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Research article

Existence and limit behavior of constraint minimizers for elliptic equations with two nonlocal terms

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Abstract: This paper is devoted to studying constraint minimizers for a class of elliptic equations with two nonlocal terms. Using the methods of constrained variation and energy estimation, we analyze the existence, non-existence, and limit behavior of minimizers for the related minimization problem. Our work extends and enriches the study of bi-nonlocal problems.

Keywords: elliptic equation; bi-nonlocal problems; constraint minimizer; limit behavior

1. Introduction

We consider the following elliptic equation with bi-nonlocal terms:

$$\begin{cases} \left(\int_\Omega |\nabla u|^2 dx\right)^s \Delta u + \sin^2 |x| u = \mu u + \left(\int_\Omega |u|^{p+2} dx\right)^r |u|^p u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $s, r > 0$ and $0 < p < \infty$, and $\Omega \subset \mathbb{R}^2$ is a bounded connected domain with a smooth boundary, and $(0, 0)$ is its inner point. The functionals $(\int_\Omega |\nabla u|^2 dx)^s$, $(\int_\Omega |u|^{p+2} dx)^r$ are two nonlocal terms, the μ is a suitable Lagrange multiplier.

In the past few decades, nonlocal problems have gained widespread attention not only in the field of mathematics; but also in concrete real-world applications. To our knowledge, the earliest non-local problem was proposed by Kirchhoff [1], such as the well-known stationary analogue equation

$$\begin{cases} u_{tt} - \left(a + b \int_\Omega |\nabla u|^2\right) \triangle u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which is used to describe the free vibrations of elastic string. After this, the nonlocal models similar to (1.2) are also presented by the forms of

$$\begin{cases} -M(\|\nabla u\|^p)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u = f(x,u)\Big(\int_\Omega g(x,u)dx\Big)^r, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

which arise in several fields: for instance, mechanical phenomena, population dynamics, plasma physics, and heat conduction; see [2–6]. Readers are also advised to refer to their references for more details on the physical aspects.

In recent years, many researchers have begun to investigate bi-nonlocal problems similar to the elliptic equation (1.1); see [7–9] and their references. The existence of various solutions in these papers was established by applying the mountain-pass theorem, concentration compactness principle, mapping theory, genus theory, Ljusternik–Schnirelman critical point theory, etc. Meanwhile, there are many interesting works involving p-Laplacian equations with bi-nonlocal terms, which are described by

$$\begin{cases} -M(\|\nabla u\|^p)\Delta_p u = \lambda|u|^{q-2}u + \mu g(x)|u|^{\gamma-2}u\Big(\int_\Omega \frac{1}{\gamma}g(x)|u|^\gamma)dx\Big)^{2r}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

see [9–11] and the references therein, in which the infinitely many solutions, non-negative solutions, and multiplicity of solutions are studied by using variational approaches. Besides, it is worth mentioning that Mao and Wang, in their paper [12], have studied the following bi-nonlocal fourth-order elliptic equation:

$$\begin{cases} (a + b\int_\Omega(|\Delta u|^2 + |\nabla u|^2)dx)(\Delta^2 u - \Delta u) = \Big(\int_\Omega \frac{1}{p}|u|^p dx\Big)^{\frac{2}{p}}|u|^{p-2}u + \lambda|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Based on the variational invariant sets of descending flow and cone theory, in [12], they obtained the existence of signed and sign-changing solutions.

Motivated by previous works, this paper is mainly concerned with the solutions of the bi-nonlocal problem (1.1) with the L^2-constraint $\int_\Omega |u|^2 dx = 1$. In fact, a simple calculation shows that the L^2-normalized solutions for (1.1) can be obtained by solving the following constraint minimization problem:

$$I(s,p,r,\beta) := \inf_{u\in\mathcal{K}} J(u), \tag{1.3}$$

where $\mathcal{K} := \left\{u \in H_0^1(\Omega),\ \int_\Omega |u|^2 = 1\right\}$ and $J(u)$ is an energy functional satisfying

$$J(u) = \frac{1}{s+1}\Big(\int_\Omega |\nabla u|^2 dx\Big)^{s+1} + \int_\Omega \sin^2|x|u^2 dx - \frac{2\beta}{(p+2)(r+1)}\Big(\int_\Omega |u|^{p+2}dx\Big)^{r+1}. \tag{1.4}$$

Some recent works involving different kinds of constrained variational problems have attracted our attention. In particular, we notice that for $s, r = 0,\ p = 2$, and $\beta > 0$ in (1.4), it is a hot research topic related to the well-known Gross–Pitaevskii functional (see [13, 14]), which is derived from the physical experimental phenomena of Bose–Einstein condensates. Roughly speaking, when the potential function $\sin^2|x|$ behaves like the types of polynomials, ring-shaped, multi-well, and periodic, in papers [15–18], the authors have established some results of constraint minimizers on the existence, nonexistence, and mass concentration behavior under the L^2-critical state. Especially for the potential being a logarithmic or homogeneous function [19, 20], the local uniqueness of the constraint minimizer is also analyzed.

In addition, for $s = 1,\ r = 0,\ \beta > 0$, and the potential function fulfilling suitable choices, (1.4) is regarded as a Kirchhoff-type energy functional, and there are many works related to studying the existence and limit behavior of constraint minimizers for (1.3). More precisely, Ye [21, 22] obtained the detailed results of existence and nonexistence for constraint minimizers when $\sin^2|x| = 0$ and Ω is replaced by the whole space. If the potential is a periodic function, Meng and Zeng [23] gave a detailed limit behavior of the minimizer. Somewhat similarly, there are many works [24–28] involved in the existence, non-existence, and limit properties of constraint minimizers when (1.1) possesses a L^2-subcritical or the L^2-critical term. Also for potential in the form of polynomials, Tang, Zeng, and their co-workers in [29, 30] obtained some results on the refined limit behavior and the uniqueness of constraint minimizers.

However, as we know, there are few articles studying the bi-nonlocal problem using constrained variational approaches. Inspired by the works mentioned above, in the present paper we are interested in the constrained minimization problem (1.3) with two nonlocal terms. More precisely, we are concerned with the existence, non-existence, and limit behavior of constraint minimizers for (1.3) by using the techniques of constrained variation and energy estimation.

Before stating our main results, we first introduce the following elliptic equation:

$$-\frac{p}{2}\Delta w + w - w^{p+1} = 0, \quad x \in \mathbb{R}^2, \ 0 < p < \infty, \tag{1.5}$$

and from [31], we know that (1.5) admits a unique (under translations) positive radially symmetric solution $w_p \in H^1(\mathbb{R}^2)$. Secondly, the following equality can be obtained directly by applying the Pohozaev identity,

$$\|\nabla w_p\|_{L^2}^2 = \|w_p\|_{L^2}^2 = \frac{2}{p+2}\|w_p\|_{L^{p+2}}^{p+2}. \tag{1.6}$$

Note from ([32], Proposition 4.1) that the solution w_p of (1.5) is exponential decay at infinity, that is,

$$|\nabla w_p(x)|, w_p(|x|) = O(|x|^{-\frac{1}{2}}e^{-|x|}) \ \text{ as } \ |x| \to \infty. \tag{1.7}$$

At last, a classical Gagliardo–Nirenberg inequality in bounded domains (see [33, 34]) is introduced as follows:

$$\|u\|_{L^{2+p}(\Omega)}^{2+p} \le C^*\|\nabla u\|_{L^2(\Omega)}^{p}\|u\|_{L^2(\Omega)}^2, \quad 0 < p < \infty, \tag{1.8}$$

where $C^* := \frac{p+2}{2\|w_p\|_{L^2}^p}$ and w_p is given by (1.5). Notice also from [33, 34] that the best constant C^* can not be attained.

