \otimes S_{0,1} \cong S_{0,1} \otimes M_0 \cong M_0 \oplus M_1, \\ N_0 \otimes S_{0,1} \cong S_{0,1} \otimes N_0 \cong N_0 \oplus N_1, \end{array}$$

by Lemma 4.2. One sees that $[M_i] = b^i d$, $b^n d = d$, $[N_i] = b^i e$ and $b^n e = e$. The statements (a) and (b) of (1) are followed.

Moreover, based on statements (a) and (b), one can get that $cd = d + bd$, $ce = e + be$. Also,

$$M_0 \otimes N_0 \cong L, \quad M_0 \otimes M_0 \cong M_0, \quad N_0 \otimes N_0 \cong N_0$$

by Lemma 4.3, then $[L] = ed$, $d^2 = d$ and $e^2 = e$.

The first claim follows.

(2) If $n$ is even, by [15, Lemma 4], it can be concluded that

$$a^i = S_i,\ b^i = S_{i+\frac{(-1)^{i+1}+1}{2}n},\ \text{for } i \in \mathbb{Z}_n \text{ and } b^n = a^n = S_0.$$

The remaining statements can be proven in a similar way of claim (1). □

**Corollary 4.5.** *For the representation ring $r(\overline{H}_{2n^2})$, we have*

*(1) if $n$ is odd, the set*

$$\left\{b^k|0 \le k \le 2n-1\right\} \bigcup \left\{c^i b^j|1 \le i \le \tfrac{n-1}{2}, 0 \le j \le n-1\right\}$$
$$\bigcup \left\{b^l d, b^l e, de|0 \le l \le n-1\right\}$$

*forms a $\mathbb{Z}$-basis of $r(\overline{H}_{2n^2})$;*

*(2) if $n$ is even, the set*

$$\left\{a^i b^j|0 \le i \le n-1, j = 0,1\right\} \bigcup \left\{c^i b^j|1 \le i < \tfrac{n}{2}, 0 \le j \le n-1\right\}$$
$$\bigcup \left\{c^{\frac{n}{2}} b^j|0 \le j < \tfrac{n}{2}\right\} \bigcup \left\{a^l d, a^l e, de|0 \le l \le n-1\right\}$$

*forms a $\mathbb{Z}$-basis of $r(\overline{H}_{2n^2})$.*

*Proof.* (1) If $n$ is odd, the set

$$\left\{b^k|0 \le k \le 2n-1\right\} \bigcup \left\{c^i b^j|1 \le i \le \frac{n-1}{2}, 0 \le j \le n-1\right\}$$

corresponds one-to-one to the set of irreducible $\overline{H}_{2n^2}$-modules

$$\{[S_i]|0 \le i \le 2n-1\} \bigcup \left\{[S_{i,j}]|0 \le i < j \le n-1\right\}$$

by [15, Corollary 2 (1)]. We have

$$[M_i] = b^i d,\ b^n d = d,\ [N_i] = b^i e,\ b^n e = e,\ [L] = ed$$

by Lemma 4.4 (1).

Similarly, the set $\left\{b^l d, b^l e, de|0 \le l \le n-1\right\}$ corresponds one-to-one to the set of irreducible $\overline{H}_{2n^2}$-modules $\{[M_l], [N_l], [L]|0 \le l \le n-1\}$.

(2) If $n$ is even, the set

$$\left\{a^i b^j|0 \le i \le n-1, j = 0,1\right\} \bigcup \left\{c^i b^j|1 \le i < \frac{n}{2}, 0 \le j \le n-1\right\} \bigcup \left\{c^{\frac{n}{2}} b^j|0 \le j < \frac{n}{2}\right\}$$

corresponds one-to-one to the set of irreducible $\overline{H}_{2n^2}$-modules

$$\{[S_i], [S_{i,j}|0 \le i \le 2n-1, 0 \le i < j \le n-1\}$$

by [15, Corollary 2 (2)].

On the other hand, we have

$$[M_i] = a^i d,\ a^n d = d,\ [N_i] = a^i e,\ a^n e = e,\ [L] = ed,$$

by Lemma 4.4 (2). Hence, the set $\{a^l d, a^l e, de | 0 \le l \le n-1\}$ corresponds one-to-one to the set of irreducible $\overline{H}_{2n^2}$-modules $\{[M_l], [N_l], [L] | 0 \le l \le n-1\}$.

Accordingly, the proof is finished. □

Now, we can prove the main result of this section.

**Theorem 4.6.** *Assume that $n \in \mathbb{N}$ and $n \ge 2$, then the representation ring $r(\overline{H}_{2n^2})$ is a commutative ring, which can be characterized by generators and relations as follows:*

*(1) if $n$ is odd, then $r(\overline{H}_{2n^2}) \cong \mathbb{Z}[y, z, \alpha, \beta]/\mathscr{I}_1$, where $\mathscr{I}_1$ is the ideal generated by the set*

$$\left\{ \begin{array}{l} y^{2n} - 1, \quad zy^n - z, \quad B_{m+1}(y,z) - y^{m+1} B_m(y,z), \\ y^n \alpha - \alpha, \quad y^n \beta - \beta, \quad z\alpha - \alpha - y\alpha, \quad z\beta - \beta - y\beta, \\ \alpha^2 - \alpha, \quad \beta^2 - \beta. \end{array} \right\}_{m := \frac{n-1}{2}};$$

*(2) if $n = 2$, then $r(\overline{H}_{2n^2}) \cong \mathbb{Z}[y, z, \alpha, \beta]/\mathscr{I}_2$, where $\mathscr{I}_2$ is the ideal generated by the set*

$$\left\{ \begin{array}{l} y^2 - 1, \quad x^2 - y^2, \quad zx - zy, \quad z - zy, \quad z^2 - x - y - xy - 1, \\ x^2\alpha - \alpha, \quad x^2\beta - \beta, \quad y\alpha - x\alpha, \quad y\beta - x\beta, \\ z\alpha - \alpha - x\alpha, \quad z\beta - \beta - x\beta,, \quad \alpha^2 - \alpha \quad \beta^2 - \beta \end{array} \right\};$$

*(3) if $n > 2$ is even, then $r(\overline{H}_{2n^2}) \cong \mathbb{Z}[y, z, \alpha, \beta]/\mathscr{I}_3$, where $\mathscr{I}_3$ is the ideal generated by the set*

$$\left\{ \begin{array}{l} x^n - 1, \quad x^2 - y^2, \quad zx - zy, \\ D_{m+1}(y,z,x) - y^{m+1} D_{m-1}(y,z,x), \quad D_m(y,z,x) - y^m D_m(y,z,x), \\ x^n\alpha - \alpha, \quad x^n\beta - \beta, \quad y\alpha - x\alpha, \quad y\beta - x\beta, \\ z\alpha - \alpha - x\alpha, \quad z\beta - \beta - x\beta, \quad \alpha^2 - \alpha, \quad \beta^2 - \beta \end{array} \right\}_{m := \frac{n}{2}}.$$

*Proof.* It is easy to see that $r(\overline{H}_{2n^2})$ is commutative.

