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*Research article*

# On optimality conditions and duality for multiobjective fractional optimization problem with vanishing constraints

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**Abstract:** The aim of this paper is to investigate the optimality conditions for a class of nonsmooth multiobjective fractional optimization problems subject to vanishing constraints. In particular, necessary and sufficient conditions for (weak) Pareto solution are presented in terms of the Clark subdifferential. Furthermore, we construct Wolfe and Mond–Weir-type dual models and derive some duality theorems by using generalized quasiconvexity assumptions. Some examples to show the validity of our conclusions are provided.

**Keywords:** multiobjective fractional optimization; vanishing constraint; locally Lipschitz function; duality theorems

## 1. Introduction

Recently, there has been a lot of attention on mathematical programming problems with vanishing constraints, which serve as a unified framework for several applications in topological optimization and optimal contral. The optimality conditions and duality theorems of these problems have been extensively researched since their introduction by Achtziger and Kanzow [1]. Mishra et al. [2] developed and analyzed dual models and obtained some duality results under differentiable assumptions. Hu and his co-authors in [3] provided some new dual models based on the dual models proposed by [2], which do not require computing the index sets. Tung [4] extended the single objective programming to multiobjective semi-infinite cases with vanishing constraints and investigated the KKT optimality conditions and duality results of the Wolfe and Mond–Weir-type dual models for this problem. Furthermore, Tung [5] established the KKT optimality conditions and the duality theorems for nonsmooth multiobjective semi-infinite optimization problems with vanishing constraints in terms of Clarke subdifferentials. By proposing new constraints for ACQ and VC-ACQ, Antczak [6] derived optimality conditions and duality results for differentiable semi-infinite multiobjective optimization problems with vanishing constraints. Additionally, Antczak [7] addressed the KKT optimality conditions for a class of nondifferentiable multiobjective programming problems with vanishing constraints under the VC-Cottle constraint qualification. However, duality results are not taken into account in [7]. Meanwhile, for directionally differentiable vector optimization problems, Antczak [8] also discussed the KKT necessary optimality conditions under both ACQ and m-ACQ; the sufficient optimality conditions and Wolfe-type duality theorems were also established under appropriate convexity hypotheses. Huang and Zhu [9] studied optimality conditions for Borwein proper efficient solutions of nonsmooth multiobjective optimization problems with vanishing constraints in terms of Clark subdifferential. Guu et al. [10] provided strong KKT sufficient optimality conditions for multiobjective semi-infinite programming problems with vanishing constraints under generalized convexity assumptions. Wang and Wang [11, 12] established optimality conditions for a class of nonsmooth interval-valued optimization problems with vanishing constraints, along with duality theorems for the corresponding dual models. The principal challenge inherent in optimization problems with vanishing constraints stems from the inclusion of a product of two functions within the constraint conditions. This situation gives rise to two notable issues: firstly, the feasible set is generally non-convex; secondly, when one of the functions in the product equals zero, the constraint properties of the other function become ineffective.

A fundamental question here is why we should study optimality conditions and duality in the framework of multiobjective fractional programming problems with vanishing constraints, as well as their corresponding Mond–Weir and Wolfe-type dual problems. We try to address this question succinctly. While many studies have been published over the past decade concerning optimization problems with vanishing constraints, there remains a scarcity of research specifically focused on multiobjective fractional programming problems with vanishing constraints (see [1–12]). Notably, the Mond–Weir and Wolfe types of dual problems have garnered significant attention in this field due to their practical applicability.

Due to the fact that in numerous optimization problems, the objective functions are expressed as quotients of two functions. There are many authors who established optimality conditions and employed the conditions to search for optimal solutions as well as duality theorems for such vector optimization problems (see [13–18]). Kim et al. [13] derived optimality conditions and duality results for nondifferentiable multiobjective fractional programming. Long [14] discussed similar results for this type of problem using $(C, \alpha, \rho, d)$-convexity. Later, under higher-order $(C, \alpha, \gamma, \rho, d)$-assumptions, Dubey et al. [15] established higher-order optimality conditions and duality results for such a problem. In addition, for nonsmooth fractional multiobjective optimization problems with equality or inequality constraints, several optimality conditions and duality theorems are studied in [16–18]. We note that there is relatively little literature on optimality conditions and duality theorems for nonsmooth multiobjective fractional programming problems with vanishing constraints.

Motivated by the above works, this paper aims to investigate nonsmooth multiobjective fractional optimization problems with vanishing constraints ( abbreviated as, (FPVC)), and establish necessary and sufficient optimality conditions for (FPVC). Subsequently, duality theorems of Wolfe type and Mond–Weir-type for (FPVC) will be formulated. The organization of this paper is outlined as follows: In Section 2, essential notions and definitions are reviewed for subsequent discussion. Section 3 focuses on the optimality conditions for the (weak) Pareto minimum of (FPVC) subject to VC-Cottle constraints. Section 4 establishes Wolfe-type and Mond–Weir type dual models for (FPVC) and studies the weak, strong and converse duality theorems between (FPVC) and its dual problems.

## 2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For any $a, b \in \mathbb{R}^n$, we define:
(i) $a < b \Leftrightarrow a_i < b_i$ for all $i = 1, 2, \ldots, n$;
(ii) $a \leqq b \Leftrightarrow a_i \leqq b_i$ for all $i = 1, 2, \ldots, n$;
(iii) $a \leq b \Leftrightarrow a_i \leqq b_i$ for all $i = 1, 2, \ldots, n$ and $a \neq b$;
(iv) $a \nleq b$ is the negation of $a \leq b$ .

Row and column vectors will be treated with the same notation in this paper when the interpretation is obvious.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke subdifferential of $f$ at $\bar{x}$ is defined as follows:

$$\partial_c f(\bar{x}) := \{\xi \in \mathbb{R}^n : \ f^\circ(\bar{x}; v) \geq \langle \xi, \ v\rangle, \ \forall v \in \mathbb{R}^n\},$$

where

$$f^\circ(\bar{x}; v) := \limsup_{(x,t)\to(\bar{x},0^+)} \frac{f(x + tv) - f(x)}{t}.$$

**Lemma 2.1.** *[19] Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ and attain its minimum at $\bar{x}$. Then $0 \in \partial_c f(\bar{x})$.*

**Lemma 2.2.** *[19] Let $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in K := \{1, \ldots, l\}$ be a locally Lipschitz function at a point $\bar{x} \in \mathbb{R}^n$. Then*

$$\partial_c(\sum_{k\in K} \lambda_k f_k)(\bar{x}) \subseteq \sum_{k\in K} \lambda_k \partial_c f_k(\bar{x}),$$

*where $\lambda_k \in \mathbb{R}$. If $f(x) := \max_{k\in K} f_k(x)$, then the function $f(x)$ is also locally Lipschitz at $\bar{x}$. In addition,*

$$\partial_c f(\bar{x}) \subset conv\{\partial_c f_k(\bar{x}) : k \in K(\bar{x})\},$$

*where $K(\bar{x}) := \{k \in K : f(\bar{x}) = f_k(\bar{x})\}$, and conv is an abbreviation for convex hull.*

**Lemma 2.3.** *[19] Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions at $\bar{x} \in \mathbb{R}^n$. Then $fg$ is a locally Lipschitz function at $\bar{x}$, and*

$$\partial_c(fg)(\bar{x}) \subset g(\bar{x})\partial_c f(\bar{x}) + f(\bar{x})\partial_c g(\bar{x}).$$

*If $g(\bar{x}) \neq 0$, $\frac{f}{g}$ is also a locally Lipschitz function at $\bar{x}$, and*

$$\partial_c\left(\frac{f}{g}\right)(\bar{x}) \subset \frac{g(\bar{x})\partial_c f(\bar{x}) - f(\bar{x})\partial_c g(\bar{x})}{g^2(\bar{x})}.$$

Accordingly, we consider multiobjective fractional optimization with vanishing constraints (FPVC) as follows:

$$\begin{aligned} \min \quad & F(x) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right) \\ \text{s.t.} \quad & h_j(x) \leqq 0, \ j \in J = \{1, \ldots, m\} \\ & U_s(x) \geqq 0, \ s \in S = \{1, \ldots, q\} \\ & U_s(x)V_s(x) \leqq 0, \ s \in S \end{aligned}$$

where $f_i, g_i, h_j, U_s, V_s : \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1, \ldots, p\}$, $j \in J$, $s \in S$, are locally Lipschitz functions. For all $i \in I$, we set $f_i(x) \geqq 0$, $g_i(x) > 0$. The set $D$ stands for the feasible set of problems (FPVC).

