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Research article

Doubly critical problems involving Sub-Laplace operator on Carnot group

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Abstract: This paper was focused on the solvability of a class of doubly critical sub-Laplacian problems on the Carnot group $\mathbb{G}$:

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2(\xi) u}{\mathrm{d}(\xi)^2} = |u|^{p-2}u + \psi^{\alpha}(\xi) \frac{|u|^{2^*(\alpha)-2}u}{\mathrm{d}(\xi)^{\alpha}}, \quad u \in S^{1,2}(\mathbb{G}).$$

Here, $p \in (1, 2^*]$, $\alpha \in (0, 2)$, $\mu \in [0, \mu_{\mathbb{G}})$, $2^* = \frac{2Q}{Q-2}$, and $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$. By means of variational techniques, we extended the arguments developed in [1]. In addition, we also established the existence result for the subelliptic system which involved sub-Laplacian and critical homogeneous terms.

Keywords: doubly critical problem; carnot group; hardy potential; pohozaev identity; variational method

1. Introduction

Recently, Filippucci et al. [1] analyzed the following quasilinear elliptic problem with multiple critical terms on the entire $\mathbb{R}^N$:

$$-\Delta_p u - \mu \frac{u}{|x|^p} = u^{p^*-1} + \frac{u^{p^*(s)-1}}{|x|^s}, \ u > 0, \quad u \in W^{1,p}(\mathbb{R}^N), \tag{1.1}$$

where $\Delta_p := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $N \geq 3$, $p \in (1, N)$, $s \in (0, p)$, $\mu \in [0, (\frac{N-p}{p})^2)$, the value $p^* = \frac{Np}{N-p}$ denotes the critical Sobolev exponents, and $p^*(s) = \frac{p(N-s)}{N-p}$ denotes the critical Hardy-Sobolev exponents. The Eq (1.1) with double critical terms induces more difficulties, and analyzing the structure of the Palais-Smale sequence approaching zero weakly and constructing a new Palais-Smale sequence at a critical value to weakly converge (PS) sequences to a nontrivial function; the authors prove that the Eq (1.1) has at least one positive solution in $W^{1,p}(\mathbb{R}^N)$. For a similar bi-harmonic problem involving two critical nonlinearities, refer to [2]. The author achieved the same result as in [1].

Later on, Ghoussoub and Shakerian [3] investigated the existence of nontrivial solutions for a fractional Laplacian problem involving critical exponents, namely,

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^{\alpha}}, \quad u \in W^{s,2}(\mathbb{R}^N), \tag{1.2}$$

where $s \in (0,1)$, $0 \leq \mu < \mu_H$, $\alpha \in (0, 2s)$, $2_s^* = \frac{2N}{N-2s}$, and $2_s^*(\alpha) = \frac{2(N-\alpha)}{N-2s}$ are the critical exponents and $\mu_H = 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$ is the best Hardy constant. Due to the nonlocal nature of the fractional Laplace operator, this problem poses more difficulties and, as a result, the authors chose not to study the problem (1.2) directly. Instead, the authors utilized Caffarelli and Silvestre's s-harmonic extension method [4] to convert (1.2) into a local problem. Again, the fundamental approach utilized by Chen [5] to demonstrate the existence of a positive solution to the following fractional Laplacian problem with both critical nonlinearities having the same singularities at origin in enter space $\mathbb{R}^N$:

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^{\alpha}} + \frac{|u|^{2_s^*(\beta)-2}u}{|x|^{\beta}}, \quad u \in W^{s,2}(\mathbb{R}^N), \tag{1.3}$$

where $s \in (0,1)$, $0 \leq \mu < \mu_H$, $\alpha, \beta \in (0, 2s)$, and $2_s^*(\cdot) = \frac{2(N-\cdot)}{N-2s}$ denotes the fractional critical Sobolev-Hardy exponent.

Subsequently, Assuncão et al. [6] extended the Eq (1.3) to the following fractional p-Laplacian problem involving critical Hardy-Sobolev terms in $\mathbb{R}^N$:

$$(-\Delta_p)^s u - \mu \frac{|u|^{p-2}u}{|x|^{sp}} = \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^{\alpha}} + \frac{|u|^{p_s^*(\beta)-2}u}{|x|^{\beta}}, \quad u \in W^{s,p}(\mathbb{R}^N), \tag{1.4}$$

where $s \in (0,1)$, $p \in (1,+\infty)$, $sp < N$, $\alpha,\beta \in (0, sp)$, $\mu \in [0, \mu_{H,p})$, and $p_s^*(\alpha) = \frac{p(N-\alpha)}{N-ps}$, $p_s^*(\beta) = \frac{p(N-\beta)}{N-ps}$ denote the critical Hardy-Sobolev exponents. Using a refined version of the concentration-compactness principle and the mountain pass theorem, the authors demonstrate that the problem (1.4) has a nontrivial weak solution in $W^{s,p}(\mathbb{R}^N)$.

We recall that the Hardy inequality on the Stratified Lie group was first introduced in the pioneering work of D'Ambrosio [7, 8], Han et al. [10], and Niu et al. [9]. With these inequalities, the subelliptic problem on the Stratified Lie group has received special attention in the past several years. For example, Lioudice [11–14] studied the version of Sobolev and Hardy-Sobolev inequalities on the Stratified Lie group and showed the existence result for the Brezis-Nirenberg type equation. Zhang [15–17] investigated the multiplicity of nontrivial solutions of subelliptic equations with critical Hardy-Sobolev exponents. In [18–20], the authors studied existence and asymptotic behavior of nontrivial solutions of a series of problems in general open subsets Ω of the Heisenberg group $\mathbb{H}^n$, possibly unbounded or even $\mathbb{H}^n$. For the results of the subellipse problem on more general homogeneous groups, we refer to [21–23] and references therein. Finally, we suggest [24] to the reader which is interested on the fractional Laplacian on the Heisenberg group.

Motivated by the results mentioned above, in this article we are interested in finding solutions to the following sub-Laplacian problem with Hardy-type potentials and critical terms on Carnot group $\mathbb{G}$:

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2(\xi)u}{\mathrm{d}(\xi)^2} = |u|^{p-2}u + \psi^{\alpha}(\xi)\frac{|u|^{2^*(\alpha)-2}u}{\mathrm{d}(\xi)^{\alpha}}, \quad \xi \in \mathbb{G}, \tag{1.5}$$

where $-\Delta_{\mathbb{G}}$ is the sub-Laplace operator on the Carnot group, $\mathrm{d}(\xi)$ is the natural gauge on $\mathbb{G}$, the weight function ψ is defined as $\psi(\xi) = |\nabla_{\mathbb{G}}\mathrm{d}(\xi)|$, the parameters $p \in (1, 2^*]$, $\alpha \in (0, 2)$, $\mu \in [0, \mu_{\mathbb{G}})$, and $2^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent, $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$ is the critical Hardy-Sobolev exponent, $\mu_{\mathbb{G}} = (\frac{Q-2}{2})^2$ is the best Hardy constant and Q denotes the homogeneous dimension of the space $\mathbb{G}$ with respect to the dilation δ_γ; see Section 2. The space $S^{1,2}(\mathbb{G})$ denotes the completion of $C_0^\infty(\mathbb{G})$ with respect to norm

$$\|u\| = \Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}u|^2 d\xi\Big)^{\frac{1}{2}}.$$

Problem (1.5) is related to the following Hardy-type inequality (see [8, 25]):

$$\mu_{\mathbb{G}} \int_{\mathbb{G}} \frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2} d\xi \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}}u|^2 d\xi, \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{1.6}$$

where $\mu_{\mathbb{G}} = (\frac{Q-2}{2})^2$ is the best constant in this context. By using (1.6), it can be shown that the operator $L := -\Delta_{\mathbb{G}} \cdot -\mu\frac{\psi^2\cdot}{\mathrm{d}(\xi)^2}$ is positive for all $\mu < \mu_{\mathbb{G}}$ and, therefore, we can define the following equivalent norm of $S^{1,2}(\mathbb{G})$:

$$\|u\|_\mu = \Big(\int_{\mathbb{G}} (|\nabla_{\mathbb{G}}u|^2 - \mu\frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}) d\xi\Big)^{\frac{1}{2}}.$$

Additionally, according to Folland and Stein [26], the following Sobolev-type inequality holds:

$$S\Big(\int_{\mathbb{G}} |u|^{2^*} d\xi\Big)^{\frac{2}{2^*}} \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}}u|^2 d\xi, \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{1.7}$$

where the best constant in (1.7) is achieved; refer to [27, 28]. However, only the explicit form of the minimizers is known for the Iwasawa-type group class. For $\alpha \in [0, 2)$, from (1.6) and (1.7), the following Sobolev-Hardy inequality holds: There exists a positive constant $C(Q, \alpha)$, depending on Q and α, such that

$$\Big(\int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi\Big)^{\frac{2}{2^*(\alpha)}} \leq C(Q, \alpha) \int_{\mathbb{G}} |\nabla_{\mathbb{G}}u|^2 d\xi, \quad \forall u \in C_0^\infty(\mathbb{G}). \tag{1.8}$$

The energy functional related to (1.5) takes the following form:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{G}} (|\nabla_{\mathbb{G}}u|^2 - \mu\frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}) d\xi - \frac{1}{p}\int_{\mathbb{G}} |u|^p d\xi - \frac{1}{2^*(\alpha)}\int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi. \tag{1.9}$$

Using the previously mentioned inequalities (1.6) and (1.8), it is straghtforward to show that the functional Φ is well-defined in $S^{1,2}(\mathbb{G})$ and $I \in C^1(S^{1,2}(\mathbb{G}), \mathbb{R})$. A function $u \in S^{1,2}(\mathbb{G})$ is said to be a nontrivial solution of (1.5) if $u \neq 0$, and $\langle \Phi'(u), \phi\rangle = 0$ for all $\phi \in S^{1,2}(\mathbb{G})$, where $\Phi'(u)$ denotes the Fréchet derivative of functional Φ at u.

Now, we can state our result.

Theorem 1.1. *Let $\alpha \in (0, 2)$ and $\mu \in (-\infty, \mu_{\mathbb{G}})$. If $u \in S^{1,2}(\mathbb{G})$ is a weak solution of* (1.5) *where $1 < p < 2^*$, then $u \equiv 0$.*

The result of Theorem 1.1 tells us that we need to discuss the existence of solutions to the Eq (1.5) at $p = 2^*$. The conclusion is as follows:

Theorem 1.2. *Let $Q \geq 3$, $\alpha \in (0,2)$, $\mu \in [0,\mu_{\mathbb{G}})$, and $p = 2^*$. Then, there exists a weak nontrivial solution $u \in S^{1,2}(\mathbb{G})$ to problem* (1.5).

Furthermore, continuing in the same spirit as problem (1.5) with $p = 2^*$, we consider the following subelliptic system with critical homogeneous terms

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \dfrac{\psi^2(\xi)u}{\mathrm{d}(\xi)^2} = \dfrac{\lambda}{2^*} H_u(u,v) + \dfrac{\eta}{2^*(\alpha)} \dfrac{\psi^\alpha(\xi) Q_u(u,v)}{\mathrm{d}(\xi)^\alpha}, & \xi \in \mathbb{G}, \\ -\Delta_{\mathbb{G}} v - \mu \dfrac{\psi^2(\xi)v}{\mathrm{d}(\xi)^2} = \dfrac{\lambda}{2^*} H_v(u,v) + \dfrac{\eta}{2^*(\alpha)} \dfrac{\psi^\alpha(\xi) Q_v(u,v)}{\mathrm{d}(\xi)^\alpha}, & \xi \in \mathbb{G}, \end{cases} \tag{1.10}$$

where $\lambda > 0$, $\eta > 0$, H_u, H_v, Q_u, and Q_v are the partial derivatives of the 2-variable C^1-functions $H(u,v)$ and $Q(u,v)$, respectively.

