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Research article

Sharp conditions for the existence of infinitely many positive solutions to q-k-Hessian equation and systems

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Abstract: In this paper, only under the q-k-Keller–Osserman conditions, we consider the existence and global estimates of innumerable radial q-k-convex positive solutions to the q-k-Hessian equation and systems. Our conditions are strictly weaker than those in previous papers.

Keywords: q-k-Hessian problems; radial q-k-convex solutions; the existence of solutions

1. Introduction and main results

The aims of this presentation are to investigate the existence of positive q-k-convex radial solutions to the q-k-Hessian equation

$$S_k(D_i(|D\omega|^{q-2}D_j\omega)) = H(|x|)f(\omega), \text{ if } x \in \mathscr{D}_{\Re} \tag{1.1}$$

and systems

$$\begin{cases} S_k(D_i(|D\omega|^{q-2}D_j\omega)) = H(|x|)f(\upsilon), & \text{if } x \in \mathscr{D}_{\Re}, \\ S_k(D_i(|D\upsilon|^{q-2}D_j\upsilon)) = L(|x|)g(\omega), & \text{if } x \in \mathscr{D}_{\Re} \end{cases} \tag{1.2}$$

and

$$\begin{cases} S_k(D_i(|D\omega|^{q-2}D_j\omega)) = H(|x|)f_1(\upsilon)f_2(\omega), & \text{if } x \in \mathscr{D}_{\Re}, \\ S_k(D_i(|D\upsilon|^{q-2}D_j\upsilon)) = L(|x|)g_1(\omega)g_2(\upsilon), & \text{if } x \in \mathscr{D}_{\Re}, \end{cases} \tag{1.3}$$

where $k \in \{1, \cdots, N\}$, $D_i(|D\omega|^{q-2}D_j\omega)$ denotes the element of row i and column j in the matrix $(D_i(|D\omega|^{q-2}D_j\omega))_{ij=1,\cdots,N}$, $q \geq 2$, $\Re \leq \infty$ and

$$\mathscr{D}_{\Re} := \begin{cases} \mathbb{R}^N, & \text{if } \Re = \infty, \\ \{x \in \mathbb{R}^N : |x| < \Re\}, & \text{if } \Re < \infty. \end{cases}$$

In this article, we assume that $H, L, f, g, f_1, f_2, g_1, g_2$ satisfy

(H_1) $H, L \in C([0,\infty),(0,\infty))$;

(H_2) $f, g \in C([0,\infty),[0,\infty))$ are increasing on $[0,\infty)$;

(H_3) $f_1, f_2, g_1, g_2 \in C([0,\infty),[0,\infty))$ are increasing on $[0,\infty)$.

Let $\mathcal{M}$ be a N-order symmetric real matrix and

$$S_k(\mathcal{M}) := S_k(\lambda(\mathcal{M})) = \sum_{1\le i_1<\cdots<i_k\le N} \lambda_{i_1}\cdots\lambda_{i_k},$$

where $\lambda = (\lambda_1,\cdots,\lambda_N)$ and $\lambda_1,\cdots,\lambda_N$ are the eigenvalues of the matrix, $\mathcal{M}$. Trudinger and Wang in [1] first introduced the operator $S_k(D_i(|D\omega|^{q-2}D_j\omega))$ to establish the local integral estimates for the gradients of k-convex functions in the study of the weak continuity of the associated k-Hessian measure with respect to convergence in measure. If $k = 1$, this operator becomes the well-known q-Laplacian operator; if $q = 2$, it is the k-Hessian operator, and it is the Laplacian operator provided $k = 1$. In particular, if $k = N$ and $q = 2$, it is the famous Monge–Ampère operator.

We first review the following: Laplacian equation

$$\Delta u = f(u) \text{ in } \Omega. \tag{1.4}$$

The study of the existence, uniqueness, and asymptotic behavior of (1.4) has a long story. If $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with C^2-boundary, Bieberbach [2] in 1916 first studied the existence, uniqueness, and asymptotic behavior of classical boundary blow-up solutions to Eq (1.4) with $f(u) = e^u$. In 1943, Rademacher [3], using the ideas of Bieberbach, proved that the results still hold for $N = 3$. If $\Omega = \mathbb{R}^N$, Wittich [4] in 1944 proved that if $N = 2$ and $f(u) = e^u$, then (1.4) has no entire solution. In 1951, Haviland [5] showed that Eq (1.4) with $\Omega = \mathbb{R}^N$ has no entire solution for $N = 3$ if and only if

$$\int_1^\infty \left(\int_0^s f(t)dt\right)^{-1/2} ds < \infty. \tag{1.5}$$

In 1955, under some additional conditions, Walter [6] generalized the above result to the N-dimension case. In 1957, Keller [7] and Osserman [8] obtained two very famous theorems:
(**i**) If Ω is a bounded domain, then (1.4) has an entire subsolution if and only if f satisfies (1.5);
(**ii**) If $\Omega = \mathbb{R}^N$, then (1.4) has an entire subsolution if and only if f satisfies

$$\int_1^\infty \left(\int_0^s f(t)dt\right)^{-1/2} ds = \infty. \tag{1.6}$$

After the works of Keller [7] and Osserman [8], the conditions (1.5) and (1.6) and their generalizations are all called Keller–Osserman conditions by many authors in the literature. When H satisfies (**H_1**), Lair [9] first using (1.6) studied existence of the radial solution to (1.1) with $q = 2$ and $k = 1$. Then, Lair and Mohammed [10] proved the existence and nonexistence of nonnegative entire large solutions to a class of semilinear elliptic equations of mixed type. When H and L satisfy (**H_1**), Lair [11] consider the following system:

$$\begin{cases} \Delta u = H(|x|)v^\alpha, & x \in \mathbb{R}^N, \\ \Delta u = L(|x|)u^\beta, & x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $N \geq 3$ and α, β are positive constants. The author showed that if $\alpha\beta < 1$, then (1.7) has an entire large solution if and only if

$$\int_0^\infty sH(s)\Big[s^{2-N}\int_0^s t^{N-3}\Big(\int_0^t \tau L(\tau)d\tau\Big)dt\Big]^\alpha ds = \infty$$

and

$$\int_0^\infty sL(s)\Big[s^{2-N}\int_0^s t^{N-3}\Big(\int_0^t \tau H(\tau)d\tau\Big)dt\Big]^\beta ds = \infty.$$

For some related insights on semilinear elliptic equations, we refer readers to [12–17].

In fact, the condition (1.6) and its generalization are usually used to study the existence of entire solutions to some nonlinear elliptic equations. In 1997, Naito and Usami [18] showed that the q-Laplacian equation

$$\nabla \cdot (|\nabla u|^{q-2}\nabla u) = f(u) \text{ in } \mathbb{R}^N$$

has a positive entire subsolution $u \in C^1(\mathbb{R}^N)$ with $|\nabla u|^{p-2}\nabla u \in C^1(\mathbb{R}^N)$ if and only if f satisfies the Keller–Osserman condition

$$\int_1^\infty \Big(\int_0^s f(t)dt\Big)^{-1/q} ds = \infty.$$

In 2010, Filippucci et al. [19] proved the more general equation

$$\nabla \cdot (|\nabla u|^{q-2}\nabla u) = f(u)|\nabla u|^\theta \text{ in } \mathbb{R}^N$$

has a nonnegative, entire, unbounded subsolution if and only if f satisfies the Keller–Osserman condition

$$\int_1^\infty \Big(\int_0^s f(t)dt\Big)^{-1/(q-\theta)} ds = \infty,$$

where $\theta \in [0, q-1)$.