**Definition 2.1.** *Let $\bar{x} \in D$,*

*(i) $\bar{x}$ is said to be a weak Pareto solution for (FPVC) if there is no other $x \in D$ such that $F(x) < F(\bar{x})$.*

*(ii) $\bar{x}$ is said to be a Pareto solution for (FPVC) if there is no other $x \in D$ such that $F(x) \leq F(\bar{x})$.*

*Now, for any feasible point $\bar{x} \in D$, we denote the following index sets:*

$$J(\bar{x}) := \{j \in J \mid h_j(\bar{x}) = 0\},$$

$$S_{+}(\bar{x}) := \{s \in S \mid U_s(\bar{x}) > 0\},$$

$$S_{0}(\bar{x}) := \{s \in S \mid U_s(\bar{x}) = 0\},$$

$$S_{+0}(\bar{x}) := \{s \in S \mid U_s(\bar{x}) > 0, V_s(\bar{x}) = 0\},$$

$$S_{+-}(\bar{x}) := \{s \in S \mid U_s(\bar{x}) > 0, V_s(\bar{x}) < 0\},$$

$$S_{0+}(\bar{x}) := \{s \in S \mid U_s(\bar{x}) = 0, V_s(\bar{x}) > 0\},$$

$$S_{00}(\bar{x}) := \{s \in S \mid U_s(\bar{x}) = 0, V_s(\bar{x}) = 0\},$$

$$S_{0-}(\bar{x}) := \{s \in S \mid U_s(\bar{x}) = 0, V_s(\bar{x}) < 0\},$$

$$S_{UV}(\bar{x}) := \{s \in S \mid U_s(\bar{x})V_s(\bar{x}) = 0\}.$$

*Obviously, $S_0(\bar{x}) = S_{0+}(\bar{x}) \cup S_{00}(\bar{x}) \cup S_{0-}(\bar{x})$, $S_+(\bar{x}) = S_{+0}(\bar{x}) \cup S_{+-}(\bar{x})$, $S_{UV}(\bar{x}) = S_0(\bar{x}) \cup S_{+0}(\bar{x})$.*

## 3. Optimality conditions

In the sequel, the KKT-necessary optimality conditions of the (weak) Pareto solution for (FPVC) are presented. Firstly, we introduce the following VC-Cottle constraint qualification given by Antczak [7].

**Definition 3.1.** *[7] The VC-Cottle constraint qualification is fulfilled at $\bar{x} \in D$ for (FPVC) if either $h_j(\bar{x}) < 0$, $\forall j \in J$, $U_s(\bar{x}) > 0$ and $V_s(\bar{x}) < 0$, $\forall s \in S$ or*

$$0 \notin conv\{\partial_c h_j(\bar{x}),\ j \in J(\bar{x}),\ -\partial_c U_s(\bar{x}),\ s \in S,\ \partial_c(V_s U_s)(\bar{x}),\ s \in S\} \quad if\ S_{00}(\bar{x}) = \emptyset,$$
$$0 \notin conv\{\partial_c h_j(\bar{x}),\ j \in J(\bar{x}),\ -\partial_c U_s(\bar{x}),\ s \in S,\ \partial_c V_s(\bar{x}),\ s \in S\} \quad if\ S_{00}(\bar{x}) \neq \emptyset.$$

**Theorem 3.1.** *Suppose that $\bar{x} \in D$ is a weak Pareto solution in (FPVC) and that the VC-Cottle constraint qualification is satisfied at $\bar{x}$. Then there exist $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$ and $\gamma^V \in \mathbb{R}^q$ such that certain conditions hold:*

$$0 \in \sum_{i=1}^{p} \alpha_i(\partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x})) + \sum_{j=1}^{m} \beta_j \partial_c h_j(\bar{x}) - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(\bar{x}) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(\bar{x}), \tag{3.1}$$

$$\beta_j h_j(\bar{x}) = 0,\ j \in J, \tag{3.2}$$

$$\alpha \geq 0,\ \beta \geqq 0, \tag{3.3}$$

$$\gamma_s^U U_s(\bar{x}) = 0,\ s \in S, \tag{3.4}$$

$$\gamma_s^V V_s(\bar{x}) = 0,\ s \in S, \tag{3.5}$$

$$\gamma_s^U = 0,\ s \in S_+(\bar{x}),\ \gamma_s^U \geqq 0,\ s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}),\ \gamma_s^U \in \mathbb{R},\ s \in S_{0+}(\bar{x}), \tag{3.6}$$

$$\gamma_s^V = 0,\ s \in S_{0+}(\bar{x}) \cup S_{00}(\bar{x}) \cup S_{0-}(\bar{x}) \cup S_{+-}(\bar{x}),\ \gamma_s^V \geqq 0,\ s \in S_{+0}(\bar{x}). \tag{3.7}$$

*where $r_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ ($\forall i \in I$).*

*Proof.* We define an auxiliary function $\Psi(x) : \mathbb{R}^n \to \mathbb{R}$, where

$$\Psi(x) := \max\left\{\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})},\ h_j(x),\ -U_s(x),\ U_s(x)V_s(x),\ i \in I,\ j \in J,\ s \in S\right\}. \tag{3.8}$$

Since $\bar{x}$ is a weak Pareto solution of (FPVC), it can be deduced that $\Psi(x) \geqq 0$ for all $x \in \mathbb{R}^n$, and that $\Psi(\bar{x}) = 0$. Which implies that $\Psi$ attains its global minimum at $\bar{x}$. It follows from Lemma 2.1 one has

$$0 \in \partial_c \Psi(\bar{x}). \tag{3.9}$$

Furthermore, since

$$\partial_c\left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right) = \partial_c\left(\frac{f_i(x)}{g_i(x)}\right), \tag{3.10}$$

From Lemma 2.2, one has

$$\begin{aligned}\partial_c\Psi(\bar{x}) \subset conv\Big\{\partial_c\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}\right),\ \partial_c h_j(\bar{x}),\ -\partial_c U_s(\bar{x}),\ \partial_c(U_sV_s)(\bar{x}) :\\ i \in I,\ j \in J(\bar{x}),\ s \in S_0(\bar{x}),\ s \in S_{UV}(\bar{x})\Big\}.\end{aligned} \tag{3.11}$$

*Case 1*. We suppose that $h_j(\bar{x}) < 0$, $\forall j \in J$, $U_s(\bar{x}) > 0$ and $V_s(\bar{x}) < 0$, $\forall s \in S$. Then, by (3.9) and (3.11), one has $0 \in conv\left\{\partial_c\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}\right) :\ i \in I\right\}$. Then there exist $\mu \in \mathbb{R}^p$, $\mu \geq 0$, $\sum\limits_{i=1}^{p} \mu_i = 1$ such that $0 \in \sum\limits_{i=1}^{p} \mu_i \partial_c\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}\right)$.

From Lemma 2.3, one has

$$\partial_c\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}\right) \subset \frac{g_i(\bar{x})\partial_c f_i(\bar{x}) - f_i(\bar{x})\partial_c g_i(\bar{x})}{g_i^2(\bar{x})}. \tag{3.12}$$

Thus,

$$0 \in \sum_{i=1}^{p} \mu_i \frac{1}{g_i(\bar{x})}\left(\partial_c f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})}\partial_c g_i(\bar{x})\right).$$

Setting $r_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ and $\alpha_i = \mu_i \frac{1}{g_i(\bar{x})}$, $\forall i \in I$, we obtain $\alpha \geq 0$ and

$$0 \in \sum_{i=1}^{p} \alpha_i\left(\partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x})\right).$$

Therefore, we have (3.1)–(3.7) by setting $\beta_j = 0,\ j \in J, \gamma_S^U = 0,\ s \in S_+(\bar{x}), \gamma_S^V = 0,\ s \in S_{+-}(\bar{x})$.