Before stating our result, we need the following assumptions.

(H_1) $H_u(u,0) = H_u(0,v) = H_v(u,0) = H_v(0,v) = Q_u(u,0) = Q_u(0,v) = Q_v(u,0) = Q_v(0,v) = 0$, where u, $v \in \mathbb{R}^+$.

(H_2) $H \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ and $Q \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ are positively homogeneous of degrees 2^* and $2^*(\alpha)$, respectively, i.e., $H(tu,tv) = t^{2^*} H(u,v)$ and $Q(tu,tv) = t^{2^*(\alpha)} Q(u,v)$ hold for all $t \geq 0$ and u, $v \in \mathbb{R}^+$.

Now, we work on the product space $W = S^{1,2}(\mathbb{G}) \times S^{1,2}(\mathbb{G})$ with respect to the norm $\|(u,v)\| = (\|u\|_\mu^2 + \|v\|_\mu^2)^{\frac{1}{2}}$, and get the following existence result for system (1.10).

Theorem 1.3. *Suppose that $\mu \in [0,\mu_{\mathbb{G}})$, $\alpha \in (0,2)$, $\lambda > 0$, $\eta > 0$, and $(H)_1$, (H_2) hold. Then, the system* (1.10) *has a nontrivial weak solution in W.*

Remark 1.1. *By Theorem 1.3, the existence of solutions to* (1.10) *is obvious in either of the following cases: (i) $\lambda = 0$, $\eta > 0$, $\alpha \geq 0$; (ii) $\lambda > 0$, $\eta = 0$, $\alpha \geq 0$; (iii) $\alpha = 0$, $\lambda > 0$, $\eta > 0$.*

The proof of Theorems 1.2 and 1.3 follow several ideas that have appeared in [1,3,6]. However, since we consider the subelliptic problem on Carnot group $\mathbb{G}$ and since problem (1.5) or (1.10) contains critical nonlinearities in the sense of the Hardy-Sobolev embeddings, it follows that the Hardy-Sobolev embedding $S^{1,2}(\mathbb{G}) \hookrightarrow L^{2^*(\alpha)}(\mathbb{G}, \frac{\psi^\alpha}{d(z)^\alpha} dz)$ $(0 \leq \alpha < 2)$ is non-compact. This poses several difficulties to prove that bounded Palais-Smale in Banach space $S^{1,2}(\mathbb{G})$ have at least a subsequence that converges strongly to a nontrivial function in this space. Clear enough, the presence of multiple Sobolev critical nonlinearities also contributes to the difficulties in the proof of the theorem. Based on some estimates proved by Zhang [15,16], we managed to overcome these difficulties and prove a refined version of the concentration-compactness principle.

The article is organized as follows. In Sections 2 and 3, some preliminary results together with our main results are verified. Meanwhile, for existence of nontrivial weak solutions, Theorems 1.2 and 1.3 will be proved in Sections 4 and 5, respectively.

2. Preliminary results

First, we will provide a brief overview of Carnot groups. For a more comprehensive treatment of this topic, please reference the monographs [29,30] and the papers [26,31]. A Carnot group $(\mathbb{G}, \circ)$, also known as a stratified Lie group, is defined as a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ is stratified. Specifically, this means that $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \oplus_{i=1}^{k} V_i$, where $[V_1, V_i] = V_{i+1}$ for $i = 1, \cdots k-1$ and $[V_1, V_k] = \{0\}$. The number k is called the step of the group $\mathbb{G}$. In this context, the symbol $[V_1, V_i]$ represents the subalgebra of $\mathfrak{g}$ generated by the commutators $[X, Y]$, where $X \in V_1$, $Y \in V_i$ and the last bracket denotes the Lie bracket of vector fields, i.e., $[X, Y] = XY - YX$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map (which we shall assume throughout), it is reasonable to assume that $\mathbb{G}$ is a homogeneous Lie group on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_k}$, where N_i denotes the dimensionality of V_i, and is equipped with a set of group-automorphisms called $\delta_\gamma : \mathbb{G} \to \mathbb{G}$. These automorphisms take the form of

$$\delta_\gamma(\xi) = \delta_\gamma(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(k)}) = (\gamma^1 \xi^{(1)}, \cdots, \gamma^k \xi^{(k)}), \quad \gamma > 0,$$

where $\xi^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, 2, \cdots, k$. Here, $N = \sum_{i=1}^{k} N_i$ is called the topological dimension of $\mathbb{G}$ and δ_γ is called the dilations of $\mathbb{G}$. Under these automorphisms $\{\delta_\gamma\}_{\gamma>0}$, the homogeneous dimension of $\mathbb{G}$ is expressed as $Q = \sum_{i=1}^{k} i \cdot \mathrm{dim} V_i$. From now on, we will assume that $Q \geq 3$ throughout this paper. It is noteworthy that if $Q \leq 3$, $\mathbb{G}$ must be the ordinary Euclidean space $\mathbb{G} = (\mathbb{R}^Q, +)$.

Let $\{X_1, \cdots, X_{N_1}\}$ be a basis of V_1, then the second-order differential operator

$$\Delta_{\mathbb{G}} := \sum_{i=1}^{N_1} X_i^2$$

is referred to as a sub-Laplacian on $\mathbb{G}$. We now use the notation of $\nabla_{\mathbb{G}} := (X_1, \cdots, X_{N_1})$ to denote the horizontal gradient, and the divergence with respect to the vector fields X_j is defined by

$$\mathrm{div}_{\mathbb{G}} h := \sum_{j=1}^{N_1} X_j h_j, \qquad \forall h = (h_1, h_2, \cdots, h_{N_1}).$$

The homogeneous norm on $\mathbb{G}$, which conforms to a fixed homogeneous structure, is a continuous function represented by $\mathrm{d} : \mathbb{G} \to [0, +\infty)$. This function is smooth away from the origin and satisfies $\mathrm{d}(\delta_\gamma(\xi)) = \gamma \mathrm{d}(\xi)$ for $\gamma > 0$, $\mathrm{d}(\xi^{-1}) = \mathrm{d}(\xi)$, $\mathrm{d}(\xi) = 0$ iff $\xi = 0$. When $Q \geq 3$, the function

$$\Gamma(\xi) = \frac{C}{\mathrm{d}(\xi)^{Q-2}}, \quad \forall \xi \in \mathbb{G}$$

is a fundamental solution of the sub-Laplacian on Carnot group $\mathbb{G}$ with the pole at 0, where $C > 0$ is a suitable constant. In addition, the left translation on $\mathbb{G}$ is defined by

$$\tau_\xi : \mathbb{G} \to \mathbb{G}, \quad \tau_\xi(\xi') = \xi \circ \xi', \quad \forall \xi, \xi' \in \mathbb{G},$$

and we can verify that $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ satisfy the following results:

$$\nabla_{\mathbb{G}}(u \circ \tau_z) = \nabla_{\mathbb{G}} u \circ \tau_z, \quad \nabla_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma \nabla_{\mathbb{G}} u \circ \delta_\gamma,$$

$$\Delta_{\mathbb{G}}(u \circ \tau_z) = \Delta_{\mathbb{G}} u \circ \tau_z, \quad \Delta_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma^2 \Delta_{\mathbb{G}} u \circ \delta_\gamma.$$

The $k\,(k \geq 2)$-step Carnot group $\mathbb{G}$ and the Euclidean space $\mathbb{R}^N$ differ in numerous essential ways. For instance, the basis level vector field on $\mathbb{G}$ is noncommutative, meaning that there exist $1 \leq i,\ j \leq m$ such that the Poisson bracket $[V_i, V_j] \neq 0$. In constrast, $\mathbb{R}^N$ is an exchange group with a step number 1, which means that for any $i,\ j = 1, 2, \cdots, \dim(V_1)$, whose Poisson brackets satisfy $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$. Therefore, there are several significant differences between the operator on Carnot group and on the Euclidean space. For example, the Laplace operator on $\mathbb{G}$ is $\Delta_{\mathbb{G}} = \sum_{i=1}^{\dim(V_1)} X_i^2$, which is a point-by-point degenerate elliptic operator. In contrast, the Laplace operator on $\mathbb{R}^N$, $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is a uniformly elliptic operator. Therefore, the study of partial differential equations on the Carnot group is of theoretical importance.

By (1.6) and (1.8), the following best Hardy-Sobolev constant is well-defined:

$$S_{\mu,\alpha} = \inf_{u\in S^{1,2}(\mathbb{G})\setminus\{0\}} \frac{\int_{\mathbb{G}}(|\nabla_{\mathbb{G}} u|^2(\xi) - \mu\frac{\psi^2|u|^2}{\mathrm{d}(\xi)^2})d\xi}{\left(\int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi\right)^{\frac{2}{2^*(\alpha)}}}. \tag{2.1}$$

For $\mu \in [0, \mu_{\mathbb{G}})$, it can be inferred from [15] that $S_{\mu,\alpha}$ is achieved by the extremal functions

$$U_{\varepsilon,\mu,\alpha}(\xi) = \varepsilon^{-\frac{Q-2}{2}} U_{\mu,\alpha}(\delta_{\frac{1}{\varepsilon}}(\xi)), \quad \forall \varepsilon > 0, \tag{2.2}$$

where $U_{\mu,\alpha}$ is a ground state solution of

$$-\Delta_{\mathbb{G}} u - \mu\frac{\psi^2(\xi)u}{\mathrm{d}(\xi)^2} = \frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)-2}u}{\mathrm{d}(\xi)^\alpha}, \quad \xi \in \mathbb{G}\setminus\{0\}. \tag{2.3}$$

Furthermore, for all $\varepsilon > 0$, the function $U_{\varepsilon,\mu,\alpha}(\xi)$ solves the Eq (2.3) and satisfies

$$\int_{\mathbb{G}} \left(|\nabla_{\mathbb{G}} U_{\varepsilon,\mu,\alpha}|^2 - \mu\frac{\psi^2(\xi)|U_{\varepsilon,\mu,\alpha}|^2}{\mathrm{d}(\xi)^2}\right)d\xi = \int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|U_{\varepsilon,\mu,\alpha}|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi = (S_{\mu,\alpha})^{\frac{Q-2}{2-\alpha}}.$$

We note that the explicit form of the Hardy-Sobolev extremals is unknown in any Carnot group, except for the trivial Euclidean case. However, the pure Sobolev extremals (when $\mu = \alpha = 0$) are known to be expressed solely in the Iwasawa-type group, as seen in [27, 32].

For $\mu \in (-\infty, \mu_{\mathbb{G}})$ and $\alpha \in (0, 2)$, (H_2) shows that the following best Hardy-Sobolev constants are well-defined:

$$S_H(\mu, 0) = \inf_{(u,v)\in W\setminus\{(0,0)\}} \frac{\int_{\mathbb{G}}(|\nabla_{\mathbb{G}} u|^2 + |\nabla_{\mathbb{G}} v|^2 - \mu\frac{\psi^2(\xi)(|u|^2+|v|^2)}{\mathrm{d}(\xi)^2})d\xi}{(\int_{\mathbb{G}} H(u, v)d\xi)^{\frac{2}{2^*}}}, \tag{2.4}$$

$$S_Q(\mu, \alpha) = \inf_{(u,v)\in W\setminus\{(0,0)\}} \frac{\int_{\mathbb{G}}(|\nabla_{\mathbb{G}} u|^2 + |\nabla_{\mathbb{G}} v|^2 - \mu\frac{\psi^2(\xi)(|u|^2+|v|^2)}{\mathrm{d}(\xi)^2})d\xi}{(\int_{\mathbb{G}} \frac{\psi^\alpha Q(u,v)}{\mathrm{d}(\xi)^\alpha} d\xi)^{\frac{2}{2^*(\alpha)}}}. \tag{2.5}$$

These constants are crucial for the study of (1.10); we then have the following result.