Through a prior energy estimation of $J(u)$, one finds that the related properties of constraint minimizers depend heavily on the exponents s, r, p, and parameter β. Denote $\beta^* := \frac{(r+1)^{r+1}}{(s+r+2)^r(s+1)}\|w_p\|_{L^2}^{2(s+1)}$ and we divide s, p, r, β into the following cases for convenience.

(c_1). $p < \frac{2(s+1)}{r+1}$; (c_2). $p = \frac{2(s+1)}{r+1}$, $0 < \beta < \beta^*$;

(c_3). $p > \frac{2(s+1)}{r+1}$; (c_4). $p = \frac{2(s+1)}{r+1}, \beta \geq \beta^*$.

Based on the notations mentioned above, we next establish the following theorem on the existence and nonexistence of constraint minimizers:

Theorem 1.1. *If (c_1) or (c_2) holds, then $I(s,p,r,\beta)$ admits at least one minimizer; If either (c_3) or (c_4) holds, then $I(s,p,r,\beta)$ has no minimizer. Furthermore, we have for $p = \frac{2(s+1)}{r+1}$ and any β with $\beta \nearrow \beta^*$, the* $\lim_{\beta\nearrow\beta^*} I(s,p,r,\beta) = I(s,p,r,\beta^*) = 0$.

Notice that for $p = \frac{2(s+1)}{r+1}$, the Theorem 1.1 presents the fact that $I(s,p,r,\beta^*)$ has no minimizer. We care about what happens to the constraint minimizers for any β with $\beta \nearrow \beta^*$, and for this, the refined energy estimation of $I(s,p,r,\beta)$ as $\beta \nearrow \beta^*$ is necessary. In effect, one knows from [15–19] that when potential $\sin^2|x|$ behaves in the forms of polynomial, logarithmic, ring-shaped, multi-well, and periodic, the key steps in estimating energy are to deal with the potential term. However, since our elliptic equation (1.1) not only contains bi-nonlocal terms, but potential is a sinusoidal function, the techniques in [15–18] are ineffective for dealing with our problem. Hence, some skills for handling the potential term are constructed in Section 3. Meanwhile, the main result of energy estimation for $I(s,p,r,\beta)$ as $\beta \nearrow \beta^*$ can be stated as follows theorem:

Theorem 1.2. *If $p = \frac{2(s+1)}{r+1}$ and for any $\beta > 0$, the $I(s,p,r,\beta)$ satisfies*

$$I(s,p,r,\beta) \approx \frac{s+2}{s+1}(\beta^*)^{-\frac{1}{s+2}}\lambda^{\frac{s+1}{s+2}}(\beta^*-\beta)^{\frac{1}{s+2}} \quad \text{as } \beta \nearrow \beta^*, \tag{1.9}$$

where $\lambda = \frac{1}{\|w_p\|_{L^2}^2}\int_{\mathbb{R}^2}|x|^2|w_p|^2dx$ and w_p is given by (1.5).

Remark that the above $f \approx g$ means $f/g \to 1$ as $\beta \nearrow \beta^*$. According to the result of Theorem 1.2, our last theorem is concerned with the exact limit behavior of constraint minimizers as $\beta \nearrow \beta^*$. In truth, we can always assume that minimizers u_β of $I(s,p,\lambda)$ are positive due to $J(u_\beta) \geq J(|u_\beta|)$ and by applying the strong maximum principle to related elliptic equations. Therefore, we only establish a detailed result on the limit behavior of positive minimizers u_β as β tends to β^* from below.

Theorem 1.3. *Assume that $p = \frac{2(s+1)}{r+1}$ and u_β is a positive minimizer of $I(s,p,r,\beta)$, then we have*

1). u_β has a unique maximum point x_β fulfilling

$$x_\beta \to x_0 \text{ as } \beta \nearrow \beta^*,\ |x_0| = n_0\pi \text{ for some } n_0 \in N \text{ and } x_0 \notin \partial\Omega.$$

2). Set $\epsilon_\beta := \left(\int_\Omega |\nabla u_\lambda|^2dx\right)^{-\frac{1}{2}}$ and define a L^2-normalized function

$$v_\beta(x) := \epsilon_\beta u_\beta(\epsilon_\beta x + x_\beta),$$

then v_β satisfies

$$v_\beta(x) \to \frac{w_p(|x|)}{\|w_p\|_{L^2}} \quad \text{strongly in } H^1(\mathbb{R}^2),$$

where w_p is given by (1.5). *Further, the ϵ_β satisfies as $\beta \nearrow \beta^*$*

$$\epsilon_\beta \approx (\beta^*\lambda)^{-\frac{1}{2(s+2)}}(\beta^*-\beta)^{\frac{1}{2(s+2)}}.$$

Comment that the limit behavior of constraint minimizers in our paper is quite different from these conclusions in [15–19]. Although the sinusoidal potential $\sin^2|x|$ may attain its minimum at an inner point or some boundary point of Ω, one can rule out the case of minimizers blow-up near the boundary. Furthermore, we also give a refined blow-up rate of minimizers as $\beta \nearrow \beta^*$, which is mainly determined by the energy power of potential term $\int_\Omega \sin^2|x|u^2dx$.

We organized the article as follows: in Section 2, the existence and non-existence of minimizers are established by variational approaches and the upper energy estimation of functional $J(u)$. Section 3 gives a refined upper and lower energy estimation of $I(s,p,r,\beta)$ when $p = \frac{2(s+1)}{r+1}$ as $\beta \nearrow \beta^*$. The proof procedures of Theorems 1.2 and 1.3 are constructed in Section 4.

2. Existence and nonexistence of minimizers

This section is devoted to proving Theorem 1.1 on the existence and nonexistence of constraint minimizers for $I(s,p,r,\beta)$, which is divided into two cases for convenience.

Case 1. *If (c_1) or (c_2) holds, then $I(s,p,r,\beta)$ admits at least one minimizer.*
Proof. Assuming that (c_1) holds, one then derives from (1.8) that for any $u \in \mathcal{K}$

$$\begin{aligned} J(u) \geq & \frac{1}{s+1}\Big(\int_\Omega |\nabla u|^2dx\Big)^{s+1} + \int_\Omega \sin^2|x|u^2dx \\ & - \frac{2\beta}{(p+2)(r+1)}\Big(\frac{p+2}{2}\Big)^{r+1}\|w_p\|_{L^2}^{-p(r+1)}\Big(\int_\Omega |\nabla u|^2dx\Big)^{\frac{p(r+1)}{2}}. \end{aligned} \tag{2.1}$$

If (c_2) holds, we also have for any $u \in \mathcal{K}$

$$\begin{aligned} J(u) &\geq \Big[\frac{1}{s+1} - \frac{\beta(s+r+2)^r}{(r+1)^{r+1}\|w_p\|_{L^2}^{2(s+1)}}\Big]\Big(\int_\Omega |\nabla u|^2dx\Big)^{s+1} + \int_\Omega \sin^2|x|u^2dx \\ &= \frac{1}{s+1}\Big(1-\frac{\beta}{\beta^*}\Big)\Big(\int_\Omega |\nabla u|^2dx\Big)^{s+1} + \int_\Omega \sin^2|x|u^2dx \end{aligned}. \tag{2.2}$$