*Case 2*. If there exists $j \in J$ such that $h_j(\bar{x}) = 0$ or $s \in S$ such that $U_s(\bar{x}) = 0$ or $V_s(\bar{x}) = 0$, then there exist $\mu \in \mathbb{R}^p$, $\mu \geqq 0$, $\beta \in \mathbb{R}^m$, $\beta \geqq 0$, $\omega \in \mathbb{R}^{S_0(\bar{x})}$, $\omega \geqq 0$ and $\upsilon \in \mathbb{R}^{S_{UV}(\bar{x})}$, $\upsilon \geqq 0$ with $\sum_{i=1}^{p} u_i + \sum_{j \in J(\bar{x})} \beta_j + \sum_{s \in S_{UV}(\bar{x})} v_s = 1$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial_c\left(\frac{f_i(\bar{x})}{g_i(\bar{x})}\right) + \sum_{j \in J(\bar{x})} \beta_j \partial_c h_j(\bar{x}) - \sum_{s \in S_0(\bar{x})} \omega_s \partial_c U_s(\bar{x}) + \sum_{s \in S_{UV}(\bar{x})} \upsilon_s \partial_c(U_sV_s)(\bar{x}). \tag{3.13}$$

Therefore, we obtain

$$0 \in \sum_{i=1}^{p} \mu_i \partial_c \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} \right) + \sum_{j=1}^{m} \beta_j \partial_c h_j(\bar{x}) - \sum_{s=1}^{q} \omega_s \partial_c U_s(\bar{x}) + \sum_{s=1}^{q} \upsilon_s \partial_c (U_s V_s)(\bar{x}), \tag{3.14}$$

where $\beta_j = 0,\ j \notin J(\bar{x})$, $\omega_s = 0,\ s \notin S_0(\bar{x})$ and $\upsilon_s = 0,\ s \notin S_{UV}(\bar{x})$. From 2.3.13 in [18], one has

$$\partial_c (U_s V_s)(\bar{x}) \subset V_s(\bar{x}) \partial_c U_s(\bar{x}) + U_s(\bar{x}) \partial_c V_s(\bar{x}). \tag{3.15}$$

Let $r_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ and $\alpha_i = \mu_i \frac{1}{g_i(\bar{x})}$ for all $i \in I$. Combining (3.12), (3.14), and (3.15), we have

$$0 \in \sum_{i=1}^{p} \alpha_i \left( \partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x}) \right) + \sum_{j=1}^{m} \beta_j \partial_c h_j(\bar{x}) - \sum_{s=1}^{q} \left( \omega_s - \upsilon_s V_s(\bar{x}) \right) \partial_c U_s(\bar{x}) + \sum_{s=1}^{q} \upsilon_s U_s(\bar{x}) \partial_c V_s(\bar{x}). \tag{3.16}$$

Now, setting $\gamma_s^U = \omega_s - \upsilon_s V_s(\bar{x})$ and $\gamma_s^V = \upsilon_s U_s(\bar{x})$ for all $s \in S$, we have

$$0 \in \sum_{i=1}^{p} \alpha_i \left( \partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x}) \right) + \sum_{j=1}^{m} \beta_j \partial_c h_j(\bar{x}) - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(\bar{x}) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(\bar{x}). \tag{3.17}$$

The proofs of (3.6) and (3.7) are coupled with Theorem 3.1 in [7]. Then, (3.4) and (3.5) hold. By the VC-Cottle constraint qualification, we have Lagrange multiplier $\alpha$ is not equal to 0 (i.e., $\alpha \geq 0$). In this case, the conditions (3.1)–(3.7) hold. □

**Remark 1.** *When $\bar{x}$ is a Pareto solution of (FPVC), the conditions* (3.1)–(3.7) *hold as well. The proof of this statement is similar to that of Theorem 1 and is thus omitted in this paper. Further, note that the conditions* (3.1)–(3.7) *are KKT necessary optimality conditions due to the fact that $\alpha \neq 0$.*

**Remark 2.** *It is noted that when $g_i(x) \equiv 1$ ($\forall i \in I$), the nonsmooth multiobjective fractional optimization problems with vanishing constraints (FPVC) transforms into the nonsmooth multiobjective optimization problems with vanishing constraints (MPVC) in [7]. Consequently, Theorem 1 in our study enhances the corresponding conclusions in [7].*

**Definition 3.2.** *The point $\bar{x} \in D$ is called an S-stationary point for (FPVC) if there exist $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$ and $\gamma^V \in \mathbb{R}^q$ not equal to 0, such that the conditions*

$$0 \in \sum_{i=1}^{p} \alpha_i (\partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x})) + \sum_{j=1}^{m} \beta_j \partial_c h_j(\bar{x}) - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(\bar{x}) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(\bar{x}), \tag{3.18}$$

$$\alpha \geq 0, \beta_j \geqq 0,\ j \in J(\bar{x}),\ \beta_j = 0,\ j \notin J(\bar{x}), \tag{3.19}$$

$$\gamma_s^U = 0,\ s \in S_+(\bar{x}),\ \gamma_s^U \geqq 0,\ s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}),\ \gamma_s^U \in \mathbb{R},\ s \in S_{0+}(\bar{x}), \tag{3.20}$$

$$\gamma_s^V = 0,\ s \in S_{0+}(\bar{x}) \cup S_{00}(\bar{x}) \cup S_{0-}(\bar{x}) \cup S_{+-}(\bar{x}),\ \gamma_s^V \geqq 0,\ s \in S_{+0}(\bar{x}), \tag{3.21}$$

*hold, where $r_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ ($\forall i \in I$).*

An example is provided to demonstrate the application of Theorem 3.1.

**Example 3.1.** *Consider the problem (FPVC) with the following parameters: $I = \{1, 2\}$, $J = \{1\}$. For all $x = (x_1, x_2) \in \mathbb{R}^2$,*

$$\begin{aligned} \min \quad & F(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}\right) \\ s.t. \quad & h_1(x) = -x_1 - x_2 \leqq 0 \\ & U_1(x) = x_2 \geqq 0 \\ & U_1(x)V_1(x) = x_2(x_1 + |x_2| - 1) \leqq 0 \end{aligned}$$

*where $f_1(x) = x_1 + x_2^2$, $f_2(x) = |x_1| + |x_2|$, $g_1(x) = 1 - x_1^2$, $g_2(x) = -3x_1^2 + x_2 + 2$, $V_1(x) = x_1 + |x_2| - 1$. We have that $D = \{(x_1, x_2) \in \mathbb{R}^2 : -x_1 - x_2 \leqq 0,\ x_2 \geqq 0,\ x_2(x_1 + |x_2| - 1) \leqq 0\}$ and $\bar{x} = (0, 0) \in D$. The sets $J(\bar{x}) = \{1\}$, $S_{0-}(\bar{x}) = \{1\}$, $S_{+0}(\bar{x}) = S_{+-}(\bar{x}) = S_{0+}(\bar{x}) = S_{00}(\bar{x}) = \emptyset$, and the parameter $(r_1, r_2) = (0, 0)$. Thus, we have*

$$\partial_c(f_1 - r_1 g_1)(\bar{x}) = \{(1, 0)\},$$

$$\partial_c(f_2 - r_2 g_2)(\bar{x}) = [-1, 1] \times [-1, 1],$$

$$\partial_c h_1(\bar{x}) = \{(-1, -1)\},$$

$$\partial_c U_1(\bar{x}) = \{(0, 1)\},$$

$$\partial_c V_1(\bar{x}) = \{1\} \times [-1, 1],$$

$$\partial_c(U_1 V_1)(\bar{x}) = \{(0, -1)\}.$$

*Since $0 \notin conv\{\partial_c h_1(\bar{x}),\ -\partial_c U_1(\bar{x}),\ \partial_c(U_1 V_1)(\bar{x})\}$ when $S_{00}(\bar{x}) = \emptyset$, the VC-Cottle constraint qualification is fulfilled at $\bar{x}$. Further, there exist $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = \frac{1}{2}$, $\gamma_1^U = 0$, $\gamma_1^V = 0$, and $\xi_1 = (1, 0) \in \partial_c(f_1 - r_1 g_1)(\bar{x})$, $\xi_2 = (0, 1) \in \partial_c(f_2 - r_2 g_2)(\bar{x})$, $\rho_1 = (-1, -1) \in \partial_c h_1(\bar{x})$, $\delta_1 = (0, 1) \in \partial_c U_1(\bar{x})$, $\nu_1 = (1, -1) \in \partial_c V_1(\bar{x})$ satisfying $\alpha_1 \xi_1 + \alpha_2 \xi_2 + \beta_1 \rho_1 - \gamma_1^U \nu_1 + \gamma_1^V \nu_2 = 0$, that is*

$$0 \in \sum_{i=1}^{2} \alpha_i(\partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x})) + \beta_1 \partial_c h_1(\bar{x}) - \gamma_1^U \partial_c U_1(\bar{x}) + \gamma_1^V \partial_c V_1(\bar{x}).$$