Theorem 2.1. *Assume that $\alpha \in (0, 2)$, $\mu \in (-\infty, \mu_{\mathbb{G}})$, and (H_2) holds. Then,*

(i) $S_H(\mu, 0) = M_H^{-1} S_{\mu,0}$ and $S_Q(\mu, \alpha) = M_Q^{-1} S_{\mu,\alpha}$, where M_H, M_Q are defined by

$$M_H := \max\{H(u, v)^{\frac{2}{2^*}} : (u, v) \in \mathbb{R}^2 \text{ and } |u|^2 + |u|^2 = 1\}; \tag{2.6}$$

$$M_Q := \max\{Q(u, v)^{\frac{2}{2^*(\alpha)}} : (u, v) \in \mathbb{R}^2 \text{ and } |u|^2 + |u|^2 = 1\}. \tag{2.7}$$

(ii) For $\mu \in [0, \mu_{\mathbb{G}})$, $S_H(\mu, 0)$ has the minimizers $(s_1 U_{\varepsilon,\mu,0}(\xi), t_1 U_{\varepsilon,\mu,0}(\xi))$, $S_Q(\mu,\alpha)$ has the minimizers $(s_2 U_{\varepsilon,\mu,\alpha}(\xi), t_2 U_{\varepsilon,\mu,\alpha}(\xi))$, where $U_{\varepsilon,\mu,\alpha}(\xi)$ are defined as in (2.2) *and (s_1, t_1), (s_2, t_2) are constants given in* (2.8), (2.9)*, respectively.*

Now, we study $S_H(\mu, 0)$, $S_Q(\mu, \alpha)$ and verify Theorem 2.1. First, we give some preliminary results.

Proposition 2.1. *([33]) Let $H \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ and $Q \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ be positively homogeneous of degrees 2^* and $2^*(\alpha)$, respectively. Then, there exist M_H, $M_Q > 0$ such that*

$$H(u, v) \leq (M_H(|u|^2 + |v|^2))^{\frac{2^*}{2}},$$
$$Q(u, v) \leq (M_Q(|u|^2 + |v|^2))^{\frac{2^*(\alpha)}{2}},$$

where M_F and M_Q are given in (2.6) *and* (2.7)*, respectively. Moreover, there exist $(s_i, t_i) \in \mathbb{R}^+ \times \mathbb{R}^+$ ($i = 1, 2$), such that M_F and M_Q are achieved respectively, that is,*

$$M_H = H(s_1, t_1)^{\frac{2}{2^*}}, \quad s_1^2 + t_1^2 = 1; \tag{2.8}$$
$$M_Q = Q(s_2, t_2)^{\frac{2}{2^*(\alpha)}}, \quad s_2^2 + t_2^2 = 1. \tag{2.9}$$

Proof of Theorem 2.1. We only show the proof for $S_Q(\mu, \alpha)$.

(i) Let $\{U_n\} \subset S^{1,2}(\mathbb{G}) \setminus \{0\}$ be a minimizing sequence for $S_{\mu,\alpha}$ and (s_2, t_2) be defined as in (2.9). Choosing $(u_n, v_n) = (s_2 U_n, t_2 U_n)$ in (2.5), we have

$$\frac{(s_2^2 + t_2^2) \int_{\mathbb{G}} (|\nabla_{\mathbb{G}} U_n|^2 - \mu \frac{\psi^2(\xi)|U_n|^2}{d(\xi)^2}) d\xi}{|Q(s_2, t_2)|^{\frac{2}{2^*(\alpha)}} \Big(\int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|U_n|^{2^*(\alpha)}}{d(\xi)^\alpha} d\xi \Big)^{\frac{2}{2^*(\alpha)}}} \geq S_Q(\mu, \alpha). \tag{2.10}$$

Taking $n \to \infty$ in (2.10), by (2.9) we have

$$S_Q(\mu, \alpha) \leq M_Q^{-1} S_{\mu,\alpha}. \tag{2.11}$$

On the other hand, let $\{(u_n, v_n)\} \subset W \backslash \{(0, 0)\}$ be a minimizing sequence for $S_Q(\mu, \alpha)$. From $Q(tu, tv) = t^{2^*(\alpha)} Q(u, v)$ and Proposition 4 of [33], we have that

$$\begin{aligned}
\int_{\mathbb{G}} \frac{\psi^\alpha(\xi) Q(u_n, v_n)}{d(\xi)^\alpha} d\xi &= \int_{\mathbb{G}} \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha \cdot 2^*(\alpha)}{2^*(\alpha)}} Q(u_n, v_n) d\xi \\
&= \int_{\mathbb{G}} Q\bigg(\Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} u_n, \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} v_n \bigg) d\xi \\
&\leq Q\bigg(\Big\| \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} u_n \Big\|_{L^{2^*(\alpha)}(\mathbb{G})}, \Big\| \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} v_n \Big\|_{L^{2^*(\alpha)}(\mathbb{G})} \bigg).
\end{aligned} \tag{2.12}$$

Set

$$\theta := \bigg[\Big\| \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} u_n \Big\|^2_{L^{2^*(\alpha)}(\mathbb{G})} + \Big\| \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} v_n \Big\|^2_{L^{2^*(\alpha)}(\mathbb{G})} \bigg]^{-\frac{1}{2}}.$$

Then,

$$\Big\| \theta \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} u_n \Big\|^2_{L^{2^*(\alpha)}(\mathbb{G})} + \Big\| \theta \Big(\frac{\psi(\xi)}{d(\xi)}\Big)^{\frac{\alpha}{2^*(\alpha)}} v_n \Big\|^2_{L^{2^*(\alpha)}(\mathbb{G})} = 1. \tag{2.13}$$

From (2.1), (2.12), (2.13), and (2.9), it follows that

$$\begin{aligned}
&\frac{\int_{\mathbb{G}}(|\nabla_{\mathbb{G}}u_n|^2+|\nabla_{\mathbb{G}}v_n|^2-\mu\frac{\psi^2(\xi)(|u_n|^2+|v_n|^2)}{\mathrm{d}(\xi)^2})d\xi}{\left(\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)Q(u_n,v_n)}{\mathrm{d}(\xi)^\alpha}d\xi\right)^{\frac{2}{\alpha+\beta}}}\\
&\geq S_{\mu,\alpha}\frac{(\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi)^{\frac{2}{2^*(\alpha)}}+(\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|v_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi)^{\frac{2}{2^*(\alpha)}}}{\left[Q\left(\left\|(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}u_n\right\|_{L^{2^*(\alpha)}(\mathbb{G})},\ \left\|(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}v_n\right\|_{L^{2^*(\alpha)}(\mathbb{G})}\right)\right]^{\frac{2}{2^*(\alpha)}}}\\
&=S_{\mu,\alpha}\frac{\left\|(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}u_n\right\|^2_{L^{2^*(\alpha)}(\mathbb{G})}+\left\|(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}v_n\right\|^2_{L^{2^*(\alpha)}(\mathbb{G})}}{\left[Q\left(\left\|(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}u_n\right\|_{L^{2^*(\alpha)}(\mathbb{G})},\ \left\|(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}v_n\right\|_{L^{2^*(\alpha)}(\mathbb{G})}\right)\right]^{\frac{2}{2^*(\alpha)}}}\\
&=S_{\mu,\alpha}\frac{\left\|\theta(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}u_n\right\|^2_{L^{2^*(\alpha)}(\mathbb{G})}+\left\|\theta(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}v_n\right\|^2_{L^{2^*(\alpha)}(\mathbb{G})}}{\left[Q\left(\left\|\theta(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}u_n\right\|_{L^{2^*(\alpha)}(\mathbb{G})},\ \left\|\theta(\frac{\psi(\xi)}{\mathrm{d}(\xi)})^{\frac{\alpha}{2^*(\alpha)}}v_n\right\|_{L^{2^*(\alpha)}(\mathbb{G})}\right)\right]^{\frac{2}{2^*(\alpha)}}}\\
&\geq\frac{1}{|Q(\alpha_2,\beta_2)|^{\frac{2}{2^*(\alpha)}}}S_{\mu,\alpha}=M_Q^{-1}S_{\mu,\alpha}.
\end{aligned}\tag{2.14}$$

Passing to the limit in the above inequality (2.14), we have

$$M_Q^{-1}S_{\mu,\alpha}\leq S_Q(\mu,\alpha),$$

which together with (2.11) implies that

$$S_Q(\mu,\alpha)=M_Q^{-1}S_{\mu,\alpha}.$$

(ii) From (i), (2.4), and (2.5), the desired result follows. □

3. Nonexistence result when $p < 2^*$

In order to prove Theorem 1.1, we first establish it under an additional assumption.

Proposition 3.1. *Let* $\alpha \in (0,2)$ *and* $\mu \in (-\infty, \mu_{\mathbb{G}})$. *If* $u \in L^p(\mathbb{G})$ *is a weak solution of* (1.5) *with* $p \in (1, 2^*)$*, then* $u \equiv 0$.

Proof. Let $\phi, \zeta \in C^1(\mathbb{G}, [0,1])$ and satisfy $\phi(t) = 1$ for $t \geq 2$, $\zeta(t) = 1$ for $t \leq 1$. Let $\eta_{\varepsilon,R}(\xi) = \phi_\varepsilon(\xi)\zeta_R(\xi)$ for $\varepsilon > 0$ and $R > 0$, where

$$\phi_\varepsilon(\xi)=\phi(\frac{\mathrm{d}(\xi)}{\varepsilon}),\qquad \zeta_R(\xi)=\zeta(\frac{\mathrm{d}(\xi)}{R}).$$

Let $u \in S^{1,2}(\mathbb{G})$ be a weak solution of (1.5) with $1 < p < 2^*$. Then, u is smooth away from the origin and $Zu\eta_{\varepsilon,R} \in C_0^1(\mathbb{G})$. By multiplying the Eq (1.5) with $Zu\eta_{\varepsilon,R}$ and integrating by parts, we get

$$-\int_{\mathbb{G}}\Delta u\cdot Zu\eta_{\varepsilon,R}d\xi=\mu\int_{\mathbb{G}}\frac{\psi^2(\xi)u}{\mathrm{d}(\xi)^2}Zu\eta_{\varepsilon,R}d\xi+\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)-2}u}{\mathrm{d}(\xi)^\alpha}Zu\eta_{\varepsilon,R}d\xi+\int_{\mathbb{G}}|u|^{p-2}uZu\eta_{\varepsilon,R}d\xi.\tag{3.1}$$

Proceeding similarly as proved in [14, Theorem 4.1], we can show that

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\text{LHS of (3.1)} = -\frac{Q-2}{2}\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2 d\xi. \tag{3.2}$$

and

$$\begin{aligned}\lim_{R\to\infty}\lim_{\varepsilon\to 0}\text{RHS of (3.1)} = &-\frac{Q-2}{2}\mu\int_{\mathbb{G}}\frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}d\xi - \frac{Q-2}{2}\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi\\ &-\frac{Q}{p}\int_{\mathbb{G}}|u|^p d\xi.\end{aligned} \tag{3.3}$$

Therefore, substituting back (3.3) and (3.2) in (3.1), we obtain

$$\frac{Q-2}{2}\left(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2 d\xi - \mu\int_{\mathbb{G}}\frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}d\xi - \int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi\right) = \frac{Q}{p}\int_{\mathbb{G}}|u|^p d\xi. \tag{3.4}$$

On the other hand, since $u \in L^p(\mathbb{G})$ is a solution of (1.5), we have

$$\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2 d\xi = \mu\int_{\mathbb{G}}\frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}d\xi + \int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi + \int_{\mathbb{G}}|u|^p d\xi,$$

which together with (3.4) implies that

$$\left(\frac{Q-2}{2} - \frac{Q}{p}\right)\int_{\mathbb{G}}|u|^p d\xi = 0. \tag{3.5}$$

As $p < 2^*$, i.e., $\frac{Q-2}{2} - \frac{Q}{p} < 0$, (3.5) implies $u \equiv 0$. This completes the proof. □

Proof of Theorem 1.1. According to Proposition 3.1, once we prove $u \in L^p(\mathbb{G})$, the proof of Theorem 1.1 follows.

