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Research article

Stabilization for a degenerate wave equation with drift and potential term with boundary fractional derivative control

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Abstract: This paper explores the boundary stabilization of a degenerate wave equation in the non-divergence form, which includes a drift term and a singular potential term. Additionally, we introduce boundary fractional derivative damping at the endpoint where divergence is absent. Using semi-group theory and the multiplier method, we establish polynomial stability, with a decay rate depending upon the order of the fractional derivative.

Keywords: degenerate wave equation; singular potentials; drift; boundary stabilization; polynomial decay; fractional derivative; non-divergence form

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1. Introduction

This paper is devoted to studying a class of degenerate and singular wave equations. These equations are characterized by degeneracy in the non-divergence form, accompanied by a drift term, and a singular potential term with fractional derivative feedback on the boundary. The system is defined as follows:

$$\begin{cases} v_{tt}(x,t) - p(x)v_{xx}(x,t) - \dfrac{\beta}{q(x)}v(x,t) - r(x)v_x(x,t) = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ v(0,t) = 0, (\eta v_x)(1,t) = -\partial_t^{\alpha,\eta} v(1,t), & t \in (0,+\infty), \end{cases} \tag{1.1}$$

with the following initial condition:

$$v(0,x) = v_0(x), v_t(0,x) = v_1(x),\ x \in (0,1),$$

where $p, q, r \in C^0[0,1], p, q > 0$ on $(0,1]$, $p(0) = q(0) = 0$, $\beta \in \mathbb{R}$, and $\dfrac{r}{p} \in L^1(0,1)$. Hence, if $p(x) = x^K$, $K > 0$, we can consider $r(x) = x^m$, $m > 0$, for any $m > K - 1$. This condition is clearly satisfied if $K < 1$. In the boundary condition, η is the well-known absolutely continuous weight function

$$\eta(x) := \exp\left\{\int_{\frac{1}{2}}^{x} \frac{r(s)}{p(s)} ds\right\}, \quad x \in [0,1]$$

introduced by Feller in a related context [1] (see also [2, 3] and the references therein). It is clear that the function $\eta : [0,1] \to \mathbb{R}$ is well defined, and we immediately find that $\eta \in C^0[0,1] \cap C^1(0,1]$ is a strictly positive function that is bounded above and below by a positive constant. Notice also that η can be extended to a function of class $C^1[0,1]$ when r degenerates at 0 not slower than p, for instance, if $p(x) = x^{n_1}$ and $r(x) = x^{n_2}$ with $n_1 \le n_2$. The notation $\partial_t^{\alpha,\tau}$ represents Caputo's fractional derivative of order $\alpha \in (0,1)$ with respect to time variable t and is defined by

$$[D^{\alpha,\tau}\omega](t) = \partial_t^{\alpha,\tau}\omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\tau(t-s)} \frac{d\omega}{ds}(s) ds, \tag{1.2}$$

where Γ denotes the Gamma function and $\tau \ge 0$.

The degeneracy of a function ϱ at $x = 0$ is measured by the parameter K_ϱ defined by

$$K_\varrho := \sup_{x \in (0,1]} \frac{x|\varrho'(x)|}{\varrho(x)}. \tag{1.3}$$

- We say that ϱ is weakly degenerate (WD) if $\varrho \in C^0[0,1] \cap C^1(0,1]$ and $K_\varrho \in (0,1)$.
- We say that ϱ is strongly degenerate (SD) if $\varrho \in C^1[0,1]$ and $K_\varrho \in [1,2)$.

We assume here that $K_p, K_q < 2$ because it is essential to the calculation that will be conducted later below. Additionally, later, we will need a condition on K_p and $K_q < 2$ such that $K_p + 2K_q \le 2$.

Prior to exploring the system discussed in this paper, it is advantageous to conduct a comprehensive literature review focusing on the study of degenerate systems. It is commonly known that investigating the standard linear theory concerning transverse waves in a string of length L under tension $\mathcal{T}$ results in the derivation of the classical wave equation:

$$\rho(x)u_{tt}(t,x) = \frac{\partial \mathcal{T}}{\partial x} u_x(t,x) + \mathcal{T}(x,t)u_{xx}(t,x),$$

where $u(t,x)$ denotes the vertical displacement of the string from the x axis at position $x \in (0,L)$ and time $t > 0$, $\rho(x)$ is the mass density of the string at position x, and $\mathcal{T}(t,x)$ denotes the tension in the string at position x and time t. Divide by $\rho(x)$, assume $\mathcal{T}$ is independent of t, and set $p(x) = \mathcal{T}(x)\rho^{-1}(x)$, $r(x) = \mathcal{T}'(x)\rho^{-1}(x)$. In this way, we obtain

$$u_{tt}(t,x) = p(x)u_{xx}(t,x) + r(x)u_x(t,x).$$

Let us consider a scenario where the density at a specific point, say $x = 0$, is notably high. In such cases, the equation degenerates at $x = 0$, as we can set $p(0) = 0$, and the remaining term becomes a drift term.

Recently, there has been a noticeable increase in interest in studying degenerate parabolic and hyperbolic equations. This indicates a substantial change in focus after a time of neglect in exploring these types of systems. These equations have attracted interest because of their applicability in several real-world situations, such as camouflage (to make operators undetectable), Levy noise phenomena, meteorology, and biology [4–7]. Diverse disciplines have been faced with demanding control and inverse problems due to the rise of degenerate partial differential equations (PDEs). The complex mathematical difficulties related to degenerate partial differential equations have been exacerbated by their extensive range of applications.

The literature provides limited insight into scenarios where the coefficient $a(x)$ (in the equation $u_{tt} - a(x)u_{xx}$) demonstrates degeneracy, despite numerous applications described by hyperbolic equations that degenerate at the spatial domain's boundary. This was investigated in the pioneering work by Alabau et al. [8] for a general function $a(x)$ and by Gueye [9] for the prototype case ($a(x) = x^{\alpha}, \alpha \in (0, 1)$). More recently, Chouaou et al. [10] examined a one-dimensional weakly degenerate wave equation with dynamic non-local boundary feedback of fractional type acting at a degenerate point. They demonstrated the absence of uniform stability and established a polynomial decay rate.

In the paper by Boutaayamou et al. [11], the authors examine a degenerate wave equation featuring drift, where the leading operator does not conform to divergence form:

$$\begin{cases} u_{tt} - a(x)u_{xx} - b(x)u_x = 0, \\ u(t, 0) = 0, \ u(t, 1) = f(t) \end{cases}$$

and they studied the boundary controllability of the system. In addition, Akil et al. [12] investigate a one-dimensional degenerate wave equation with degenerate damping. The equation includes a drift term and a leading operator in non-divergence form. The same authors in [13] investigate the stability of a transmission problem that involves a degenerate wave equation and a heat equation. The problem is analyzed under the Coleman–Gurtin heat conduction law, or Gurtin–Pipkin law, which includes a memory effect.

Lately, there has been a significant focus on addressing challenges related to controllability in parabolic and hyperbolic situations, which include both degenerate and singular terms. In physics, biology, and mathematical finance, degenerate parabolic equations with singular terms are often used to describe a wide range of problems. For the physics and biology problems, several notable studies have been conducted. For instance, in [14], the authors investigate the boundary controllability of a system comprising two coupled degenerate and singular parabolic equations, with control applied to only one of the equations. In [15], a model representing the interaction between two species, u and v, is examined. The authors establish Carleman estimates and observability inequalities for the associated non-homogeneous adjoint problem, using the Carleman estimates provided in [16] for a single equation. A parabolic problem with degeneracy occurring within the interior of the spatial domain and subject to Neumann boundary conditions is analyzed in [17], where new observability inequalities are derived. Subsequently, [18] addresses the case in non-divergence form. In [19], the authors explore the null controllability of a single population model. Additionally, the work in [20] focuses on degenerate diffusion operators relevant to population biology. Finally, [21] examines the null controllability of the heat equation perturbed by a singular inverse-square potential, a topic pertinent to quantum mechanics and combustion theory. For mathematical finance problems refer to [22].