*Hence, the conditions of Theorem 1 are met.*

**Definition 3.3.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function.*

*(i) $f$ is said to be generalized quasiconvex at $\bar{x}$ if, for each $x \in \mathbb{R}^n$,*

$$f(x) \leqq f(\bar{x}) \Longrightarrow \langle \eta,\ x - \bar{x} \rangle \leqq 0, \forall \eta \in \partial_c f(\bar{x}).$$

*(ii) $f$ is said to be strictly generalized quasiconvex at $\bar{x}$ if, for each $x \in \mathbb{R}^n$ with $x \neq \bar{x}$,*

$$f(x) \leqq f(\bar{x}) \Longrightarrow \langle \eta,\ x - \bar{x} \rangle < 0, \forall \eta \in \partial_c f(\bar{x}).$$

**Lemma 3.1.** *[8] Let $f_0$ be strictly generalized quasiconvex and $f_1, f_2, \ldots, f_s$ be generalized quasiconvex at $\bar{x}$. If $\lambda_0 > 0$ and $\lambda_l \geqq 0,\ l = 1, \ldots, s$, then $\sum_{l=0}^{s} \lambda_l f_l$ is strictly generalized quasiconvex at $\bar{x}$.*

Let $\bar{x} \in D$ be an S-stationary point for (FPVC). According to Definition 3.2, if there exist $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$ and $\gamma^V \in \mathbb{R}^q$ not equal to 0, such that (3.18)–(3.21) are fulfilled at $\bar{x}$, then we introduce the following denotations:

$$S_{0+}^{U+}(\bar{x}) := \{s \in S_{0+}(\bar{x}) \mid \gamma_s^U > 0\},$$
$$S_{0+}^{U-}(\bar{x}) := \{s \in S_{0+}(\bar{x}) \mid \gamma_s^U < 0\},$$
$$S_{+0}^{V+}(\bar{x}) := \{s \in S_{+0}(\bar{x}) \mid \gamma_s^V > 0\}.$$

**Theorem 3.2.** *Let $\bar{x} \in D$ be an S-stationary point for (FPVC). Suppose that the conditions* (3.18)–(3.21) *are fulfilled at $\bar{x}$ and the following assumptions are satisfied:*

*(a)* $D_{V+} := \bigcup_{t \in S_{+0}(\bar{x})}\{x \in D \backslash \{\bar{x}\} \mid V_s > 0\} = \emptyset$ *or* $S_{+0}^{V+}(\bar{x}) = \emptyset$,

*(b)* $S_{0+}^{U-}(\bar{x}) = \emptyset$.

*Additionally, it is assumed that the functions $f_i,\ i \in I,\ h_j,\ j \in J(\bar{x}),\ -g_i,\ i \in I,\ -U_s,\ s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}) \cup S_{0+}^{U+}(\bar{x})$ and $V_s,\ s \in S_{+0}(\bar{x})$ are generalized quasiconvex at $\bar{x}$. Among the functions $f_i - r_i g_i,\ i \in I,\ h_j,\ j \in J(\bar{x}),\ -U_s$ and $V_s,\ s \in S$, at least one is strictly generalized quasiconvex at $\bar{x}$. Then, $\bar{x}$ is a weak Pareto solution of (FPVC).*

*Proof.* Given that $\bar{x} \in S$ is an S-stationary point for (FPVC), it follows from Definition 3.2 that there exist $\alpha \in \mathbb{R}^p, \beta \in \mathbb{R}^m, \gamma^U \in \mathbb{R}^q$ and $\gamma^V \in \mathbb{R}^q$ such that

$$0 \in \sum_{i=1}^{p} \alpha_i(\partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x})) + \sum_{j=1}^{m} \beta_j \partial_c h_j(\bar{x}) - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(\bar{x}) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(\bar{x}),$$

and (3.19)–(3.21) hold. Then, there are $\xi_i \in \partial_c f_i(\bar{x}) - r_i \partial_c g_i(\bar{x}),\ i \in I, \rho_j \in \partial_c h_j(\bar{x}),\ j \in J, \delta_s \in \partial_c U_s(\bar{x})$ and $\nu_s \in \partial_c V_s(\bar{x}),\ s \in S$, such that

$$0 = \sum_{i=1}^{p} \alpha_i \xi_i + \sum_{j=1}^{m} \beta_j \rho_j - \sum_{s=1}^{q} \gamma_s^U \delta_s + \sum_{s=1}^{q} \gamma_s^V \nu_s. \tag{3.22}$$

Assuming the contrary, if $\bar{x}$ is not a weak Pareto solution of (FPVC), then there exists $\tilde{x} \in S$ that satisfies

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} - \frac{f_i(\bar{x})}{g_i(\bar{x})} < 0.$$

Therefore, one has

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} - \frac{f_i(\bar{x})}{g_i(\bar{x})} < 0 \Longleftrightarrow f_i(\tilde{x}) - r_i g_i(\tilde{x}) < 0,$$

where $r_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ $(\forall i \in I)$. Thus, there exists $\alpha \in \mathbb{R}^p,\ \alpha \geq 0$, such that

$$\sum_{i=1}^{p} \alpha_i(f_i(\tilde{x}) - r_i g_i(\tilde{x})) < 0 = \sum_{i=1}^{p} \alpha_i \left(f_i(\bar{x}) - r_i g_i(\bar{x})\right). \tag{3.23}$$

By $\tilde{x} \in S$ and Definition 3.2, we have

$$\sum_{j=1}^{m} \beta_i h_j(\tilde{x}) \leqq \sum_{j=1}^{m} \beta_i h_j(\bar{x}). \tag{3.24}$$

According to the conditions $(a)$ and $(b)$, one has

$$-\sum_{s=1}^{q}\gamma_s^U U_s(\tilde{x}) \leqq -\sum_{s=1}^{q}\gamma_s^U U_s(\bar{x}), \tag{3.25}$$

$$\sum_{s=1}^{q}\gamma_s^V V_s(\tilde{x}) \leqq \sum_{s=1}^{q}\gamma_s^V V_s(\bar{x}). \tag{3.26}$$

Thus, combining (3.23)–(3.26), we have

$$\begin{aligned}&\sum_{i=1}^{p}\alpha_i(f_i(\tilde{x})-r_i g_i(\tilde{x}))+\sum_{j=1}^{m}\beta_i h_j(\tilde{x})-\sum_{s=1}^{q}\gamma_s^U U_s(\tilde{x})+\sum_{s=1}^{q}\gamma_s^V V_s(\tilde{x})\\ <&\sum_{i=1}^{p}\alpha_i(f_i(\bar{x})-r_i g_i(\bar{x}))+\sum_{j=1}^{m}\beta_i h_j(\bar{x})-\sum_{s=1}^{q}\gamma_s^U U_s(\bar{x})+\sum_{s=1}^{q}\gamma_s^V V_s(\bar{x}).\end{aligned} \tag{3.27}$$

By the generalized quasiconvex hypotheses of the functions $f_i$ and $-g_i$, $\forall i \in I$, it can be deduced that the function $f_i - r_i g_i(\forall i \in I)$ is generalized quasiconvex at $\bar{x}$, where $r_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} \geqq 0$ for all $i \in I$. By applying Lemma 2.2 , it follows that

$$\sum_{i=1}^{p}\alpha_i(f_i(x)-r_i g_i(x))+\sum_{j=1}^{m}\beta_i h_j(x)-\sum_{s=1}^{q}\gamma_s^U U_s(x)+\sum_{s=1}^{q}\gamma_s^V V_s(x)$$

is strictly generalized quasiconvex at $\bar{x}$, and

$$\begin{aligned}&\sum_{i=1}^{p}\alpha_i\xi_i+\sum_{j=1}^{m}\beta_j\rho_j-\sum_{s=1}^{q}\gamma_s^U\delta_s+\sum_{s=1}^{q}\gamma_s^V\nu_s\\ \in&\partial_c\left(\sum_{i=1}^{p}\alpha_i(f_i(\bar{x})-r_i g_i(\bar{x}))+\sum_{j=1}^{m}\beta_i h_j(\tilde{x})-\sum_{s=1}^{q}\gamma_s^U U_s(\bar{x})+\sum_{s=1}^{q}\gamma_s^V V_s(\bar{x})\right).\end{aligned}$$