Next, we review the k-Hessian equation

$$S_k(D^2u) = f(u) \text{ in } \mathbb{R}^N. \tag{1.8}$$

In 2005, Jin et al. [20] proved that if $f(u) = u^\gamma$ with $\gamma > k$, then (1.8) has no entire subsolution. In 2010, Ji and Bao [21] made an important contribution to this problem, i.e., they showed that Eq (1.8) has an entire k-convex positive subsolution if and only if f satisfies the Keller–Osserman condition

$$\int_1^\infty \Big(\int_0^s f(t)dt\Big)^{-1/(k+1)} ds = \infty. \tag{1.9}$$

If f is a continuous and nondecreasing function on $\mathbb{R}$ and has a positive lower bound, Dai [22] in 2020 generalized the work of Ji and Bao [21] to a more general Hessian-type equation. When H, L, f, g satisfy $(\mathbf{H_1})$–$(\mathbf{H_2})$, $q = 2$, $k \in \{1, \cdots, N\}$ and $\Re = \infty$, Zhang and Zhou [23] in 2015 studied the existence of radial solutions to (1.1) and (1.2) by using the following integral conditions:

$$\int_1^\infty (f(\tau))^{-1/k}d\tau = \infty \text{ and } \int_1^\infty (f(\tau) + g(\tau))^{-1/k}d\tau = \infty. \tag{1.10}$$

Moreover, under some additional conditions, they also considered the existence of entire positive bounded radial solutions when (1.10) is false. In 2021, Bhattacharya and Mohammed [24] investigated a class of k-Hessian equations with lower-order terms on unbounded domains. Especially, they obtained the Phragmén–Lindelöf and Liouville type results. Let $\Re = \infty$ and $q = 2$, and H and f satisfy $(\mathbf{H_1})$–$(\mathbf{H_2})$. If H further satisfies

$$\mathscr{H}(s) := s^{k-1}H(s) - s^{k-1-N}(N-k)^2 \int_0^s t^{N-1}H(t)dt,\ s > 0 \tag{1.11}$$

and there exists some positive constant s_0 such that

$$\int_{s_0}^{\infty} \big(\min_{t\in[s_0,s]} \mathscr{H}(t)\big)^{1/(k+1)} ds = \infty,$$

Zhang and Xia [25] in 2023 showed that Eq (1.1) (with $q = 2$) has a large radial convex solution if and only if (1.9) holds. A similar result of existence was also obtained in [26]. When $f(u)$ is replaced by $b(x)f(u)$ in (1.8), where $b \in C(\mathbb{R}^N)$ is positive in $\mathbb{R}^N$ and $f \in C^1(0,\infty)$ is a nonnegative, nondecreasing function, $f(0) = 0$ and

$$\int_1^{\infty} f^{1/k}(s)ds = \infty,$$

Li and Bao [27] in 2024 showed a necessary and sufficient condition for the existence of nonradial, entire large solutions. Moreover, they also studied the asymptotic behavior of entire solutions at infinity. With regard to the other works of Monge–Ampère type equation (system), we refer readers to [28–32]. For more general Hessian type equation (system), we refer readers to [33–39].

Now, let us return to (1.1)–(1.3). As far as we know, the q-k Hessian equation (system) has rarely been investigated in previous literature. When $H(|\cdot|) \equiv 1$ in $\mathscr{D}_{\Re}$ with $\Re = \infty$, the sufficient and necessary condition for the existence of the entire subsolution to Eq (1.1) was given via the Keller–Osserman condition

$$\int_1^{\infty} \left(\int_0^t f(\tau)d\tau\right)^{-1/((q-1)k+1)} dt = \infty \tag{1.12}$$

by Bao and Feng [40] for $q \geq 2$ and $k \in \{1, \cdots, N\}$. Recently, the results in [23] were generalized by Fan et al. [41] and Kan and Zhang [42] to the cases of q-k Hessian equation and system. In particular, Kan and Zhang [42] showed that if H, L, f, and g satisfy $(\mathbf{H_1})$–$(\mathbf{H_2})$ and $\Re = \infty$, then (1.1) has an entire positive q-k-convex radial solution provided f satisfies

$$\int_1^{\infty} (f(\tau))^{-1/(q-1)k} d\tau = \infty; \tag{1.13}$$

and (1.2) has an entire positive q-k-convex radial solution provided f and g satisfy

$$\int_1^{\infty} (f(\tau) + g(\tau))^{-1/(q-1)k} d\tau = \infty. \tag{1.14}$$

Fan et al. [41] showed that if H, L satisfy $(\mathbf{H_1})$, f_1, f_2, g_1, g_2 satisfy $(\mathbf{H_3})$, and $\Re = \infty$, then (1.3) has a radial solution provided f_1, f_2, g_1, g_2 satisfy

$$\int_1^{\infty} (f_1(\tau)f_2(\tau) + g_1(\tau)g_2(\tau))^{-1/(q-1)k} d\tau = \infty. \tag{1.15}$$

Especially under some additional conditions, they further investigated the result of the existence in entire bounded solutions when (1.15) holds. Recently, by using (1.12), the result of existence to (1.1) was investigated by Feng and Zhang in [43]. Specifically, they showed that if $\Re = 1$, H satisfies $(\mathbf{H_1})$ and the following condition

$(\mathbf{C_1})$ there are two positive constants d_1, d_2 and some function $L \in \Lambda$ such that

$$d_1 L(1-s) \leq H(s) \leq d_2 L(1-s),\ \forall\, s < 1 \text{ near } 1,$$

where Λ denotes the set of functions L that satisfy $L \in C^1(0,\infty)$, $L > 0$, $L' < 0$,

$$\lim_{s\to 0^+} L(s) = \infty \text{ and } \int_0^1 \left(\int_t^1 L(\tau)d\tau\right)^{1/(q-1)k} dt = \infty,$$

f satisfies the following conditions:

$(\mathbf{C_2})$ $f \in C(0,\infty)$ is positive and increasing and is local Lipschitz on $(0,\infty)$; moreover, f satisfies (1.12);

$(\mathbf{C_3})$ let c_0 be a positive constant,

$$\Upsilon(t) := \int_{c_0}^t \left(((q-1)k+1)\int_0^\tau f(\varsigma)d\varsigma\right)^{-1/((q-1)k+1)} d\tau$$

and

$$\lim_{t\to\infty} -\frac{\Upsilon''(t)\Upsilon(t)}{(\Upsilon'(t))^2} \text{ exists,}$$

then Eq (1.1) has innumerable radial q-k-convex boundary blow-up solutions that are positive in $\mathscr{D}_\Re$. For further insights on q-Mange–Ampère equation and q-k Hessian type equation, we refer the readers to [44, 45].