In fact, one can get from (2.1) and (2.2) that for any sequence $\{u_n\} \subseteq \mathcal{K}$, the functional $J(u_n)$ is bounded uniformly from below. Therefore, there is a minimizing sequence $\{u_n\} \subseteq \mathcal{K}$ such that

$$I(s,p,r,\beta) = \lim_{n\to\infty} J(u_n). \tag{2.3}$$

Since $\sin^2|x| \geq 0$, it is easy to deduce from (2.1) and (2.2) that $\int_\Omega |\nabla u_n|^2dx$ is bounded uniformly for n, that is, $\{u_n\}$ bounded in $\mathcal{K}$. The well-known Rellich's compactness ([35], Theorem 1.9) $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < +\infty$, yields that there exists a $u_0 \in \mathcal{K}$ such that $\{u_n\}$ admits a subsequence $\{u_k\}$ fulfilling as $k \to \infty$

$$u_k \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega), \quad u_k \to u_0 \text{ strongly in } L^q(\Omega),\ 1 \leq q < \infty. \tag{2.4}$$

Using (2.4) and weakly lower semi-continuity, one has

$$\liminf_{k\to\infty}\Big(\int_\Omega |\nabla u_k|^2dx\Big)^{s+1} \geq \Big(\int_\Omega |\nabla u_0|^2dx\Big)^{s+1},$$

and for any fixed $\beta > 0$

$$\lim_{k\to\infty}\frac{2\beta}{(p+2)(r+1)}\Big(\int_\Omega |u_k|^{p+2}dx\Big)^{r+1} = \frac{2\beta}{(p+2)(r+1)}\Big(\int_\Omega |u_0|^{p+2}dx\Big)^{r+1}.$$

It then yields that

$$I(s,p,r,\beta)=\liminf_{k\to\infty}J(u_k)\geq J(u_0)\geq I(s,p,r,\beta).$$

The above inequality shows that $J(u_0)=I(s,p,r,\beta)$, hence u_0 is a minimizer of $I(s,p,r,\beta)$. We then complete the proof of existence for the minimizer.

Case 2. *Either* (c_3) *or* (c_4) *holds, then* $I(s,p,r,\beta)$ *has no minimizer.*
Proof. Since Ω is a bounded connected domain and contains $(0,0)$ as an inner point, there is a finite circular region $B_{2R}(0)\subset\Omega$. Choosing a cut-off function $\varphi(x)\in C_0^\infty(\mathbb{R}^2)$ satisfies $0\leq\varphi(x)\leq 1$, $\varphi(x)=1$ for $|x|\leq R$, $\varphi(x)=0$ for $|x|>2R$, and $|\nabla\varphi(x)|\leq 2$ for $x\in\mathbb{R}^2$. Define a test function

$$u_\tau(x):=\frac{A_{\tau,R}\tau}{\|w_p\|_{L^2}}\varphi(\frac{x}{R})w_p(\tau x),\ x\in\Omega,\ \tau>0, \tag{2.5}$$

where w_p is given by (1.5) and $A_{\tau,R}>0$ is chosen so that $\int_\Omega|u_\tau(x)|^2dx=1$. Notice that the u_τ is well-defined in $H_0^1(\Omega)$ for any $\tau>0$. Using (1.7), a direct calculation yields that

$$1\leq A_{\tau,R}^2\leq 1+O(\tau^{-\infty})\quad\text{and}\ \lim_{\tau\to\infty}A_{\tau,R}=1\quad\text{as}\ \tau\to\infty. \tag{2.6}$$

The function $g(\tau)=O(\tau^{-\infty})$ means that $\lim_{\tau\to\infty}g(\tau)\tau^\iota=0$ for any $\iota>0$. Combining (1.7), (2.5) and (2.6), we obtain

$$\frac{1}{s+1}(\int_\Omega|\nabla u_\tau|^2dx)^{s+1}=\frac{1}{s+1}\frac{A_{\tau,R}^{2(s+1)}\tau^{2(s+1)}}{\|w_p\|_{L^2}^{2(s+1)}}(\int_{\mathbb{R}^2}|\nabla w_p|^2dx)^{s+1}+O(\tau^{-\infty}), \tag{2.7}$$

and

$$\begin{aligned}&\frac{2\beta}{(p+2)(r+1)}(\int_\Omega|u_\tau|^{p+2}dx)^{r+1}\\=&\frac{2\beta}{(p+2)(r+1)}\frac{A_{\tau,R}^{(p+2)(r+1)}\tau^{p(r+1)}}{\|w_p\|_{L^2}^{(p+2)(r+1)}}\Big(\int_{\mathbb{R}^2}|w_p|^{p+2}dx\Big)^{r+1}+O(\tau^{-\infty})\end{aligned}. \tag{2.8}$$

Since $0\leq\sin^2|x|\leq 1$ and $\sin^2|\frac{x}{\tau}|\approx|\frac{x}{\tau}|^2$ as $\tau\to\infty$ in $B_{\sqrt{\tau R}}(0)$, we hence have

$$\begin{aligned}\int_\Omega\sin^2|x|u_\tau^2dx&\leq\frac{A_{\tau,R}^2}{\|w_p\|_{L^2}^2}\Big[\int_{B_{\sqrt{\tau R}}(0)}\sin^2|\frac{x}{\tau}|w_p^2dx+\int_{B_{2\tau R}(0)\setminus B_{\sqrt{\tau R}}(0)}w_p^2dx\Big]\\&=\frac{A_{\tau,R}^2}{\|w_p\|_{L^2}^2}\Big[\tau^{-2}(1+o(1))\int_{B_{\sqrt{\tau R}}(0)}|x|^2w_p^2dx+\int_{B_{2\tau R}(0)\setminus B_{\sqrt{\tau R}}(0)}w_p^2dx\Big]\\&\leq\frac{A_{\tau,R}^2}{\|w_p\|_{L^2}^2}\Big[\tau^{-2}\int_{\mathbb{R}^2}|x|^2w_p^2dx+Ce^{-2\sqrt{\tau R}}+o(\tau^{-2})\Big]\\&\leq A_{\tau,R}^2\lambda\tau^{-2}+o(\tau^{-2})\quad\text{as}\ \tau\to\infty,\end{aligned} \tag{2.9}$$

where $\lambda = \frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |x|^2 |w_p|^2 dx$. Together with (2.7)–(2.9), one derives from (1.4) that

$$\begin{aligned} I(s,p,r,\beta) \le & J(u_\tau) \\ = & \frac{1}{s+1} \frac{A_{\tau,R}^{2(s+1)} \tau^{2(s+1)}}{\|w_p\|_{L^2}^{2(s+1)}} \Big(\int_{\mathbb{R}^2} |\nabla w_p|^2 dx \Big)^{s+1} + A_{\tau,R}^2 \lambda \tau^{-2} \\ & - \frac{2\beta}{(p+2)(r+1)} \frac{A_{\tau,R}^{(p+2)(r+1)} \tau^{p(r+1)}}{\|w_p\|_{L^2}^{(p+2)(r+1)}} \Big(\int_{\mathbb{R}^2} |w_p|^{p+2} dx \Big)^{r+1} + o(\tau^{-2}) \end{aligned}. \tag{2.10}$$

Under the assumption of (c_3), we get from (2.10) that

$$I(s,p,r,\beta) \le J(u_\tau) \to -\infty \quad \text{as } \tau \to \infty,$$

which yields that $I(s,p,r,\beta)$ has no minimizer.