Therefore,

$$\left\langle\sum_{i=1}^{p}\alpha_i\xi_i+\sum_{j=1}^{m}\beta_j\rho_j-\sum_{s=1}^{q}\gamma_s^U\delta_s+\sum_{s=1}^{q}\gamma_s^V\nu_s,\ \tilde{x}-\bar{x}\right\rangle<0,$$

which contradicts (3.22). □

**Example 3.2.** *In Example 3.1, the functions* $f_1$, $f_2$, $h_1$, $-g_1$, $-g_2$, $-U_1$ *and* $V_1$ *are generalized quasiconvex at* $\bar{x}$ *on* $D$, $g_i(x) > 0$ *and* $r_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} \geqq 0$, $\forall i \in I$. *One can see that* $\bar{x} = (0,0)$ *is an S-stationary point of the problem in Example 3.1. Since* $S_{+0}(\bar{x}) = S_{0+}(\bar{x}) = \emptyset$, *the conditions* $(a)$ *and* $(b)$ *are satisfied. Furthermore, we can verify that* $h_1$ *is strictly generalized quasiconvex at* $\bar{x}$. *In fact, for any* $x = (x_1, x_2) \neq \bar{x}$ *satisfying* $h_1(x) \leqq h_1(\bar{x}) = 0$, *then* $x_1 + x_2 > 0$, *and so* $\langle \eta,\ x - \bar{x}\rangle = -x_1 - x_2 < 0$, *where* $\eta \in \partial_c h_1(\bar{x}) = \{(-1,-1)\}\}$. *Therefore,* $\bar{x} = (0,0)$ *is a weak Pareto solution.*

## 4. Duality theorems

The aim of this section is to consider the Wolfe and Mond–Weir-type dual problems for (FPVC). We prove the duality results between (FPVC) and its dual problems under the generalized quasiconvexity and strictly generalized quasiconvexity assumptions imposed on the functions involved.

Let $y \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^p_+ \backslash \{0\}$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$ and $\gamma^V \in \mathbb{R}^q$. The vector Lagrange function $\Phi$ is defined as follows:

$$\Phi(y, \alpha, \beta, \gamma^U, \gamma^V) = \left(\frac{f_1(y)}{g_1(y)}, \ldots, \frac{f_p(y)}{g_p(y)}\right) + \left(\sum_{j=1}^{m} \beta_j h_j(y) - \sum_{s=1}^{q} \gamma_s^U U_s(y) + \sum_{s=1}^{q} \gamma_s^V V_s(y)\right) e,$$

where $e = (1, \cdots, 1) \in \mathbb{R}^p$.

For any $\bar{x} \in D$, the Wolfe-type dual model $(D_W(\bar{x}))$ associated with the problem (FPVC) is defined as:

$$\begin{aligned}
(D_W(\bar{x})) \quad \mathbb{R}^p_+ - \max \quad & \Phi(y, \alpha, \beta, \gamma^U, \gamma^V) \\
\text{s.t.} \quad & 0 \in \sum_{i=1}^{p} \alpha_i(\partial_c f_i(y) - r_i \partial_c g_i(y)) + \sum_{j=1}^{m} \beta_j \partial_c h_j(y) \\
& - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(y) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(y), \\
& \alpha \geq 0, \ \sum_{i=1}^{p} \alpha_i = 1, \ \beta_j \geqq 0, \ j \in J, \\
& \gamma_s^U = \upsilon_s U_s(\bar{x}), \ \upsilon_s \geqq 0, \ s \in S, \\
& \gamma_s^V = \omega_s - \upsilon_s V_s(\bar{x}), \ \omega_s \geqq 0, \ s \in S.
\end{aligned}$$

Let

$$\Omega_W(\bar{x}) = \{(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) : verifying\ the\ constraints\ of\ (D_W(\bar{x}))\},$$

denote the feasible set of $(D_W(\bar{x}))$.

The other Wolfe-type dual model, which does not rely on $\bar{x}$, is

$$\begin{aligned}
(D_W) \quad \mathbb{R}^p_+ - \max \quad & \Phi(y, \alpha, \beta, \gamma^U, \gamma^V) \\
\text{s.t.} \quad & (y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_W
\end{aligned}$$

where the sets $\Omega_W = \bigcap_{\bar{x} \in S} \Omega_W(\bar{x})$.

**Definition 4.1.** *The point $(\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}^U, \tilde{\gamma}^V, \tilde{\upsilon}, \tilde{\omega}) \in \Omega_W$ is said to be a Pareto solution of $(D_W)$, if there is no $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_W$ satisfying*

$$\Phi(\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}^U, \tilde{\gamma}^V) \leq \Phi(y, \alpha, \beta, \gamma^U, \gamma^V).$$

In what follows, weak, strong, and converse duality theorems between (FPVC) and the Wolfe type duality problem $(D_W)$ are given.

**Theorem 4.1.** *(Weak duality) Let $\bar{x} \in D$ and $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_W$ be any feasible solutions for (FPVC) and $(D_W)$, respectively. If $\Phi(\cdot, \alpha, \beta, \gamma^U, \gamma^V)$ is strictly generalized quasiconvex at $y \in \mathbb{R}^n$, then*

$$F(\bar{x}) \nleq \Phi(y, \alpha, \beta, \gamma^U, \gamma^V).$$

*Proof.* Suppose, contrary to the result, that

$$F(\bar{x}) \leq \Phi(y, \alpha, \beta, \gamma^U, \gamma^V).$$

That is

$$F(\bar{x}) \leq F(y) + \sum_{j=1}^{m} \beta_j h_j(y) - \sum_{s=1}^{q} \gamma_s^U U_s(y) + \sum_{s=1}^{q} \gamma_s^V V_s(y), \tag{4.1}$$

By $\bar{x} \in D$, it holds that

$$h_j(\bar{x}) = 0,\ \beta_j \geqq 0,\ j \in J(\bar{x}),$$
$$h_j(\bar{x}) < 0,\ \beta_j = 0,\ j \notin J(\bar{x}),$$
$$-U_s(\bar{x}) < 0,\ \gamma_s^U = 0,\ s \in S_{+}(\bar{x}),$$
$$U_s(\bar{x}) = 0,\ \gamma_s^U \geqq 0,\ s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}),$$
$$U_s(\bar{x}) = 0,\ \gamma_s^U \in \mathbb{R},\ s \in S_{0+}(\bar{x}),$$
$$V_s(\bar{x}) > 0,\ \gamma_s^V = 0, s \in S_{0+}(\bar{x}),$$
$$V_s(\bar{x}) = 0,\ \gamma_s^V \geqq 0, s \in S_{00}(\bar{x}) \cup S_{+0}(\bar{x}),$$
$$V_s(\bar{x}) < 0,\ \gamma_s^V = 0, s \in S_{0-}(\bar{x}) \cup S_{+-}(\bar{x}).$$

Thus,

$$\sum_{j=1}^{m} \beta_j h_j(\bar{x}) - \sum_{s=1}^{q} \gamma_s^U U_s(\bar{x}) + \sum_{s=1}^{q} \gamma_s^V V_s(\bar{x}) \leqq 0. \tag{4.2}$$

In (4.1) and (4.2), we have

$$\Phi(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V) \leq \Phi(y, \alpha, \beta, \gamma^U, \gamma^V).$$

By utilizing the strictly generalized quasiconvex at $y \in \mathbb{R}^n$ of $\Phi(\cdot, \alpha, \beta, \gamma^U, \gamma^V)$, it can be deduced that there exist $\bar{\xi}_i \in \partial_c f_i(y) - r_i \partial_c g_i(y),\ i \in I, \bar{\rho}_j \in \partial_c h_j(y),\ j \in J, \bar{\delta}_s \in \partial_c U_s(y),\ \bar{\nu}_s \in \partial_c V_s(y),\ s \in S$, such that

$$\left\langle \sum_{i=1}^{p} \alpha_i \bar{\xi}_i + \sum_{j=1}^{m} \beta_j \bar{\rho}_j - \sum_{s=1}^{q} \gamma_s^U \bar{\delta}_s + \sum_{s=1}^{q} \gamma_s^V \bar{\nu}_s,\ \bar{x} - y \right\rangle < 0.$$

This contradicts the constraint of $(D_W)$. □

**Theorem 4.2.** *(Strong duality) Let $\bar{x} \in D$ be a weak Pareto solution of problem (FPVC), and suppose the VC-Cottle constraint qualification is fulfilled at $\bar{x}$., then there exist Lagrange multipliers $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$, $\gamma^V \in \mathbb{R}^q$, $\upsilon \in \mathbb{R}^q$ and $\omega \in \mathbb{R}^q$ such that $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is feasible in $(D_W)$ and $F(\bar{x}) = \Phi(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V)$. If $\Phi(\cdot, \alpha, \beta, \gamma^U, \gamma^V)$ is strictly generalized quasiconvex at $y \in \mathbb{R}^n$, then $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is a Pareto solution of $(D_W)$.*

*Proof.* From the assumptions that $\bar{x} \in D$ and the VC-Cottle constraint qualification holds, there exist $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$ and $\gamma^V \in \mathbb{R}^q$ such that the necessary optimality conditions (Theorem 3.1) are fulfilled. Then, by the definition of $\Omega_W$ and (3.1)–(3.7), we conclude that $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is feasible in $(D_W)$ and

$$\sum_{j=1}^{m} \beta_j h_j(\bar{x}) - \sum_{s=1}^{q} \gamma_s^U U_s(\bar{x}) + \sum_{s=1}^{q} \gamma_s^V V_s(\bar{x}) = 0.$$

Thus, $F(\bar{x}) = \Phi(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V)$.