(1) By Corollary 4.5, if $n$ is odd, then $r(\overline{H}_{2n^2})$ is generated by $b, c, d$ and $e$. Hence, there is a unique ring epimorphism

$$\Phi : \mathbb{Z}[y, z, \alpha, \beta] \to r(\overline{H}_{2n^2})$$

such that

$$\Phi(y) = b = [S_{n+1}],\ \ \Phi(z) = c = [S_{0,1}],\ \ \Phi(\alpha) = d = [M_0],\ \ \Phi(\beta) = e = [N_0].$$

We note that

$$b^{2n} = 1,\ \ cb^n = c,\ \ B_{m+1}(b,c) = b^{m+1} B_m(b,c),$$

$$b^n d = d,\ \ b^n e = e,\ \ cd = d + bd,\ \ ce = e + be,\ \ d^2 = d,\ \ e^2 = e,$$

by Lemma 2.4, and Lemma 4.4, we have

$$\begin{aligned}&\Phi(y^{2n}-1)=0, \quad \Phi(zy^n - z) = 0, \quad \Phi\left(B_{m+1}(y,z) - y^{m+1}B_m(y,z)\right) = 0,\\ &\Phi(y^n\alpha - \alpha) = 0, \quad \Phi(y^n\beta - \beta) = 0, \quad \Phi(z\alpha - \alpha - y\alpha) = 0,\\ &\Phi(z\beta - \beta - y\beta) = 0, \quad \Phi(\alpha^2 - \alpha) = 0, \quad \Phi(\beta^2 - \beta) = 0,\end{aligned}$$

where $m := \frac{n-1}{2}$. It follows that $\Phi(\mathscr{I}_1) = 0$, and $\Phi$ induces a ring epimorphism

$$\overline{\Phi} : \mathbb{Z}[y,z,\alpha,\beta]/\mathscr{I}_1 \to r(\overline{H}_{2n^2})$$

such that

$$\overline{\Phi}(\overline{v}) = \Phi(v), \ \ \text{for all } v \in \mathbb{Z}[y,z,\alpha,\beta],$$

where $\overline{v} = \pi(v)$, and $\pi$ is a natural epimorphism $\mathbb{Z}[y,z,\alpha,\beta] \to \mathbb{Z}[y,z,\alpha,\beta]/\mathscr{I}_1$.

We note that the ring $r(\overline{H}_{2n^2})$ is the free $\mathbb{Z}$-module of rank $4n+1+\frac{n(n-1)}{2}$, with the $\mathbb{Z}$-basis

$$\begin{aligned}&\left\{b^k | 0 \le k \le 2n-1\right\} \bigcup \left\{c^i b^j | 1 \le i \le \tfrac{n-1}{2}, 0 \le j \le n-1\right\}\\ &\bigcup \left\{b^l d, b^l e, de | 0 \le l \le n-1\right\}.\end{aligned}$$

So we can define a $\mathbb{Z}$-module homomorphism:

$$\Psi : r(\overline{H}_{2n^2}) \to \mathbb{Z}[y,z,\alpha,\beta]/\mathscr{I}_1,$$

$$\Psi(c^i b^j) = \overline{z}^i \overline{y}^j, \quad \Psi(b^k) = \overline{y}^k, \quad \Psi(b^l d) = \overline{y}^l \overline{\alpha},$$

$$\Psi(b^l e) = \overline{y}^l \overline{\beta}, \quad \Psi(de) = \overline{\alpha}\overline{\beta},$$

where $0 \le k \le 2n-1, 1 \le i \le \frac{n-1}{2}, 0 \le j \le n-1$ and $0 \le l \le n-1$.

On the other hand, as the $\mathbb{Z}$-module, $\mathbb{Z}[y,z,\alpha,\beta]/\mathscr{I}_1$ is generated by elements

$$\left\{\overline{z}^i\overline{y}^j, \overline{y}^k, \overline{y}^l\overline{\alpha}, \overline{y}^l\overline{\beta}, \overline{\alpha}\overline{\beta} \middle| 0 \le k \le 2n-1, 1 \le i \le \frac{n-1}{2}, 0 \le j \le n-1, 0 \le l \le n-1\right\}.$$

Let

$$\overline{a} \in \left\{\overline{z}^i\overline{y}^j, \overline{y}^k, \overline{y}^l\overline{\alpha}, \overline{y}^l\overline{\beta}, \overline{\alpha}\overline{\beta} \middle| 0 \le k \le 2n-1, 1 \le i \le \frac{n-1}{2}, 0 \le j \le n-1, 0 \le l \le n-1\right\},$$

it is straightforward to check that $\Psi\overline{\Phi}(\overline{a}) = \overline{a}$. Hence $\Psi\overline{\Phi} = \text{id}$, which implies that $\overline{\Phi}$ is a monomorphism, and hence $\overline{\Phi}$ is an isomorphism.

The proofs of the remaining statements of the theorem are similar to the above. □

## 5. The representations of $\widehat{H}_{2n^2}$

In this section, the representations and representation ring of the $\Delta$-associative algebra $\widehat{H}_{2n^2}$ are given. Due to the similar research methods used as in Section 4, we directly provide relevant conclusions and omit their proofs.

By Proposition 3.8, it follows that $\widehat{H}_{2n^2} = \widehat{A}_0 \oplus \widehat{A}_1 \oplus \widehat{A}_2$ as algebras, then

$$\text{ind-}\widehat{H}_{2n^2} = \text{ind-}\widehat{A}_0 \cup \text{ind-}\widehat{A}_1 \cup \text{ind-}\widehat{A}_2.$$

where ind-$\widehat{A}_i$ can be viewed as the subset of ind-$\widehat{H}_{2n^2}$ in the natural way. As an algebra, we note that

$$\widehat{A}_0 \cong H_{2n^2}, \quad \widehat{A}_1 \cong \underbrace{H_2 \oplus H_2 \oplus \cdots \oplus H_2}_{n \text{ copies}}, \quad \widehat{A}_2 \cong \mathbb{K}[h]/(h^2),$$

where $H_2$ is the 4-dimensional Sweedler's algebra.