Assuming that (c_4) holds as well as $p = \frac{2(s+1)}{r+1}$, $\beta > \beta^*$, one derives from (1.6), (2.6), and (2.10) as $\tau \to \infty$

$$I(s,p,r,\beta) \le J(u_\tau) = \frac{1}{s+1} \Big[1 - \frac{\beta}{\beta^*} \Big] \tau^{2(s+1)} + \lambda \tau^{-2} + o(\tau^{-2}), \tag{2.11}$$

which also presents $I(s,p,r,\beta) \le J(u_\tau) \to -\infty$, and hence $I(s,p,r,\beta)$ has no minimizer. For the other case of $p = \frac{2(s+1)}{r+1}$ and $\beta = \beta^*$, one can gain from (2.2) and (2.11) that $I(s,p,r,\beta^*) = 0$. We next prove that $I(s,p,r,\beta^*)$ has no minimizer by a contradiction. Suppose that there exists a $\hat{u} \in \mathcal{K}$ such that $\hat{u}$ is a minimizer of $I(s,p,r,\beta^*)$. We then derive from (1.4), (2.2), and (2.11) that

$$\int_\Omega \sin |x|^2 \hat{u}^2 dx = \frac{1}{s+1} \Big(\int_\Omega |\nabla \hat{u}|^2 dx \Big)^{s+1} - \frac{2\beta^*}{(p+2)(r+1)} \Big(\int_\Omega |\hat{u}|^{p+2} dx \Big)^{r+1} = 0,$$

which gives

$$\frac{1}{s+1} \Big(\int_\Omega |\nabla \hat{u}|^2 dx \Big)^{s+1} = \frac{2\beta^*}{(p+2)(r+1)} \Big(\int_\Omega |\hat{u}|^{p+2} dx \Big)^{r+1}. \tag{2.12}$$

However, this is impossible due to the fact that the best constant C^* in (1.8) can-not be attained. Thus, the non-existence proof of the minimizer for $I(s,p,r,\beta^*)$ has finished.

At last, assume that $p = \frac{2(s+1)}{r+1}$ and for any β with $\beta \nearrow \beta^*$. Taking $\tau = (\beta^* - \beta)^{-\frac{1}{4(s+1)}}$, one obtains from (2.2) and (2.11) that as $\beta \nearrow \beta^*$

$$0 \le \lim_{\beta \nearrow \beta^*} I(s,p,r,\beta) \le J(u_\tau) \le \frac{1}{(s+1)\beta^*} (\beta^* - \beta)^{\frac{1}{2}} + \lambda (\beta^* - \beta)^{\frac{1}{2(s+1)}} \to 0,$$

which, together with $I(s,p,r,\beta^*) = 0$, gives

$$\lim_{\beta \nearrow \beta^*} I(s,p,r,\beta) = I(s,p,r,\beta^*) = 0.$$

So far, we have completed the proof of Theorem 1.1.

3. Upper and lower energy estimations

In this section, we mainly care about how the energy changes of $I(s, p, r, \beta)$ for $p = \frac{2(s+1)}{r+1}$ as $\beta \nearrow \beta^*$. To achieve our goals, we begin with the upper energy estimation of $I(s, p, r, \beta)$, which is stated as the following lemma:

Lemma 3.1. *For* $p = \frac{2(s+1)}{r+1}$ *and* $0 < \beta < \beta^* = \frac{(r+1)^{r+1}}{(s+r+2)^r(s+1)}\|w_p\|_{L^2}^{2(s+1)}$, *the* $I(s, p, r, \beta)$ *satisfies*

$$\limsup_{\beta \nearrow \beta^*} I(s, p, r, \beta) \leq \frac{s+2}{s+1}(\beta^*)^{-\frac{1}{s+2}} \lambda^{\frac{s+1}{s+2}} (\beta^* - \beta)^{\frac{1}{s+2}} [1 + o(1)], \tag{3.1}$$

where $\lambda = \frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |x|^2 |w_p|^2 dx > 0$ *and* w_p *is given by* (1.5).

Proof. Repeating the proof procedure in (2.11), we obtain $\tau \to \infty$

$$I(s, p, r, \beta) \leq J(u_\tau) \leq \frac{1}{(s+1)\beta^*}(\beta^* - \beta)\tau^{2(s+1)} + \lambda\tau^{-2} + o(\tau^{-2}). \tag{3.2}$$

Define a function

$$f(\tau) := \frac{1}{(s+1)\beta^*}(\beta^* - \beta)\tau^{2(s+1)} + \lambda\tau^{-2},$$

and let $f'(\tau) = 0$, then we have

$$\tau^{2(s+2)} = \beta^* \lambda (\beta^* - \beta)^{-1}.$$

Taking $\tau = (\beta^* \lambda)^{\frac{1}{2(s+2)}} (\beta^* - \beta)^{-\frac{1}{2(s+2)}}$ and putting it into (3.2), we get the upper energy estimation of Lemma 3.1.

For the sake of estimating lower energy, we assume that u_β is a positive minimizer of $I(s, p, r, \beta)$ and x_β is its local maximum point. Set a L^2-normalized function

$$v_\beta(x) := \epsilon_\beta u_\beta(\epsilon_\beta x + x_\beta), \quad x \in \Omega, \tag{3.3}$$

and ϵ_β is defined by

$$\epsilon_\beta := \Big(\int_\Omega |\nabla u_\beta|^2 dx \Big)^{-\frac{1}{2}}. \tag{3.4}$$

We establish some indispensable conclusions on ϵ_β and v_β as $\beta \nearrow \beta^*$, which are described by the following Claims 1–5.

Claim 1. *Denote* $\Omega_\beta := \{x | (\epsilon_\beta x + x_\beta) \in \Omega\}$, *then we have* $\epsilon_\beta \to 0$ *as* $\beta \nearrow \beta^*$. *Moreover,* $\int_{\Omega_\beta} |\nabla v_\beta|^2 dx = 1$ *and* $\big(\int_{\Omega_\beta} |v_\beta|^{p+2} dx \big)^{r+1} \to \frac{s+r+2}{\beta(s+1)}$ *as* $\beta \nearrow \beta^*$.

Since $\int_\Omega |u_\beta|^2 = 1$, one can rule out $\epsilon_\beta \to \infty$ by Rellich's compactness ([35], Theorem 1.9). We next shows that $\epsilon_\beta \to 0$ as $\beta \nearrow \beta^*$. If not, then there exists a sequence $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$, such that $\{u_{\beta_k}\}$ is bounded uniformly in $\mathcal{K}$. Repeating the existence proof of constraint minimizer in Theorem 1.1, one obtains that $I(s, p, r, \beta^*)$ has at least one minimizer. However, this is a contradiction due to the Theorem 1.1 presents a fact that $I(s, p, r, \beta^*)$ has no minimizer. Thus, we declare that $\epsilon_\beta \to 0$ as $\beta \nearrow \beta^*$.

In truth, (3.3) and (3.4) just give

$$\int_{\Omega_\beta} |\nabla v_\beta|^2 dx = \epsilon_\beta^{-2} \int_\Omega |\nabla u_\beta|^2 dx = 1.$$

Together with (1.8) and $\lim_{\beta\nearrow\beta^*} I(s,p,r,\beta) = 0$ in Theorem 1.1, one further deduces that for $p = \frac{2(s+1)}{r+1}$,

$$\begin{aligned}
0 &\le \frac{1}{s+1}\Big(\int_\Omega |\nabla u_\beta|^2 dx\Big)^{s+1} - \frac{2\beta}{(p+2)(r+1)}\Big(\int_\Omega |u_\beta|^{p+2} dx\Big)^{r+1} \\
&= \frac{1}{s+1}\epsilon_\beta^{-2(s+1)} - \epsilon_\beta^{-2(s+1)}\frac{2\beta}{(p+2)(r+1)}\Big(\int_{\Omega_\beta} |v_\beta|^{p+2} dx\Big)^{r+1} \qquad , \\
&\le I(s,p,r,\beta) \to 0 \quad \text{as } \beta \nearrow \beta^*
\end{aligned} \tag{3.5}$$

which then yields that

$$\Big(\int_{\Omega_\beta} |v_\beta|^{p+2} dx\Big)^{r+1} \to \frac{s+r+2}{\beta(s+1)} \quad \text{as } \beta \nearrow \beta^*.$$

Claim 2. *There exists a finite circular region $B_{2R}(0) \subset \Omega_\beta$ and a constant $\theta > 0$ satisfying*

$$\liminf_{\beta\nearrow\beta^*} \int_{B_{2R}(0)} |v_\beta|^2 dx \ge \theta > 0. \tag{3.6}$$