Now, let $\eta_{\varepsilon,R} \in C_0^\infty(\mathbb{G}\setminus\{0\})$ be a cutoff function as in the proof of Proposition 3.1. Choosing $\eta_{\varepsilon,R}u$ as the test function, we get

$$\int_{\mathbb{G}}\nabla_{\mathbb{G}}u\nabla_{\mathbb{G}}(\eta_{\varepsilon,R}u)d\xi = \mu\int_{\mathbb{G}}\frac{\psi^2(\xi)|u|^2\eta_{\varepsilon,R}}{\mathrm{d}(\xi)^2}d\xi + \int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}\eta_{\varepsilon,R}}{\mathrm{d}(\xi)^\alpha}d\xi + \int_{\mathbb{G}}|u|^p\eta_{\varepsilon,R}d\xi. \tag{3.6}$$

Hence,

$$\int_{\mathbb{G}}|u|^p\eta_{\varepsilon,R}d\xi \le \mu\int_{\mathbb{G}}\frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}d\xi + \int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi + \int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2 d\xi + \int_{\mathbb{G}}|u||\nabla_{\mathbb{G}}u||\nabla_{\mathbb{G}}\eta_{\varepsilon,R}|d\xi. \tag{3.7}$$

Since $u \in S^{1,2}(\mathbb{G})$, there exists a constant $C > 0$ such that $\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2 d\xi \le C$. Then, based on the Hardy inequality and the Sobolev-Hardy inequality, we can conclude that $\int_{\mathbb{G}}\frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}d\xi \le C_1$ and $\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi \le C_2$, where $C_1 > 0$ and $C_2 > 0$ are constants. In order to prove $u \in L^p(\mathbb{G})$, our aim is to show that $\int_{\mathbb{G}}|u||\nabla_{\mathbb{G}}u||\nabla_{\mathbb{G}}\eta_{\varepsilon,R}|d\xi$ are uniformly bounded by a constant independent of ε and R. To see this,

$$\begin{aligned}\int_{\mathbb{G}}|u||\nabla_{\mathbb{G}}u||\nabla_{\mathbb{G}}\eta_{\varepsilon,R}|d\xi &= \int_{\mathbb{G}}|u||\nabla_{\mathbb{G}}u||\zeta_R\nabla_{\mathbb{G}}\phi_\varepsilon + \phi_\varepsilon\nabla_{\mathbb{G}}\zeta_R|d\xi\\ &\le \int_{\varepsilon\le \mathrm{d}(\xi)\le 2\varepsilon}|u||\nabla_{\mathbb{G}}u|\frac{c|\nabla_{\mathbb{G}}\mathrm{d}(\xi)|}{|\varepsilon|}d\xi + \int_{R\le \mathrm{d}(\xi)\le 2R}|u||\nabla_{\mathbb{G}}u|\frac{c|\nabla_{\mathbb{G}}\mathrm{d}(\xi)|}{R}d\xi\\ &= 2c\int_{\varepsilon\le \mathrm{d}(\xi)\le 2\varepsilon}|u||\nabla_{\mathbb{G}}u|\frac{\psi(\xi)}{\mathrm{d}(\xi)}d\xi + 2c\int_{R\le \mathrm{d}(\xi)\le 2R}|u||\nabla_{\mathbb{G}}u|\frac{\psi(\xi)}{\mathrm{d}(\xi)}d\xi.\end{aligned} \tag{3.8}$$

Here, we use the fact that $\frac{1}{\varepsilon} \leq \frac{2}{\mathrm{d}(\xi)}$ in the first integral and $\frac{1}{R} \leq \frac{2}{\mathrm{d}(\xi)}$ in the second integral. By the Hölder inequality and the Hardy-Sobolev inequality, for $u \in S^1(\mathbb{G})$, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned}\int_{\varepsilon\leq \mathrm{d}(\xi)\leq 2\varepsilon} |u||\nabla_{\mathbb{G}} u|\frac{\psi(\xi)}{\mathrm{d}(\xi)}d\xi &\leq \int_{\mathbb{G}} \frac{\psi(\xi)|u|}{\mathrm{d}(\xi)}|\nabla_{\mathbb{G}} u|d\xi \\ &\leq \left(\int_{\mathbb{G}} \frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 d\xi\right)^{\frac{1}{2}} \leq C_1 < +\infty,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\int_{R\leq \mathrm{d}(\xi)\leq 2R} |u||\nabla_{\mathbb{G}} u|\frac{\psi(\xi)}{\mathrm{d}(\xi)}d\xi &\leq \int_{\mathbb{G}} \frac{\psi(\xi)|u|}{\mathrm{d}(\xi)}|\nabla_{\mathbb{G}} u|d\xi \\ &\leq \left(\int_{\mathbb{G}} \frac{\psi^2(\xi)|u|^2}{\mathrm{d}(\xi)^2}d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 d\xi\right)^{\frac{1}{2}} \leq C_2 < +\infty.\end{aligned} \tag{3.10}$$

So, from (3.9), (3.10), (3.8), and (3.7), we get $\int_{\mathbb{G}} |u|^p \eta_{\varepsilon,R} d\xi \leq C$, where C is a positive constant independent of ε and R. Therefore, letting $\varepsilon \to 0$ and $R \to \infty$, we obtain that $u \in L^p(\mathbb{G})$. Hence, the Theorem 1.1 follows. □

4. Proof of Theorem 1.2

This section is devoted to proving the Theorem 1.2. To begin with, we use the following mountain pass lemma of Ambrosetti and Rabinowitz [34] to prove Theorem 1.2.

Lemma 4.1. *Let $(E, \|\cdot\|_E)$ be a Banach space and $I \in C^1(E, \mathbb{R})$, satisfying the following conditions:*

(i) $I(0) = 0$.
(ii) *There exist $a > 0$, $R > 0$ such that $I(u) \geq a$ for all $u \in E$ with $\|u\|_E = R$.*
(iii) *There exists $u_0 \in E\backslash\{0\}$ such that $\limsup_{t\to\infty} I(tu_0) < 0$.*

Let $t_0 > 0$ be a real number such that $\|t_0u_0\|_E > R$ and $I(t_0u_0) < 0$. Define

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } \gamma(1) = t_0u_0\}.$$

Then, $c \geq a > 0$ and there exists a (PS)-sequence $\{u_n\} \subset E$ for I at the level c, i.e.,

$$\lim_{n\to\infty} I(u_n) = c_u, \quad \lim_{n\to\infty} I'(u_n) = 0 \text{ strongly in } E'.$$

Proposition 4.1. *Let $\mu \in [0, \mu_{\mathbb{G}})$, $\alpha \in (0, 2)$, and $p = 2^*$. Then, there is a (PS)-sequence $\{u_n\} \subset S^{1,2}(\mathbb{G})$ for Φ at some $c \in (0, c^*)$, i.e.,*

$$\lim_{n\to\infty} \Phi(u_n) = c \text{ and } \lim_{n\to\infty} \Phi'(u_n) = 0 \text{ strongly in } (S^{1,2}(\mathbb{G}))',$$

where

$$c^* := \min\Big\{\frac{1}{Q}(S_{\mu,0})^{\frac{Q}{2}}, \quad \frac{2-\alpha}{2(Q-\alpha)}(S_{\mu,\alpha})^{\frac{Q-\alpha}{2-\alpha}}\Big\}. \tag{4.1}$$

The proof of Proposition 4.1 follows from the next results.

Lemma 4.2. *The energy functional Φ verifies the hypotheses of Lemma 4.1 for any $u \in S^{1,2}(\mathbb{G})\backslash\{0\}$.*

Proof. Clearly, $\Phi \in C^1(S^{1,2}(\mathbb{G}), \mathbb{R})$ and $\Phi(0) = 0$. By (2.1), we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\|u\|_\mu^2 - \frac{1}{2^*(S_{\mu,0})^{\frac{2^*}{2}}}\|u\|_\mu^{2^*} - \frac{1}{2^*(\alpha)(S_{\mu,\alpha})^{\frac{2^*(\alpha)}{2}}}\|u\|_\mu^{2^*(\alpha)} \\ &= \Big(\frac{1}{2} - \frac{\|u\|_\mu^{2^*-2}}{2^*(S_{\mu,0})^{\frac{2^*}{2}}} - \frac{\|u\|_\mu^{2^*(\alpha)-2}}{2^*(\alpha)(S_{\mu,\alpha})^{\frac{2^*(\alpha)}{2}}}\Big)\|u\|_\mu^2.\end{aligned} \tag{4.2}$$

Since $2^* > 2$ and $2^*(\alpha) > 2$ for all $\alpha \in (0, 2)$, there exist $R > 0$ and $a > 0$ such that $\Phi(u) \geq a$ for all $u \in S^{1,2}(\mathbb{G})$ with $\|u\|_\mu = R$ small enough.

Let $u_0 \in S^{1,2}(\mathbb{G})\backslash\{0\}$. For $t > 0$, we have

$$\Phi(tu_0) := \frac{t^2}{2}\|u_0\|_\mu^2 - \frac{t^{2^*}}{2^*}\int_{\mathbb{G}} |u_0|^{2^*} d\xi - \frac{t^{2^*(\alpha)}}{2^*(\alpha)}\int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u_0|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi,$$

which implies that $\Phi(tu_0) \to -\infty$ as $t \to +\infty$. So, there exists $t_{u_0} > 0$ such that $\|t_{u_0}u_0\|_\mu > R$ and $\Phi(tu_0) < 0$ for all $t > t_{u_0}$.

Now, we can define

$$c_{u_0} := \inf_{\gamma \in \Gamma_{u_0}} \sup_{t \in [0,1]} \Phi(\gamma(t)),$$

where $\Gamma_{u_0} := \{\gamma \in C([0, 1], S^{1,2}(\mathbb{G})) : \gamma(0) = 0 \text{ and } \gamma(1) = t_{u_0}u_0\}$. Consequently, Φ possesses the hypotheses of Lemma 4.1. □

From Lemmas 4.1 and 4.2 , for $u \in S^{1,2}(\mathbb{G}) \setminus \{0\}$, we define

$$c_u := \inf_{\gamma \in \Gamma_u} \sup_{t \in [0,1]} \Phi(\gamma(t)),$$

where

$$\Gamma_u := \{\gamma \in C([0, 1], S^{1,2}(\mathbb{G})) : \gamma(0) = 0 \text{ and } \gamma(1) = t_u u\}.$$

Then, $c_u \geq a > 0$ for $u \in S^{1,2}(\mathbb{G}) \setminus \{0\}$, and there is a (PS)-sequence $\{u_n\} \subset S^{1,2}(\mathbb{G})\backslash\{0\}$ for Φ at level c_u, that is,

$$\lim_{n\to\infty} \Phi(u_n) = c_u \text{ and } \lim_{n\to\infty} \Phi'(u_n) = 0 \text{ strongly in } (S^{1,2}(\mathbb{G}))'.$$

Lemma 4.3. *Let $\mu \in [0, \mu_{\mathbb{G}})$, $\alpha \in (0, 2)$, and $p = 2^*$. Then, there exists a $u \in S^{1,2}(\mathbb{G}) \setminus \{0\}$ such that $0 < c_u < c^*$, where c^* is defined in* (4.1).