Inspired by the above works, in this paper, we prove the existence of innumerable positive q-k-convex radial solutions (including boundary blow-up solutions) to Eq (1.1), the systems (1.2), and (1.3) by using the Keller–Osserman conditions (1.12),

$$\int_1^\infty \left(\int_0^t f(\tau) + g(\tau)d\tau\right)^{-1/((q-1)k+1)} dt = \infty \tag{1.16}$$

and

$$\int_1^\infty \left(\int_0^t f_1(\tau)f_2(\tau) + g_1(\tau)g_2(\tau)d\tau\right)^{-1/((q-1)k+1)} dt = \infty, \tag{1.17}$$

respectively. We omit the hypothesis $(\mathbf{C_3})$ in [43] and our assumptions on f and H are weaker than the ones in [43]. Moreover, we note the conditions (1.12), (1.16), and (1.17) are strictly weaker than the conditions (1.13)–(1.15), respectively (the reasons are given by Remark 2.4 and Proposition 3.4).

2. The main results

Theorem 2.1. *Let H, f satisfy* ($\mathbf{H_1}$)–($\mathbf{H_2}$) *and* (1.12) *hold, then for any $a_0 \in \mathbb{R}^+$, Eq* (1.1) *has a radial q-k-convex positive solution ω satisfying*

$$a_0 + (f(a_0))^{1/(q-1)k}\mathcal{H}(s) \leq \omega(s) \leq \mathcal{T}_0^{-1}(s\mathfrak{A}(s)),\ s \in [0, \Re), \tag{2.1}$$

where

$$\mathcal{H}(s) := \int_0^s \left[t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1}H(\tau)d\tau\right]^{1/(q-1)k} dt,\ s \in [0, \Re), \tag{2.2}$$

$$\mathfrak{A}(s) := \left(\frac{(p-1)k+1}{(p-1)k}\binom{N-1}{k-1}^{-1} k\right)^{1/((q-1)k+1)} \max_{t\in[0,s]}(t^{k-1}H(t))^{1/((q-1)k+1)}, \tag{2.3}$$

$\mathcal{T}_0^{-1}$ is the inverse of $\mathcal{T}_0$ given by

$$\mathcal{T}_0(t) := \int_{a_0}^t \left(\int_0^\tau f(\varsigma)d\varsigma\right)^{-1/((q-1)k+1)} d\tau,\ t \geq a_0. \tag{2.4}$$

In particular, if $\mathcal{H}(\Re) = \infty$, then $\omega(\Re) = \infty$.

Remark 2.2. *If $\Re < \infty$, then $\mathcal{H}(\Re) = \infty$ is equivalent to*

$$\int_0^{\Re} \left(\int_0^\tau H(\varsigma)d\varsigma\right)^{1/(q-1)k} d\tau = \infty.$$

Theorem 2.3. *Let $\Re = \infty$, H, f satisfy* ($\mathbf{H_1}$)–($\mathbf{H_2}$) *and* (1.12) *be false. If $\mathscr{H}(s) > 0$ for $s \in (0, \infty)$ and there exists some positive constant s_0 such that*

$$\int_{s_0}^\infty (\min_{t\in[s_0,s]} \mathscr{H}(t))^{1/((q-1)k+1)} ds = \infty,$$

where $\mathscr{H}$ is given by (1.11)*. Then* (1.1) *has no radial q-k-convex positive large solution.*

Remark 2.4. *From Proposition 3.4* (*see page* 8)*, we see that if* (1.13) *holds, then* (1.12) *holds. But, the converse of the result is not true. A basic example is*

$$f(s) = s^{(q-1)k}(\ln s)^{(q-1)k+1},\ s \geq s_0 \text{ for some large constant } s_0 > 1.$$

By a simple calculation, we see that

$$\int_{s_0}^s f(\tau)d\tau \sim \frac{s^{(q-1)k+1}(\ln s)^{(q-1)k+1}}{(q-1)k+1},\ s \to \infty.$$

This implies that $\int_1^\infty (f(\tau))^{-1/(q-1)k} d\tau < \infty$ and (1.12) *holds. So, the condition* (1.12) *is strictly weaker than* (1.13).

Theorem 2.5. *Let H, L, f, and g satisfy* $(\mathbf{H_1})$–$(\mathbf{H_2})$ *and* (1.16) *hold, then for any $a_0 \in \mathbb{R}^+$,* (1.2) *has a radial q-k-convex positive solution (ω, υ) satisfying*

$$\frac{a_0}{2} + (f(\frac{a_0}{2}))^{1/(q-1)k}\mathcal{H}(s) \leq \omega(s) \leq \mathcal{T}_1^{-1}(s\mathfrak{B}(s)),\ s \in [0, \mathfrak{R}),$$
$$\frac{a_0}{2} + (g(\frac{a_0}{2}))^{1/(q-1)k}\mathcal{L}(s) \leq \upsilon(s) \leq \mathcal{T}_1^{-1}(s\mathfrak{B}(s)),\ s \in [0, \mathfrak{R}),$$

where $\mathcal{H}$ is given by (2.2) *and*

$$\mathcal{L}(s) := \int_0^s \left[t^{k-N}\int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1}L(\tau)d\tau\right]^{1/(q-1)k} dt,\ s \in [0, \mathfrak{R}), \tag{2.5}$$

$$\begin{aligned}\mathfrak{B}(s) := &\Big(\frac{(q-1)k+1}{(q-1)k}\binom{N-1}{k-1}^{-1}k\Big)^{1/((q-1)k+1)}\Big(\max_{t\in[0,s]}(t^{k-1}H(t))^{1/((q-1)k+1)} \\ &+ \max_{t\in[0,s]}(t^{k-1}L(t))^{1/((q-1)k+1)}\Big),\end{aligned} \tag{2.6}$$

$\mathcal{T}_1^{-1}$ is the inverse of $\mathcal{T}_1$ given by

$$\mathcal{T}_1(t) := \int_{a_0}^t \Big(\int_0^\tau f(\varsigma) + g(\varsigma)d\varsigma\Big)^{-1/((q-1)k+1)} d\tau. \tag{2.7}$$

In particular, if $\mathcal{H}(\mathfrak{R}) = \infty$, then $\omega(\mathfrak{R}) = \infty$; if $\mathcal{L}(\mathfrak{R}) = \infty$, then $\upsilon(\mathfrak{R}) = \infty$.