In quantum physics, a singular potential term is of utmost importance due to its ability to reach infinite values or display unconventional characteristics at specified points or areas in space. The singular potential term's sign in the wave equation can have a significant influence on the system's physical interpretation and behavior. A negative sign in the wave function causes it to diverge from the concentration of the potential, while a positive sign makes it an attracting potential. The controllability of systems with regular degenerate coefficients is investigated in the studies [23–25]. Research has expanded its scope to encompass non-smooth degenerate coefficient systems [17,26] and has investigated issues of controllability in degenerate and singular coefficient systems [27, 28]. In addition, research has investigated the wave equation with Kelvin–Voigt damping that degenerates near the interface. This study has been examined in the 1-dimensional case in [29, 30] and in the multidimensional situation in [31].

A recent study by Allal et al. in [32] addresses the issue of null controllability in wave equations characterized by both degeneracy and singularity, with a specific emphasis on cases involving pure powers. Besides, Fragnelli et al. [33] introduce results for a degenerate hyperbolic equation in non-divergence form with drift, where both the degeneracy and singularity are characterized by more general functions. Specifically, they establish a controllability result considering a boundary control acting on the non-degenerate point. Also, they studied in another work [34] the stability of the same system with boundary damping. Subsequently, Akil et al. [35] investigated the stability of this same system with localized singular damping. The purpose of this study is to fill the research gap regarding the stabilization of degenerate wave equations that have both drift and singular potential terms, as well as boundary fractional derivative feedback damping. This forms the primary novelty of this paper, as we delve into the degenerate wave equation with a drift and a singular potential term, alongside boundary fractional derivative damping. The inclusion of such damping is crucial from both theoretical and practical perspectives, describing memory and hereditary properties in various materials [36]. For instance, in viscoelasticity, materials like soils, concrete, rubber, biological tissue, and polymers exhibit elastic solid and viscous fluid-like responses [37–39]. In our context, fractional dissipation may represent an active boundary damping strategy designed to stabilize the system. Various systems incorporating control mechanisms based on fractional derivatives have been explored in studies such as [40–43].

Thus, the primary novelty of this work lies in investigating the degenerate wave equation with both a drift term and a singular potential term, coupled with boundary fractional derivative feedback damping. In summary, this introduction has outlined the problem statement, reviewed relevant literature, discussed the physical implications of the study, and highlighted its novelty. The subsequent sections will delve into the methodology, results, and conclusions of our research.

This paper is structured as follows: In Section 2, we present preliminary results and reformulate the system (1.1) into an augmented system by coupling the degenerate wave equation with a suitable diffusion equation. The well-posedness of our problem is demonstrated through semigroup theory. In Section 3, we establish the strong stability of the system. Finally, we derive a polynomial energy decay rate whose order depends on the fractional derivative's order, following the Borichev–Tomilov result and multiplier techniques.

2. Preliminary results

In this section, we will introduce several Hilbert spaces essential for our analysis, and we will reformulate the system into an augmented model. Using the semi-group approach, we will establish the well-posedness of the system.

2.1. Functional spaces

We start by setting the function σ as

$$\sigma(x) := \frac{p(x)}{\eta(x)}, \tag{2.1}$$

which is a continuous function in $[0,1]$, independent of the possible degeneracy of p. Moreover, observe that if v is a sufficiently smooth function, e.g., $v \in W^{2,1}_{loc}(0,1)$, then we can write $\Lambda v := pv_{xx}+rv_x$ as

$$\Lambda v = \sigma(\eta v_x)_x.$$

Using the definition of σ, the system (1.1) can be rewritten as

$$\begin{cases} v_{tt}(x,t) - \sigma(\eta v_x)_x(x,t) - \dfrac{\beta}{q(x)} v(x,t) = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ v(0,t) = 0, (\eta v_x)(1,t) = -\partial_t^{\alpha,\tau} v(1,t), & t \in (0,+\infty), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in (0,1). \end{cases} \tag{2.2}$$

We introduce the following weighted spaces:

$$\begin{cases} L^2_{\frac{1}{\sigma}}(0,1) := \left\{ v \in L^2(0,1); \|v\|_{\frac{1}{\sigma}} < +\infty \right\}, \quad \langle v, z\rangle_{\frac{1}{\sigma}} := \displaystyle\int_0^1 \frac{1}{\sigma} v\bar{z}dx, \quad \text{for every} \quad v,z \in L^2_{\frac{1}{\sigma}}(0,1), \\ H^1_{\frac{1}{\sigma}}(0,1) := L^2_{\frac{1}{\sigma}}(0,1) \cap H^1(0,1), \quad \langle v,z\rangle_{1,\frac{1}{\sigma}} := \langle v,z\rangle_{\frac{1}{\sigma}} + \displaystyle\int_0^1 \eta v_x \bar{z}_x dx, \quad \text{for every} \quad v,z \in H^1_{\frac{1}{\sigma}}(0,1), \\ H^2_{\frac{1}{\sigma}}(0,1) := \left\{ v \in H^1_{\frac{1}{\sigma}}(0,1); \Lambda v \in L^2_{\frac{1}{\sigma}}(0,1) \right\}, \quad \langle v,z\rangle_2 := \langle v,z\rangle_{1,\frac{1}{\sigma}} + \langle \Lambda v, \Lambda z\rangle_{\frac{1}{\sigma}}. \end{cases}$$

The previous inner products induce the related respective norms given by

$$\|v\|^2_{\frac{1}{\sigma}} = \int_0^1 \frac{1}{\sigma}|v|^2 dx, \quad \|v\|^2_{1,\frac{1}{\sigma}} = \|v\|^2_{\frac{1}{\sigma}} + \int_0^1 \eta |v_x|^2 dx \quad \text{and} \quad \|u\|_2^2 = \|v\|^2_{1,\frac{1}{\sigma}} + \int_0^1 \sigma |(\eta v_x)_x|^2 dx.$$

Also, we define the following Hilbert spaces:

$$H^1_{\frac{1}{\sigma},L}(0,1) = \left\{ v \in H^1_{\frac{1}{\sigma}}(0,1); v(0) = 0 \right\}, \quad \text{and} \quad H^2_{\frac{1}{\sigma},L}(0,1) := \left\{ v \in H^1_{\frac{1}{\sigma},L}(0,1); \Lambda v \in L^2_{\frac{1}{\sigma}}(0,1) \right\}$$

endowed with the previous inner products and the previous norms. In the following, we will denote by $\|\cdot\|$ the usual norm in $L^2(0,1)$, i.e., $\|\cdot\| := \|\cdot\|_{L^2(0,1)}$.

Proposition 2.1. *(see Proposition 2.2. in [35]) Assume that $\frac{r}{p} \in L^2(0,1)$. Then, there exists $\bar{C}'$ such that for all $v \in H^1_{\frac{1}{\sigma},L}(0,1)$*

$$\int_0^1 |v|^2 \frac{1}{\sigma(x)} dx \leq \bar{C}' \int_0^1 \eta |v_x|^2 dx. \tag{2.3}$$

Also, if $K_p + K_q \leq 2$ and $v \in H^1_{\frac{1}{\sigma},L}(0,1)$, then $\frac{v}{\sqrt{q}} \in L^2_{\frac{1}{\sigma}}(0,1)$, and there exists a constant $\bar{C} > 0$ such that

$$\int_0^1 \frac{1}{\sigma(x)q(x)} |v|^2 dx \leq \bar{C} \int_0^1 \eta |v_x|^2 dx, \tag{2.4}$$

where $\bar{C}' = \max_{x\in[0,1]} \frac{1}{\eta} \left(\frac{4 \max_{x\in[0,1]} \eta(x)}{p(1)} \right)$ *and* $\bar{C} = \frac{1}{q(1)} \bar{C}'$.

Let C'_H and C_H be the best constants in (2.3) and (2.4), respectively. From (2.3), it is evident that $||v||_1 := \int_0^1 \eta |v_x|^2 dx$ and $||v||_{1,\frac{1}{\sigma}}$ are equivalent in the functional space $H^1_{\frac{1}{\sigma},L}(0,1)$. Now, we will introduce some assumptions that are essential to defining an equivalent norm on $H^1_{\frac{1}{\sigma},L}(0,1)$ and subsequently presenting the energy.