Combining (2.2) and $\lim_{\beta\nearrow\beta^*} I(s,p,r,\beta) = 0$ in Theorem 1.1, then we get

$$\int_\Omega \sin^2|x||u_\beta|^2 dx = \int_{\Omega_\beta} \sin^2|\epsilon_\beta x + x_\beta||v_\beta|^2 dx \to 0 \quad \text{as } \beta \nearrow \beta^*. \tag{3.7}$$

Since u_β is a positive minimizer of (1.3), it fulfills

$$\begin{cases}
-\Big(\int_\Omega |\nabla u_\beta|^2 dx\Big)^s \Delta u_\beta + \sin^2|x| u_\beta = \mu_\beta u_\beta + \beta\Big(\int_\Omega |u_\beta|^{p+2} dx\Big)^r |u_\beta|^p u_\beta, & x \in \Omega, \\
u_\beta = 0, & x \in \partial\Omega,
\end{cases}$$

with $\int_\Omega |u_\beta|^2 dx = 1$. Multiplying the equation by u_β and integrating over Ω, one has

$$\Big(\int_\Omega |\nabla u_\beta|^2 dx\Big)^{s+1} + \int_\Omega \sin^2|x| u_\beta^2 dx = \mu_\beta + \beta\Big(\int_\Omega |u|^{p+2} dx\Big)^{r+1},$$

which, together with (1.3), gives $I(s,p,r,\beta) = J(u_\beta)$ and

$$\mu_\beta = I(s,p,r,\beta) + \frac{s}{s+1}\Big(\int_\Omega |\nabla u_\beta|^2 dx\Big)^{s+1} - \frac{\beta(s+r+1)}{s+r+2}\Big(\int_\Omega |u_\beta|^{p+2} dx\Big)^{r+1}. \tag{3.8}$$

(3.8) and Claim 1 yield that

$$\mu_\beta \epsilon_\beta^{2(s+1)} \to -\frac{r+1}{s+1} \quad \text{as } \beta \nearrow \beta^*, \tag{3.9}$$

as well as v_β fulfills

$$\begin{cases}
-\Delta v_\beta + \epsilon_\beta^{2(s+1)} \sin^2|\epsilon_\beta x + x_\beta| v_\beta = \mu_\beta \epsilon_\beta^{2(s+1)} v_\beta + \beta\Big(\int_{\Omega_\beta} |v_\beta|^{p+2} dx\Big)^r v_\beta^{p+1}, & x \in \Omega_\beta, \\
v_\beta = 0, & x \in \partial\Omega_\beta,
\end{cases} \tag{3.10}$$

where $\Omega_\beta := \{x|\epsilon_\beta x + x_\beta \in \Omega\}$. A fact shows that v_β attains its local maximum at $x = 0$ due to x_β being the local maximum of u_β. Hence, we deduce from Claim 1, (3.9), and (3.10) that

$$v_\beta(0) \geq \theta > 0 \quad \text{as } \beta \nearrow \beta^*. \tag{3.11}$$

Furthermore, we have

$$-\Delta v_\beta - c(x)v_\beta \leq 0, \quad x \in \Omega_\beta, \tag{3.12}$$

where $\theta > 0$ is a constant and $c(x) = \beta(\int_{\Omega_\beta} |v_\beta|^{p+2}dx)^r v_\beta^p$. In fact, one can claim that $0 \notin \partial\Omega_\beta$. If this is not true, then from (3.10) we know $v_\beta(0) = 0$ for $0 \in \partial\Omega_\beta$, which is a contradiction with (3.11). By applying Theorem 4.1 in [36], one derives from (3.12) that there exists a finite circular region $B_{2R}(0) \subset \Omega_\beta$ such that

$$\max_{B_R(0)} v_\beta \leq C\Big(\int_{B_{2R}(0)} |v_\beta|^2 dx\Big)^{\frac{1}{2}}, \tag{3.13}$$

where C is a suitable positive constant. (3.11) and (3.13) then yield that

$$\liminf_{\beta\nearrow\beta^*} \int_{B_{2R}(0)} |v_\beta|^2 dx \geq \theta > 0. \tag{3.14}$$

Hence Claim 2 is holding.

Claim 3. *For any $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$ as $k \to \infty$, the local maximum sequence $\{x_{\beta_k}\}$ of u_{β_k} has a subsequence (still denoted by x_{β_k}) satisfying*

$$x_{\beta_k} \to x_0 \in \bar{\Omega}, \quad \textit{as } \beta_k \nearrow \beta^*. \tag{3.15}$$

Furthermore, $\Omega_{\beta^*} := \lim_{k\to\infty} \Omega_{\beta_k} = \lim_{k\to\infty}\{x|\epsilon_{\beta_k}x + x_{\beta_k} \in \Omega\} = \mathbb{R}^2$.

Because Ω is a bounded domain, $\{x_{\beta_k}\}$ admits a subsequence satisfying

$$x_{\beta_k} \to x_0 \in \bar{\Omega}, \quad \text{as } \beta_k \nearrow \beta^*.$$

We next prove that $\Omega_{\beta^*} = \mathbb{R}^2$. In view of the fact $\int_{\Omega_{\beta_k}} |\nabla v_{\beta_k}|^2 dx = 1$, by passing the weak limit to (3.10), there exists a function $0 \leq v_0 \in H_0^1(\Omega_{\beta^*})$ such that

$$\begin{cases} -\Delta v_0 + \dfrac{r+1}{s+1}v_0 - (\beta^*)^{\frac{1}{r+1}}(\dfrac{s+r+2}{s+1})^{\frac{r}{r+1}} v_0^{p+1} = 0, & x \in \Omega_{\beta^*}, \\ v_0 = 0, & x \in \partial\Omega_{\beta^*}. \end{cases} \tag{3.16}$$

If x_0 is an inner point of Ω, then one has $\Omega_{\beta^*} = \mathbb{R}^2$ due to $\epsilon_{\beta_k} \to 0$ in Claim 1. If $x_0 \in \partial\Omega$, we declare that

$$\liminf_{k\to\infty} \frac{|x_{\beta_k} - x_0|}{\epsilon_{\beta_k}} \to \infty, \tag{3.17}$$

which also yields $\Omega_{\beta^*} = \mathbb{R}^2$. Assume that (3.17) is false, that is, $\liminf_{k\to\infty} \frac{|x_{\beta_k}-x_0|}{\epsilon_{\beta_k}} \leq C$. Up to translation and rotation, one might set

$$\liminf_{k\to\infty} \frac{x_0 - x_{\beta_k}}{\epsilon_{\beta_k}} = y_0 := (0, -\alpha), \tag{3.18}$$

where $\alpha \in \mathbb{R}$ is a positive constant. The (3.18) then gives

$$\Omega_{\beta^*} = \lim_{k\to\infty}\{x|\epsilon_{\beta_k}x + x_{\beta_k} \in \Omega\} = \mathbb{R}^2_{-\alpha} := \mathbb{R} \times (-\alpha, +\infty).$$

By (3.16), one has

$$\begin{cases} -\Delta v_0 + \frac{r+1}{s+1}v_0 - (\beta^*)^{\frac{1}{r+1}}(\frac{s+r+2}{s+1})^{\frac{r}{r+1}}v_0^{p+1} = 0, & x \in \mathbb{R}^2_{-\alpha}, \\ v_0 = 0, & x \in \partial\mathbb{R}^2_{-\alpha}. \end{cases} \tag{3.19}$$

However, the nonexistence result in [37] shows that $v_0 \equiv 0$, which contradicts Claim 2. Therefore, (3.17) is holding, and the proof of Claim 3 is completed.