Theorem 2.6. *Let H, L satisfy* $(\mathbf{H_1})$*, f_1, f_2, g_1, and g_2 satisfy* $(\mathbf{H_3})$*, then for any $a_0 \in \mathbb{R}^+$,* (1.3) *has a radial q-k-convex positive solution (ω, υ) satisfying*

$$\frac{a_0}{2} + (f_1(\frac{a_0}{2})f_2(\frac{a_0}{2}))^{1/(q-1)k}\mathcal{H}(s) \leq \omega(s) \leq \mathcal{T}_2^{-1}(s\mathfrak{B}(s)),\ s \in [0, \mathfrak{R}),$$
$$\frac{a_0}{2} + (g_1(\frac{a_0}{2})g_2(\frac{a_0}{2}))^{1/(q-1)k}\mathcal{L}(s) \leq v(s) \leq \mathcal{T}_2^{-1}(s\mathfrak{B}(s)),\ s \in [0, \mathfrak{R}),$$

where $\mathcal{H}$, $\mathcal{L}$, and $\mathfrak{B}$ are given by (2.2), (2.5)*, and* (2.6)*, and $\mathcal{T}_2^{-1}$ is the inverse of $\mathcal{T}_2$ given by*

$$\mathcal{T}_2(t) := \int_{a_0}^t \Big(\int_0^\tau f_1(\varsigma)f_2(\varsigma) + g_1(\varsigma)g_2(\varsigma)d\varsigma\Big)^{-1/((q-1)k+1)} d\tau. \tag{2.8}$$

In particular, if $\mathcal{H}(\mathfrak{R}) = \infty$, then $\omega(\mathfrak{R}) = \infty$; if $\mathcal{L}(\mathfrak{R}) = \infty$, then $\upsilon(\mathfrak{R}) = \infty$.

Remark 2.7. *By the same argument as Remark 2.4, we see that* (1.16) *is strictly weaker than* (1.14)*, and* (1.17) *is strictly weaker than* (1.15)*.*

3. Preliminary results

Definition 3.1. *The q-k-convex function in $\mathscr{D}_\mathfrak{R}$ is defined as below: if*

$$\begin{aligned}\omega \in \Phi^{q,k}(\mathscr{D}_\mathfrak{R}) := \{&\omega \in C^2(\mathscr{D}_\mathfrak{R} \setminus \{\boldsymbol{0}\}) \cap C^1(\mathscr{D}_\mathfrak{R}) : |D\omega|^{q-2}D\omega \in C^1(\mathscr{D}_\mathfrak{R}),\\ &\text{the eigenvalue } \lambda = (\lambda_1, \cdots, \lambda_N) \text{ of } (D_i(|D\omega|^{q-2}D_j\omega))_{i,j=1,\cdots,N}\\ &\text{belongs to} \in \Gamma_k\},\end{aligned}$$

where $\Gamma_k := \{\lambda \in \mathbb{R}^N : S_i(\lambda) > 0,\ i = 1, \cdots, k\}$. Especially, if $\omega \in \Phi^{2,k}(\mathscr{D}_\mathfrak{R}) \cap C^2(\mathscr{D}_\mathfrak{R})$, then ω is the k-convex function.

By Lemmas 1 and 2 and Corollary 1 of Fan et al. in [41], we obtain the following lemma:

Lemma 3.2. *Let H, L, f, and g satisfy* $(\mathbf{H_1})$–$(\mathbf{H_2})$*, a_0 be a positive constant, and $\zeta_0, \zeta, \eta \in C^0[0,R) \cap C^1(0,R)$ satisfy the following equation and system:*

$$\zeta_0'(s) = \left(s^{k-N}\int_0^s \binom{N-1}{k-1}^{-1} kt^{N-1}H(t)f(\zeta_0(t))dt\right)^{1/(q-1)k}, s \in (0,\Re),\ \zeta_0(0) = a_0$$

and

$$\begin{cases}\zeta'(s) = (s^{k-N}\int_0^s \binom{N-1}{k-1}^{-1} kt^{N-1}H(t)f(\eta(t))dt)^{1/(q-1)k}, s \in (0,\Re),\\ \eta'(s) = (s^{k-N}\int_0^s \binom{N-1}{k-1}^{-1} kt^{N-1}L(t)g(\zeta(t))dt)^{1/(q-1)k}, s \in (0,\Re),\\ \zeta(0) = \eta(0) = \frac{a_0}{2},\end{cases} \tag{3.1}$$

then $\zeta_0, \zeta, \eta \in C^2(0,\Re) \cap C^1[0,\Re)$ satisfy $\zeta_0(0) = a_0$, $\zeta_0'(0) = 0$,

$$\begin{aligned}&\binom{N-1}{k-1}\left(\frac{(\zeta_0'(s))^{q-1}}{s}\right)^{k-1}((\zeta_0'(s))^{q-1})' + \binom{N-1}{k}\left(\frac{(\zeta_0'(s))^{q-1}}{s}\right)^{k}\\ &= H(s)f(\zeta_0(s)), s \in (0,\Re),\end{aligned} \tag{3.2}$$

$$\begin{cases}\binom{N-1}{k-1}(\frac{(\zeta'(s))^{q-1}}{s})^{k-1}((\zeta'(s))^{q-1})' + \binom{N-1}{k}(\frac{(\zeta'(s))^{q-1}}{s})^{k} = H(s)f(\eta(s)), s \in (0,\Re),\\ \binom{N-1}{k-1}(\frac{(\eta'(s))^{q-1}}{s})^{k-1}((\eta'(s))^{q-1})' + \binom{N-1}{k}(\frac{(\eta'(s))^{q-1}}{s})^{k} = L(s)g(\zeta(s)), s \in (0,\Re),\\ \zeta(0) = \eta(0) = \frac{a_0}{2},\ \zeta'(0) = \eta'(0) = 0,\end{cases} \tag{3.3}$$

and $\omega_0(x) = \zeta_0(s)$ and $(\omega(x), \upsilon(x)) = (\zeta(s), \eta(s))$ are, respectively, the radial q-k-convex solutions to the Eq (1.1) *and system* (1.2)*.*

Remark 3.3. *In Lemma* 3.2*, if $f(\eta(s))$ is replaced by $f_1(\eta(s))f_2(\zeta(s))$ and $g(\zeta(s))$ is replaced by $g_1(\zeta(s))g_2(\eta(s))$ in* (3.1) *and* (3.3)*, where f_1, f_2, g_1, g_2 are given by* $(\mathbf{H_3})$*, then by Lemmas* 1 *and* 2 *of Fan et al. [41] we see that this conclusion still holds.*

Proposition 3.4. *Let $h \in C([0,\infty),[0,\infty))$ be increasing on* $(0,\infty)$*. If*

$$\int_1^\infty \left(\int_0^t h(\tau)d\tau\right)^{-1/((q-1)k+1)} dt < \infty,\ \text{then}\ \int_1^\infty \frac{dt}{(h(t))^{1/(q-1)k}} < \infty.$$

Proof. The proof is divided into two steps.