Hypothesis 1.

a) *The constant $\beta \in \mathbb{R}$ satisfies the condition $\beta < \frac{1}{C_H}$.*

b) *The assumption $\frac{r}{p} \in L^1(0,1)$ is satisfied. Also, p is (WD) or (SD), q is (WD), and such that*

$$K_p + 2K_q \leq 2.$$

We consider in $H^1_{\frac{1}{\sigma},L}(0,1)$ the following inner product:

$$\langle v, z\rangle_* := \int_0^1 \eta v_x \bar{z}_x dx - \beta \int_0^1 \frac{v\bar{z}}{\sigma q} dx \tag{2.5}$$

which induces the following norm:

$$||v||_*^2 := \int_0^1 \eta |v_x|^2 dx - \beta \int_0^1 \frac{|v|^2}{\sigma q} dx$$

and by using Proposition 2.1 (for details, one can see [35]), we have that the next equivalence holds.

Corollary 2.1. *Under Hypothesis 1, the two norms $|| \cdot ||_1$ and $|| \cdot ||_*$ are equivalent in $H^1_{\frac{1}{\sigma},L}(0,1)$.*

2.2. Augmented model and well-posedness

In this part, we are concerned with studying the well-posedness of (1.1) by using a semigroup approach. But first, we aim to reformulate the system (2.2) into an augmented model by coupling the degenerate wave equation with a suitable diffusion equation. First, we recall the following theorem presented in [40, 44].

Theorem 2.1. *Let μ be the function defined as*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}, \quad \xi \in \mathbb{R}, \alpha \in (0,1).$$

The relation between the 'input' $\mathcal{V}$ and the 'output' O of the following system

$$\begin{aligned} \partial_t\varphi(\xi,t) + (\xi^2+\tau)\varphi(\xi,t) - \mathcal{V}(t)\mu(\xi) &= 0, \quad (\xi,t) \in \mathbb{R}\times(0,\infty), \tau \geq 0, && (2.6)\\ \varphi(\xi,0) &= 0, \quad \xi \in \mathbb{R}, && (2.7)\\ O(t) - \gamma\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}}\varphi(\xi,t)d\xi &= 0, \quad t \in (0,\infty), && (2.8) \end{aligned}$$

is given by

$$O = I^{1-\alpha,\tau}\mathcal{V}, \tag{2.9}$$

where

$$[I^{\alpha,\tau}\mathcal{V}](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\theta)^{\alpha-1}e^{-\tau(t-\theta)}\mathcal{V}(\theta)d\theta \quad and \quad \gamma = \pi^{-1}\sin(\alpha\pi).$$

Lemma 2.1. *(see Lemma 2.1 in [41]) Let $\alpha \in (0,1)$, $\tau \geq 0$, then the following integrals*

$$\begin{aligned} &\mathtt{N}_0(\tau,\alpha) = \kappa(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1+\xi^2+\tau}d\xi \quad , \quad \mathtt{N}_1(\tau,\alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(1+\xi^2+\tau)^2}d\xi \\ &and \quad \mathtt{N}_2(\tau,\alpha) = \int_0^{+\infty} \frac{\xi^{2\alpha+1}}{(1+\xi^2+\tau)^2}d\xi \end{aligned} \tag{2.10}$$

are well defined.

Lemma 2.2. *(see Lemma 2.7 in [41]) Let $\alpha \in (0,1)$, $\tau \geq 0$, and $\lambda \in \mathbb{R}$, then*

$$\begin{aligned} \mathtt{N}_3(\lambda,\tau,\alpha) &= \int_{\mathbb{R}} \frac{|\xi|^{\alpha+\frac{1}{2}}}{\left(|\lambda|+\xi^2+\tau\right)^2}d\xi = c_1\left(|\lambda|+\tau\right)^{\frac{\alpha}{2}-\frac{5}{4}}, \\ \mathtt{N}_4(\lambda,\tau) &= \left(\int_{\mathbb{R}} \frac{1}{(|\lambda|+\xi^2+\tau)^2}d\xi\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}\frac{1}{(|\lambda|+\tau)^{\frac{3}{4}}}, \end{aligned}$$

and

$$\mathtt{N}_5(\lambda,\tau) = \left(\int_{\mathbb{R}} \frac{\xi^2}{\left(|\lambda|+\xi^2+\tau\right)^4}d\xi\right)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4}\frac{1}{(|\lambda|+\tau)^{\frac{5}{4}}}$$

where $c_1 = \int_1^{\infty} \frac{(y-1)^{\frac{\alpha}{2}-\frac{1}{4}}}{y^2}dy.$

Now, we can reformulate the system (2.2). Utilizing theorem 2.1 and considering the input $\mathcal{V}(t) = v_t(1,t)$, we can express the output O using Eq (1.2) as:

$$O(t) = I^{1-\alpha,\tau}v_t(1,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\theta)^{-\alpha}e^{-\tau(t-\theta)}\partial_\theta v(1,\theta)d\theta = \partial_t^{\alpha,\tau}v(1,t).$$

Therefore, by taking the input $\mathcal{V}(t) = v_t(1,t)$ and using the above equation, the system (2.2) can be reformulated into the following augmented model:

$$\begin{cases} v_{tt}(x,t) - \sigma(\eta v_x)_x(x,t) - \dfrac{\beta}{q(x)} v(x,t) = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \partial_t \varphi(\xi,t) + (|\xi|^2+\tau)\varphi(\xi,t) - v_t(1,t)\mu(\xi) = 0, & (\xi,t) \in \mathbb{R}\times(0,\infty), \\ (\eta v_x)(1,t) = -\gamma \displaystyle\int_{\mathbb{R}} \mu(\xi)\varphi(\xi,t)d\xi, & (\xi,t) \in \mathbb{R}\times(0,\infty), \\ v(0,t) = 0, & t \in (0,+\infty), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in (0,1). \end{cases} \tag{S}$$

Let (v, v_t, φ) be a regular solution of (S). The energy associated with the system is given by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 \left(\frac{1}{\sigma}|v_t(x,t)|^2 + \eta |v_x(x,t)|^2 - \frac{\beta}{\sigma q}|v(x,t)|^2 \right) dx + \frac{\gamma}{2}\int_{\mathbb{R}} |\varphi(\xi,t)|^2 d\xi. \tag{2.11}$$

We note that the energy of the system is positive by using Hypothesis 1 and Corollary 2.1.

Lemma 2.3. *Let $\Upsilon = (v, v_t, \varphi)$ be a regular solution of the system* (S). *Then, the energy $\mathcal{E}(t)$ satisfies the following estimation:*

$$\frac{d}{dt}\mathcal{E}(t) = -\gamma \int_{\mathbb{R}} (|\xi|^2+\tau)|\varphi(\xi,t)|^2 d\xi \leq 0.$$

Proof. Multiplying $(S)_1$ by $\frac{1}{\sigma}\overline{v}_t$, integrating over $(0,1)$, and using integration by parts lead to

$$\int_0^1 \frac{1}{\sigma} v_{tt}(x,t)\overline{v}_t(x,t)dx - \int_0^1 (\eta v_x)_x(x,t)\overline{v}_t(x,t)dx - \int_0^1 \frac{\beta}{\sigma q} v(x,t)\overline{v}_t(x,t)dx = 0.$$

Then,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{1}{\sigma}|v_t(x,t)|^2 dx + \frac{1}{2}\frac{d}{dt}\int \eta|v_x(x,t)|^2 dx - \Re\left([\eta v_x(x,t)\overline{v}_t(x,t)]_0^1 \right) - \frac{1}{2}\frac{d}{dt}\int_0^1 \frac{\beta}{\sigma q}|v(x,t)|^2 dx = 0.$$

Hence, using the boundary conditions in the above equation yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{1}{\sigma}|v_t(x,t)|^2 dx + \frac{1}{2}\frac{d}{dt}\int \eta|v_x(x,t)|^2 dx + \Re\left(\gamma \overline{v}_t(1,t) \int_{\mathbb{R}} \mu(\xi)\overline{\varphi}(\xi,t)d\xi \right) \\ &-\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{\beta}{\sigma q}|v(x,t)|^2 dx = 0. \end{aligned} \tag{2.12}$$

Multiplying $(S)_2$ by $\gamma\overline{\varphi}$ and integrating over $\mathbb{R}$ leads to

$$\frac{\gamma}{2}\frac{d}{dt}\int_{\mathbb{R}} |\varphi(\xi,t)|^2 d\xi + \gamma\int_{\mathbb{R}} (\xi^2+\tau)|\varphi(\xi,t)|^2 d\xi - \Re\left(\gamma v_t(1,t)\int_{\mathbb{R}} \mu(\xi)\overline{\varphi}(\xi,t)d\xi \right) = 0. \tag{2.13}$$

Combining (2.12) and (2.13) and using (2.11), we obtain

$$\frac{d}{dt}\mathcal{E}(t) = -\gamma \int_{\mathbb{R}} (|\xi|^2+\tau)|\varphi(\xi,t)|^2 d\xi \leq 0.$$

Thus, the proof is complete. □

Therefore, we can conclude that the system (S) is dissipative in the sense that its energy is a non increasing function with respect to the time variable t.