Suppose, on the contrary, that $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is not a Pareto solution of $(D_W)$, then we have $(\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma^U}, \tilde{\gamma^V}, \tilde{\upsilon}, \tilde{\omega})$ such that

$$\Phi(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V) \leq \Phi(\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma^U}, \tilde{\gamma^V}).$$

Then, $F(\bar{x}) \leq \Phi(\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma^U}, \tilde{\gamma^V})$, which contradicts Theorem 4.1. □

**Theorem 4.3.** *(Converse duality) Suppose that $\bar{x} \in D$ is a feasible solution of (FPVC), $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is a weak Pareto solution of $(D_W)$, and the inequalities*

$$\begin{cases} \beta_j h_j(y) \geqq 0, & \forall\ j \in J \\ -\gamma_s^U U_s(y) \geqq 0, & \forall\ s \in S \\ \gamma_s^V V_s(y) \geqq 0, & \forall\ s \in S \end{cases} \tag{4.3}$$

*hold, such that $y \in D$. If one of the following assumptions is fulfilled:*

*(i) $\Phi(\cdot, \alpha, \beta, \gamma^U, \gamma^V)$ is strictly generalized quasiconvex at $y$;*

*(ii) $f_i \geqq 0$, $-g_i < 0 (i \in I)$ are strictly generalized quasiconvex at $y$, $h_j$, $j \in J(\bar{x})$, $-U_s$, $s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}) \cup S_{0+}^{U+}(\bar{x})$ and $V_s$, $s \in S_{+0}(\bar{x})$ are generalized quasiconvex at $\bar{x}$,*
*then $y$ is a Pareto solution in $(FPVC)$.*

*Proof.* Suppose, on the contrary, that $y \in D$ is not a Pareto solution in $(FPVC)$. Then, there exists $\tilde{y} \in D$ such that

$$F(\tilde{y}) \leq F(y). \tag{4.4}$$

For the assumption $(i)$, since $\tilde{y}$ and $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ are feasible points for (FPVC) and $(D_W)$, respectively, combined with (4.2) and (4.3), one gets

$$\sum_{j=1}^{m} \beta_j h_j(\tilde{y}) - \sum_{s=1}^{q} \gamma_s^U U_s(\tilde{y}) + \sum_{s=1}^{q} \gamma_s^V V_s(\tilde{y}) \leqq \sum_{j=1}^{m} \beta_j h_j(y) - \sum_{s=1}^{q} \gamma_s^U U_s(y) + \sum_{s=1}^{q} \gamma_s^V V_s(y),$$

Hence,

$$\Phi(\tilde{y}, \alpha, \beta, \gamma^U, \gamma^V) \leqq \Phi(y, \alpha, \beta, \gamma^U, \gamma^V).$$

Due to the fact that $\Phi(\cdot, \alpha, \beta, \gamma^U, \gamma^V)$ is strictly generalized quasiconvex at $y$, there exist $\bar{\xi}_i \in \partial_c f_i(y) - r_i \partial_c g_i(y)$, $i \in I$, $\bar{\rho}_j \in \partial_c h_j(y)$, $j \in J$, $\bar{\delta}_s \in \partial_c U_s(y)$ and $\bar{\nu}_s \in \partial_c V_s(y)$, $s \in S$, such that

$$\left\langle \sum_{i=1}^{p} \alpha_i \bar{\xi}_i + \sum_{j=1}^{m} \beta_j \bar{\rho}_j - \sum_{s=1}^{q} \gamma_s^U \bar{\delta}_s + \sum_{s=1}^{q} \gamma_s^V \bar{\nu}_s,\ \tilde{y} - y \right\rangle < 0.$$

This contradicts the constraint of $(D_W)$.

For the assumption (*ii*), since $\tilde{y}$ and $(y,\alpha,\beta,\gamma^U,\gamma^V,\upsilon,\omega)$ are feasible points for (FPVC) and $(D_W)$ respectively, by (4.3), we have

$$\begin{aligned}&\beta_j h_j(\tilde{y}) \leqq \beta_j h_j(y), \quad \forall\ j \in J\\&-\gamma_s^U U_s(\tilde{y}) \leqq -\gamma_s^U U_s(y), \quad \forall\ s \in S\\&\gamma_s^V V_s(\tilde{y}) \leqq \gamma_s^V G_i(y). \quad \forall\ s \in S\end{aligned}$$

Thus

$$\begin{cases} h_j(\tilde{y}) \leqq h_j(y),\ \forall\ j \in J(\tilde{y})\\ -U_s(\tilde{y}) \leqq -U_s(y),\ \forall\ s \in S_{00}(\tilde{y}) \cup S_{0-}(\tilde{y}) \cup S_{0+}^{U+}(\tilde{y})\\ -U_s(\tilde{y}) \geqq -U_s(y),\ \forall\ s \in S_{0+}^{U-}(\tilde{y})\\ V_s(\tilde{y}) \leqq V_s(y).\ \forall\ s \in S_{+0}(\tilde{y}) \end{cases} \tag{4.5}$$

Using the generalized quasiconvex of the functions in assumption (*ii*) and (4.5), the inequalities

$$\left\langle \bar{\rho}_j,\ \tilde{y}-y \right\rangle \leqq 0,\ \beta_j \geqq 0,\ \forall \bar{\rho}_j \in \partial_c h_j(y),\ j \in J(\tilde{y}),$$

$$\left\langle -\bar{\delta}_s,\ \tilde{y}-y \right\rangle \leqq 0,\ \gamma_s^U \geqq 0,\ \forall \bar{\delta}_s \in \partial_c U_s(y),\ s \in S_{00}(\tilde{y}) \cup S_{0-}(\tilde{y}) \cup S_{0+}^{U+}(\tilde{y}),$$

$$\left\langle -\bar{\delta}_s,\ \tilde{y}-y \right\rangle \geqq 0,\ \gamma_s^U < 0,\ \forall \bar{\delta}_s \in \partial_c U_s(y),\ s \in S_{0+}^{U-}(\tilde{y}),$$

$$\langle \bar{\nu}_s,\ \tilde{y}-y \rangle \leqq 0,\ \gamma_s^V \geqq 0,\ \forall \bar{\nu}_s \in \partial_c V_s(y),\ s \in S_{+0}(\tilde{y}),$$

hold, that is

$$\left\langle \sum_{j=1}^{m} \beta_j \bar{\rho}_j - \sum_{s=1}^{q} \gamma_s^U \bar{\delta}_s + \sum_{s=1}^{q} \gamma_s^V \bar{\nu}_s,\ \tilde{y}-y \right\rangle \leqq 0.$$

Since $0 \in \sum_{i=1}^{p} \alpha_i(\partial_c f_i(y) - r_i \partial_c g_i(y)) + \sum_{j=1}^{m} \beta_j \partial_c h_j(y) - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(y) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(y)$, there exists $\bar{\xi}_i \in \partial_c f_i(y) - r_i \partial_c g_i(y),\ i \in I$, such that

$$\left\langle \sum_{i=1}^{p} \alpha_i \bar{\xi}_i,\ \tilde{y}-y \right\rangle \geqq 0. \tag{4.6}$$