It is noted that the algebra $H_2$ and $\mathbb{k}[y]/(y^d)$ are Nakayama algebras, and their representations are also easy to construct; for examples, see [21, 30, 31]. Therefore, we can give a complete list of finite dimensional indecomposable representations in ind-$\widehat{A}_1$ and ind-$\widehat{A}_2$ as follows.

(1) Let $\widehat{M}_i, i \in \mathbb{Z}_n$. The 1-dimensional irreducible $\widehat{H}_{2n^2}$-module with basis $\widehat{v}^{(i)}$, the actions of $\widehat{H}_{2n^2}$ on $\widehat{M}_i$ are

$$x.\widehat{v}^{(i)} = 0, \quad y.\widehat{v}^{(i)} = q^i\widehat{v}^{(i)}, \quad z.\widehat{v}^{(i)} = 0.$$

(2) Let $\widehat{N}_i, i \in \mathbb{Z}_n$. The 1-dimensional irreducible $\widehat{H}_{2n^2}$-module with basis $\widehat{w}^{(i)}$, the actions of $\widehat{H}_{2n^2}$ on $\widehat{N}_i$ are

$$x.\widehat{w}^{(i)} = q^i\widehat{w}^{(i)}, \quad y.\widehat{w}^{(i)} = 0, \quad z.\widehat{w}^{(i)} = 0.$$

(3) Let $L_i, i \in \mathbb{Z}_n$. The 2-dimensional indecomposable $\widehat{H}_{2n^2}$-module with basis $\widehat{v}_1^{(i)}, \widehat{v}_2^{(i)}$, the actions of $\widehat{H}_{2n^2}$ on $L_i$ are

$$\begin{aligned} &x.\widehat{v}_1^{(i)} = q^i v_1^{(i)}, && y.\widehat{v}_1^{(i)} = 0, && z.\widehat{v}_1^{(i)} = \widehat{v}_2^{(i)}, \\ &x.\widehat{v}_2^{(i)} = 0, && y.\widehat{v}_2^{(i)} = q^i\widehat{v}_2^{(i)}, && z.\widehat{v}_2^{(i)} = 0. \end{aligned}$$

(4) Let $P_i, i \in \mathbb{Z}_n$. The 2-dimensional indecomposable $\widehat{H}_{2n^2}$-module with basis $\widehat{w}_1^{(i)}, \widehat{w}_2^{(i)}$, the actions of $\widehat{H}_{2n^2}$ on $P_i$ are

$$\begin{aligned} &x.\widehat{w}_1^{(i)} = 0, && y.\widehat{w}_1^{(i)} = q^i\widehat{w}_1^{(i)}, && z.\widehat{w}_1^{(i)} = \widehat{w}_2^{(i)}, \\ &x.\widehat{w}_2^{(i)} = q^i\widehat{w}_2^{(i)}, && y.\widehat{w}_2^{(i)} = 0, && z.\widehat{w}_2^{(i)} = 0. \end{aligned}$$

In fact,

$$\text{ind-}\widehat{A}_1 = \{\widehat{M}_i,\ \widehat{N}_i,\ L_i,\ P_i | i \in \mathbb{Z}_n\}.$$

(5) Let $Q$ be a 1-dimensional irreducible $\widehat{H}_{2n^2}$-module with basis $\widehat{u}$, the actions of $\widehat{H}_{2n^2}$ on $Q$ are

$$x.\widehat{u} = 0, \quad y.\widehat{u} = 0, \quad z.\widehat{u} = 0.$$

(6) Let $R$ be the 2-dimensional indecomposable $\widehat{H}_{2n^2}$-module with basis $\widehat{u}_1, \widehat{u}_2$, the actions of $\widehat{H}_{2n^2}$ on $R$ are

$$x.\widehat{u}_1 = 0, \quad y.\widehat{u}_1 = 0, \quad z.\widehat{u}_1 = \widehat{u}_2,$$
$$x.\widehat{u}_2 = 0, \quad y.\widehat{u}_2 = 0, \quad z.\widehat{u}_2 = 0.$$

In fact,

$$\text{ind-}\widehat{A}_2 = \{Q,\ R\}.$$

Therefore, we have

**Proposition 5.1.** *The set*

$$\mathscr{G} = \{S_m, S_{i,j}, \widehat{M}_s, \widehat{N}_s, L_s, P_s, Q, R \mid m \in \mathbb{Z}_{2n}, 0 \le i < j \le n-1, s \in \mathbb{Z}_n\}$$

*forms a complete list of non-isomorphic indecomposable $\widehat{H}_{2n^2}$-modules.*

Also, we have several lemmas as follows. As an example, we only give the proof of the first one.

**Lemma 5.2.** *For all $i, i' \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$ and $0 \le s < t \le n-1$, we have*

*(1)* $L_i \otimes S_m \cong \widehat{M}_{m+i(\text{mod } n)} \oplus \widehat{N}_{m+i(\text{mod } n)}$;
*(2)* $S_m \otimes L_i \cong L_{m+i(\text{mod } n)}$;
*(3)* $L_i \otimes S_{s,t} \cong \widehat{M}_{i+s(\text{mod } n)} \oplus \widehat{M}_{i+t(\text{mod } n)} \oplus \widehat{N}_{i+s(\text{mod } n)} \oplus \widehat{N}_{i+t(\text{mod } n)}$;
*(4)* $S_{s,t} \otimes L_i \cong L_{i+s(\text{mod } n)} \oplus L_{i+t(\text{mod } n)}$;
*(5)* $L_i \otimes L_{i'} \cong \widehat{M}_{i+i'(\text{mod } n)} \oplus \widehat{N}_{i+i'(\text{mod } n)} \oplus Q \oplus Q$;
*(6)* $L_i \otimes P_{i'} \cong \widehat{M}_{i+i'(\text{mod } n)} \oplus \widehat{N}_{i+i'(\text{mod } n)} \oplus Q \oplus Q$;
*(7)* $P_{i'} \otimes L_i \cong R \oplus \widehat{M}_{i+i'(\text{mod } n)} \oplus \widehat{N}_{i+i'(\text{mod } n)}$;
*(8)* $L_i \otimes Q \cong Q \otimes L_i \cong Q \oplus Q$;
*(9)* $L_i \otimes R \cong R \otimes L_i \cong Q \oplus Q \oplus Q \oplus Q$.