In the following, we will demonstrate that under appropriate conditions, Eq (2.2) has solutions that are regular, ensuring the well-definedness of the associated energy. First, we introduce the energy Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = H^1_{\frac{1}{\sigma},L}(0,1) \times L^2_{\frac{1}{\sigma}}(0,1) \times L^2(\mathbb{R}).$$

For $\Upsilon_1 = (v_1, z_1, \varphi_1)^\top$ and $\Upsilon_2 = (v_2, z_2, \varphi_2)^\top$ we define the following inner product in $\mathcal{H}$

$$\langle \Upsilon_1, \Upsilon_2 \rangle_{\mathcal{H}} = \int_0^1 \left(\frac{1}{\sigma} z_1 \overline{z_2} + \eta (v_1)_x \overline{(v_2)_x} - \frac{\beta}{\sigma q} v_1 \overline{v_2} \right) dx + \gamma \int_{\mathbb{R}} \varphi_1 \overline{\varphi_2} d\xi,$$

and endowed with the associated norm $\|\Upsilon\|^2_{\mathcal{H}} = \int_0^1 \left(\frac{1}{\sigma}|z|^2 + \eta|v_x|^2 - \frac{\beta}{\sigma q}|v|^2 \right) dx + \gamma \int_{\mathbb{R}} |\varphi(\xi)|^2 d\xi$. Moreover, consider the unbounded linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Upsilon = \begin{pmatrix} z \\ \sigma(\eta v_x)_x + \dfrac{\beta}{q(x)} v \\ -(\xi^2 + \tau)\varphi(\xi) + \mu(\xi) z(1) \end{pmatrix}$$

for all $\Upsilon = (v, z, \varphi)^\top \in D(\mathcal{A})$, where

$$D(\mathcal{A}) = \left\{ \begin{array}{l} \Upsilon = (v, z, \varphi) \in \mathcal{H};\ v \in H^2_{\frac{1}{\sigma},L}(0,1), z \in H^1_{\frac{1}{\sigma},L}(0,1), |\xi|\varphi \in L^2(\mathbb{R}), \\ -(\xi^2 + \tau)\varphi(\xi) + \mu(\xi) z(1) \in L^2(\mathbb{R}),\ (\eta v_x)(1) = -\gamma \displaystyle\int_{\mathbb{R}} \mu(\xi)\varphi(\xi) d\xi \end{array} \right\}.$$

In $D(\mathcal{A})$, we require $v \in H^2_{\frac{1}{\sigma},L}(0,1)$, which is ensured by $v \in H^1_{\frac{1}{\sigma},L}(0,1)$ giving $\frac{v}{q} \in L^2_{\frac{1}{\sigma}}(0,1)$, and under the condition $K_p + 2K_q \leq 2$ is stated in Hypothesis 1. In fact, using the definition of σ, we have

$$\begin{aligned} \int_0^1 \frac{|v|^2}{\sigma q^2} dx &= \int_0^1 \eta \frac{|v|^2}{p q^2} dx \leq \max_{x\in[0,1]} \eta \int_0^1 \frac{|v|^2}{p q^2} dx \leq \frac{\max_{x\in[0,1]} \eta}{p(1) q^2(1)} \int_0^1 \frac{|v|^2}{x^{K_p + 2K_q}} dx \\ &\leq \frac{\max_{x\in[0,1]} \eta}{p(1) q^2(1)} \int_0^1 \frac{|v|^2}{x^2} dx \leq \frac{4 \max_{x\in[0,1]} \eta(x)}{p(1) q^2(1)} \int_0^1 |v_x|^2 dx. \end{aligned}$$

Thus, we can rewrite (S) as the following evolution equation:

$$\Upsilon_t = \mathcal{A}\Upsilon, \quad \Upsilon(0) = \Upsilon_0, \text{ where } \quad \Upsilon_0 = (v_0, v_1, 0)^\top. \tag{2.14}$$

Proposition 2.2. *The unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof. For all $\Upsilon = (v, z, \varphi)^\top \in D(\mathcal{A})$, we have

$$\Re\left(\langle \mathcal{A}\Upsilon, \Upsilon \rangle_{\mathcal{H}} \right) = -\gamma \int_{\mathbb{R}} (|\xi|^2 + \tau)|\varphi(\xi, t)|^2 d\xi \leq 0, \tag{2.15}$$

which implies that $\mathcal{A}$ is dissipative. Now, let $\mathtt{F} = (f_1, f_2, f_3)^\top \in \mathcal{H}$. We need to prove the existence of $\Upsilon = (v, z, \varphi)^\top \in D(\mathcal{A})$ unique solution of the equation

$$(I - \mathcal{A})\Upsilon = \mathtt{F}. \tag{2.16}$$

Equation (2.16) is equivalent to the following

$$\begin{cases} v - z = f_1, \\ z - \sigma(\eta v_x)_x - \dfrac{\beta}{q} v = f_2, \\ \varphi + (\xi^2 + \tau)\varphi - \mu(\xi) z(1) = f_3. \end{cases} \tag{2.17}$$

From $(2.17)_3$, we have

$$\varphi = \frac{f_3(\xi)}{\xi^2 + \tau + 1} + \frac{\mu(\xi) z(1)}{\xi^2 + \tau + 1}. \tag{2.18}$$

Combining $(2.17)_1$ and $(2.17)_2$, we obtain

$$v - \sigma(\eta v_x)_x - \frac{\beta}{q} v = f_2 + f_1. \tag{2.19}$$

Let $\psi \in H^1_{\frac{1}{\sigma},L}(0,1)$. Multiplying (2.19) by $\dfrac{1}{\sigma}\overline{\psi}$ and integrating over $(0,1)$, we obtain

$$\begin{aligned} &\int_0^1 \frac{1}{\sigma} v\overline{\psi} dx + \int_0^1 \eta v_x \overline{\psi}_x dx - \int_0^1 \frac{\beta}{\sigma q} v\overline{\psi} dx + \gamma \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \tau + 1} d\xi z(1)\overline{\psi}(1) \\ &= \int_0^1 \frac{1}{\sigma}(f_1 + f_2)\overline{\psi} dx - \gamma \int_{\mathbb{R}} \frac{\mu(\xi)}{\xi^2 + \tau + 1} f_3(\xi) d\xi \overline{\psi}(1). \end{aligned} \tag{2.20}$$

From $(2.17)_1$, we have

$$z(1) = v(1) - f_1(1).$$

Inserting the above equation in (2.20), we obtain

$$\begin{aligned} &\int_0^1 \frac{1}{\sigma} v\overline{\psi} dx + \int_0^1 \eta v_x \overline{\psi}_x dx - \int_0^1 \frac{\beta}{\sigma q} v\overline{\psi} dx + \gamma \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \tau + 1} d\xi v(1)\overline{\psi}(1) \\ &= \int_0^1 \frac{1}{\sigma}(f_1 + f_2)\overline{\psi} dx - \gamma \int_{\mathbb{R}} \frac{\mu(\xi)}{\xi^2 + \tau + 1} f_3(\xi) d\xi \overline{\psi}(1) + \gamma \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \tau + 1} d\xi f_1(1)\overline{\psi}(1). \end{aligned}$$