Claim 4. *For any $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$ as $k \to \infty$, the x_{β_k} is the unique maximum point of u_{β_k} as well as v_{β_k} fulfills*

$$\lim_{k\to\infty} v_{\beta_k} = \frac{w_p(|x|)}{||w_p||_{L^2}} \quad \textit{strongly in } H^1(\mathbb{R}^2). \tag{3.20}$$

Using Claim 3 and (3.19), we have

$$-\Delta v_0 + \frac{r+1}{s+1}v_0 - (\beta^*)^{\frac{1}{r+1}}(\frac{s+r+2}{s+1})^{\frac{r}{r+1}}v_0^{p+1} = 0, \quad \text{in } \mathbb{R}^2. \tag{3.21}$$

After this, one can say that $v_0 > 0$ by applying the strong maximum principle. Taking $p = \frac{2(s+1)}{r+1}$ in (1.5), it then gives a fact that the v_0 (under rescaling) behaves like

$$v_0(x) = \frac{1}{||w_p||_{L^2}}w_p(|x - y_0|),$$

for some $y_0 \in \mathbb{R}^2$ and $||v_0||_2^2 = 1$. Applying the Hölder and Sobolev inequalities, we know that

$$||u||_{L^q} \leq C||u||^{\gamma}_{L^2}||u||^{1-\gamma}_{H^1},$$

for any $u \in H^1(\mathbb{R}^2)$ with $q \in (2, \infty)$ and $\gamma \in (0, 1)$, which then yields that $v_{\beta_k} \to v_0$ strongly in $L^q(\mathbb{R}^2)$ with $q \in [2, \infty)$ as $k \to \infty$. One therefore concludes from (3.10) and (3.21) that

$$\lim_{k\to\infty} ||\nabla v_{\beta_k}||^2_{L^2} = ||\nabla v_0||^2_{L^2}. \tag{3.22}$$

Because $\sin^2|\epsilon_{\beta_k}x + x_{\beta_k}|$ is locally Lipschitz continuous in Ω_{β_k}, by the method of proving Theorem 1.2 in [17], one then deduces from (3.10) that $v_{\beta_k} \in C^{2,\alpha}_{loc}(\Omega_{\beta_k})$, $\alpha \in (0, 1)$. Therefore, we have $v_0 \in C^2_{loc}(\mathbb{R}^2)$, and v_0 fulfills

$$v_{\beta_k} \to v_0 \quad \text{in } C^2_{loc}(\mathbb{R}^2) \text{ as } k \to \infty. \tag{3.23}$$

A well-known result is that the solution w_p of (1.5) admits 0 as its unique (up to translations) critical point, which then yields from (3.23) that 0 is a unique critical point of v_0. Therefore,

$$v_0(x) = \frac{1}{||w_p||_{L^2}}w_p(|x|). \tag{3.24}$$

In view of (3.20) and Claim 2, we know $v_{\beta_k} \to 0$ uniformly in k as $|x| \to \infty$, which then yields that local maximum points of v_{β_k} stay in a finite circular region $B_\gamma(0)$. Taking γ small enough, it thus infers from Lemma 4.2 in [38] that 0 is the unique critical point of v_{β_k} for k large enough. The above conclusion, together with (3.3), gives that x_{β_k} is the unique maximum point of u_{β_k} as $k \to \infty$. We thus complete the proof of Claim 4.

Claim 5. *For any* $\{\beta_k\}$ *with* $\beta_k \nearrow \beta^*$ *as* $k \to \infty$*, the unique maximum point* x_{β_k} *of* u_{β_k} *satisfies*

$$x_{\beta_k} \to x_0 \text{ as } \beta_k \nearrow \beta^*, \ |x_0| = n_0\pi \text{ for some } n_0 \in N \text{ and } x_0 \notin \partial\Omega.$$

If $|x_0| \neq n\pi$ for any $n \in N$, then $\sin^2|x_0| > 0$. By applying Claim 2 and Fatou's lemma, there exists a positive constant $\mathcal{E}$ such that

$$\begin{aligned} &\liminf_{k\to\infty} \int_{\Omega_{\beta_k}} \sin^2|\epsilon_{\beta_k}x + x_{\beta_k}||v_{\beta_k}(x)|^2dx \\ \geq &\int_{B_{2R}(0)} \liminf_{k\to\infty} \sin^2|\epsilon_{\beta_k}x + x_{\beta_k}||v_{\beta_k}(x)|^2dx \geq \mathcal{E} > 0 \end{aligned},$$

which is a contradiction with (3.7). Thus, there exists a $n_0 \in N$ such that $|x_0| = n_0\pi$.

In the following part, we shall prove $x_0 \notin \partial\Omega$, which comes true by establishing a contradiction. In point of fact, we may assume that $(0,0) \neq x_0 \in \partial\Omega$ due to $(0,0)$ being an inner point of Ω. For any $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$ as $k \to \infty$, we first claim that

$$\liminf_{k\to\infty} \frac{|x_0| - |x_{\beta_k}|}{\epsilon_{\beta_k}} = \liminf_{k\to\infty} \frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}} \to \infty \, (n_0 \neq 0). \tag{3.25}$$

Set

$$x_{\beta_k} := (x_{\beta_{1,k}}, x_{\beta_{2,k}}) \text{ and } x_0 := (m_1, m_2) \in \partial\Omega,$$

where m_1 and m_2 satisfy

$$m_1^2 + m_2^2 = n_0^2\pi^2 \text{ for some } n_0 \in N^+.$$

Without loss of generality, we only consider the case of $m_1, m_2 > 0$ because the other cases are essentially the same. On basis of $x_{\beta_{1,k}} \to m_1 > 0$, $x_{\beta_{2,k}} \to m_2 > 0$ as $k \to \infty$, one easily knows that there exist constants $r_1, r_2 > 0$ and C_1, C_2 satisfying

$$m_1 - x_{\beta_{1,k}} = C_1\epsilon_{\beta_k}^{r_1} \text{ and } m_2 - x_{\beta_{2,k}} = C_2\epsilon_{\beta_k}^{r_2} \text{ as } k \to \infty. \tag{3.26}$$

As a matter of fact, one can show that

$$r := \min\{r_1, r_2\} < 1. \tag{3.27}$$

If not, that is, $r \geq 1$ in (3.27), it then follows from (3.26) that there exists a constant $M_1 > 0$ such that

$$\begin{aligned} \liminf_{k\to\infty} \frac{|x_{\beta_k} - x_0|}{\epsilon_{\beta_k}} &= \liminf_{k\to\infty} \frac{\sqrt{(m_1 - x_{\beta_{1,k}})^2 + (m_2 - x_{\beta_{2,k}})^2}}{\epsilon_{\beta_k}} \\ &= \liminf_{k\to\infty} \frac{\sqrt{C_1^2\epsilon_{\beta_k}^{2r_1} + C_2^2\epsilon_{\beta_k}^{2r_2}}}{\epsilon_{\beta_k}} \\ &= \liminf_{k\to\infty} \frac{M_1\epsilon_{\beta_k}^{r}(1 + o(1))}{\epsilon_{\beta_k}} \leq M_1 \end{aligned}, \tag{3.28}$$

which contradicts (3.17). Therefore, the above (3.27) holds. By (3.27), one can calculate that there is a constant $M_2 > 0$ such that

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{|x_0| - |x_{\beta_k}|}{\epsilon_{\beta_k}} = \liminf_{k\to\infty} \frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}} \\
&= \liminf_{k\to\infty} \frac{n_0\pi - \sqrt{m_1^2 + m_2^2 + C_1^2\epsilon_{\beta_k}^{2r_1} + C_2^2\epsilon_{\beta_k}^{2r_2} - 2m_1C_1\epsilon_{\beta_k}^{r_1} - 2m_2C_2\epsilon_{\beta_k}^{r_2}}}{\epsilon_{\beta_k}} \\
&= \liminf_{k\to\infty} \frac{n_0\pi - \sqrt{(m_1^2 + m_2^2)(1 - M_2\epsilon_{\beta_k}^{r} + o(\epsilon_{\beta_k}^{r}))^2}}{\epsilon_{\beta_k}} \\
&= \liminf_{k\to\infty} \frac{n_0\pi\epsilon_{\beta_k}^{r}(M_2 + o(1))}{\epsilon_{\beta_k}} \to \infty
\end{aligned} \tag{3.29}$$

Therefore, (3.25) is holding.