*Proof.* (1) In $\mathscr{G}$, considering the tensor product $L_i \otimes S_m$ for $i \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$, we have

$$x.(\widehat{v}_1^{(i)} \otimes v^m) = q^{i+m}\widehat{v}_1^{(i)} \otimes v^m, \quad y.(\widehat{v}_1^{(i)} \otimes v^m) = 0, \quad z.(\widehat{v}_1^{(i)} \otimes v^m) = 0,$$
$$x.(\widehat{v}_2^{(i)} \otimes v^m) = 0, \quad y.(\widehat{v}_2^{(i)} \otimes v^m) = q^{i+m}\widehat{v}_2^{(i)} \otimes v^m, \quad z.(\widehat{v}_2^{(i)} \otimes v^m) = 0.$$

Hence, $L_i \otimes S_m \cong \widehat{N}_{i+m(\text{mod } n)} \oplus \widehat{M}_{i+m(\text{mod } n)}$.

(2) Considering the tensor product $S_m \otimes L_i$ for $i \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$, we have

$$x.(v^m \otimes \widehat{v}_1^{(i)}) = q^{i+m} v^m \otimes \widehat{v}_1^{(i)}, \quad x.(v^m \otimes \widehat{v}_2^{(i)}) = 0,$$
$$y.(v^m \otimes \widehat{v}_1^{(i)}) = 0, \quad y.(v^m \otimes \widehat{v}_2^{(i)}) = q^{i+m} v^m \otimes \widehat{v}_2^{(i)},$$
$$z.(v^m \otimes \widehat{v}_1^{(i)}) = \frac{1}{n}\sigma(m) q^{\frac{m^2}{2}} \sum_{i',j'=1}^{n} q^{-i'j'+mi'+ij'} v^m \otimes \widehat{v}_2^{(i)}, \quad z.(v^m \otimes \widehat{v}_2^{(i)}) = 0.$$

Let $\pi_1 = v^m \otimes \widehat{v}_1^{(i)}$, $\quad \pi_2 = \frac{1}{n}\sigma(m) q^{\frac{m^2}{2}} \sum\limits_{i',j'=1}^{n} q^{-i'j'+mi'+ij'} v^m \otimes \widehat{v}_2^{(i)}$, then

$$x.\pi_1 = q^{i+m}\pi_1, \quad y.\pi_1 = 0, \quad z.\pi_1 = \pi_2,$$
$$x.\pi_2 = 0, \quad y.\pi_2 = q^{i+m}\pi_2, \quad z.\pi_2 = 0.$$

Thus, $S_m \otimes L_i \cong L_{i+m(\text{mod } n)}$.

(3) Considering the tensor product $L_i \otimes S_{s,t}$ for $i \in \mathbb{Z}_n$ and $0 \le s < t \le n-1$, we have

$$\begin{array}{lll} x.(\widehat{v}_1^{(i)} \otimes v_1^{st}) = q^{i+s}\widehat{v}_1^{(i)} \otimes v_1^{st}, & y.(\widehat{v}_1^{(i)} \otimes v_1^{st}) = 0, & z.(\widehat{v}_1^{(i)} \otimes v_1^{st}) = 0, \\ x.(\widehat{v}_1^{(i)} \otimes v_2^{st}) = q^{i+t}\widehat{v}_1^{(i)} \otimes v_2^{st}, & y.(\widehat{v}_1^{(i)} \otimes v_2^{st}) = 0, & z.(\widehat{v}_1^{(i)} \otimes v_2^{st}) = 0, \\ x.(\widehat{v}_2^{(i)} \otimes v_1^{st}) = 0, & y.(\widehat{v}_2^{(i)} \otimes v_1^{st}) = q^{i+s}\widehat{v}_2^{(i)} \otimes v_1^{st}, & z.(\widehat{v}_2^{(i)} \otimes v_1^{st}) = 0, \\ x.(\widehat{v}_2^{(i)} \otimes v_2^{st}) = 0, & y.(\widehat{v}_2^{(i)} \otimes v_2^{st}) = q^{i+t}\widehat{v}_2^{(i)} \otimes v_2^{st}, & z.(\widehat{v}_2^{(i)} \otimes v_2^{st}) = 0. \end{array}$$

Hence, $L_i \otimes S_{s,t} \cong \widehat{M}_{i+s(\text{mod } n)} \oplus \widehat{M}_{i+t(\text{mod } n)} \oplus \widehat{N}_{i+s(\text{mod } n)} \oplus \widehat{N}_{i+t(\text{mod } n)}$.

(4) Considering the tensor product $S_{s,t} \otimes L_i$ for all $i \in \mathbb{Z}_n$ and $0 \le s < t \le n-1$, we have

$$\begin{array}{ll} x.(v_1^{st} \otimes \widehat{v}_1^{(i)}) = q^{i+s} v_1^{st} \otimes \widehat{v}_1^{(i)}, & y.(v_1^{st} \otimes \widehat{v}_1^{(i)}) = 0, \\ z.(v_1^{st} \otimes \widehat{v}_1^{(i)}) = \frac{1}{n} \sum_{i',j'=1}^{n} q^{-i'j'+i't+ti} v_2^{st} \otimes \widehat{v}_2^{(i)}, & \\ x.(v_2^{st} \otimes \widehat{v}_1^{(i)}) = q^{i+t} v_2^{st} \otimes \widehat{v}_1^{(i)}, & y.(v_2^{st} \otimes \widehat{v}_1^{(i)}) = 0, \\ z.(v_1^{st} \otimes \widehat{v}_1^{(i)}) = \frac{1}{n} \sum_{i',j'=1}^{n} q^{-i'j'+st+i's+ti} v_1^{st} \otimes \widehat{v}_2^{(i)}, & \\ x.(v_1^{st} \otimes \widehat{v}_2^{(i)}) = 0, & y.(v_1^{st} \otimes \widehat{v}_2^{(i)}) = q^{i+t} v_1^{st} \otimes \widehat{v}_2^{(i)}, \\ z.(v_1^{st} \otimes \widehat{v}_2^{(i)}) = 0, & \\ x.(v_2^{st} \otimes \widehat{v}_2^{(i)}) = 0, & y.(v_2^{st} \otimes \widehat{v}_2^{(i)}) = q^{i+s} v_2^{st} \otimes \widehat{v}_2^{(i)}, \\ z.(v_2^{st} \otimes \widehat{v}_2^{(i)}) = 0. & \end{array}$$

Let

$$\begin{array}{ll} \pi_3 = v_1^{st} \otimes \widehat{v}_1^{(i)}, & \pi_4 = \frac{1}{n} \sum_{i',j'=1}^{n} q^{-i'j'+i't+ti} v_2^{st} \otimes \widehat{v}_2^{(i)}, \\ \pi_5 = v_2^{st} \otimes \widehat{v}_1^{(i)}, & \pi_6 = \frac{1}{n} \sum_{i',j'=1}^{n} q^{-i'j'+st+i's+ti} v_1^{st} \otimes \widehat{v}_2^{(i)}, \end{array}$$

then

$$\begin{array}{lll} x.\pi_3 = q^{i+s}\pi_3, & y.\pi_3 = 0, & z.\pi_3 = \pi_4, \\ x.\pi_4 = 0, & y.\pi_4 = q^{i+s}\pi_4, & z.\pi_4 = 0, \\ x.\pi_5 = q^{i+t}\pi_5, & y.\pi_5 = 0, & z.\pi_5 = \pi_6, \\ x.\pi_6 = 0, & y.\pi_6 = q^{i+t}\pi_6, & z.\pi_6 = 0, \end{array}$$

it follows that $S_{s,t} \otimes L_i \cong L_{i+s(\text{mod } n)} \oplus L_{i+t(\text{mod } n)}$.