Step 1. We show that for any positive constant $M > 0$, there exists $t_* > 0$ such that for any $t \geq t_*$,

$$\frac{h(t)}{t^{(q-1)k}} \geq M. \tag{3.4}$$

Otherwise, there exist a positive constant $c_0 > 0$ and an increasing sequence $\{t_i\}_{i=0}^\infty$ of real numbers satisfying $\lim_{i\to\infty} t_i = \infty$ and $2t_{i-1} \leq t_i$, $i = 1, 2, \cdots$, such that $\frac{h(t_i)}{t_i^{(q-1)k}} \leq c_0$. This, together with

$$\int_0^t h(\tau)d\tau \leq th(t) \leq t_i h(t_i),\ t \in [0,t_i]$$

shows that

$$\infty > \int_{t_0}^{\infty}\Big(\int_0^t h(\tau)d\tau\Big)^{-1/((q-1)k+1)}dt = \sum_{i=i}^{\infty}\int_{t_{i-1}}^{t_i}\Big(\int_0^t h(\tau)d\tau\Big)^{-1/((q-1)k+1)}dt$$
$$\geq \sum_{i=i}^{\infty}\int_{t_{i-1}}^{t_i}(th(t))^{-1/((q-1)k+1)}dt$$
$$\geq \sum_{i=1}^{\infty}(t_ih(t_i))^{-1/((q-1)k+1)}(t_i - t_{i-1})$$
$$\geq \sum_{i=1}^{\infty}c_0^{-\frac{1}{(q-1)k+1}}(1-(t_{i-1}/t_i)) = \infty.$$

This is a contradiction. So, the first step is finished.
Step 2. By (3.4), we see that

$$\int_0^t h(\tau)d\tau \leq th(t) \leq \frac{(h(t))^{\frac{((q-1)k+1)}{(q-1)k}}}{M^{1/k}},\ t \geq t_*.$$

So, we obtain

$$\int_{t_*}^{\infty}\Big(\int_0^t h(\tau)d\tau\Big)^{-1/((q-1)k+1)}dt \geq M^{-1/k}\int_{t_*}^{\infty}\frac{dt}{(h(t))^{1/(q-1)k}}.$$

The proof is finished.

4. Proof of Theorem 2.1

Proof. Let $\mathcal{T}_0$ be given by (2.4). Since

$$\mathcal{T}_0'(t) = \Big(\int_0^t f(\tau)d\tau\Big)^{-1/((q-1)k+1)} > 0,\ t \geq a_0,$$

we can obtain that $\mathcal{T}_0$ has the inverse $\mathcal{T}_0^{-1}$, which is increasing on $[0,\infty)$ with

$$\mathcal{T}_0^{-1}(0) = a_0 \text{ and } \mathcal{T}_0^{-1}(\infty) := \lim_{t\to\infty}\mathcal{T}_0^{-1}(t) = \infty. \tag{4.1}$$

We consider the following initial value problem:

$$\begin{cases}\binom{N-1}{k-1}\big(\frac{(\omega'(s))^{q-1}}{s}\big)^{k-1}((\omega'(s))^{q-1})' + \binom{N-1}{k}\big(\frac{(\omega'(s))^{q-1}}{s}\big)^{k} \\ = k^{-1}\binom{N-1}{k-1}s^{1-N}(s^{N-k}(\omega'(s))^{(q-1)k})' = H(s)f(\omega(s)),\ s\in(0,\Re), \\ u(0) = a_0,\ u'(0) = 0.\end{cases} \tag{4.2}$$

Problem (4.2) is equivalent to the integral equation

$$\omega(s) = a_0 + \int_0^s\Big(t^{k-N}\int_0^t\binom{N-1}{k-1}^{-1}k\tau^{N-1}H(\tau)f(\omega(\tau))d\tau\Big)^{1/(q-1)k}dt,\ s\in[0,\Re).$$

Now, by constructing some iterative approximation sequence, we prove the existence of q-k-convex solutions to problem (4.2). We assume that $\{\omega_m\}$ is the sequence of positive continuous functions defined by

$$\omega_1(s) = a_0,$$
$$\omega_2(s) = a_0 + \int_0^s \left(t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) f(\omega_1(\tau)) d\tau \right)^{1/(q-1)k} dt,$$
$$\ldots$$
$$\omega_m(s) = a_0 + \int_0^s \left(t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) f(\omega_{m-1}(\tau)) d\tau \right)^{1/(q-1)k} dt,$$
$$\ldots$$

The conditions $(\mathbf{H_1})$–$(\mathbf{H_2})$ imply that

$$\omega_m'(s) = \left(s^{k-N} \int_0^s \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) f(\omega_{m-1}(\tau)) d\tau \right)^{1/(q-1)k} > 0, \ s > 0$$

and

$$\omega_m(s) > a_0 + (f(a_0))^{1/(q-1)k} \mathcal{H}(s). \tag{4.3}$$

So, we see that ω_m is a positive increasing function and $\{\omega_m\}$ is an increasing sequence. These facts, together with (4.2), imply that for any $s \in (0, \Re)$, we have

$$\begin{aligned}(s^{N-k}(\omega_m'(s))^{(q-1)k})' &= \binom{N-1}{k-1}^{-1} ks^{N-1} H(s) f(\omega_{m-1}(s)) \\ &\leq \binom{N-1}{k-1}^{-1} ks^{N-1} H(s) f(\omega_m(s)), \ m \geq 1\end{aligned}$$

and

$$(s^{N-k}(\omega_m'(s))^{(q-1)k})' \omega_m'(s) \leq \binom{N-1}{k-1}^{-1} ks^{N-1} H(s) f(\omega_m(s)) \omega_m'(s), \ m \geq 1. \tag{4.4}$$

For any $\Re \in (0, \Re)$, we set

$$\mathfrak{H}_\Re := \max_{0 \leq s \leq \Re} \binom{N-1}{k-1}^{-1} ks^{k-1} H(s). \tag{4.5}$$

This fact, combined with (4.4), shows that

$$((q-1)k+1)(\omega_m')^{(q-1)k} \omega_m'' \leq \frac{(q-1)k+1}{(q-1)k} \mathfrak{H}_\Re f(\omega_m) \omega_m' \text{ on } (0, \Re]. \tag{4.6}$$

Moreover, by direct calculation, we see that

$$\lim_{s \to 0} (\omega_m'(s))^{(q-1)k} \omega_m''(s) = 0.$$

Integrating (4.6) from τ ($\tau \in (0, \Re)$) to s and letting $\tau \to 0$, we obtain

$$(\omega_m'(s))^{(q-1)k+1} \leq \frac{(q-1)k+1}{(q-1)k} \mathfrak{H}_\Re \int_{a_0}^{\omega_m(s)} f(t) dt, \ s \in [0, \Re]. \tag{4.7}$$

Furthermore, we arrive at

$$\begin{aligned}\mathcal{T}_0(\omega_m(\Re)) &\leq \int_{a_0}^{\omega_m(\Re)}\left(\int_{a_0}^{t} f(\tau)d\tau\right)^{-1/((q-1)k+1)} dt\\ &\leq \left(\frac{(q-1)k+1}{(q-1)k}\mathfrak{H}_{\Re}\right)^{1/((q-1)k+1)}\Re = \mathfrak{A}(\Re)\Re,\end{aligned}$$

where $\mathfrak{A}$ is given by (2.3). It is clear that $\mathcal{T}_0(\omega_m) \leq \mathfrak{A}(\Re)\Re$ on $[0, \Re]$. It follows from (4.1) that

$$\omega_m \leq \mathcal{T}_0^{-1}(\mathfrak{A}(\Re)\Re) \text{ on } [0, \Re]. \tag{4.8}$$

This implies that $\{\omega_m\}$ is a uniformly bounded sequence on $[0, \Re]$ for any $\Re \in [0, \Re)$. On the other hand, it follows from (4.7) and (4.8) that $\{\omega_m'\}$ is also uniformly bounded on $[0, \Re]$. We conclude by Arzela–Ascoli's theorem that there is a subsequence of $\{\omega_m\}$, denoted by itself, such that $\omega_m \to \omega$ on $[0, \Re]$. The arbitrariness of $\Re$ and Lemma 3.2 imply that ω is a positive q-k-convex solution to problem (4.2). It follows from (4.3) and (4.8) that (2.1) holds. The proof is finished.