Based on Claims 3 and 4, one further deduces that

$$\liminf_{k\to\infty} \int_{\Omega_{\beta_k}} |\nabla v_{\beta_k}|^2 dx = \int_{\mathbb{R}^2} |\nabla v_0|^2 dx = \frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |\nabla w_p|^2 dx, \tag{3.30}$$

and

$$\liminf_{k\to\infty} \int_{\Omega_{\beta_k}} |v_{\beta_k}|^{p+2} dx = \int_{\mathbb{R}^2} |v_0|^{p+2} dx = \frac{1}{\|w_p\|_{L^{p+2}}^{p+2}} \int_{\mathbb{R}^2} |w_p|^{p+2} dx. \tag{3.31}$$

By the fact that $\sin^2|x| = \sin^2(n_0\pi - |x|)$ for all $x \in \Omega$, one then deduces from Claim 2 and (3.25) that as $k \to \infty$

$$\begin{aligned}
&\liminf_{k\to\infty} \epsilon_{\beta_k}^{-2} \int_{\Omega_{\beta_k}} \sin^2|\epsilon_{\beta_k}x + x_{\beta_k}||v_{\beta_k}|^2 dx \\
&= \liminf_{k\to\infty} \epsilon_{\beta_k}^{-2} \int_{B_{2R}(0)} \sin^2(n_0\pi - |\epsilon_{\beta_k}x + x_{\beta_k}|)|v_{\beta_k}|^2 dx \\
&= \liminf_{k\to\infty} \epsilon_{\beta_k}^{-2} \int_{B_{2R}(0)} (n_0\pi - |\epsilon_{\beta_k}x + x_{\beta_k}|)^2(1 + o(1))|v_{\beta_k}|^2 dx \\
&\geq \liminf_{k\to\infty} \int_{B_{2R}(0)} (\frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}} + \frac{|x_{\beta_k}| - |\epsilon_{\beta_k}x + x_{\beta_k}|}{\epsilon_{\beta_k}})^2(1 + o(1))|v_{\beta_k}|^2 dx \\
&\geq \liminf_{k\to\infty} \int_{B_{2R}(0)} (\frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}} - \frac{|\epsilon_{\beta_k}x|}{\epsilon_{\beta_k}})^2(1 + o(1))|v_{\beta_k}|^2 dx \geq (\mathcal{P}_1 - 2R)^2\theta
\end{aligned} \tag{3.32}$$

where $R, \theta > 0$ are constants and $\mathcal{P}_1$ is an arbitrarily large constant. For $p = \frac{2(s+1)}{r+1}$ and $\beta_k \nearrow \beta^*$, combining (1.6), (3.30)–(3.32), and Claim 3, a direct calculation deduces that

$$\begin{aligned}
&\liminf_{k\to\infty} I(s, p, r, \beta_k) = \liminf_{k\to\infty} J(u_{\beta_k}) \\
&= \liminf_{k\to\infty} \Big[\frac{1}{s+1}\epsilon_{\beta_k}^{-2(s+1)}(\int_{\Omega_{\beta_k}} |\nabla v_{\beta_k}|^2 dx)^{2(s+1)} + \int_{\Omega_{\beta_k}} \sin^2|\epsilon_{\beta_k}x + x_{\beta_k}|v_{\beta_k}^2 dx \\
&\quad - \epsilon_{\beta_k}^{-p(r+1)} \frac{2\beta_k}{(p+2)(r+1)}(\int_{\Omega_{\beta_k}} |v_{\beta_k}|^{p+2} dx)^{r+1}\Big] \\
&\geq \frac{1}{s+1}(1 - \frac{\beta_k}{\beta^*})\epsilon_{\beta_k}^{-2(s+1)} + (\mathcal{P}_1 - 2R)^2\theta\epsilon_{\beta_k}^2 \geq \mathcal{P}_2(\beta^* - \beta_k)^{\frac{1}{s+2}}
\end{aligned} \tag{3.33}$$

where $\mathcal{P}_2$ is an arbitrarily large constant. However, this contradicts the energy upper bound in Lemma 3.1. Hence, one concludes that $x_0 \notin \partial\Omega$. So far, we have completed the proof of Claim 5.

In virtue of Claims 1–5, we next establish the lower energy estimation of $I(s, p, r, \beta_k)$ for any $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$, which can be rendered by the following lemma.

Lemma 3.2. *For $p = \frac{2(s+1)}{r+1}$ and any sequence $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$ as $k \to \infty$, then there exists a subsequence $\{\beta_k\}$ (still denoted by $\{\beta_k\}$) such that the $I(s, p, r, \beta_k)$ fulfills*

$$\liminf_{\beta \nearrow \beta^*} I(s, p, r, \beta_k) \geq \frac{s+2}{s+1} (\beta^*)^{-\frac{1}{s+2}} \lambda^{\frac{s+1}{s+2}} (\beta^* - \beta_k)^{\frac{1}{s+2}}, \tag{3.34}$$

where $\lambda = \frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |x|^2 |w_p|^2 dx$ and w_p is given by (1.5).

Proof. Assuming that $\{u_{\beta_k}\}$ is a positive minimizer sequence and x_{β_k} is its unique maximum point. Define ϵ_{β_k}, v_{β_k} similar to (3.3) and (3.4). Repeating the proof procedures of Claims 1–4, we have

$$\liminf_{k \to \infty} \int_{\Omega_{\beta_k}} |\nabla v_{\beta_k}|^2 dx = \frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |\nabla w_p|^2 dx, \tag{3.35}$$

and

$$\liminf_{k \to \infty} \int_{\Omega_{\beta_k}} |v_{\beta_k}|^{p+2} dx = \frac{1}{\|w_p\|_{L^{p+2}}^{p+2}} \int_{\mathbb{R}^2} |w_p|^{p+2} dx. \tag{3.36}$$

Claims 4 and 5 show that there exists an inner point $x_0 \in \Omega$ such that the unique maximum point x_{β_k} satisfying as $k \to \infty$

$$x_{\beta_k} \to x_0, \ |x_0| = n_0\pi \text{ for some } n_0 \in N.$$

Similar to the calculation of (3.32), one obtains that

$$\begin{aligned} &\liminf_{k \to \infty} \epsilon_{\beta_k}^{-2} \int_{\Omega_{\beta_k}} \sin^2 |\epsilon_{\beta_k} x + x_{\beta_k}| |v_{\beta_k}|^2 dx \\ \geq &\liminf_{k \to \infty} \int_{B_{\epsilon_{\beta_k}^{-\frac{1}{2}}}(0)} \Big(\frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}} + \frac{|x_{\beta_k}| - |\epsilon_{\beta_k} x + x_{\beta_k}|}{\epsilon_{\beta_k}}\Big)^2 (1 + o(1)) |v_{\beta_k}|^2 dx \end{aligned}. \tag{3.37}$$