The remaining statements can be proven similarly. □

**Lemma 5.3.** *For $i, i' \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$ and $0 \le s < t \le n-1$, we have*

*(1)* $\widehat{M}_i \otimes S_m \cong S_m \otimes \widehat{M}_i \cong \widehat{M}_{m+i(\text{mod } n)}$;
*(2)* $\widehat{M}_i \otimes S_{s,t} \cong S_{s,t} \otimes \widehat{M}_i \cong \widehat{M}_{i+s(\text{mod } n)} \oplus \widehat{M}_{i+t(\text{mod } n)}$;
*(3)* $\widehat{M}_i \otimes \widehat{M}_{i'} \cong \widehat{M}_{i+i'(\text{mod } n)}$;
*(4)* $\widehat{M}_i \otimes N_{i'} \cong N_{i'} \otimes \widehat{M}_i \cong Q$;
*(5)* $\widehat{M}_i \otimes L_{i'} \cong L_{i'} \otimes \widehat{M}_i \cong Q \oplus \widehat{M}_{i+i'(\text{mod } n)}$;

(6) $\widehat{M}_i \otimes P_{i'} \cong P_{i'} \otimes \widehat{M}_i \cong Q \oplus \widehat{M}_{i+i'(\text{mod } n)}$;
(7) $\widehat{M}_i \otimes Q \cong Q \otimes \widehat{M}_i \cong Q$;
(8) $\widehat{M}_i \otimes R \cong R \otimes \widehat{M}_i \cong Q \oplus Q$.

**Lemma 5.4.** *For all $i, i' \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$ and $0 \leq s < t \leq n-1$, we have*

(1) $\widehat{N}_i \otimes S_m \cong S_m \otimes \widehat{N}_i \cong \widehat{N}_{m+i(\text{mod } n)}$;
(2) $\widehat{N}_i \otimes S_{s,t} \cong S_{s,t} \otimes \widehat{N}_i \cong \widehat{N}_{i+s(\text{mod } n)} \oplus \widehat{N}_{i+t(\text{mod } n)}$;
(3) $\widehat{N}_i \otimes \widehat{N}_{i'} \cong \widehat{N}_{i+i'(\text{mod } n)}$;
(4) $\widehat{N}_i \otimes L_{i'} \cong L_{i'} \otimes \widehat{N}_i \cong \widehat{N}_{i+i'(\text{mod } n)} \oplus Q$;
(5) $\widehat{N}_i \otimes P_{i'} \cong P_{i'} \otimes \widehat{N}_i \cong Q \oplus \widehat{N}_{i+i'(\text{mod } n)}$;
(6) $\widehat{N}_i \otimes Q \cong Q \otimes \widehat{N}_i \cong Q$;
(7) $\widehat{N}_i \otimes R \cong R \otimes \widehat{N}_i \cong Q \oplus Q$.

**Lemma 5.5.** *For all $i, i' \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$ and $0 \leq s < t \leq n-1$, we have*

(1) $P_i \otimes S_m \cong P_{m+i(\text{mod } n)}$;
(2) $S_m \otimes P_i \cong \widehat{M}_{m+i(\text{mod } n)} \oplus \widehat{N}_{m+i(\text{mod } n)}$;
(3) $P_i \otimes S_{s,t} \cong \widehat{P}_{i+s(\text{mod } n)} \oplus \widehat{P}_{i+t(\text{mod } n)}$;
(4) $S_{s,t} \otimes P_i \cong M_{i+s(\text{mod } n)} \oplus M_{i+t(\text{mod } n)} \oplus N_{i+s(\text{mod } n)} \oplus N_{i+t(\text{mod } n)}$;
(5) $P_i \otimes P_{i'} \cong \widehat{M}_{i+i'(\text{mod } n)} \oplus \widehat{N}_{i+i'(\text{mod } n)} \oplus Q \oplus Q$;
(6) $P_i \otimes Q \cong Q \otimes P_i \cong Q \oplus Q$;
(7) $P_i \otimes R \cong R \otimes P_i \cong Q \oplus Q \oplus Q \oplus Q$.

**Lemma 5.6.** *For all $i \in \mathbb{Z}_n$, $m \in \mathbb{Z}_{2n}$ and $0 \leq s < t \leq n-1$, we have*

(1) $Q \otimes S_m \cong S_m \otimes Q \cong Q$;
(2) $Q \otimes S_{s,t} \cong S_{s,t} \otimes Q \cong Q \oplus Q$;
(3) $Q \otimes Q \cong Q$;
(4) $Q \otimes R \cong R \otimes Q \cong Q \oplus Q$;
(5) $R \otimes S_m \cong S_m \otimes R \cong Q \oplus Q$;
(6) $R \otimes S_{ij} \cong S_{ij} \otimes R \cong Q \oplus Q \oplus Q \oplus Q$;
(7) $R \otimes R \cong Q \oplus Q \oplus Q \oplus Q$.

Let

$$a = [s_1],\ b = [s_{n+1}],\ c = [s_{0,1}],\ d' = [\widehat{M}_0],\ e' = [\widehat{N}_0],\ f' = [L_0],\ g' = [P_0].$$

Then, we have the following result.

**Lemma 5.7.** *For all $i, i' \in \{1, 2, \cdots, n-1\}$, the following statements hold in $r(\widehat{H}_{2n^2})$.*

(1) *If n is odd, then*
  (a) $[\widehat{M}_i] = b^i d' = d' b^i$ *and* $[\widehat{M}_0] = b^n d' = d' b^n = d'$;
  (b) $[\widehat{N}_i] = b^i e' = e' b^i$ *and* $[\widehat{N}_0] = b^n e' = e' b^n = e'$;
  (c) $[L_i] = b^i f'$, *and* $[L_0] = b^n f' = f'$;
  (d) $[P_i] = g' b^i$, *and* $[P_0] = g' b^n = g'$;
  (e) $[Q] = e' d' = d' e'$;

(f) $[R] = g'f' - d' - e'$;
(g) $f'b = bg' = b(d' + e')$;
(h) $cd' = d'c = d' + bd'$, $ce' = e'c = e' + be'$, $cf' = f' + bf'$, $f'c = cg' = d' + e' + bd' + be'$ and $g'c = g' + g'b$;
(i) $d'^2 = d'$, $e'^2 = e'$, $f'g' = g'^2 = f'^2 = d' + e' + 2e'd'$, $d'f' = f'd' = d'g' = g'd' = d' + e'd'$, and $e'f' = f'e' = e'g' = g'e' = e' + e'd'$.