5. Proof of Theorem 2.3

Proof. Suppose ω is a positive q-k-convex radial large solution. We will derive a contradiction. By Lemma 3.2, we see that

$$\omega'(s) = \left(s^{k-N}\int_0^s \binom{N-1}{k-1}^{-1} kt^{N-1}H(t)f(\omega(t))dt\right)^{1/(q-1)k} > 0,\ s \in (0, \Re),$$

i.e.,

$$(\omega'(s))^{(q-1)k} = s^{k-N}\int_0^s \binom{N-1}{k-1}^{-1} kt^{N-1}H(t)f(\omega(t))dt.$$

Furthermore, we have

$$\begin{aligned}\binom{N-1}{k}\left(\frac{(\omega'(s))^{q-1}}{s}\right)^k &\leq s^{-N}f(\omega(s))\frac{\binom{N-1}{k}}{\binom{N-1}{k-1}}k\int_0^s t^{N-1}H(t)dt\\ &= (N-k)^2 s^{-N}f(\omega(s))\int_0^s t^{N-1}H(t)dt.\end{aligned} \tag{5.1}$$

Since ω satisfies Eq (3.2), we obtain by (5.1) that

$$(\omega'(s))^{(q-1)(k-1)}((\omega'(s))^{q-1})' \geq \frac{f(\omega(s))}{\binom{N-1}{k-1}}\mathscr{H}(s),\ s \in (0, \Re),$$

where $\mathscr{H}$ is given by (1.11). Multiplying both sides of the above inequality by $\omega'(s)$, we have

$$(q-1)(\omega'(s))^{(q-1)k}\omega''(s) \geq \frac{f(\omega(s))\omega'(s)}{\binom{N-1}{k-1}}\mathscr{H}(s),\ s \in (0, \Re),$$

i.e.,

$$\frac{q-1}{(q-1)k+1}((\omega'(s))^{(q-1)k+1})' \geq \frac{f(\omega(s))\omega'(s)}{\binom{N-1}{k-1}}\mathscr{H}(s),\ s \in (0, \Re).$$

Integrating this inequality from s_0 to s, we obtain

$$(\omega'(s))^{(q-1)k+1} \geq \frac{(q-1)k+1}{(q-1)\binom{N-1}{k-1}}(\min_{t\in[s_0,s]} \mathscr{H}(t)) \int_{\omega(s_0)}^{\omega(s)} f(t)dt.$$

This implies that

$$\begin{aligned}&\int_{\omega(s_0)}^{\infty}\Big(\int_{\omega(s_0)}^{s} f(t)dt\Big)^{-1/((q-1)k+1)} ds\\ &\geq \Big(\frac{(q-1)k+1}{(q-1)\binom{N-1}{k-1}}\Big)^{1/((q-1)k+1)} \int_{s_0}^{\infty} (\min_{t\in[s_0,s]} \mathscr{H}(t))^{1/((q-1)k+1)} ds = \infty\end{aligned}$$

which is a contradiction to (1.12).

6. Proof of Theorem 2.5

Proof. Let $\mathcal{T}_1$ be given by (2.7). It is clear that

$$\mathcal{T}_1'(t) = \Big(\int_0^t f(\tau) + g(\tau)d\tau\Big)^{-1/((q-1)k+1)} > 0,\ t \geq a_0.$$

It follows that $\mathcal{T}_1$ has the inverse $\mathcal{T}_1^{-1}$, which is increasing on $[0, \infty)$ with

$$\mathcal{T}_1^{-1}(0) = a_0 \text{ and } \mathcal{T}_1^{-1}(\infty) := \lim_{t\to\infty} \mathcal{T}_1^{-1}(t) = \infty.$$

We consider the following system:

$$\begin{cases}\binom{N-1}{k-1}\big(\frac{(\omega'(s))^{q-1}}{s}\big)^{k-1}(\omega'(s))^{q-1} + \binom{N-1}{k}\big(\frac{(\omega'(s))^{q-1}}{s}\big)^{k} = H(s)f(\upsilon), & s \in (0, \Re),\\ \binom{N-1}{k-1}\big(\frac{(\upsilon'(s))^{q-1}}{s}\big)^{k-1}(\upsilon'(s))^{q-1} + \binom{N-1}{k}\big(\frac{(\upsilon'(s))^{q-1}}{s}\big)^{k} = L(s)g(\omega), & s \in (0, \Re),\\ \omega(0) = \frac{a_0}{2},\ \upsilon(0) = \frac{a_0}{2} \text{ and } \omega'(0) = \upsilon'(0) = 0.\end{cases} \tag{6.1}$$

System (6.1) is equivalent to the integral system

$$\begin{cases}\omega(s) = \frac{a_0}{2} + \int_0^s [t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau)f(\upsilon(\tau))d\tau]^{1/(q-1)k}dt, & s \in [0, \Re),\\ \upsilon(s) = \frac{a_0}{2} + \int_0^s [t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} L(\tau)g(\omega(\tau))d\tau]^{1/(q-1)k}dt, & s \in [0, \Re).\end{cases}$$

By a similar argument as in the proof of Theorem 2.1, we construct the iterative approximation sequence $\{(\omega_m, \upsilon_m)\}$ as below:

$$\begin{cases}\omega_0(s) = \frac{a_0}{2},\\ \upsilon_0(s) = \frac{a_0}{2}\end{cases}$$

and

$$\begin{cases}\omega_m(s) = \frac{a_0}{2} + \int_0^s [t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) f(\upsilon_{m-1}(\tau)) d\tau]^{1/(q-1)k} dt,\\ \upsilon_m(s) = \frac{a_0}{2} + \int_0^s [t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} L(\tau) g(\omega_{m-1}(\tau)) d\tau]^{1/(q-1)k} dt.\end{cases}$$

From $(\mathbf{H_1})$–$(\mathbf{H_2})$, we obtain

$$\begin{cases}\omega_m'(s) = \left[s^{k-N} \int_0^s \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) f(\upsilon_{m-1}(\tau)) d\tau\right]^{1/(q-1)k} > 0,\ s > 0,\\ \upsilon_m'(s) = \left[s^{k-N} \int_0^s \binom{N-1}{k-1}^{-1} k\tau^{N-1} L(\tau) g(\omega_{m-1}(\tau)) d\tau\right]^{1/(q-1)k} > 0,\ s > 0\end{cases}$$

and

$$\omega_m(s) > a_0/2 + (f(a_0/2))^{1/(q-1)k} \mathcal{H}(s),\ \ \upsilon_m(s) > a_0/2 + (g(a_0/2))^{1/(q-1)k} \mathcal{L}(s).$$