Proof. Let $u(\xi) = U_{\varepsilon,\mu,0}(\xi)$ be the extremal function of $S_{\mu,0}$ as in (2.2). By the definition of c_u, we get

$$0 < c_u \leq \sup_{t\geq 0} \Phi(tu) \leq \sup_{t \geq 0} f(t), \tag{4.3}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$f(t) := \frac{t^2}{2}\|u\|_\mu^2 - \frac{t^{2^*}}{2^*}\int_{\mathbb{G}} |u|^{2^*} d\xi.$$

Note that

$$\sup_{t\geq 0} f(t) = \frac{1}{Q}\left(\frac{\|u\|_\mu^2}{(\int_{\mathbb{G}} |u|^{2^*} d\xi)^{\frac{1}{2^*}}}\right)^{\frac{2^*}{2^*-2}} = \frac{1}{Q}(S_{\mu,0})^{\frac{Q}{2}},$$

this and (4.3) imply that

$$0 < c_u \leq \frac{1}{Q}(S_{\mu,0})^{\frac{Q}{2}}.$$

Now, we will show that the equality does not hold in (4.3). Otherwise, we would have that $\sup_{t\geq 0}\Phi(tu) \leq \sup_{t\geq 0} f(t)$. Let t_1, $t_2 > 0$ where $\sup_{t\geq 0}\Phi(tu)$ and $\sup_{t\geq 0} f(t)$ are attained, respectively. We get

$$f(t_2) = \Phi(t_1 u) = f(t_1) - \frac{t_1^{2^*(\alpha)}}{2^*(\alpha)}\int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi,$$

which implies that $f(t_2) < f(t_1)$ since $u \neq 0$ and $t_1 > 0$. This contradicts the fact that t_2 is the unique maximum point of f. Thus,

$$c_u \leq \sup_{t\geq 0}\Phi(tu) < \sup_{t\geq 0} f(t) = \frac{1}{Q}(S_{\mu,0})^{\frac{Q}{2}}.$$

Similarly,

$$\begin{aligned} c_u &\leq \sup_{t\geq 0}\Phi(tu) < \sup_{t\geq 0}\Big(\frac{t^2}{2}\|u\|_\mu^2 - \frac{t^{2^*(\alpha)}}{2^*(\alpha)}\int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi\Big) \\ &= \frac{2-\alpha}{2(Q-\alpha)}(S_{\mu,\alpha})^{\frac{Q-\alpha}{2-\alpha}}. \end{aligned}$$

This completes the proof of Lemma 4.3. □

Proof of Proposition 4.1. From Lemmas 4.1, 4.2, and 4.3, it follows the conclusions of Proposition 4.1 for a suitable $u \in S^{1,2}(\mathbb{G})\backslash\{0\}$. □

Proposition 4.2. *Let $\mu \in [0, \mu_{\mathbb{G}})$, $\alpha \in (0, 2)$, and $p = 2^*$, and let $\{u_n\} \subset S^{1,2}(\mathbb{G})$ be a $(PS)_c$-sequence at some $c \in (0, c^*)$. If $u_n \rightharpoonup 0$ weakly in $S^{1,2}(\mathbb{G})$ as $n \to \infty$, then there exists $\varepsilon_0 > 0$ such that for $r > 0$, one of the following limits is valid:*

$$\lim_{n\to\infty}\int_{B_d(0,r)} |u_n|^{2^*} d\xi = 0, \ or \ \lim_{n\to\infty}\int_{B_d(0,r)} |u_n|^{2^*} d\xi \geq \varepsilon_0,$$

where $B_d(0, r)$ denotes the ball with center at 0 and radius r with respect to the gauge d.

Lemma 4.4. *Let $\mu \in [0, \mu_{\mathbb{G}})$, $\alpha \in (0, 2)$, and $p = 2^*$, and let $\{u_n\}$ be a $(PS)_c$-sequence for Φ with $c \in (0, c^*)$. If $u_n \rightharpoonup 0$ in $S^{1,2}(\mathbb{G})$ as $n \to \infty$, then for every compact subset $\Omega \subset\subset \mathbb{G} \setminus \{0\}$, up to a subsequence, we have*

$$\lim_{n\to\infty}\int_\Omega \frac{\psi^2(\xi)|u_n|^2}{d(\xi)^2} d\xi = 0, \quad \lim_{n\to\infty}\int_\Omega \frac{\psi^\alpha(\xi)|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha} d\xi = 0, \tag{4.4}$$

and

$$\lim_{n\to\infty}\int_\Omega |\nabla_{\mathbb{G}} u_n|^2 d\xi = 0, \quad \lim_{n\to\infty}\int_\Omega |u_n|^{2^*} d\xi = 0. \tag{4.5}$$

Proof. Let $\Omega \subset\subset \mathbb{G} \setminus \{0\}$ be a fixed compact subset. Since the embedding $S^{1,2}(\mathbb{G}) \hookrightarrow L^p(\Omega)$ is compact for any $p \in [1, 2^*)$, and $\frac{\psi^2(\xi)}{\mathrm{d}(\xi)^2}$, $\frac{\psi^\alpha(\xi)}{\mathrm{d}(\xi)^\alpha}$ are bounded on Ω, (4.4) follows at once being $2^*(\alpha) < 2^*$ and $u_n \rightharpoonup 0$ in $S^{1,2}(\mathbb{G})$.

Now, we verify (4.5). Arguing as the proof of Proposition 2 in [1], let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ be a cutoff function satisfying $\mathrm{supp}\phi \subset\subset \mathbb{G} \setminus \{0\}$, $0 \le \phi \le 1$, and $\phi = 1$ for all $z \in \Omega$. Then, from (4.4) we have

$$\begin{aligned}
o_n(1) &= \langle \Phi'(u_n), \phi^2 u_n \rangle \\
&= \int_{\mathbb{G}} \nabla_{\mathbb{G}} u_n \nabla_{\mathbb{G}} (\phi^2 u_n) d\xi - \mu \int_{\mathbb{G}} \phi^2 \frac{\psi^2(\xi)|u_n|^2}{\mathrm{d}(\xi)^2} d\xi - \int_{\mathbb{G}} \phi^2 |u_n|^{2^*} d\xi - \int_{\mathbb{G}} \phi^2 \frac{\psi^\alpha(\xi)|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi \\
&= \int_{\mathbb{G}} \phi^2 |\nabla_{\mathbb{G}} u_n|^2 d\xi + \int_{\mathbb{G}} 2\phi u_n \nabla_{\mathbb{G}} u_n \nabla_{\mathbb{G}} \phi d\xi - \int_{\mathbb{G}} \phi^2 |u_n|^{2^*} d\xi \\
&= \int_{\mathbb{G}} |\nabla_{\mathbb{G}} (\phi u_n)|^2 d\xi - \int_{\mathbb{G}} |u_n \nabla_{\mathbb{G}} \phi|^2 d\xi - \int_{\mathbb{G}} \phi^2 |u_n|^{2^*} d\xi,
\end{aligned} \tag{4.6}$$

where $o_n(1)$. From now on, it is such that $o_n(1) \to 0$ as $n \to \infty$. By the Hölder inequatity and $u_n \rightharpoonup 0$ in $S^{1,2}(\mathbb{G})$, we have

$$\lim_{n\to\infty} \int_{\mathbb{G}} |u_n \nabla_{\mathbb{G}} \phi|^2 d\xi = \lim_{n\to\infty} \int_{\mathrm{supp}\phi} |u_n \nabla_{\mathbb{G}} \phi|^2 d\xi = 0. \tag{4.7}$$

Combining with (4.6) and (4.7), there holds

$$\begin{aligned}
\int_{\mathbb{G}} |\nabla_{\mathbb{G}} (\phi u_n)|^2 d\xi &= \int_{\mathbb{G}} \phi^2 |u_n|^{2^*} d\xi + o_n(1) \\
&= \int_{\mathbb{G}} |u_n|^{2^*-2} (\phi^2 |u_n|^2) d\xi + o_n(1) \\
&\le \Big(\int_{\mathbb{G}} |u_n|^{2^*} d\xi \Big)^{\frac{2^*-2}{2^*}} \Big(\int_{\mathbb{G}} |\phi u_n|^{2^*} d\xi \Big)^{\frac{2}{2^*}} + o_n(1) \\
&\le \Big(\int_{\mathbb{G}} |u_n|^{2^*} d\xi \Big)^{\frac{2^*-2}{2^*}} \frac{1}{S_{\mu,0}} \int_{\mathbb{G}} |\nabla_{\mathbb{G}} (\phi u_n)|^2 d\xi + o_n(1),
\end{aligned}$$

that is,

$$\left(1 - \frac{1}{S_{\mu,0}} \Big(\int_{\mathbb{G}} |u_n|^{2^*} d\xi \Big)^{\frac{2^*-2}{2^*}} \right) \int_{\mathbb{G}} |\nabla_{\mathbb{G}} (\phi u_n)|^2 d\xi \le o_n(1). \tag{4.8}$$

On the other hand,

$$\begin{aligned}
c + o_n(1) &= \Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle \\
&= \Big(\frac{1}{2} - \frac{1}{2^*} \Big) \int_{\mathbb{G}} |u_n|^{2^*} d\xi + \Big(\frac{1}{2} - \frac{1}{2^*(\alpha)} \Big) \int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi \\
&\ge \frac{1}{Q} \int_{\mathbb{G}} |u_n|^{2^*} d\xi,
\end{aligned} \tag{4.9}$$

which yields that

$$\int_{\mathbb{G}} |u_n|^{2^*} d\xi \le cQ + o_n(1), \tag{4.10}$$

Consequently, this together with (4.8) implies that

$$\Big(1-\frac{(cQ)^{\frac{2^*-2}{2^*}}}{S_{\mu,0}}\Big)\int_{\mathbb{G}}|\nabla_{\mathbb{G}}(\phi u_n)|^2 d\xi \le o_n(1). \tag{4.11}$$

If $\lim_{n\to\infty}\int_{\mathbb{G}}|\nabla_{\mathbb{G}}(\phi u_n)|^2 d\xi \neq 0$, it follows from (4.11) that

$$c \ge \frac{1}{Q}(S_{\mu,0})^{\frac{2^*}{2^*-2}} = \frac{1}{Q}(S_{\mu,0})^{\frac{Q}{2}} \ge c^*.$$

Then, we have $\lim_{n\to\infty}\int_{\mathbb{G}}|\nabla_{\mathbb{G}}(\phi u_n)|^2 d\xi = 0$, which this and $\phi|_{\Omega} = 1$ imply that

$$\lim_{n\to\infty}\int_{\Omega}|\nabla_{\mathbb{G}}u_n|^2 d\xi = 0$$

Therefore, the above equality and Sobolev embedding yield $\lim_{n\to\infty}\int_{\Omega}|u_n|^{2^*}d\xi = 0$, and Lemma 4.4 is proved. □

Remark 4.1. *From* (4.9)*, we will get that*

$$\int_{\mathbb{G}}\frac{\psi^{\alpha}(\xi)|u_n|^{2^*(\alpha)}}{d(\xi)^{\alpha}}d\xi \le \frac{2(Q-\alpha)}{2-\alpha}c + o_n(1).$$