The problem above has the following form:

$$\mathcal{S}(v, \psi) = L(\psi), \quad \forall\, \psi \in H^1_{\frac{1}{\sigma},L}(0,1), \tag{2.21}$$

where $\mathcal{S}(v, \psi) : H^1_{\frac{1}{\sigma},L}(0,1) \times H^1_{\frac{1}{\sigma},L}(0,1) \to \mathbb{R}$ is defined by

$$\mathcal{S}(v, \psi) = \int_0^1 \frac{1}{\sigma} v\overline{\psi} dx + \int_0^1 \eta v_x \overline{\psi}_x dx - \int_0^1 \frac{\beta}{\sigma q} v\overline{\psi} dx + \gamma \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2 + \tau + 1} d\xi v(1)\overline{\psi}(1)$$

and $L(\psi): H^1_{\frac{1}{\sigma},L}(0,1) \to \mathbb{R}$ is defined by

$$L(\psi) = \int_0^1 \frac{1}{\sigma}(f_1 + f_2)\overline{\psi}dx - \gamma \int_{\mathbb{R}} \frac{\mu(\xi)}{\xi^2+\tau+1} f_3(\xi)d\xi\overline{\psi}(1) + \gamma \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\xi^2+\tau+1} d\xi f_1(1)\overline{\psi}(1).$$

It is easy to see that $\mathcal{S}$ is a sesquilinear, continuous, and coercive form and that L is a continuous form on $H^1_{\frac{1}{\sigma},L}(0,1)$. Then, using the Lax–Milgram Theorem, we deduce that there exists $v \in H^1_{\frac{1}{\sigma},L}(0,1)$ unique solution of the variational problem (2.21). Now, taking $z := v - f_1$, we obtain $z \in H^1_{\frac{1}{\sigma},L}(0,1)$. It remains to prove that $\Upsilon \in D(\mathcal{A})$ and solve (2.16). To this end, we have that Eq (2.21) holds for every $\psi \in C_c^\infty(0,1)$, thus

$$-(\eta v_x)_x - \frac{\beta}{\sigma q}v = \frac{1}{\sigma}(f_2 - z) \quad \text{a.e. in} \quad (0,1).$$

Hence, $-\sigma(\eta v_x)_x \in L^2_{\frac{1}{\sigma}}(0,1)$, i.e., $\Lambda u \in L^2_{\frac{1}{\sigma}}(0,1)$. Now, we define

$$\varphi = \frac{f_3(\xi)}{\xi^2+\tau+1} + \frac{\mu(\xi)z(1)}{\xi^2+\tau+1}. \tag{2.22}$$

We need to prove that $\varphi, |\xi|\varphi \in L^2(\mathbb{R})$. From (2.18), using Lemma 2.1 and the fact that $f_3 \in L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |\varphi|^2 dx \leq 2\int_{\mathbb{R}} \frac{|f_3(\xi)|^2}{(\xi^2+\tau+1)^2}d\xi + 2\int_{\mathbb{R}} \frac{\mu^2(\xi)}{(\xi^2+\tau+1)^2}d\xi|z(1)|^2 \leq 2\int_{\mathbb{R}} \frac{|f_3(\xi)|^2}{(\tau+1)^2}d\xi + 2\mathtt{N}_1|z(1)|^2 < \infty$$

and

$$\begin{aligned}\int_{\mathbb{R}} |\xi\varphi|^2 dx &\leq 2\int_{\mathbb{R}} \frac{\xi^2|f_3(\xi)|^2}{(\xi^2+\tau+1)^2}d\xi + 2\int_{\mathbb{R}} \frac{\xi^{2\alpha+1}}{(\xi^2+\tau+1)^2}d\xi|z(1)|^2 \\ &\leq 2\max_{\xi\in\mathbb{R}} \frac{\xi^2}{(\xi^2+\tau+1)^2}\int_{\mathbb{R}} |f_3(\xi)|^2 d\xi + \mathtt{N}_2|z(1)|^2 \\ &\leq \frac{1}{2(\tau+1)}\int_{\mathbb{R}} |f_3(\xi)|^2 d\xi + \mathtt{N}_2|z(1)|^2 < \infty.\end{aligned} \tag{2.23}$$

Based on this, it can be deduced that $|\xi|\varphi \in L^2(\mathbb{R})$. From (2.22) we have $-(\xi^2+\tau)\varphi(\xi) + \mu(\xi)z(1) = \varphi(\xi) - f_3(\xi) \in L^2(\mathbb{R})$ and we have that $(2.17)_3$ holds. Finally, taking into account that (2.21) holds for every $\psi \in H^1_{\frac{1}{\sigma},L}(0,1)$ and integrating by parts, we obtain that

$$\int_0^1 \frac{1}{\sigma}(v - \sigma(\eta v_x)_x - \frac{\beta}{q}v)\overline{\psi}dx + (\eta v)x(1)\overline{\psi(1)} + \gamma\int_{\mathbb{R}} \mu(\xi)\varphi(\xi)d\xi\overline{\psi(1)} = \int_0^1 \frac{1}{\sigma}(f_2 + f_1)\overline{\psi}dx$$

for all $\psi \in H^1_{\frac{1}{\sigma},L}(0,1)$. Thus, using the fact that (2.19) is valid, from the above equation we can deduce that

$$(\eta v)x(1) = -\gamma\int_{\mathbb{R}} \mu(\xi)\varphi(\xi)d\xi.$$

Thus, we finally reach the conclusion that $\Upsilon \in D(\mathcal{A})$, and consequently, $(v, z, \varphi) \in D(\mathcal{A})$ is the unique solution of (2.16). Then, using the results in [45] (Theorems 4.5 and 4.6), we deduce that $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for all $\lambda > 0$. Thus, $\mathcal{A}$ is m-dissipative, and the proof is complete. □

According to the Lumer–Phillips Theorem (see [45]), Proposition 2.2 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $(\mathcal{T}(t))_{t\geq 0} = (e^{t\mathcal{A}})_{t\geq 0}$ in $\mathcal{H}$, which gives the well-posedness of (2.14). Then, we have the following result:

Theorem 2.2. *For any $\Upsilon_0 \in \mathcal{H}$, problem* (2.14) *admits a unique weak solution satisfying*

$$\Upsilon(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

Moreover, if $\Upsilon_0 \in D(\mathcal{A})$, (2.14) *admits a unique strong solution Υ satisfying*

$$\Upsilon(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

Lemma 2.4. *(See [46, Lemma 2.2] and [12, Lemma 2.5]) Assume $\frac{r}{p} \in L^1(0,1)$. Then, we have:*

1) If p is (WD) or (SD), then $\lim_{x\to 0} z(x)v_x(x) = 0$, for all $v \in H^2_{\frac{1}{\sigma},L}(0,1)$ and for all $z \in H^1_{\frac{1}{\sigma},L}(0,1)$.

2) If p is (WD) or (SD), then $xv_x(\eta v_x)_x \in L^1(0,1)$, for all $v \in H^2_{\frac{1}{\sigma},L}(0,1)$.

3) If $K_p \leq 1$, then $\lim_{x\to 0} x|v_x|^2 = 0$, for all $v \in H^2_{\frac{1}{\sigma},L}(0,1)$.

4) If $K_p > 1$ and $\dfrac{xr}{p} \in L^\infty(0,1)$, then $\lim_{x\to 0} x|v_x|^2 = 0$, for all $v \in H^2_{\frac{1}{\sigma},L}(0,1)$.

5) If p is (WD) or (SD), then $\lim_{x\to 0} \dfrac{x}{p}|v(x)|^2 = 0$, for all $v \in H^1_{\frac{1}{\sigma},L}(0,1)$.

6) Assume b) in Hypothesis 1. If $v \in H^1_{\frac{1}{\sigma}}(0,1)$, then $\lim_{x\to 0} \dfrac{x}{\sigma q}|v(x)|^2 = 0$.