So, we see that ω_m and υ_m are positive increasing functions, and $\{\omega_m\}$ and $\{\upsilon_m\}$ are increasing sequences. Furthermore, we have that for any $s \in (0, \Re)$, there hold

$$\begin{aligned}(s^{N-k} (\omega_m')^{(q-1)k})' &= \binom{N-1}{k-1}^{-1} k s^{N-1} H(s) f(\upsilon_{m-1}(s))\\ &\leq \binom{N-1}{k-1}^{-1} k s^{N-1} H(s) f(\upsilon_m(s)),\ m \geq 1\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}(s^{N-k} (\upsilon_m')^{(q-1)k})' &= \binom{N-1}{k-1}^{-1} k s^{N-1} L(s) g(\omega_{m-1}(s))\\ &\leq \binom{N-1}{k-1}^{-1} k s^{N-1} L(s) g(\omega_m(s)),\ m \geq 1.\end{aligned} \tag{6.3}$$

For an arbitrary $\Re \in (0, \Re)$, we define

$$\mathfrak{H}_\Re := \max_{0 \leq s \leq \Re} \binom{N-1}{k-1}^{-1} k s^{k-1} H(s) \text{ and } \mathfrak{L}_\Re := \max_{0 \leq s \leq \Re} \binom{N-1}{k-1}^{-1} k s^{k-1} L(s). \tag{6.4}$$

These facts, combined with (6.2) and (6.3), show that

$$\begin{aligned}&((q-1)k+1)(\omega_m')^{(q-1)k} \omega_m''\\ &\leq \frac{((q-1)k+1)}{(q-1)k} \mathfrak{H}_\Re f(\upsilon_m) \omega_m'\\ &\leq \frac{((q-1)k+1)}{(q-1)k} \mathfrak{H}_\Re \big(f(\omega_m + \upsilon_m) + g(\omega_m + \upsilon_m)\big)(\omega_m' + \upsilon_m') \text{ on } (0, \Re]\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}&((q-1)k+1)(\upsilon_m')^{(q-1)k} \upsilon_m''\\ &\leq \frac{((q-1)k+1)}{(q-1)k} \mathfrak{L}_\Re g(\omega_m) \upsilon_m'\\ &\leq \frac{((q-1)k+1)}{(q-1)k} \mathfrak{L}_\Re \big(f(\omega_m + \upsilon_m) + g(\omega_m + \upsilon_m)\big)(\omega_m' + \upsilon_m') \text{ on } (0, \Re].\end{aligned} \tag{6.6}$$

Moreover, by direct calculation, we see that

$$\lim_{s\to 0}(\omega_m'(s))^{(q-1)k}\omega_m''(s) = 0 \text{ and } \lim_{s\to 0}(\upsilon_m'(s))^{(q-1)k}\upsilon_m''(s) = 0. \tag{6.7}$$

Integrating (6.5) and (6.6) from τ ($\tau \in (0,\Re)$) to s and letting $\tau \to 0$, we obtain

$$(\omega_m'(s))^{(q-1)k+1} \le \frac{(q-1)k+1}{(q-1)k}\mathfrak{H}_\Re \int_{a_0}^{\omega_m(s)+\upsilon_m(s)} f(t)+g(t)dt,\ s\in[0,\Re]$$

and

$$(\upsilon_m'(s))^{(q-1)k+1} \le \frac{(q-1)k+1}{(q-1)k}\mathfrak{L}_\Re \int_{a_0}^{\omega_m(s)+\upsilon_m(s)} f(t)+g(t)dt,\ s\in[0,\Re].$$

Furthermore, we arrive at

$$\begin{aligned}\mathcal{T}_1(\omega_m(\Re)+\upsilon_m(\Re)) &\le \int_{a_0}^{\omega_m(\Re)+\upsilon_m(\Re)}\Big(\int_{a_0}^{t} f(\tau)+g(\tau)d\tau\Big)^{-1/((q-1)k+1)}dt\\ &\le \mathfrak{B}(\Re)\Re,\end{aligned}$$

where $\mathfrak{B}$ is given by (2.6). It is clear that

$$\mathcal{T}_1(\omega_m+\upsilon_m) \le \mathfrak{B}(\Re)\Re \text{ on } [0,\Re],\text{ i.e., } \omega_m+\upsilon_m \le \mathcal{T}_1^{-1}(\mathfrak{B}(\Re)\Re) \text{ on } [0,\Re].$$

The rest of the proof is similar to the one in Theorem 2.1, so we omit it here. The proof is finished.

7. Proof of Theorem 2.6

Proof. Let $\mathcal{T}_2$ be given by (2.8). We have

$$\mathcal{T}_2'(t) = \Big(\int_0^t f_1(t)f_2(t)+g_1(t)g_2(t)\Big)^{-1/((q-1)k+1)} > 0,\ t\ge a_0.$$

It is clear that $\mathcal{T}_2$ has the inverse $\mathcal{T}_2^{-1}$, which is increasing on $[0,\infty)$ with

$$\mathcal{T}_2^{-1}(0) = a_0 \text{ and } \mathcal{T}_2^{-1}(\infty) := \lim_{t\to\infty}\mathcal{T}_2^{-1}(t) = \infty.$$

As the proof of Theorem 2.5, we consider the system:

$$\begin{cases}\binom{N-1}{k-1}\Big(\frac{(\omega'(s))^{q-1}}{s}\Big)^{k-1}(\omega'(s))^{q-1} + \binom{N-1}{k}\Big(\frac{(\omega'(s))^{q-1}}{s}\Big)^{k} & \\ = H(s)f_1(\upsilon)f_2(\omega), & s\in(0,\Re),\\ \binom{N-1}{k-1}\Big(\frac{(\upsilon'(s))^{q-1}}{s}\Big)^{k-1}(\upsilon'(s))^{q-1} + \binom{N-1}{k}\Big(\frac{(\upsilon'(s))^{q-1}}{s}\Big)^{k} & \\ = L(s)g_1(\omega)g_2(\upsilon), & s\in(0,\Re),\\ \omega(0)=\frac{a_0}{2},\ \upsilon(0)=\frac{a_0}{2} \text{ and } \omega'(0)=\upsilon'(0)=0. &\end{cases} \tag{7.1}$$