By $F(\tilde{y}) - F(y) \le 0 \Longleftrightarrow f_i(\tilde{y}) - \hat{r}_i g_i(\tilde{y}) \le 0$, where $\hat{r}_i = \frac{f_i(y)}{g_i(y)},\ \forall i \in I$. Hence, there exists $\alpha \in \mathbb{R}^p,\ (\alpha \ge 0)$, such that

$$\sum_{i=1}^{p} \alpha_i (f_i(\tilde{y}) - \hat{r}_i g_i(\tilde{y})) \le 0 = \sum_{i=1}^{p} \alpha_i \left(f_i(y) - \hat{r}_i g_i(y)\right). \tag{4.7}$$

For all $i \in I$, the functions $f_i \geqq 0$ and $-g_i < 0$ are strictly generalized quasiconvex at $y$ and $\hat{r}_i = \frac{f_i(y)}{g_i(y)} \geqq 0$, it follows that $f_i - \hat{r}_i * g_i$ $(\forall i \in I)$ is strictly generalized quasiconvex at $y$. Then, there exists $\bar{\xi}_i \in \partial_c f_i(y) - r_i \partial_c g_i(y)$ $(\forall i \in I)$ such that

$$\left\langle \sum_{i=1}^{p} \alpha_i \bar{\xi}_i,\ \tilde{y}-y \right\rangle < 0.$$

This contradicts (4.6). □

For $\bar{x} \in D$, the Mond–Weir type dual model $(D_{MW}(\bar{x}))$ is follows:

$$\begin{aligned} (D_{MW}(\bar{x})) \quad \mathbb{R}^p_+ - \max \quad & F(y) \\ \text{s.t.} \quad & 0 \in \sum_{i=1}^{p} \alpha_i(\partial_c f_i(y) - r_i \partial_c g_i(y)) + \sum_{j=1}^{m} \beta_j \partial_c h_j(y) \\ & - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(y) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(y), \\ & \alpha \geq 0, \ \sum_{i=1}^{p} \alpha_i = 1, \\ & \beta_j h_j(y) = 0, \ \beta_j \geqq 0, \ j \in J, \\ & -\gamma_s^U U_s(y) \geqq 0, \ \gamma_s^V V_s(y) \geqq 0, \ s \in S, \\ & \gamma_s^U = \upsilon_s U_s(\bar{x}), \ \upsilon_s \geqq 0, \ s \in S, \\ & \gamma_s^V = \omega_s - \upsilon_s V_s(\bar{x}), \ \omega_s \geqq 0, \ s \in S. \end{aligned}$$

Let

$$\Omega_{MW}(\bar{x}) = \{(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) : verifying\ the\ constraints\ of\ (D_{MW}(\bar{x}))\},$$

denote the feasible set of $(D_{MW}(\bar{x}))$.

The other Mond–Weir type dual model, which does not rely on $\bar{x}$, is

$$\begin{aligned} (D_{MW}) \quad \mathbb{R}^p_+ - \max \quad & F(y) \\ \text{s.t.} \quad & (y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_{MW} \end{aligned}$$

where $\Omega_{MW} = \bigcap_{\bar{x} \in S} \Omega_{MW}(\bar{x})$.

**Definition 4.2.** *The point $(\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}^U, \tilde{\gamma}^V, \tilde{\upsilon}, \tilde{\omega}) \in \Omega_{MW}$ is said to be a Pareto solution of $(D_{MW})$, if there is no $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_{MW}$ satisfying*

$$F(\tilde{y}) \leq F(y).$$

Next, we present the duality theorems between (FPVC) and the Mond–Weir type dual problem $(D_{MW})$.

**Theorem 4.4.** *(Weak duality) Let $\bar{x} \in D$ and $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_{MW}$ be any feasible solutions for (FPVC) and $(D_{MW})$, respectively. If one of the following assumptions is fulfilled:*

*(i) $f_i \geqq 0$, $-g_i < 0 (i \in I)$ are strictly generalized quasiconvex at $y$, $\sum_{j=1}^{m} \beta_j h_j(\cdot) - \sum_{s=1}^{q} \gamma_s^U U_s(\cdot) + \sum_{s=1}^{q} \gamma_s^V V_s(\cdot)$ is generalized quasiconvex at $y$;*

*(ii) $f_i \geqq 0$, $-g_i < 0 (i \in I)$ are strictly generalized quasiconvex at $y$, $h_j$, $j \in J(\bar{x})$, $-U_s$, $s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}) \cup S_{0+}^{U+}(\bar{x})$ and $V_s$, $s \in S_{+0}(\bar{x})$ are generalized quasiconvex at $\bar{x}$,*
*then*

$$F(\bar{x}) \nleq F(y).$$

*Proof.* Suppose, contrary to the result, that

$$F(\bar{x}) \leq F(y).$$

By the strictly generalized quasiconvex at $y$ of $f_i$ and $-g_i$ ($\forall i \in I$), we have that $f_i - \hat{r}_i g_i(\forall i \in I)$ is strictly generalized quasiconvex at $y$, where $\hat{r}_i = \frac{f_i(y)}{g_i(y)} \geqq 0$. Then, there exists $\bar{\xi}_i \in \partial_c f_i(y) - r_i \partial_c g_i(y)$ for all $i \in I$ such that

$$\left\langle \sum_{i=1}^{p} \alpha_i \bar{\xi}_i,\ \tilde{y} - y \right\rangle < 0. \tag{4.8}$$

For the assumption $(i)$, since $\bar{x} \in D$ and $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_{MW}$, it follows that

$$\sum_{j=1}^{m} \beta_j h_j(\bar{x}) - \sum_{s=1}^{q} \gamma_s^U U_s(\bar{x}) + \sum_{s=1}^{q} \gamma_s^V V_s(\bar{x}) \leqq \sum_{j=1}^{m} \beta_j h_j(y) - \sum_{s=1}^{q} \gamma_s^U U_s(y) + \sum_{s=1}^{q} \gamma_s^V V_s(y).$$

Since $\sum_{j=1}^{m} \beta_j h_j(\cdot) - \sum_{s=1}^{q} \gamma_s^U U_s(\cdot) + \sum_{s=1}^{q} \gamma_s^V V_s(\cdot)$ is generalized quasiconvex at $y$, there are $\bar{\rho}_j \in \partial_c h_j(y),\ j \in J$, $\bar{\delta}_s \in \partial_c U_s(y),\ \bar{\nu}_s \in \partial_c V_s(y),\ s \in S$, such that

$$\left\langle \sum_{j=1}^{m} \beta_j \bar{\rho}_j - \sum_{s=1}^{q} \gamma_s^U \bar{\delta}_s + \sum_{s=1}^{q} \gamma_s^V \bar{\nu}_s,\ \bar{x} - y \right\rangle \leqq 0.$$

According to the condition

$$0 \in \sum_{i=1}^{p} \alpha_i (\partial_c f_i(y) - r_i \partial_c g_i(y)) + \sum_{j=1}^{m} \beta_j \partial_c h_j(y) - \sum_{s=1}^{q} \gamma_s^U \partial_c U_s(y) + \sum_{s=1}^{q} \gamma_s^V \partial_c V_s(y),$$

there exists $\bar{\xi}_i \in \partial_c f_i(y) - r_i \partial_c g_i(y),\ i \in I$ satisfying

$$\left\langle \sum_{i=1}^{p} \alpha_i \bar{\xi}_i,\ \bar{x} - y \right\rangle \geqq 0,$$

which contradicts (4.8).