3. Strong stability

This section is devoted to studying the strong stability of system (S), whether p is weakly or strongly degenerate. We will present this result under some conditions related to p, q, and r. For this reason and in order to avoid any inconvenience, we start by denoting the following:

$$\mathcal{P}_0 := \left\| x\frac{p'-r}{p} \right\|_{L^\infty(0,1)}, \quad Q_0 := \left\| x\frac{q'}{q} \right\|_{L^\infty(0,1)} \quad \text{and} \quad \mathcal{R}_0 := \left\| x\frac{r}{p} \right\|_{L^\infty(0,1)}. \tag{3.1}$$

Hypothesis 2. *Assume Hypothesis 1 holds. Also, assume that*

- *If $\beta > 0$: $\mathcal{P}_0 + Q_0 < 1 + \dfrac{K_p}{2}$ and $\mathcal{R}_0 < 1 - \dfrac{K_p}{2}$*
- *If $\beta < 0$: $\mathcal{P}_0 < 1 + \dfrac{K_p}{2}$ and $\mathcal{R}_0 < 1 - \dfrac{K_p}{2} + 4\beta C_H$.*

Example 3.1. *To demonstrate that the conditions outlined in Hypothesis 2 hold for a specific choice of the functions p, q, and r, we provide examples for each case.*

- *If $\beta > 0$:*

  - *If p is weakly degenerate, consider the following example: Let $p(x) = x^{\frac{1}{2}}$, $r(x) = \frac{1}{4}$, and $q(x) = x^{\frac{1}{4}}$. It is evident that $K_p = \frac{1}{2}$ and $K_q = \frac{1}{4}$, hence $K_p + 2K_q \leq 2$. In this case, we find $\mathcal{P}_0 = \frac{1}{2}$, $\mathcal{Q}_0 = \frac{1}{4}$, and $\mathcal{R}_0 = \frac{1}{4}$. A straightforward verification confirms that the conditions outlined in Hypothesis 2 hold true.*
  - *If p is strongly degenerate, consider the following example: Let $p(x) = x^{\frac{3}{2}}$, $r(x) = \frac{1}{16}x$, and $q(x) = x^{\frac{1}{8}}$. It is evident that $K_p + 2K_q \leq 2$. In this case, we find $\mathcal{P}_0 = \frac{3}{2}$, $\mathcal{Q}_0 = \frac{1}{8}$, and $\mathcal{R}_0 = \frac{1}{16}$. A straightforward verification confirms that the conditions outlined in Hypothesis 2 hold true.*
- *If $\beta < 0$:*
  - *If p is weakly degenerate, consider this example: Let $p(x) = x^{\frac{1}{2}}$, $r(x) = \frac{1}{4}$, and $q(x)$ be any functions such that $K_p + 2K_q \leq 2$, with $\beta = \frac{-1}{16C_H}$. Here, we find $\mathcal{P}_0 = \frac{1}{2}$ and $\mathcal{R}_0 = \frac{1}{4}$. Notably, $\mathcal{P}_0 < \frac{5}{4}$ and $\mathcal{R}_0 < \frac{1}{2}$. Thus, the conditions outlined in Hypothesis 2 are satisfied.*
  - *If p is strongly degenerate, consider the following example: Let $p(x) = x^{\frac{3}{2}}$, $r(x) = \frac{1}{8}x$, $\beta = \frac{-1}{36C_H}$ and $q(x)$ be chosen to be any function such that $K_p + 2K_q \leq 2$. In this case, we find $\mathcal{P}_0 = \frac{3}{2}$ and $\mathcal{R}_0 = \frac{1}{8}$. Notably, $\mathcal{P}_0 < \frac{7}{4}$ and $\mathcal{R}_0 < \frac{5}{36}$. Thus, the conditions outlined in Hypothesis 2 are met.*

Theorem 3.1. *The C_0- semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable in $\mathcal{H}$, i.e., for all $\Upsilon_0 \in \mathcal{H}$, the solution of* (2.14) *satisfies the following:*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}\Upsilon_0\|_{\mathcal{H}} = 0.$$

To prove Theorem 3.1, we need to satisfy two conditions based on the Arendt–Batty Theorem [47]:

- The operator $\mathcal{A}$ has no pure imaginary eigenvalues.
- The set $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable.

Lemma 3.1. *If* $\tau = 0$, *then the operator $\mathcal{A}$ is not invertible, and consequently,* $0 \in \sigma(\mathcal{A})$.

Proof. Let $\mathbb{F}_0 = \left(\sin(\frac{\pi x}{2}),\, 0\,, 0\right) \in \mathcal{H}$, in fact $\sin(\frac{\pi x}{2}) \in H^1_{\frac{1}{\sigma}}(0,1)$, since from the equivalence between norms $\|\cdot\|_1$ and $\|\cdot\|_{H^1_{\frac{1}{\sigma}}(0,1)}$, and the fact that η is bounded from above and below, then

$$\left\|\sin(\frac{\pi x}{2})\right\|_{H^1_{\frac{1}{\sigma}}(0,1)} \leq \frac{\pi^2}{4}\int_0^1 \eta(x)\cos^2(\frac{\pi x}{2})dx \leq \max_{x\in(0,1)} \eta(x)\frac{\pi^2}{8}$$

and thus $\mathbb{F}_0 \in \mathcal{H}$. Assume there exists $\mathcal{U}_0 = (z_0, v_0, \varphi_0) \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\mathcal{U}_0 = \mathbb{F}_0$. In this case, $\varphi_0(\xi) = |\xi|^{\frac{2\alpha-5}{2}}$. This implies that, $\varphi_0 \notin L^2(\mathbb{R})$ for $0 < \alpha < 1$. □

Corollary 3.1. *For the case when $\tau > 0$, we can prove that $-\mathcal{A}$ is a surjective operator. In fact, letting $F = (f_1, f_2, f_3) \in \mathcal{H}$, we need to prove that there exists a unique solution $\Upsilon = (v, z, \varphi)^\top \in D(\mathcal{A})$ of the equation*

$$-\mathcal{A}\Upsilon = F.$$

Equivalently,

$$\begin{cases} -z = f_1, \\ -\sigma(\eta v_x)_x - \dfrac{\beta}{q}v = f_2, \\ (\xi^2+\tau)\varphi - \mu(\xi)z(1) = f_3. \end{cases} \tag{3.2}$$

and by proceeding in a similar way as in the computations in Proposition 2.2, we reach that $\Upsilon \in D(\mathcal{A})$ *is a unique solution of* $-\mathcal{A}\Upsilon = F$*, and consequently,* $-\mathcal{A}$ *is surjective, and since* $\rho(\mathcal{A})$ *is an open set of* $\mathbb{C}$*, we can consequently deduce that* $0 \in \rho(\mathcal{A})$*.*

The proof of Theorem 3.1 will rely on the subsequent proposition. We note that $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Proposition 3.1. *We have*

$$\begin{cases} i\mathbb{R} \subseteq \rho(\mathcal{A}) & \text{if} \quad \tau \neq 0, \\ i\mathbb{R}^* \subseteq \rho(\mathcal{A}) & \text{if} \quad \tau = 0. \end{cases} \tag{3.3}$$

Proof. We will prove Proposition 3.1 through a contradiction argument. Note that, according to Lemma 3.1, for $\tau = 0$, it follows that $0 \in \sigma(\mathcal{A})$, hence $0 \notin \rho(\mathcal{A})$. Furthermore, as indicated in Corollary 3.1, we have $0 \in \rho(\mathcal{A})$. Thus, our focus narrows to demonstrating that $i\mathbb{R}^* \subseteq \rho(\mathcal{A})$. Now, suppose that $i\mathbb{R}^* \nsubseteq \rho(\mathcal{A})$, then there exists $\delta \in \mathbb{R}^*$ such that $i\delta \notin \rho(\mathcal{A})$. According to Remark A.3 in [48] and page 25 in [49] , there exists $\{\zeta_n, \Upsilon^n = (v^n, z^n, \varphi^n)^\top\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$, such that

$$\zeta_n \to \delta \quad \text{as} \quad n \to \infty \quad \text{and} \quad |\zeta_n| < |\delta|, \tag{3.4}$$

$$\|\Upsilon^n\|_{\mathcal{H}} = \|(v^n, z^n, \varphi^n)^\top\|_{\mathcal{H}} = 1, \tag{3.5}$$

and

$$(i\zeta_n I - \mathcal{A})\Upsilon^n = F_n := (f_n^1, f_n^2, f_n^3(\xi)) \to 0 \quad \text{in} \quad \mathcal{H}, \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Equivalently, we obtain

$$\begin{cases} i\zeta_n v^n - z^n = f_n^1, \\ i\zeta_n z^n - \sigma(\eta v_x^n)_x - \dfrac{\beta}{q} v^n = f_n^2, \\ (i\zeta_n + |\xi|^2 + \tau)\varphi^n(\xi) - \mu(\xi) z^n(1) = f_n^3(\xi), \quad \forall\, \xi \in \mathbb{R}. \end{cases} \tag{3.7}$$