(2) *If $n$ is even, then*

(a) $[\widehat{M}_i] = a^i d' = d'a^i$ and $[\widehat{M}_0] = a^n d' = d'a^n = d'$;
(b) $[\widehat{N}_i] = a^i e' = e'a^i$ and $[\widehat{N}_0] = a^n e' = e'a^n = e'$;
(c) $[L_i] = a^i f'$ and $[L_0] = a^n f' = f'$;
(d) $[P_i] = g'a^i$ and $[P_0] = g'a^n = g'$;
(e) $[Q] = e'd' = d'e'$;
(f) $[R] = g'f' - d' - e'$;
(g) $f'a = ag' = a(d' + e')$;
(h) $bd' = d'b = ad'$, $be' = e'b = ae'$, $bf' = af'$, $bg' = f'b = f'a$ and $g'b = g'a$;
(i) $cd' = d'c = d' + ad'$, $ce' = e'c = e' + ae'$, $cf' = f' + af'$, $f'c = cg' = d' + e' + ad' + ae'$ and $g'c = g' + g'a$;
(j) $d'^2 = d'$, $e'^2 = e'$, $f'g' = g'^2 = f'^2 = d' + e' + 2e'd'$, $d'f' = f'd' = d'g' = g'd' = d' + e'd'$ and $e'f' = f'e' = e'g' = g'e' = e' + e'd'$.

**Corollary 5.8.** *(1) If $n$ is odd, the set*

$$\begin{array}{l}\left\{b^k|1 \le k \le 2n\right\} \bigcup \left\{c^i b^j|1 \le i \le \frac{n-1}{2}, 1 \le j \le n\right\} \\ \bigcup \left\{b^s d', b^s e', b^s f', g'b^s, de, g'f' - d' - e'|1 \le s \le n\right\}\end{array}$$

*forms a $\mathbb{Z}$-basis of $r(\widehat{H}_{2n^2})$.*
*(2) If $n$ is even, the set*

$$\begin{array}{l}\left\{a^i b^j|1 \le i \le n, j = 1, n\right\} \bigcup \left\{c^i b^j|1 \le i < \frac{n}{2}, 1 \le j \le n\right\} \\ \bigcup \left\{c^{\frac{n}{2}} b^j|1 \le j < \frac{n}{2}, j = n\right\} \bigcup \left\{a^s d', a^s e', a^s f', g'a^s, d'e', g'f' - d' - e'|1 \le s \le n\right\}\end{array}$$

*forms a $\mathbb{Z}$-basis of $r(\widehat{H}_{2n^2})$.*

*Proof.* By Lemmas 5.3–5.7, we see that $[S_0]$ is no longer the identity of $r(\widehat{H}_{2n^2})$. By Lemma 2.3, $[S_0] = [S_{2n}]$, and if $n$ is odd, the set

$$\left\{b^k|1 \le k \le 2n\right\} \bigcup \left\{c^i b^j|1 \le i \le \frac{n-1}{2}, 1 \le j \le n\right\}$$

corresponds one-to-one to the set of irreducible $\widehat{H}_{2n^2}$-modules

$$\left\{[S_i]|1 \le i \le 2n\right\} \bigcup \left\{[S_{i,j}]|0 \le i < j \le n-1\right\};$$

if $n$ is even, the set

$$\left\{a^i b^j|1 \le i \le n, j = 1, n\right\} \bigcup \left\{c^i b^j|1 \le i < \frac{n}{2}, 1 \le j \le n\right\} \bigcup \left\{c^{\frac{n}{2}} b^j|1 \le j < \frac{n}{2}, j = n\right\}$$

corresponds one-to-one to the set of irreducible $\widehat{H}_{2n^2}$-modules

$$\left\{[S_i],[S_{i,j}]|1\leq i\leq 2n,0\leq i<j\leq n-1\right\}.$$

On the other hand, if $n$ is odd, we have

$$\begin{aligned}&[\widehat{M}_s]=b^sd'=d'b^s,\text{ and }[\widehat{M}_0]=b^nd'=d'b^n=d',\\&[\widehat{N}_s]=b^se'=e'b^s,\text{ and }[\widehat{N}_0]=b^ne'=e'b^n=e',\\&[L_s]=b^sf',\text{ and }[L_0]=b^nf'=f',\\&[P_s]=g'b^s,\text{ and }[P_0]=g'b^n=g',\\&[Q]=e'd'=d'e',\ [R]=g'f'-d'-e'\end{aligned}$$

by Lemma 5.7, where $1\leq s\leq n-1$. Hence, the set

$$\left\{b^sd',b^se',b^sf',g'b^s,d'e',g'f'-d'-e'|1\leq s\leq n\right\}$$

corresponds one-to-one to the set of indecomposable $\widehat{H}_{2n^2}$-modules

$$\left\{[\widehat{M}_s],[\widehat{N}_s],[L_s],[P_s],[Q],[R]\middle|0\leq s\leq n-1\right\}.$$

The remaining can be proven in a similar ways. □

By Lemma 5.3–5.7, we see that $r(\widehat{H}_{2n^2})$ is a noncommutative ring without an identity. Let $r^*(\widehat{H}_{2n^2})$ be the ring with the identity extended from the ring $r(\widehat{H}_{2n^2})$ in the natural way. For the general definition one can refer to Section 2. Therefore, $r^*(\widehat{H}_{2n^2})$ is a ring with the identity $(1,0)$ and

$$r(\widehat{H}_{2n^2})\cong\{(0,\alpha)|\alpha\in r(\widehat{H}_{2n^2})\}\subseteq r^*(\widehat{H}_{2n^2}).$$

Now, we can explicitly describe $r^*(\widehat{H}_{2n^2})$ by generators with relations.