For the assumption $(ii)$, since $\bar{x} \in D$ and $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega) \in \Omega_{MW}$, it holds that

$$\begin{cases} h_j(\bar{x}) \leqq h_j(y),\ \forall\ j \in J(\bar{x}) \\ -U_s(\bar{x}) \leqq -U_s(y),\ \forall\ s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}) \cup S_{0+}^{U+}(\bar{x}) \\ -U_s(\bar{x}) \geqq -U_s(y),\ \forall\ s \in S_{0+}^{U-}(\bar{x}) \\ V_s(\bar{x}) \leqq V_s(y).\ \forall\ s \in S_{+0}(\bar{x}) \end{cases} \tag{4.9}$$

By the generalized quasiconvex of the functions in conditions $(ii)$ and (4.9), the inequalities

$$\left\langle \bar{\rho}_j,\ \bar{x} - y \right\rangle \leqq 0,\ \beta_j \geqq 0,\ \forall \bar{\rho}_j \in \partial_c h_j(y),\ j \in J(\bar{x})$$

$$\left\langle -\bar{\delta}_s,\ \bar{x} - y \right\rangle \leqq 0,\ \gamma_s^U \geqq 0,\ \forall \bar{\delta}_s \in \partial_c U_s(y),\ s \in S_{00}(\bar{x}) \cup S_{0-}(\bar{x}) \cup S_{0+}^{U+}(\bar{x})$$

$$\left\langle -\bar{\delta}_s,\ \bar{x} - y \right\rangle \geqq 0,\ \gamma_s^U < 0,\ \forall \bar{\delta}_s \in \partial_c U_s(y),\ s \in S_{0+}^{U-}(\bar{x})$$

$$\langle \bar{\nu}_s,\ \bar{x} - y \rangle \leqq 0,\ \gamma_s^U \geqq 0,\ \forall \bar{\delta}_s \in \partial_c U_s(y),\ s \in S_{+0}(\bar{x})$$

hold, that is

$$\left\langle \sum_{j=1}^{m} \beta_j \bar{\rho}_j - \sum_{s=1}^{q} \gamma_s^U \bar{\delta}_s + \sum_{s=1}^{q} \gamma_s^V \bar{\nu}_s,\ \bar{x} - y \right\rangle \leqq 0.$$

The rest of the proof is omitted because it is consistent with assumption ($i$). □

**Theorem 4.5.** *Let $\bar{x} \in D$ be a weak Pareto solution of the problem (FPVC). The VC-Cottle constraint qualification holds at $\bar{x}$. Then, there exist $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$, $\gamma^V \in \mathbb{R}^q$, $\bar{\upsilon} \in \mathbb{R}^q$ and $\bar{\beta} \in \mathbb{R}^q$ such that $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is feasible in $(D_{MW})$. If the assumptions of Theorem 4.4 are satisfied, then $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is a Pareto solution of $(D_{MW})$.*

*Proof.* From the assumption that $\bar{x} \in D$ and the VC-Cottle constraint qualification holds at $\bar{x}$, there exist $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^m$, $\gamma^U \in \mathbb{R}^q$, and $\gamma^V \in \mathbb{R}^q$, such that necessary optimality conditions (Theorem 3.1) are fulfilled. Then, by the definitions of $\Omega_{MW}$ and (3.1)–(3.7), we conclude that $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is feasible in $(D_{MW})$.

Suppose, on the contrary, that $(\bar{x}, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ is not a Pareto solution of $(D_W)$, then we get $(\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma^U}, \tilde{\gamma^V}, \tilde{\upsilon}, \tilde{\omega}) \in \Omega_{MW}$ such that

$$F(\bar{x}) \leq F(\tilde{y}),$$

which contradicts Theorem 4.4. □

**Theorem 4.6.** *(Converse duality) Let $\bar{x} \in D$ be feasible in (FPVC) and $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ be a weak Pareto solution in $(D_{MW})$ such that $y \in D$. If the hypotheses of Theorem 4.4 hold, then $y$ is a Pareto solution in $(FPVC)$.*

*Proof.* Suppose on the contrary that $y$ is not a Pareto solution in $(FPVC)$. Then there exists $\tilde{y} \in D$ such that

$$F(\tilde{y}) \leq F(y). \tag{4.10}$$

Since $\tilde{y} \in D$ and $(y, \alpha, \beta, \gamma^U, \gamma^V, \upsilon, \omega)$ are feasible points for (FPVC) and $(D_{MW})$, respectively, it holds that $F(\tilde{y}) \nleq F(y)$ by Theorem 4.4, which contradicts to (4.10). □

## 5. Concluding remarks

The optimality conditions and duality results for the problem (FPVC) with both inequality and vanishing constraints are presented. Utilizing the Clarke subdifferential, the necessary KKT optimality conditions are derived under the VC-Cottle constraint. By assuming generalized quasiconvexity and strictly generalized quasiconvexity, sufficient optimality conditions and duality theorems are established. The results in this paper improve the existing ones in [7]. In further research, it will be interesting to consider the second-order optimality conditions for (FPVC).

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflicts of interest.

## References


1. W. Achtziger, C. Kanzow, Mathematical programs with vanishing constraints: optimality conditions and constraint qualifications, *Math. Program.*, **114** (2008), 69–99. https://doi.org/10.1007/s10107-006-0083-3
2. S. K. Mishra, V. Singh, V. Laha, On duality for mathematical programs with vanishing constraints, *Ann. Oper. Res.*, **243** (2016), 249–272. https://doi.org/10.1007/s10479-015-1814-8
3. Q. J. Hu, J. G. Wang, Y. Chen, New dualities for mathematical programs with vanishing constraints, *Ann. Oper. Res.*, **287** (2020), 233–255. https://doi.org/10.1007/s10479-019-03409-6
4. L. T. Tung, Karush–Kuhn–Tucker optimality conditions and duality for multiobjective semi-infinite programming with vanishing constraints, *Ann. Oper. Res.*, **311** (2022), 1307–1334. https://doi.org/10.1007/s10479-020-03742-1
5. L. T. Tung, Karush-Kuhn-Tucker optimality conditions and duality for nonsmooth multiobjective semi-infinite programming problems with vanishing constraints, *Appl. Set-Valued Anal. Optim.*, **4** (2022). https://doi.org/10.23952/asvao.4.2022.1.01
6. T. Antczak, Optimality conditions and Mond–Weir duality for a class of differentiable semi-infinite multiobjective programming problems with vanishing constraints, *4OR-Q. J. Oper. Res.*, **20** (2022), 417–442. https://doi.org/10.1007/s10288-021-00482-1
7. T. Antczak, Optimality results for nondifferentiable vector optimization problems with vanishing constraints, *J. Appl. Anal. Comput.*, **13** (2023), 2613–2629. https://doi.org/10.11948/20220465
8. T. Antczak, On directionally differentiable multiobjective programming problems with vanishing constraints, *Ann. Oper. Res.*, **328** (2023), 1181–1212. https://doi.org/10.1007/s10479-023-05368-5
9. H. Huang, H. Zhu, Stationary condition for Borwein proper efffcient solutions of nonsmooth multiobjective problems with vanishing constraints, *Mathematics*, **10** (2022), 4569. https://doi.org/10.3390/math10234569
10. S. M. Guu, Y. Singh, S. K. Mishra, On strong KKT type sufficient optimality conditions for multiobjective semi-infinite programming problems with vanishing constraints, *J. Inequal. Appl.*, **2017** (2017), 282. https://doi.org/10.1186/s13660-017-1558-x
11. H. J. Wang, H. H. Wang, Optimality conditions and duality theorems for interval-valued optimization problems with vanishing constraints, *Oper. Res. Trans.*, **27** (2023), 87–102.

12. H. J. Wang, H. H. Wang, Duality theorems for nondifferentiable semi-infinite interval-valued optimization problems with vanishing constraints, *J. Inequal. Appl.*, **182** (2021). https://doi.org/10.1186/s13660-021-02717-5

13. D. S. Kim, S. J. Kim, M. H. Kim, Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems, *J. Optim. Theory Appl.*, **129** (2006), 131–146. https://doi.org/10.1007/s10957-006-9048-1

14. X. J. Long, Optimality conditions and duality for nondifferentiable multiobjective fractional programming problems with $(C, \alpha, \rho, d)$-convexity, *J. Optim. Theory Appl.*, **148** (2011), 197–208. https://doi.org/10.1007/s10957-010-9740-z

15. R. Dubey, S. K. Gupta, M. A. Khan, Optimality and duality results for a nondifferentiable multiobjective fractional programming problem, *J. Inequal. Appl.*, **345** (2015). https://doi.org/10.1186/s13660-015-0876-0

16. T. D. Chuong, Nondifferentiable fractional semi-infinite multiobjective optimization problems, *Oper. Res. Lett.*, **44** (2016), 260–266. https://doi.org/10.1016/j.orl.2016.02.003

17. X. K. Sun, X. Feng, K. L. Teo, Robust optimality, duality and saddle points for multiobjective fractional semi-infinite optimization with uncertain data, *Optim. Lett.*, **16** (2022), 1457–1476. https://doi.org/10.1007/s11590-021-01785-2

18. S. T. Van, D. D. Hang, Optimality and duality in nonsmooth multiobjective fractional programming problem with constraints, *4OR-Q. J. Oper. Res.*, **20** (2022), 105–137. https://doi.org/10.1007/s10288-020-00470-x

19. F. H. Clarke, Nonsmooth analysis and optimization, in *Proceedings of the International Congress of Mathematicians*, **5** (1983), 847–853. https://doi.org/10.1137/1.9781611971309