Inserting $(3.7)_1$ into $(3.7)_2$, we obtain

$$\zeta_n^2 v^n + \sigma(\eta v_x^n)_x + \frac{\beta}{q} v^n = -(f_n^2 + i\zeta_n f_n^1). \tag{3.8}$$

For clarity, we break down the computational analysis into several steps.
Step 1. The aim of this step is to show that the solution (v^n, z^n, φ^n) of (3.7) satisfies the following:

$$\int_{\mathbb{R}} (\xi^2 + \tau)|\varphi^n(\xi)|^2 d\xi \xrightarrow[n\to\infty]{} 0, \quad |z^n(1)| \xrightarrow[n\to\infty]{} 0, \quad \text{and} \quad \int_{\mathbb{R}} |\varphi^n(\xi)|^2 d\xi \xrightarrow[n\to\infty]{} 0. \tag{3.9}$$

First, taking the inner product of (3.6) with Υ^n in $\mathcal{H}$ and using the fact that $\|F_n\|_{\mathcal{H}} \xrightarrow[n\to\infty]{} 0$ and $\|\Upsilon^n\|_{\mathcal{H}} = 1$, we obtain

$$\gamma \int_{\mathbb{R}} (|\xi|^2 + \tau)|\varphi^n(\xi)|^2 d\xi = -\Re(\langle \mathcal{A}\Upsilon^n, \Upsilon^n\rangle_{\mathcal{H}}) \leq \|F_n\|_{\mathcal{H}}\|\Upsilon^n\|_{\mathcal{H}} \xrightarrow[n\to\infty]{} 0.$$

In fact,

$$\Re\langle \mathcal{A}\Upsilon^n, \Upsilon^n\rangle_{\mathcal{H}} = \Re\left(\int_0^1 (\eta v_x^n)_x \overline{z^n} dx\right) + \Re\left(\int_0^1 \frac{\beta}{\sigma q} v^n \overline{z^n} dx\right) + \Re\left(\int_0^1 \eta z_x^n \overline{v_x^n} dx\right) - \Re\left(\int_0^1 \frac{\beta}{\sigma q} z^n \overline{v^n} dx\right)$$

$$
\begin{aligned}
&-\gamma\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2d\xi+\Re\left(\gamma z^n(1)\int_{\mathbb{R}}\mu(\xi)\overline{\varphi^n(\xi)}d\xi\right)\\
&=-\Re\left(\int_0^1\eta v_x^n\overline{z_x^n}dx\right)+\Re\left((\eta v_x^n)(1)\overline{z^n(1)}\right)+\Re\left(\int_0^1\frac{\beta}{\sigma q}v^n\overline{z^n}dx\right)+\Re\left(\int_0^1\eta z_x^n\overline{v_x^n}dx\right)\\
&-\Re\left(\int_0^1\frac{\beta}{\sigma q}z^n\overline{v^n}dx\right)-\gamma\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2d\xi+\Re\left(\gamma z^n(1)\int_{\mathbb{R}}\mu(\xi)\overline{\varphi^n(\xi)}d\xi\right)
\end{aligned}
$$

Using the fact that $(\eta v_x^n)(1)=-\gamma\int_{\mathbb{R}}\mu(\xi)\varphi^n(\xi)d\xi$ in the above equation, we obtain

$$
\Re\langle\mathcal{A}\Upsilon^n,\,\Upsilon^n\rangle_{\mathcal{H}}=-\Re\left(\gamma\overline{z^n(1)}\int_{\mathbb{R}}\mu(\xi)\varphi^n(\xi)d\xi\right)-\gamma\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2d\xi+\Re\left(\gamma z^n(1)\int_{\mathbb{R}}\mu(\xi)\overline{\varphi^n(\xi)}d\xi\right).
$$

Hence,

$$
\Re\langle\mathcal{A}\Upsilon^n,\,\Upsilon^n\rangle_{\mathcal{H}}=-\gamma\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2d\xi.
$$

Thus, taking the inner product of (3.6) with Υ^n in $\mathcal{H}$

$$
\gamma\int_{\mathbb{R}}(|\xi|^2+\tau)|\varphi^n(\xi)|^2d\xi=\Re(\langle(i\zeta_nI-\mathcal{A})\Upsilon^n,\Upsilon^n\rangle_{\mathcal{H}})=-\Re(\langle\mathcal{A}\Upsilon^n,\Upsilon^n\rangle_{\mathcal{H}})\leq\|F_n\|_{\mathcal{H}}\|\Upsilon^n\|_{\mathcal{H}}\xrightarrow[n\to\infty]{}0.
$$

Thus, we obtain the first limit in (3.9). From $(3.7)_3$, we have that

$$
\mu(\xi)|z^n(1)|\leq(|\zeta_n|+|\xi|^2+\tau)|\varphi^n(\xi)|+|f_n^3(\xi)|.
$$

Multiplying the above equation by $\frac{|\xi|}{(|\zeta_n|+|\xi|^2+\tau)^2}$ and integrating over $\mathbb{R}$ leads to

$$
\mathtt{N}_3(\zeta,\tau,\alpha)|z^n(1)|\leq\mathtt{N}_4(\zeta,\tau)\left(\int_{\mathbb{R}}|\xi\varphi^n(\xi)|^2d\xi\right)^{\frac{1}{2}}+\mathtt{N}_5(\zeta,\tau)\left(\int_{\mathbb{R}}|f_3^n(\xi)|^2d\xi\right)^{\frac{1}{2}},
$$

where $\mathtt{N}_3(\zeta,\tau,\alpha),\mathtt{N}_4(\zeta,\tau),$ and $\mathtt{N}_5(\zeta,\tau)$ are defined in Lemma 2.2.
Applying Young's inequality to the above equation leads to

$$
\begin{aligned}
|z^n(1)|^2&\leq2\frac{\mathtt{N}_4^2}{\mathtt{N}_3^2}\int_{\mathbb{R}}|\xi\varphi^n(\xi)|^2d\xi+2\frac{\mathtt{N}_5^2}{\mathtt{N}_3^2}\int_{\mathbb{R}}|f_3^n(\xi)|^2d\xi\\
&\leq\frac{1}{c_1(|\zeta_n|+\tau)^{\alpha-1}}\int_{\mathbb{R}}|\xi\varphi^n(\xi)|^2d\xi dx+\frac{\sqrt{\pi}}{4}\frac{1}{c_1\,(|\zeta_n|+\tau)^{\alpha}}\int_{\mathbb{R}}|f_3^n(\xi)|^2d\xi.
\end{aligned}
$$

By taking the limit in the preceding inequality and utilizing the first limit provided in (3.9), along with the fact that $\|F^n\|\xrightarrow[n\to\infty]{}0$ and that $\zeta_n\xrightarrow[n\to\infty]{}\delta$, we deduce the convergence of the second limit stated in (3.9).