**Theorem 5.9.** *Assume that $n\in\mathbb{N}$ and $n\geq 2$, we have the following statements:*

*(1) if $n$ is odd, then $r^*(\widehat{H}_{2n^2})\cong\mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle/\mathscr{I}_1$, where $\mathscr{I}_1$ is the ideal generated by the following set*

$$T_1=\left\{\begin{array}{l}y^{2n+1}-y,\ yz-zy,\ zy^n-z,\ B_{m+1}(y,z)-y^{m+1}B_m(y,z),\\ y\alpha-\alpha y;\ y^n\alpha-\alpha,\ y\beta-\beta y,\ y^n\beta-\beta,\ y^n\gamma-\gamma,\ \delta y^n-\delta,\\ \alpha\beta-\beta\alpha,\ \gamma y-y\delta,\ y\delta-y(\alpha+\beta),\ z\alpha-\alpha z,\\ \alpha z-\alpha-y\alpha,\ z\beta-\beta z,\ \beta z-\beta-y\beta,\ z\gamma-\gamma-y\gamma,\ \gamma z-z\delta,\\ z\delta-\alpha-\beta-y\alpha-y\beta,\ \delta z-\delta-\delta y,\ \alpha^2-\alpha,\ \beta^2-\beta,\ \gamma\delta-\delta^2,\\ \gamma^2-\delta^2,\ \delta^2-\alpha-\beta-2\alpha\beta,\ \alpha\gamma-\gamma\alpha,\ \gamma\alpha-\alpha\delta,\ \alpha\delta-\delta\alpha,\\ \delta\alpha-\alpha-\alpha\beta,\ \beta\gamma-\gamma\beta,\ \gamma\beta-\beta\delta,\ \beta\delta-\delta\beta,\ \delta\beta-\beta-\alpha\beta.\end{array}\right\}_{m:=\frac{n-1}{2}};$$

*(2) if $n=2$, then $r^*(\widehat{H}_{2n^2})\cong\mathbb{Z}\langle x,y,z,\alpha,\beta,\gamma,\delta\rangle/\mathscr{I}_2$, where $\mathscr{I}_2$ is the ideal generated by the following*

*set*

$$T_2 = \left\{ \begin{array}{l} y^3-y,\ x^3-x,\ xy-yx,\ xz-zx,\ yz-zy, \\ x^2-y^2,\ zx-zy,\ z-zy,\ z^2-x-y-xy-y^2, \\ x\alpha-\alpha x,\ x^2\alpha-\alpha,\ x\beta-\beta x, x^2\beta-\beta,\ x^2\gamma-\gamma,\ \delta x^2-\delta,\ \alpha\beta-\beta\alpha, \\ \gamma x - x(\alpha+\beta),\ x\delta-\gamma x,\ y\alpha-\alpha y,\ y\alpha-x\alpha,\ y\beta-\beta y,\ y\beta-x\beta, \\ y\gamma-x\gamma,\ y\delta-\gamma y,\ \gamma y-\gamma x,\ \delta y-\delta x,\ z\alpha-\alpha z,\ z\alpha-\alpha-x\alpha, \\ z\beta-\beta z,\ z\beta-\beta-x\beta,\ z\gamma-\gamma-x\gamma,\ \gamma z-z\delta,\ z\delta-\alpha-\beta-x\alpha-x\beta, \\ \delta z-\delta-\delta x,\ \alpha^2-\alpha,\ \beta^2-\beta,\ \gamma\delta-\delta^2,\ \gamma^2-\delta^2,\ \delta^2-\alpha-\beta-2\alpha\beta, \\ \alpha\gamma-\gamma\alpha,\ \gamma\alpha-\alpha\delta, \alpha\delta-\delta\alpha,\ \delta\alpha-\alpha-\alpha\beta,\ \beta\gamma-\gamma\beta, \\ \gamma\beta-\beta\delta,\ \beta\delta-\delta\beta,\ \delta\beta-\beta-\alpha\beta. \end{array} \right\};$$

*(3) if $n>2$ is even, then $r^*(\widehat{H}_{2n^2}) \cong \mathbb{Z}\langle x,y,z,\alpha,\beta,\gamma,\delta\rangle / \mathscr{J}_3$, where $\mathscr{I}_3$ is the ideal generated by the following set*

$$T_3 = \left\{ \begin{array}{l} x^{n+1}-x,\ y^{n+1}-y,\ xy-yx,\ xz-zx,\ yz-zy,\ x^2-y^2,\ zx-zy, \\ D_{m+1}(y,z,x)-y^{m+1}D_{m-1}(y,z,x), \quad D_{m+1}(y,z,x)-y^m D_m(y,z,x), \\ x\alpha-\alpha x,\ x^n\alpha-\alpha,\ x\beta-\beta x, x^n\beta-\beta,\ x^n\gamma-\gamma,\ \delta x^n-\delta,\ \alpha\beta-\beta\alpha, \\ \gamma x - x(\alpha+\beta),\ x\delta-\gamma x,\ y\alpha-\alpha y,\ y\alpha-x\alpha,\ y\beta-\beta y,\ y\beta-x\beta, \\ y\gamma-x\gamma,\ y\delta-\gamma y,\ \gamma y-\gamma x,\ \delta y-\delta x,\ z\alpha-\alpha z,\ z\alpha-\alpha-x\alpha, \\ z\beta-\beta z,\ z\beta-\beta-x\beta,\ z\gamma-\gamma-x\gamma,\ \gamma z-z\delta,\ z\delta-\alpha-\beta-x\alpha-x\beta, \\ \delta z-\delta-\delta x,\ \alpha^2-\alpha,\ \beta^2-\beta,\ \gamma\delta-\delta^2,\ \gamma^2-\delta^2,\ \delta^2-\alpha-\beta-2\alpha\beta, \\ \alpha\gamma-\gamma\alpha,\ \gamma\alpha-\alpha\delta, \alpha\delta-\delta\alpha,\ \delta\alpha-\alpha-\alpha\beta,\ \beta\gamma-\gamma\beta, \\ \gamma\beta-\beta\delta,\ \beta\delta-\delta\beta,\ \delta\beta-\beta-\alpha\beta. \end{array} \right\}_{m:=\frac{n}{2}}.$$

*Proof.* It is easy to see that $r^*(\widehat{H}_{2n^2})$ is a noncommutative ring.

(1) Let $\pi : \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle \to \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle / \mathscr{J}_1$ be the natural epimorphisms and $\overline{a} = \pi(a)$ for any $a \in \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle$.