Let $r > 0$ be fixed. From Lemma 4.4, the following quantities are well-defined:

$$\begin{aligned}
\beta &:= \limsup_{n\to\infty}\int_{B_{\mathrm{d}}(0,r)}\Big(|\nabla_{\mathbb{G}}u_n|^2 - \mu\frac{\psi^2(\xi)|u_n|^2}{\mathrm{d}(\xi)^2}\Big)d\xi;\\
\gamma &:= \limsup_{n\to\infty}\int_{B_{\mathrm{d}}(0,r)}|u_n|^{2^*}d\xi;\\
\nu &:= \limsup_{n\to\infty}\int_{B_{\mathrm{d}}(0,r)}\frac{\psi^{\alpha}(\xi)|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^{\alpha}}d\xi.
\end{aligned} \tag{4.12}$$

Lemma 4.5. *Let $\{u_n\} \subset S^{1,2}(\mathbb{G})$ be a $(PS)_c$-sequence for Φ with $c \in (0, c^*)$. If $u_n \rightharpoonup 0$ in $S^{1,2}(\mathbb{G})$ as $n \to \infty$, then*

$$S_{\mu,0}\cdot\gamma^{\frac{2}{2^*}} \le \beta, \quad S_{\mu,\alpha}\cdot\nu^{\frac{2}{2^*(\alpha)}} \le \beta, \quad \textit{and} \ \ \beta \le \gamma + \nu. \tag{4.13}$$

Proof. Let $\phi \in C_0^{\infty}(\mathbb{G})$ be a cutoff function such that $0 \le \phi \le 1$ and $\phi|_{B_{\mathrm{d}}(0,r)} \equiv 1$. Then,

$$S_{\mu,0}\Big(\int_{\mathbb{G}}|\phi u_n|^{2^*}d\xi\Big)^{\frac{2}{2^*}} \le \|\phi u_n\|_{\mu}^2.$$

As $n \to \infty$, Lemma 4.4 implies that

$$S_{\mu,0}\Big(\int_{B_{\mathrm{d}}(0,r)}|u_n|^{2^*}d\xi\Big)^{\frac{2}{2^*}} \le \int_{B_{\mathrm{d}}(0,r)}\Big(|\nabla_{\mathbb{G}}u_n|^2 - \mu\frac{\psi^2(\xi)|u_n|^2}{\mathrm{d}(\xi)^2}\Big)d\xi + o_n(1).$$

Consequently, $S_{\mu,0}\cdot\gamma^{\frac{2}{2^*}} \le \beta$. The second inequality in (4.13) can be verified similarly.

Notice that $\phi u_n \in S^{1,2}(\mathbb{G})$ and $\lim_{n\to\infty}\langle\Phi'(u_n), \phi u_n\rangle = 0$. Via a similar argument as in (4.6), we get that

$$o_n(1) = \int_{\mathbb{G}}\phi|\nabla_{\mathbb{G}}u_n|^2 d\xi - \mu\int_{\mathbb{G}}\phi\frac{\psi^2(\xi)|u_n|^2}{\mathrm{d}(\xi)^2}d\xi - \int_{\mathbb{G}}\phi|u_n|^{2^*}d\xi - \int_{\mathbb{G}}\phi\frac{\psi^{\alpha}(\xi)|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^{\alpha}}d\xi,$$

and the definitions of ϕ and (4.12) deduce that $\beta \le \gamma + \nu$. Lemma 4.5 is verified. □

Proof of Proposition 4.2. From (4.13), it follows that $S_{\mu,0} \cdot \gamma^{\frac{2}{2^*}} \leq \beta \leq \gamma + \nu$, which implies that $S_{\mu,0} \cdot \gamma^{\frac{2}{2^*}} - \gamma \leq \nu$, that is,

$$\gamma^{\frac{2}{2^*}}(S_{\mu,0} - \gamma^{1-\frac{2}{2^*}}) \leq \nu. \tag{4.14}$$

On the other hand, from (4.10) and $c < c^*$, we have that

$$\gamma \leq cQ < c^*Q \leq (S_{\mu,0})^{\frac{Q}{2}} = (S_{\mu,0})^{\frac{1}{1-\frac{2}{2^*}}}.$$

So, $S_{\mu,0} - \gamma^{1-\frac{2}{2^*}} > 0$, namely, there is a constant $C_1 = C_1(\mu, c, Q) > 0$ such that

$$\gamma^{\frac{2}{2^*}} \leq C_1\nu. \tag{4.15}$$

Similarly,

$$\nu^{\frac{2}{2^*(\alpha)}} \leq C_2\gamma, \tag{4.16}$$

for some constant $C_2 = C_2(\mu, c, \alpha, Q) > 0$. Then, combining with (4.15) and (4.16), there holds

$$\text{either } \gamma = \nu = 0, \text{ or } \min\{\gamma, \nu\} \geq \varepsilon_0,$$

where $\varepsilon_0 = \varepsilon_0(Q, \mu, \alpha)$ is a positive constant. This completes the proof of Proposition 4.2. □

Remark 4.2. *The Proposition 4.2 states that every $(PS)_c$-sequence $\{u_n\} \subset S^{1,2}(\mathbb{G})$ for Φ with $c \in (0, c^*)$ such that $u_n \rightharpoonup 0$ weakly in $S^{1,2}(\mathbb{G})$ as $n \to \infty$ verifies one of the following limits:*

$$\lim_{n\to\infty} \int_{B_d(0,r)} |u_n|^{2^*} d\xi = 0 \ \ or \ \lim_{n\to\infty} \int_{B_d(0,r)} |u_n|^{2^*} d\xi \geq \varepsilon_0 > 0$$

with arbitrary $r > 0$ and a constant ε_0 independent on r.

Proof of Theorem 1.2. Let $\{u_n\}$ be a $(PS)_c$-sequence for Φ with $c \in (0, c^*)$ such that $u_n \rightharpoonup 0$ in $S^{1,2}(\mathbb{G})$ as $n \to \infty$. Then, we have that

$$\|u_n\|_\mu^2 = \int_{\mathbb{G}} |u_n|^{2^*} d\xi + \int_{\mathbb{G}} \psi^\alpha(\xi) \frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi + o_n(1), \tag{4.17}$$

and

$$\begin{aligned} c + o_n(1) &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n \rangle \\ &= \frac{1}{Q} \int_{\mathbb{G}} |u_n|^{2^*} d\xi + \frac{2-\alpha}{2(Q-\alpha)} \int_{\mathbb{G}} \psi^\alpha(\xi) \frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi. \end{aligned} \tag{4.18}$$

Now, we claim that $\limsup_{n\to\infty} \int_{\mathbb{G}} |u_n|^{2^*} d\xi > 0$. Arguing by contradiction, we assume that $\int_{\mathbb{G}} |u_n|^{2^*} d\xi = o_n(1)$. Then, (4.17) and (4.18) imply that

$$\|u_n\|_\mu^2 = \int_{\mathbb{G}} \frac{\psi^\alpha(\xi)|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi + o_n(1), \tag{4.19}$$

$$c + o_n(1) = \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n \rangle = \frac{2-\alpha}{2(Q-\alpha)} \int_{\mathbb{G}} \psi^\alpha(\xi) \frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha} d\xi. \tag{4.20}$$

From (4.19) and the definition of $S_{\mu,\alpha}$, we get that

$$S_{\mu,\alpha}\Big(\int_{\mathbb{G}}\psi^\alpha(\xi)\frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi\Big)^{\frac{2}{2^*(\alpha)}} \le \|u_n\|_\mu^2 = \int_{\mathbb{G}}\psi^\alpha(\xi)\frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi + o(1),$$

that is,

$$\Big(\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi\Big)^{\frac{2}{2^*(\alpha)}}\left(S_{\mu,\alpha}-\Big(\int_{\mathbb{G}}\psi^\alpha(\xi)\frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi\Big)^{\frac{2^*(\alpha)-2}{2^*(\alpha)}}\right) \le o_n(1). \tag{4.21}$$

On the other hand, (4.20) and $c < c^*$ yield that

$$\int_{\mathbb{G}}\psi^\alpha(\xi)\frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi < \frac{2(Q-\alpha)}{2-\alpha}c^* + o_n(1) \le (S_{\mu,\alpha})^{\frac{2^*(\alpha)}{2^*(\alpha)-2}} + o_n(1), \tag{4.22}$$

which together with (4.21) implies that $\int_{\mathbb{G}}\psi^\alpha(\xi)\frac{|u_n|^{2^*(\alpha)}}{\mathrm{d}(\xi)^\alpha}d\xi = o_n(1)$, a contradiction with (4.18) and $c > 0$.

Set $\varepsilon_1 = \min\{\frac{\varepsilon_0}{2}, \limsup_{n\to\infty}\int_{\mathbb{G}}|u_n|^{2^*}d\xi\}$, where ε_0 is given in Proposition 4.2. Let $\varepsilon \in (0, \varepsilon_1)$. From Proposition 4.2 up to a subsequence still denoted by $\{u_n\}$, for $n \in \mathbb{N}$, there exists $r_n > 0$ such that

$$\int_{B_{\mathrm{d}}(0,r_n)}|u_n|^{2^*}d\xi = \varepsilon, \quad \forall\, n \in \mathbb{N}. \tag{4.23}$$

Let $\widehat{u_n}(\xi) = r_n^{\frac{Q-2}{2}}u_n(\delta_{r_n}(\xi))$. Then, $\widehat{u_n} \in S^{1,2}(\mathbb{G})$ satisfies

$$\int_{B_{\mathrm{d}}(0,1)}|\widehat{u_n}|^{2^*}d\xi = \int_{B_{\mathrm{d}}(0,r_n)}|u_n|^{2^*}d\xi = \varepsilon, \quad \forall n \in \mathbb{N}. \tag{4.24}$$

Moreover, it is easy to see that $\{\widehat{u_n}\}$ is again a (PS)-sequence of the type given in Proposition 4.2. So, we have that

$$c + o_n(1) = \Phi(\widehat{u_n}) - \frac{1}{2^*(\alpha)}\langle\Phi'(\widehat{u_n}), \widehat{u_n}\rangle \ge \left(\frac{1}{2} - \frac{1}{2^*(\alpha)}\right)\|\widehat{u_n}\|_\mu^2,$$

which implies that $\{\widehat{u_n}\}$ is bounded in $S^{1,2}(\mathbb{G})$. Then, up to a subsequence, there exists $\widehat{u} \in S^{1,2}(\mathbb{G})$ such that $\widehat{u_k} \rightharpoonup \widehat{u}$ weakly in $S^{1,2}(\mathbb{G})$, $L^{2^*(\alpha)}(\mathbb{G}, \frac{\psi^\alpha(\xi)}{\mathrm{d}(\xi)^\alpha}d\xi)$, and $L^{2^*}(\mathbb{G})$ as $n \to +\infty$. So, for any $\phi \in S^{1,2}(\mathbb{G})$, we have

$$\begin{aligned}
o_n(1) &= \langle\Phi'(\widehat{u_n}), \phi\rangle \\
&= \int_{\mathbb{G}}\nabla_{\mathbb{G}}\widehat{u_n}\nabla_{\mathbb{G}}\phi d\xi - \mu\int_{\mathbb{G}}\psi^2(\xi)\frac{\widehat{u_n}\phi}{d(\xi)^2}d\xi - \int_{\mathbb{G}}|\widehat{u_n}|^{2^*-2}\widehat{u_n}\phi d\xi - \int_{\mathbb{G}}\psi^\alpha(\xi)\frac{|\widehat{u_n}|^{2^*(\alpha)-2}\widehat{u_n}\phi}{\mathrm{d}(\xi)^\alpha}d\xi \\
&= o_n(1) + \int_{\mathbb{G}}\nabla_{\mathbb{G}}\widehat{u}\nabla_{\mathbb{G}}\phi d\xi - \mu\int_{\mathbb{G}}\psi^2(\xi)\frac{\widehat{u}\phi}{d(\xi)^2}d\xi - \int_{\mathbb{G}}|\widehat{u}|^{2^*-2}\widehat{u}\phi d\xi - \int_{\mathbb{G}}\frac{\psi^\alpha(\xi)|\widehat{u}|^{2^*(\alpha)-2}\widehat{u}\phi}{\mathrm{d}(\xi)^\alpha}d\xi \\
&= \langle\Phi'(\widehat{u}), \phi\rangle + o_n(1),
\end{aligned}$$

which concludes that $\widehat{u} \in S^{1,2}(\mathbb{G})$ is a solution of problem (1.5). In addition, if $\widehat{u} \equiv 0$, Proposition 4.2 implies that either

$$\lim_{n\to\infty}\int_{B_{\mathrm{d}}(0,1)}|\widehat{u_n}|^{2^*}d\xi = 0, \text{ or } \lim_{n\to\infty}\int_{B_{\mathrm{d}}(0,1)}|\widehat{u_n}|^{2^*}d\xi \ge \varepsilon_0,$$

which contradicts (4.24) as $0 < \varepsilon < \frac{\varepsilon_0}{2}$. Then, $\widehat{u} \ne 0$ and the proof of Theorem 1.2 is complete. □