System (7.1) is equivalent to the integral system:

$$\begin{cases}\omega(s) = \frac{a_0}{2} + \int_0^s [t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) f_1(\upsilon(\tau)) f_2(\omega(\tau)) d\tau]^{1/(q-1)k} dt, \\ \upsilon(s) = \frac{a_0}{2} + \int_0^s [t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} L(\tau) g_1(\omega(\tau)) g_2(\upsilon(\tau)) d\tau]^{1/(q-1)k} dt,\end{cases}$$

where $s \in [0, \Re)$. As the proof of Theorem 2.5, we construct the iterative approximation sequence $\{(\omega_m, \upsilon_m)\}$ as below:

$$\begin{cases}\omega_0(s) = \frac{a_0}{2}, \\ \upsilon_0(s) = \frac{a_0}{2}\end{cases}$$

and

$$\begin{cases}\omega_m(s) = \dfrac{a_0}{2} + \displaystyle\int_0^s \Bigg[t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) \\ \qquad \times f_1(\upsilon_{m-1}(\tau)) f_2(\omega_{m-1}(\tau)) d\tau\Bigg]^{1/(q-1)k} dt, \\ \upsilon_m(s) = \dfrac{a_0}{2} + \displaystyle\int_0^s \Bigg[t^{k-N} \int_0^t \binom{N-1}{k-1}^{-1} k\tau^{N-1} L(\tau) \\ \qquad \times g_1(\omega_{m-1}(\tau)) g_2(\upsilon_{m-1}(\tau)) d\tau\Bigg]^{1/(q-1)k} dt.\end{cases}$$

From $(\mathbf{H_1})$ and $(\mathbf{H_3})$, we have

$$\begin{cases}\omega_m'(s) = \Bigg[s^{k-N} \displaystyle\int_0^s \binom{N-1}{k-1}^{-1} k\tau^{N-1} H(\tau) \\ \qquad \times f_1(\upsilon_{m-1}(\tau)) f_2(\omega_{m-1}(\tau)) d\tau\Bigg]^{1/(q-1)k} > 0,\ s > 0, \\ \upsilon_m'(s) = \Bigg[s^{k-N} \displaystyle\int_0^s \binom{N-1}{k-1}^{-1} k\tau^{N-1} L(\tau) \\ \qquad \times g_1(\omega_{m-1}(\tau)) g_2(\upsilon_{m-1}(\tau)) d\tau\Bigg]^{1/(q-1)k} > 0,\ s > 0\end{cases}$$

and

$$\begin{cases}\omega_m(s) > a_0/2 + (f_1(a_0/2) f_2(a_0/2))^{1/(q-1)k} \mathcal{H}(s), \\ \upsilon_m(s) > a_0/2 + (g_1(a_0/2) g_2(a_0/2))^{1/(q-1)k} \mathcal{L}(s).\end{cases}$$

So, we have that ω_m and υ_m are increasing functions, and $\{\omega_m\}$ and $\{\upsilon_m\}$ are increasing sequences. Furthermore, we obtain that for any $s \in (0, \Re)$, there hold

$$\begin{aligned}(s^{N-k}(\omega_m')^{(q-1)k})' &= \binom{N-1}{k-1}^{-1} k s^{N-1} H(s) f_1(\upsilon_{m-1}(s)) f_2(\omega_{m-1}(s)) \\ &\leq \binom{N-1}{k-1}^{-1} k s^{N-1} H(s) f_1(\upsilon_m(s)) f_2(\omega_m(s)),\ m \geq 1\end{aligned}$$

and

$$\begin{aligned}(s^{N-k}(\upsilon_m')^{(q-1)k})' &= \binom{N-1}{k-1}^{-1} k s^{N-1} L(s) g_1(\omega_{m-1}(s)) g_1(\upsilon_{m-1}(s)) \\ &\leq \binom{N-1}{k-1}^{-1} k s^{N-1} L(s) g_1(\omega_m(s)) g_2(\upsilon_m(s)),\ m \geq 1.\end{aligned}$$

The above facts imply that for any $\Re \in (0, \Re)$, we have

$$\begin{aligned}
&((q-1)k+1)(\omega_m')^{(q-1)k}\omega_m'' \\
&\leq \frac{((q-1)k+1)}{(q-1)k}\mathfrak{H}_{\Re} f_1(\upsilon_m) f_2(\omega_m)\omega_m' \\
&\leq \frac{((q-1)k+1)}{(q-1)k}\mathfrak{H}_{\Re}\big(f_1(\omega_m+\upsilon_m) f_2(\omega_m+\upsilon_m) \\
&+ g_1(\omega_m+\upsilon_m) g_2(\omega_m+\upsilon_m)\big)(\omega_m'+\upsilon_m') \text{ on } (0, \Re]
\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}
&((q-1)k+1)(\upsilon_m')^{(q-1)k}\upsilon_m'' \\
&\leq \frac{((q-1)k+1)}{(q-1)k}\mathfrak{L}_{\Re} g_1(\omega_m) g_2(\upsilon_m)\upsilon_m' \\
&\leq \frac{((q-1)k+1)}{(q-1)k}\mathfrak{L}_{\Re}\big(f_1(\omega_m+\upsilon_m) f_2(\omega_m+\upsilon_m) \\
&+ g_1(\omega_m+\upsilon_m) g_2(\omega_m+\upsilon_m)\big)(\omega_m'+\upsilon_m') \text{ on } (0, \Re],
\end{aligned} \tag{7.3}$$

where $\mathfrak{H}_{\Re}$ and $\mathfrak{L}_{\Re}$ are defined as shown in (6.4). Moreover, by a direct calculation, we see that (6.7) holds here. Integrating (7.2) and (7.3) from τ ($\tau \in (0, \Re)$) to s and letting $\tau \to 0$, we obtain

$$(\omega_m'(s))^{(q-1)k+1} \leq \frac{(q-1)k+1}{(q-1)k}\mathfrak{H}_{\Re}\int_{a_0}^{\omega_m(s)+\upsilon_m(s)} f_1(t) f_2(t) + g_1(t) g_2(t) dt,\ s \in [0, \Re]$$

and

$$(\upsilon_m'(s))^{(q-1)k+1} \leq \frac{(q-1)k+1}{(q-1)k}\mathfrak{L}_{\Re}\int_{a_0}^{\omega_m(s)+\upsilon_m(s)} f_1(t) f_2(t) + g_1(t) g_2(t) dt,\ s \in [0, \Re].$$

Furthermore, we have

$$\begin{aligned}
&\mathcal{T}_2(\omega_m(\Re)+\upsilon_m(\Re)) \\
&\leq \int_{a_0}^{\omega_m(\Re)+\upsilon_m(\Re)} \left(\int_{a_0}^{t} f_1(\tau) f_2(\tau) + g_1(\tau) g_2(\tau) d\tau\right)^{-1/((q-1)k+1)} dt \\
&\leq \mathfrak{B}(\Re)\Re,
\end{aligned}$$

where $\mathfrak{B}$ is given by (2.6). It is clear that

$$\mathcal{T}_2(\omega_m+\upsilon_m) \leq \mathfrak{B}(\Re)\Re \text{ on } [0, \Re], \text{ i.e., } \omega_m+\upsilon_m \leq \mathcal{T}_2^{-1}(\mathfrak{B}(\Re)\Re) \text{ on } [0, \Re].$$

The rest of the proof is similar to the one in Theorem 2.1, so we omit it here. The proof is finished.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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