By Lemma 5.7(1) and Corollary 5.8(1), if $n$ is odd, $r(\widehat{H}_{2n^2})$ is generated as a ring by $b,c,d',e',f',g'$, and any element of $r(\widehat{H}_{2n^2})$ can be written as

$$\sum_{k_1+k_2+\cdots+k_6>0} \rho_{k_1k_2\cdots k_6} b^{k_1}c^{k_2}d'^{k_3}e'^{k_4}f'^{k_5}g'^{k_6}$$

where $\rho_{k_1k_2\cdots k_6}, k_1,k_2,\cdots,k_6 \in \mathbb{Z}$. Therefore,

$$r^*(\widehat{H}_{2n^2}) = \left\{ (k, \sum_{k_1+k_2+\cdots+k_6>0} \rho_{k_1k_2\cdots k_6} b^{k_1}c^{k_2}d'^{k_3}e'^{k_4}f'^{k_5}g'^{k_6}) \middle| \rho_{k_1k_2\cdots k_6}, k_1,k_2,\cdots,k_6,k \in \mathbb{Z} \right\}.$$

We defined a $\mathbb{Z}$-map

$$\Phi : \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle \to r^*(\widehat{H}_{2n^2})$$

as

$$\Phi(1) = (1,0),\ \ \Phi(y) = (0,b),\ \ \Phi(z) = (0,c),\ \ \Phi(\alpha) = (0,d'),$$
$$\Phi(\beta) = (0,e'),\ \ \Phi(\gamma) = (0,f'),\ \ \Phi(\delta) = (0,g').$$

It is easy to get that $\Phi$ can be extended to a ring epimorphism in the natural way. In fact, any $\upsilon \in \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle$ can be written as

$$\upsilon = k1_{\mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle} + \sum_{k_1+k_2+\cdots+k_6>0} \rho_{k_1k_2\cdots k_6} y^{k_1} z^{k_2} \alpha^{k_3} \beta^{k_4} \gamma^{k_5} \delta^{k_6},$$

where $\rho_{k_1k_2\cdots k_6}, k_1, k_2, \cdots, k_6, k \in \mathbb{Z}$. Thus,

$$\Phi(\upsilon) = (k, \sum_{k_1+k_2+\cdots+k_6>0} \rho_{k_1k_2\cdots k_6} b^{k_1} c^{k_2} d'^{k_3} e'^{k_4} f'^{k_5} g'^{k_6}).$$

By Lemma 5.7(1), one sees that $\Phi(\omega) = (0,0)$ for all $\omega \in T_1$. Hence, $\Phi(\mathscr{J}_1) = (0,0)$ and $\Phi$ induces a unique ring epimorphism

$$\overline{\Phi} : \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle / \mathscr{J}_1 \to r^*(\widehat{H}_{2n^2})$$

with $\overline{\Phi}(\overline{\nu}) = \Phi(\nu)$ for all $\nu \in \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle$.

We note that the ring $r^*(\widehat{H}_{2n^2})$ is the free $\mathbb{Z}$-module of rank $6n+3+\frac{n(n-1)}{2}$, with the $\mathbb{Z}$-basis

$$\begin{aligned}&\{(1,0)\} \cup \{(0,b^k) | 1 \le k \le 2n\} \cup \{(0,c^i b^j) | 1 \le i \le \tfrac{n-1}{2}, 1 \le j \le n\} \\ &\cup \{(0,b^s d'),(0,b^s e'),(0,b^s f'),(0,g' b^s),(0,de),(0,g'f'-d'-e') | 1 \le s \le n\}.\end{aligned}$$

So we can define a $\mathbb{Z}$-module homomorphism:

$$\Psi : r^*(\widehat{H}_{2n^2}) \to \mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle / \mathscr{J}_1$$

by

$$\begin{gathered}\Psi((1,0)) = \overline{1}, \quad \Psi((0,b^k)) = \overline{y}^k, \quad \Psi((0,c^i b^j)) = \overline{z}^i \overline{y}^j, \quad \Psi((0,b^s d') = \overline{y}^s \overline{\alpha}, \\ \Psi((0,b^s e')) = \overline{y}^s \overline{\beta}, \quad \Psi((0,b^s f')) = \overline{y}^s \overline{\gamma}, \quad \Psi((0,g' b^s)) = \overline{\delta} \overline{y}^s, \\ \Psi((0,de)) = \overline{\alpha}\overline{\beta}, \quad \Psi((0,g'f'-d'-e')) = \overline{\gamma}\overline{\delta} - \overline{\alpha} - \overline{\beta},\end{gathered}$$

where $1 \le k \le 2n, 1 \le i \le \frac{n-1}{2}, 1 \le j \le n$ and $1 \le s \le n$. Obviously, as the $\mathbb{Z}$-module, $\mathbb{Z}\langle y,z,\alpha,\beta,\gamma,\delta\rangle / \mathscr{J}_1$ is generated by the set

$$\begin{aligned}S = \ &\{\overline{1}\} \cup \{\overline{y}^k | 1 \le k \le 2n\} \cup \{\overline{z}^i \overline{y}^j | 1 \le i \le \tfrac{n-1}{2}, 1 \le j \le n\} \\ &\cup \left\{\overline{y}^s \overline{\alpha}, \overline{y}^s \overline{\beta}, \overline{y}^s \overline{\gamma}, \overline{\delta} \overline{y}^s, \overline{\alpha}\overline{\beta}, \overline{\gamma}\overline{\delta} - \overline{\alpha} - \overline{\beta}, | 1 \le s \le n\right\}.\end{aligned}$$

For any $\overline{a} \in S$, we see that $\Psi\overline{\Phi}(\overline{a}) = \overline{a}$. Hence $\Psi\overline{\Phi} = \mathrm{id}$, which implies that $\overline{\Phi}$ is a monomorphism, and hence $\overline{\Phi}$ is an isomorphism.

The proofs of the remaining statements are similar. □

**Remark 5.10.** *By Theorems 4.6 and 5.9, as the rings, it is easy to see that*

*(1) if n is odd, then*

$$r(H_{2n^2}) \cong r(\overline{H}_{2n^2}) / \langle \alpha, \beta \rangle \cong r^*(\widehat{H}_{2n^2}) / \langle \alpha, \beta, \gamma, \delta, y^{2n} - 1 \rangle;$$

*(2) if n is even, then*

$$r(H_{2n^2}) \cong r(\overline{H}_{2n^2}) / \langle \alpha, \beta \rangle \cong r^*(\widehat{H}_{2n^2}) / \langle \alpha, \beta, \gamma, \delta, y^n - 1, x^n - 1 \rangle.$$

## 6. Conclusions

We have described the representation rings of two classes of Δ-associative algebras, $\overline{H}_{2n^2}$ and $\widehat{H}_{2n^2}$, extended from Hopf algebra $H_{2n^2}$ of Kac-Paljutkin type. It may be interesting to consider the category of representations of these representation rings, as this could be helpful in understanding the invariants of the more general tensor categories. It is challenging to consider non-Hermitian linear systems over these rings, similar to non-Hermitian quaternion linear systems over the quaternion algebra. Readers are referred to related works, such as [32–34].

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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