5. Proof of Theorem 1.3

In this section, we show that system (1.10) has a nontrivial weak solution. Observe that the corresponding functional of (1.10) can be written as

$$I(u,v)=\frac{1}{2}\int_{\mathbb{G}}\Big[|\nabla_{\mathbb{G}}u|^2+|\nabla_{\mathbb{G}}v|^2-\mu\frac{\psi^2(\xi)(|u|^2+|v|^2)}{\mathrm{d}(\xi)^2}\Big]d\xi-\frac{\lambda}{2^*}\int_{\mathbb{G}}H(u,v)d\xi-\frac{\eta}{2^*(\alpha)}\int_{\mathbb{G}}\frac{\psi^\alpha(\xi)Q(u,v)}{\mathrm{d}(\xi)^\alpha}d\xi.$$

By the standard arguments, we can verify $I\in C^1(W,\mathbb{R})$. A critical point of functional I in W is a weak solution to (1.10). We say that a pair of functions $(u,v)\in W$ is a nontrivial solution of (1.10) if $(u,v)\neq(0,0)$ and $\langle I'(u,v),(\phi_1,\phi_2)\rangle=0$ for all $(\phi_1,\phi_2)\in W$.

We point out that the proof of Lemma 4.2 provides us with a tool to show that the functional I has a mountain pass geometrical, that is,

(i) $I(0,0)=0$

(ii) There exist $R,\rho>0$ such that $I(u,v)\geq\rho>0$ for $(u,v)\in W\backslash\{(0,0)\}$ with $\|(u,v)\|_W=R$.

(iii) There exists $(u_0,v_0)\in W\backslash\{(0,0)\}$ such that $\lim\limits_{t\to\infty}I(t(u_0,v_0))<0$.

Define

$$c:=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}I(\gamma(t))\geq\rho>0.$$

where $\Gamma:=\{\gamma\in C([0,1],W):\gamma(0)=0 \text{ and } I(\gamma(1))<0\}$. Then, there exists a sequence $\{(u_n,v_n)\}\subset W$ such that

$$\lim_{n\to\infty}I(u_n,v_n)=c,\ \lim_{n\to\infty}I'(u_n,v_n)=0 \text{ strongly in } W^{-1},$$

where $c\in(0,c^{**})$ and

$$c^{**}:=\min\Big\{\frac{1}{Q}\lambda^{\frac{2-Q}{2}}S_H(\mu,0)^{\frac{Q}{2}},\quad\frac{2-\alpha}{2(Q-\alpha)}\eta^{\frac{2-Q}{2-\alpha}}S_Q(\mu,\alpha)^{\frac{Q-\alpha}{2-\alpha}}\Big\}.$$

Proposition 5.1. *Let $\{(u_n,v_n)\}\subset W$ be a $(PS)_c$-sequence for I with $c\in(0,c^{**})$. If $(u_n,v_n)\rightharpoonup(0,0)$ weakly in W as $n\to\infty$, then there exists $\tilde{\varepsilon}_0>0$ such that for all $r>0$, either*

$$\lim_{n\to\infty}\int_{B_d(0,r)}H(u_n,v_n)d\xi=0,\ or\ \lim_{n\to\infty}\int_{B_d(0,r)}H(u_n,v_n)d\xi\geq\tilde{\varepsilon}_0.$$

Proof. The argument used is similar to that of Section 4, and for completeness we give the following argument. We first show the following results held for any compact subset $\Omega\subset\mathbb{G}\setminus\{0\}$:

$$\lim_{n\to\infty}\int_\Omega\frac{\psi^2(\xi)|u_n|^2}{\mathrm{d}(\xi)^2}d\xi=\lim_{n\to\infty}\int_\Omega\frac{\psi^2(\xi)|v_n|^2}{\mathrm{d}(\xi)^2}d\xi=0,\tag{5.1}$$

$$\lim_{n\to\infty}\int_\Omega\frac{\psi^\alpha(\xi)Q(u_n,v_n)}{\mathrm{d}(\xi)^\alpha}d\xi=0,\tag{5.2}$$

$$\lim_{n\to\infty}\int_\Omega(|\nabla_{\mathbb{G}}u_n|^2+|\nabla_{\mathbb{G}}v_n|^2)d\xi=0,\tag{5.3}$$

$$\lim_{n\to\infty}\int_\Omega H(u_n,v_n)d\xi=0.\tag{5.4}$$

Arguing as Lemma 4.4, for $\Omega \subset\subset \mathbb{G}\setminus\{0\}$, (5.1) and (5.2) follow from the properties of the homogeneous function in Proposition 2.1, the compact embedding $S^{1,2}(\mathbb{G}) \hookrightarrow L^p(\Omega)$ for $p \in [1, 2^*)$, and the fact that $\frac{\psi^2(\xi)}{\mathrm{d}(\xi)^2}$, $\frac{\psi^\alpha(\xi)}{\mathrm{d}(\xi)^\alpha}$ are bounded on $\Omega\backslash\{0\}$. Thus, it remains to show that (5.3) and (5.4) hold.

Let $\phi \in C_0^\infty(\mathbb{G}\setminus\{0\})$ be a cutoff function such that $\mathrm{supp}\phi \subset\subset \mathbb{G}\setminus\{0\}$, $0 \le \phi \le 1$, and $\phi|_\Omega = 1$. Note that the weak convergence of $\{u_n\}$ and $\{v_n\}$ in $S^{1,2}(\mathbb{G})$ implies the boundedness. Then,

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u_n||\nabla_{\mathbb{G}}(\phi^2)||u_n| d\xi \le \|\nabla_{\mathbb{G}} u_n\|_2 \|u_n\|_{L^2(\mathrm{supp}|\nabla_{\mathbb{G}}\phi|)} = o_n(1),$$

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} v_n||\nabla_{\mathbb{G}}(\phi^2)||v_n| d\xi \le \|\nabla_{\mathbb{G}} v_n\|_2 \|v_n\|_{L^2(\mathrm{supp}|\nabla_{\mathbb{G}}\phi|)} = o_n(1),$$

and

$$\int_{\mathbb{G}} (|\phi\nabla_{\mathbb{G}} u_n|^2 + |\phi\nabla_{\mathbb{G}} v_n|^2) d\xi = \int_{\mathbb{G}} (|\nabla_{\mathbb{G}}(\phi u_n)|^2 + |\nabla_{\mathbb{G}}(\phi v_n)|^2) d\xi + o_n(1).$$

From the latest inequalities and (5.1), (5.2), we get that

$$\begin{aligned} o_n(1) &= \langle I'(u_n, v_n), (\phi^2 u_n, \phi^2 v_n)\rangle \\ &= \int_{\mathbb{G}} (|\phi\nabla_{\mathbb{G}} u_n|^2 + |\phi\nabla_{\mathbb{G}} v_n|^2) d\xi - \lambda \int_{\mathbb{G}} \phi^2 H(u_n, v_n) d\xi \\ &+ \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u_n||\nabla_{\mathbb{G}}(\phi^2)||u_n| d\xi + \int_{\mathbb{G}} |\nabla_{\mathbb{G}} v_n||\nabla_{\mathbb{G}}(\phi^2)||v_n| d\xi + o_n(1) \\ &= \int_{\mathbb{G}} (|\phi\nabla_{\mathbb{G}} u_n|^2 + |\phi\nabla_{\mathbb{G}} v_n|^2) d\xi - \lambda \int_{\mathbb{G}} \phi^2 H(u_n, v_n) d\xi + o_n(1) \\ &= \int_{\mathbb{G}} (|\nabla_{\mathbb{G}}(\phi u_n)|^2 + |\nabla_{\mathbb{G}}(\phi v_n)|^2) d\xi - \lambda \int_{\mathbb{G}} \phi^2 (H(u_n, v_n) d\xi + o_n(1) \\ &\ge \|\phi u_n\|_\mu^2 + \|\phi v_n\|_\mu^2 - \lambda \int_{\mathbb{G}} \phi^2 H(u_n, v_n) d\xi + o_n(1), \end{aligned}$$

which implies that

$$\begin{aligned} &\|(\phi u_n, \phi v_n)\|_W^2 \\ &\le \lambda \int_{\mathbb{G}} \phi^2 H(u_n, v_n) d\xi + o_n(1) \\ &\le \lambda\Big(\int_{\mathbb{G}} H(u_n, v_n) d\xi\Big)^{\frac{2^*-2}{2^*}} \Big(\int_{\mathbb{G}} H(\phi u_n, \phi v_n) d\xi\Big)^{\frac{2}{2^*}} + o_n(1) \\ &\le \lambda\Big(\int_{\mathbb{G}} H(u_n, v_n) d\xi\Big)^{\frac{2^*-2}{2^*}} S_H(\mu, 0)^{-1} \|(\phi u_n, \phi v_n)\|_W^2 + o_n(1), \end{aligned}$$

and, therefore,

$$\Big(1 - \lambda\Big(\int_{\mathbb{G}} H(u_n, v_n) d\xi\Big)^{\frac{2^*-2}{2^*}} S_H(\mu, 0)^{-1}\Big) \|(\phi u_n, \phi v_n)\|_W^2 \le o_n(1). \tag{5.5}$$

In addition, since $c + o_n(1) = I(u_n, v_n) - \frac{1}{2}\langle I'(u_n, v_n), (u_n, v_n)\rangle \ge \frac{\lambda}{Q} \int_{\mathbb{G}} H(u_n, v_n) d\xi$ and the upper bounded on c yields

$$\lim_{n\to\infty} \|(\phi u_n, \phi v_n)\|_W^2 = 0, \tag{5.6}$$

Consequently, (5.6) and (2.4) imply that

$$\int_{\mathbb{G}} H(\phi u_n, \phi v_n) d\xi \leq \frac{cQ}{\lambda} + o_n(1),$$

which together with (5.5) implies that

$$\lim_{n\to\infty} \int_{\mathbb{G}} H(\phi u_n, \phi v_n) = 0.$$

Then, the definition of ϕ implies that (5.3) and (5.4) hold.