Actually, one can declare that $\frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}}$ is bounded uniformly as $k \to +\infty$. If not, then $\frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}} \to \infty$ as $k \to +\infty$, repeating the proof of (3.33), we also obtain a contradiction with Lemma 3.1. Thus, the $\{\beta_k\}$ exists a subsequence (still denoted by $\{\beta_k\}$) such that $\frac{n_0\pi - |x_{\beta_k}|}{\epsilon_{\beta_k}} \to y_0$ for some $y_0 \in \mathbb{R}^2$. It then derives from (3.37) and the definition of λ in (1.9) that

$$\begin{aligned} &\liminf_{k \to \infty} \epsilon_{\beta_k}^{-2} \int_{\Omega_{\beta_k}} \sin^2 |\epsilon_{\beta_k} x + x_{\beta_k}| |v_{\beta_k}|^2 dx \\ \geq &\frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |x + y_0|^2 |w_p|^2 dx \\ = &\frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |x|^2 |w_p(|x - y_0|)|^2 dx \\ \geq &\frac{1}{\|w_p\|_{L^2}^2} \int_{\mathbb{R}^2} |x|^2 |w_p(|x|)|^2 dx = \lambda \end{aligned}, \tag{3.38}$$

since w_p satisfies (1.5) and is also a radial decreasing function. Combining (3.35), (3.36), and (3.38), we have

$$\liminf_{k\to\infty} I(s,p,r,\beta_k) = \liminf_{k\to\infty} F(u_{\beta_k}) \geq \frac{1}{s+1}(1-\frac{\beta_k}{\beta^*})\epsilon_{\beta_k}^{-2(s+1)} + \lambda\epsilon_{\beta_k}^2. \tag{3.39}$$

Set a function

$$f(\epsilon_{\beta_k}) := \frac{1}{s+1}(1-\frac{\beta_k}{\beta^*})\epsilon_{\beta_k}^{-2(s+1)} + \lambda\epsilon_{\beta_k}^2, \tag{3.40}$$

and $f(\epsilon_{\beta_k})$ achieves its unique minimum at

$$\epsilon_{\beta_k} = (\lambda\beta^*)^{-\frac{1}{2(s+2)}}(\beta^*-\beta_k)^{\frac{1}{2(s+2)}} \text{ as } k\to\infty. \tag{3.41}$$

Taking ϵ_{β_k} into (3.39), it then yields that Lemma 3.2 is holding.

4. Proof of main theorems

In light of previous Claims 1–5, Lemmas 3.1 and 3.2, in this section we shall give the proof of Theorems 1.2 and 1.3. For $p = \frac{2(s+1)}{r+1}$ and $0 < \beta < \beta^*$, we assume that u_β is a positive minimizer of $I(s,p,r,\beta)$ and x_β being its unique maximum point. Defined ϵ_β, v_β the same as (3.3) and (3.4), in the following we begin with the proof Theorem 1.2.

Proof of Theorem 1.2. Repeating the proof process of Lemma 3.1, one obtains that, when $p = \frac{2(s+1)}{r+1}$ and for any β with $\beta \nearrow \beta^*$, the $I(s,p,r,\beta)$ satisfies

$$\limsup_{\beta\nearrow\beta^*} I(s,p,r,\beta) \leq \frac{s+2}{s+1}(\beta^*)^{-\frac{1}{s+2}}\lambda^{\frac{s+1}{s+2}}(\beta^*-\beta)^{\frac{1}{s+2}}[1+o(1)]. \tag{4.1}$$

Hence, the upper energy estimation of $I(s,p,r,\beta)$ in Theorem 1.2 is holding.

For the lower energy estimation, similar to the proof of Lemma 3.2, for any sequence $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$, passing a subsequence if necessary (still denoted by $\{\beta_k\}$), we obtain that $I(s,p,r,\beta_k)$ satisfies

$$\liminf_{\beta_k\nearrow\beta^*} I(s,p,r,\beta_k) \geq \frac{s+2}{s+1}(\beta^*)^{-\frac{1}{s+2}}\lambda^{\frac{s+1}{s+2}}(\beta^*-\beta_k)^{\frac{1}{s+2}}. \tag{4.2}$$

In fact, the lower energy in (4.2) holds for any sequence $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$. Argue by contradiction: suppose that there exists a sequence $\{\beta_k'\}$ with $\beta_k' \nearrow \beta^*$ such that (4.2) is not true. Repeating the proof of Lemma 3.2, we also derive that the $\{\beta_k'\}$ admits a subsequence, making sure that (4.2) is holding, which leads to a contradiction. Thus, (4.2) holds for any sequence $\{\beta_k\}$ with $\beta_k \nearrow \beta^*$. Furthermore, one easily knows that (4.2) is essentially true for any β with $\beta \nearrow \beta^*$, that is, for $p = \frac{2(s+1)}{r+1}$ and $\beta \nearrow \beta^*$ the $I(s,p,r,\beta)$ satisfies

$$\liminf_{\beta\nearrow\beta^*} I(s,p,r,\beta) \geq \frac{s+2}{s+1}(\beta^*)^{-\frac{1}{s+2}}\lambda^{\frac{s+1}{s+2}}(\beta^*-\beta)^{\frac{1}{s+2}}. \tag{4.3}$$

Together with (4.1) and (4.3), we have

$$I(s,p,r,\beta) \approx \frac{s+2}{s+1}(\beta^*)^{-\frac{1}{s+2}}\lambda^{\frac{s+1}{s+2}}(\beta^*-\beta)^{\frac{1}{s+2}} \quad \text{as } \beta\nearrow\beta^*, \tag{4.4}$$

which thus completes the proof of Theorem 1.2.

Proof of Theorem 1.3. For $p = \frac{2(s+1)}{r+1}$ and any β with $\beta \nearrow \beta^*$, repeating the proof of Claims 1–5 in Section 3, one deduces that the v_β fulfills

$$\lim_{\beta\nearrow\beta^*} v_\beta(x) = \lim_{\beta\nearrow\beta^*} \epsilon_\beta u_\beta(\epsilon_\beta x + x_\beta) = \frac{w_p(|x|)}{\|w_p\|_{L^2}}, \tag{4.5}$$

strongly in $H^1(\mathbb{R}^2)$ and the unique maximum point x_β satisfies

$$x_{\beta_k} \to x_0 \text{ as } \beta_k \nearrow \beta^*,\ |x_0| = n_0\pi \text{ for some } n_0 \in N \text{ and } x_0 \notin \partial\Omega.$$

Similar to the proof (3.41), we obtain that the above ϵ_β in (4.5) behaves like

$$\epsilon_\beta \approx (\lambda\beta^*)^{-\frac{1}{2(s+2)}}(\beta^* - \beta)^{\frac{1}{2(s+2)}} \text{ as } \beta \nearrow \beta^*.$$

So far, we have finished the proof of Theorem 1.3.

5. Conclusions and outlook

In this paper, we have studied the constraint minimizers of the minimization problem (1.3), which is related to the elliptic equation (1.1) with two nonlocal terms. By applying the methods of constrained variation and energy estimation, the existence, non-existence, and limit behavior of constraint minimizers for (1.3) are analyzed. In detail, we first gave the existence and nonexistence results of constraint minimizers for (1.3) according to the classification of s, p, r, β. Secondly, for $p = \frac{2(s+1)}{r+1}$, the refined energy estimation of $I(s, p, r, \beta)$ is established as $\beta \nearrow \beta^*$. At last, when $p = \frac{2(s+1)}{r+1}$ as $\beta \nearrow \beta^*$, we not only proved that the mass of minimizer concentrates at a minimum point x_0 of $\sin|x|$ (i.e., $\sin|x_0| = 0$), but also ruled out x_0 being a boundary point of Ω. Besides, one then presented the concrete limit behavior of the positive minimizer u_β as β tends to β^* from below.

However, the local uniqueness of the constraint minimizer for (1.3) is hard to deal with as $\beta \nearrow \beta^*$. We will try our best to overcome this problem in future work.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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