Now, let us define

$$\kappa = \limsup_{n\to\infty} \int_{B_{\mathrm{d}}(0,r)} (|\nabla_{\mathbb{G}} u_n|^2 + |\nabla_{\mathbb{G}} v_n|^2 - \mu \frac{\psi^2(\xi)(|u_n|^2 + |v_n|^2)}{\mathrm{d}(\xi)^2}) d\xi. \tag{5.7}$$

$$\tau = \limsup_{n\to\infty} \int_{B_{\mathrm{d}}(0,r)} H(u_n, v_n) d\xi, \tag{5.8}$$

$$\omega = \limsup_{n\to\infty} \int_{B_{\mathrm{d}}(0,r)} \frac{\psi^\alpha(\xi) Q(u_n, v_n)}{\mathrm{d}(\xi)^\alpha} d\xi, \tag{5.9}$$

where $r > 0$ is fixed. From Lemma 4.5, we can deduce that the above quantities are well-defined and independent of r. If $(u_n, v_n) \rightharpoonup (0,0)$ weakly in $\mathcal{H}$ as $n \to \infty$, we have the following results:

$$S_H(\mu, 0) \cdot \tau^{\frac{2}{2^*}} \leq \kappa, \quad S_Q(\mu, t) \cdot \omega^{\frac{2}{2^*(t)}} \leq \kappa, \quad \text{and} \quad \kappa \leq \lambda\tau + \eta\omega. \tag{5.10}$$

From (5.10), it follows that

$$S_H(\mu, 0)\tau^{\frac{2}{2^*}} \leq \kappa \leq \lambda\tau + \eta\omega,$$

which implies that

$$\tau^{\frac{2}{2^*}} \left(S_H(\mu, 0) - \lambda \tau^{\frac{2^*-2}{2^*}} \right) \leq \eta\omega. \tag{5.11}$$

On the other hand, since $\frac{\lambda}{Q} \int_{\mathbb{G}} H(u_n, v_n) d\xi \leq c + o_n(1)$, we get that $\lambda\tau \leq cQ < c^{**}Q < \lambda^{\frac{2-Q}{2}} S_H(\mu, 0)^{\frac{2^*}{2^*-2}}$, and (5.11) yields that there exists a constant $C_1 = C_1(\mu, c, \lambda, \eta) > 0$ such that

$$\tau^{\frac{2}{2^*}} \leq C_1 \omega. \tag{5.12}$$

Similarly, there exists $C_2 = C_2(\mu, c, \alpha, \lambda, \eta) > 0$ such that

$$\omega^{\frac{2}{2^*(\alpha)}} \leq C_2 \tau. \tag{5.13}$$

Based on inequalities (5.12) and (5.13), we can find a constant $\tilde{\varepsilon}_0 = \varepsilon_0(Q, \mu, c, \alpha) > 0$ such that either $\tau = \omega = 0$ or $\min\{\tau, \omega\} \geq \tilde{\varepsilon}_0$. This proves Proposition 5.1. □

Proof of Theorem 1.3. Choosing the sequence $\{(u_n, v_n)\} \subset W$ defined as in Proposition 5.1, proceeding as in proof of Theorem 1.2, we have $\limsup_{n\to\infty} \int_{\mathbb{G}} H(u_n, v_n) d\xi > 0$. Then, there exists $\tilde{\varepsilon}_1 = \min\{\Lambda, \frac{\tilde{\varepsilon}_0}{2}\}$, such that for $\varepsilon \in (0, \tilde{\varepsilon}_1)$, there exists a positive real sequence $\{r_n\}$ such that

$$\widetilde{u_n} = r_n^{\frac{Q-2}{2}} u_n(\delta_{r_n}(\xi)), \quad \widetilde{v_n} = r_n^{\frac{Q-2}{2}} v_n(\delta_{r_n}(\xi)) \in S^{1,2}(\mathbb{G})$$

is again a $(PS)_c$-sequence of the type given in Proposition 5.1 and satisfies

$$\int_{B_{\mathrm{d}}(0,1)} H(\widetilde{u}_n, \widetilde{v}_n) d\xi = \varepsilon, \quad \forall n \in \mathbb{N}.$$

Moreover, for the $(PS)_c$ sequence $\{(\widetilde{u}_n, \widetilde{v}_n)\}$, we get

$$c + o_n(1) = I(\widetilde{u}_n, \widetilde{v}_n) - \frac{1}{2^*(\alpha)} \langle I'(\widetilde{u}_n, \widetilde{v}_n), (\widetilde{u}_n, \widetilde{v}_n) \rangle \geq \left(\frac{1}{2} - \frac{1}{2^*(\alpha)} \right) \| (\widetilde{u}_n, \widetilde{v}_n) \|_W^2,$$

which implies that $\{(\tilde{u}_n, \tilde{v}_n)\}$ is bounded in W. Up to a subsequence, there exist $\tilde{u}, \tilde{v} \in S^{1,2}(\mathbb{G})$ such that $\widetilde{u}_n \rightharpoonup \tilde{u}, \widetilde{v}_n \rightharpoonup \tilde{v}$ weakly in $S^{1,2}(\mathbb{G})$ as $n \to \infty$. Similar to the proof of Theorem 1.2, we can show that $(\tilde{u}, \tilde{v}) \not\equiv (0, 0)$. Thus, there exists a nontrivial weak solution to system (1.10).

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. R. Filippucci, P. Pucci, F. Robert, On a p-Laplace equation with multiple critical nonlinearities, *J. Math. Pures Appl.*, **91** (2009), 156–177. https://doi.org/10.1016/j.matpur.2008.09.008
2. M. Bhakta, Semilinear elliptic equation with biharmonic operator and multiple critical nonlinearities, *Adv. Nonlinear Stud.*, **15** (2015), 835–848. https://doi.org/10.1515/ans-2015-0405
3. N. Ghoussoub, S. Shakerian, Borderline variational problems involving fractional Laplacians and critical singularities, *Adv. Nonlinear Stud.*, **15** (2015), 527–555. https://doi.org/10.1515/ans-2015-0302
4. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differ. Equations*, **32** (2007), 1245–1260. https://doi.org/10.1080/03605300600987306
5. W. Chen, Fractional elliptic problems with two critical Sobolov-Hardy exponents, *Electron. J. Differ. Equations*, **2018** (2018), 1–12.
6. R. B. Assuncão, J. C. Silva, O. H. Miyagaki, A Fractional p-Laplacian Problem with Multiple Critical Hardy-Sobolev Nonlinearities, *Milan J. Math.*, **88** (2020), 65–97. https://doi.org/10.1007/s00032-020-00308-5
7. L. D'Ambrosio, Some Hardy inequalities on the Heisenberg group, *Differ. Equations*, **40** (2004), 552–564. https://doi.org/10.1023/B:DIEQ.0000035792.47401.2a
8. L. D'Ambrosio, Hardy-type inequalities related to degenerate elliptic differential operators, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **4** (2005), 451–486. https://doi.org/10.2422/2036-2145.2005.3.04

9. Y. Han, P. Niu, Hardy-Sobolev type inequalities on the H-type group, *Manuscripta Math.*, **118** (2005), 235–252. https://doi.org/10.1007/s00229-005-0589-7
10. P. Niu, H. Zhang, Y. Wang, Hardy type and Rellich type inequalities on the Heisenberg group, *Proc. Amer. Math. Soc.*, **129** (2001), 3623–3630. https://doi.org/10.1090/S0002-9939-01-06011-7
11. A. Loiudice, L^p-weak regularity and asymptotic behavior of solutions for critical equations with singular potentials on Carnot groups, *Nonlinear Differ. Equation Appl.*, **17** (2010), 575–589. https://doi.org/10.1007/s00030-010-0069-y
12. A. Loiudice, Critical growth problems with singular nonlinearities on Carnot groups, *Nonlinear Anal.*, **126** (2015), 415–436. https://doi.org/10.1016/j.na.2015.06.010
13. A. Loiudice, Local behavior of solutions to subelliptic problems with Hardy potential on Carnot groups, *Mediterr. J. Math.*, **15** (2018), 81. https://doi.org/10.1007/s00009-018-1126-8
14. A. Loiudice, Critical problems with Hardy potential on Stratified Lie groups, *Adv. Differ. Equations*, **28** (2023), 1–33. https://doi.org/10.57262/ade028-0102-1
15. J. Zhang, Sub-elliptic problems with multiple critical Sobolev-Hardy exponents on Carnot groups, *Manuscripta Math.*, **172** (2023), 1–29. https://doi.org/10.1007/s00229-022-01406-x
16. J. Zhang, On the existence and multiplicity of solutions for a class of sub-Laplacian problems involving critical Sobolev-Hardy exponents on Carnot groups, *Appl. Anal.*, **102** (2023), 4209–4229. https://doi.org/10.1080/00036811.2022.2107910
17. J. Zhang, Sub-elliptic systems involving critical Hardy-Sobolev exponents and sign-changing weight functions on Carnot groups, *J. Nonlinear Var. Anal.*, **8** (2024), 199–231. https://doi.org/10.23952/jnva.8.2024.2.02
18. S. Bordoni, P. Pucci, Schrödinger-Hardy systems involving two Laplacian operators in the Heisenberg group, *Bull. Sci. Math.*, **146** (2018), 50–88. https://doi.org/10.1016/j.bulsci.2018.03.001
19. S. Bordoni, R. Filippucci, P. Pucci, Existence problems on Heisenberg groups involving Hardy and critical terms, *J. Geometric Anal.*, **30** (2020), 1887–1917. https://doi.org/10.1007/s12220-019-00295-z
20. P. Pucci, Critical Schrödinger-Hardy systems in the Heisenberg group, *Discrete Contin. Dyn. Syst. Ser. S*, **12** (2019), 375–400. https://doi.org/10.3934/dcdss.2019025
21. M. Ruzhansky, D. Suragan, N. Yessirkegenov, Caffarelli-Kohn-Nirenberg and Sobolev type inequalities on stratified Lie groups, *Nonlinear Differ. Equation Appl.*, **24** (2017), 56. https://doi.org/10.1007/s00030-017-0478-2
22. M. Ruzhansky, D. Suragan, Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups, *Adv. Math.*, **308** (2017), 483–528. https://doi.org/10.1016/j.aim.2016.12.013
23. M. Ruzhansky, D. Suragan, Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups, *Adv. Math.*, **317** (2017), 799–822. https://doi.org/10.1016/j.aim.2017.07.020
24. L. Roncal, S. Thangavelu, Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group, *Adv. Math.*, **302** (2016), 106–158. https://doi.org/10.1016/j.aim.2016.07.010
25. N. Garofalo, E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier*, **40** (1990), 313–356. https://doi.org/10.5802/aif.1215

26. G. B. Folland, E. M. Stein, Hardy spaces on Homogeneous groups, in *Mathematical Notes Vol. 28*, Princeton University Press, Princeton, 1982. https://doi.org/10.1515/9780691222455
27. N. Garofalo, D. Vassilev, Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups, *Math. Ann.*, **318** (2000), 453–516. https://doi.org/10.1007/s002080000127
28. D. Vassilev, Existence of solutions and regularity near the characteristic boundary for sub-Laplacian equations on Carnot groups, *Pacific J. Math.*, **227** (2006), 361–397. https://doi.org/10.2140/pjm.2006.227.361
29. A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer, Berlin, 2007. https://doi.org/10.1007/978-3-540-71897-0
30. M. Ruzhansky, D. Suragan, *Hardy Inequalities on Homogeneous Groups*, Birkhäuser, Cham, 2019. https://doi.org/10.1007/978-3-030-02895-4
31. G. B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Mat.*, **13** (1975), 161–207. https://doi.org/10.1007/BF02386204
32. S. P. Ivanov, D. N. Vassilev, *Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem*, Word Scientific, Singapore, 2011. https://doi.org/10.1142/7647
33. D. C. de Morais Filho, M. A. S. Souto, Systems of p-Laplacian equations involving homogeneous nonlinearities with critical Sobolev exponent degrees, *Commun. Part. Differ. Equations*, **24** (1999), 1537–1553. https://doi.org/10.1080/03605309908821473
34. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7

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