Now, if $\tau\neq0$, then it is clear that from the first limit in (3.9), we have

$$
\int_{\mathbb{R}}|\varphi^n(\xi)|^2d\xi\xrightarrow[n\to\infty]{}0.
$$

For the case when $\tau = 0$, multiplying $(3.7)_3$ by $\frac{1}{i\zeta_n}\overline{\varphi}$ and integrating over $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}} |\varphi^n|^2 d\xi = -\frac{1}{i\zeta_n}\int_{\mathbb{R}} |\xi\varphi^n|^2 d\xi + \frac{1}{i\zeta_n}\int_{\mathbb{R}} \mu(\xi)\overline{\varphi^n} d\xi\, z^n(1) + \frac{1}{i\zeta_n}\int_{\mathbb{R}} f_n^3(\xi)\overline{\varphi^n} d\xi. \tag{3.10}$$

Applying Cauchy–Schwarz inequality, the first limit in (3.9), and using the fact that $\|F^n\| \xrightarrow[n\to\infty]{} 0$, $\zeta_n \xrightarrow[n\to\infty]{} \delta$ and φ^n is bounded in $L^2(\mathbb{R})$, we obtain

$$\frac{1}{|\zeta_n|}\int_{\mathbb{R}} |\xi\varphi^n(\xi)|^2 d\xi \xrightarrow[n\to\infty]{} 0, \tag{3.11}$$

$$\left|\frac{1}{i\zeta_n}\int_{\mathbb{R}} f_n^3(\xi)\overline{\varphi^n}(\xi) d\xi\right| \le \frac{1}{|\zeta_n|}\left(\int_{\mathbb{R}} |f_n^3(\xi)|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\varphi^n(\xi)|^2 d\xi\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0, \tag{3.12}$$

and

$$\begin{aligned}\left|\frac{1}{i\zeta_n}\int_{\mathbb{R}} \mu(\xi)\overline{\varphi^n}(\xi) d\xi\, z^n(1)\right| &\le \frac{|z^n(1)|}{\zeta_n}\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}}}{\sqrt{1+\xi^2}}\sqrt{1+\xi^2}|\varphi^n(\xi)| d\xi \\ &\le \frac{|z^n(1)|}{\zeta_n}\mathtt{M}_1\left(\int_{\mathbb{R}} (1+\xi^2)|\varphi^n(\xi)|^2 d\xi\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0,\end{aligned} \tag{3.13}$$

where $\mathtt{M}_1 = \left(\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1+\xi^2}\right)^{\frac{1}{2}}$ and we have that

$$\frac{|\xi|^{2\alpha-1}}{\xi^2+1} \underset{0}{\sim} \frac{1}{\xi^{1-2\alpha}} \quad \text{and} \quad \frac{|\xi|^{2\alpha-1}}{|\xi|^2+1} \underset{+\infty}{\sim} \frac{1}{|\xi|^{3-2\alpha}}, \tag{3.14}$$

thus, $\mathtt{M}_1$ is well defined since $0 < \alpha < 1$. Therefore, by using Eqs (3.11)–(3.13) into (3.10), we can infer that for $\tau \ge 0$, the following holds:

$$\int_{\mathbb{R}} |\varphi^n(\xi)|^2 dx \xrightarrow[n\to\infty]{} 0.$$

Step 2. The aim of this step is to show that the solution (v^n, z^n, φ^n) of (3.7) satisfies the following:

$$|\zeta_n v^n(1)| \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad |(\eta v_x^n)(1)| \xrightarrow[n\to\infty]{} 0. \tag{3.15}$$

From $(3.7)_1$, we have

$$|\zeta_n v^n(1)| \le |z^n(1)| + |f_n^1(1)|.$$

Utilizing the fact that $|f_n^1(1)| = |\int_0^1 (f_n^1)_x dx| \le \sqrt{\max_{x\in[0,1]} \eta^{-1}}\left(\int_0^1 \eta|(f_n^1)_x|^2 dx\right)^{\frac{1}{2}} \le \sqrt{\max_{x\in[0,1]} \eta^{-1}}\|F\|_{\mathcal{H}} \xrightarrow[n\to\infty]{} 0$ and the second limit in (3.9) in the above equation, we obtain

$$|\zeta_n v^n(1)| \xrightarrow[n\to\infty]{} 0.$$

Now, from the boundary conditions, we have that

$$|(\eta v_x^n)(1)|^2 = \left|-\gamma\int_{\mathbb{R}} \mu(\xi)\varphi(\xi) d\xi\right|^2 \le \gamma^2\left|\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}}}{\sqrt{\xi^2+1}}\sqrt{\xi^2+1}\varphi(\xi) d\xi\right|^2 \le \gamma^2\mathtt{M}_1^2\int_{\mathbb{R}} (\xi^2+1)|\varphi(\xi)|^2 d\xi \tag{3.16}$$

Thus, by using the first limit in (3.9) and the fact that $\mathtt{M}_1$ is well defined, we deduce that $|(\eta v_x^n)(1)| \xrightarrow[n\to\infty]{} 0$.

Step 3. The aim of this step is to show that the solution (v^n, z^n, φ^n) of (3.7) satisfies the following:

$$\begin{aligned}
&\left(1+\frac{K_p}{2}\right)\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2 dx + \left(1-\frac{K_p}{2}\right)\int_0^1 \eta|v_x^n|^2 dx + \left(1+\frac{K_p}{2}\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2 dx \\
&= \int_0^1 \frac{x}{\sigma}\left(\frac{p'-r}{p}\right)|\zeta_n v^n|^2 dx + \int_0^1 \frac{xr}{p}\eta|v_x^n|^2 dx + \int_0^1 \frac{\beta x}{\sigma q}\left(\frac{p'-r}{p}\right)|v^n|^2 dx \\
&+ \int_0^1 \frac{\beta x q'}{\sigma q^2}|v^n|^2 dx + 2\Re\left(\int_0^1 \left(f_n^2 + i\zeta_n f_n^1\right)\frac{x}{\sigma}\overline{v_x^n}dx\right) - \frac{K_p}{2}\Re\left(i\int_0^1 \frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}dx\right) \\
&+ \frac{1}{\sigma(1)}|\zeta_n v^n(1)|^2 + \eta(1)|v_x^n(1)|^2 + \frac{\beta}{\sigma(1)q(1)}|v^n(1)|^2 - \frac{K_p}{2}\Re\left(\int_0^1 \frac{1}{\sigma}f_n^2\overline{v^n}dx\right) - \frac{K_p}{2}(\eta v_x^n)(1)\overline{v^n}(1).
\end{aligned} \tag{3.17}$$

First, multiplying (3.8) by $-\frac{2x}{\sigma}\overline{v_x^n}$, integrating over (0, 1), and taking the real part, we obtain

$$\begin{aligned}
&\int_0^1 \left(\frac{x}{\sigma}\right)'|\zeta_n v^n|^2 dx - \frac{1}{\sigma(1)}|\zeta_n v^n(1)|^2 + \lim_{x\to 0}\frac{x}{\sigma}|\zeta_n v^n|^2 - 2\Re\left(\int_0^1 (\eta v_x^n)_x x\overline{v_x^n}dx\right) + \int_0^1 \left(\frac{\beta x}{\sigma q}\right)'|v^n|^2 dx \\
&- \frac{\beta}{\sigma(1)q(1)}|v^n(1)|^2 + \lim_{x\to 0}\frac{\beta x}{\sigma q}|v^n|^2 = 2\Re\left(\int_0^1 \left(f^2 + i\zeta_n f_n^1\right)\frac{x}{\sigma}\overline{v_x^n}dx\right).
\end{aligned} \tag{3.18}$$

We have that $\left(\frac{x}{\sigma}\right)' = \frac{1}{\sigma} - \frac{x}{\sigma}\left(\frac{p'-r}{p}\right)$, $\eta' = \frac{r}{p}\eta$ and $\left(\frac{x}{\sigma q}\right)' = \frac{1}{\sigma q} - \frac{x}{\sigma q}\left(\frac{p'-r}{p}\right) - \frac{xq'}{\sigma q^2}$; thus, we obtain

$$\int_0^1 \left(\frac{x}{\sigma}\right)'|\zeta_n v^n|^2 dx = \int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2 dx - \int_0^1 \frac{x}{\sigma}\left(\frac{p'-r}{p}\right)|\zeta_n v^n|^2 dx,$$

$$\begin{aligned}
&-2\Re\left(\int_0^1 (\eta v_x^n)_x x\overline{v_x^n}dx\right) = 2\Re\left(\int_0^1 \eta v_x^n (x\overline{v_x^n})_x dx\right) - 2\left[\eta x|v_x^n|^2\right]_0^1 = 2\int_0^1 \eta|v_x^n|^2 dx \\
&- \int_0^1 (x\eta)'|v_x^n|^2 dx - \left[\eta x|v_x^n|^2\right]_0^1 = \int_0^1 \eta|v_x^n|^2 dx - \int_0^1 \frac{xr}{p}\eta|v_x^n|^2 dx - \eta(1)|v_x^n(1)|^2 + \lim_{x\to 0} x\eta|v_x^n|^2
\end{aligned}$$

and

$$\int_0^1 \left(\frac{\beta x}{\sigma q}\right)'|v^n|^2 dx = \int_0^1 \frac{\beta}{\sigma q}|v^n|^2 dx - \int_0^1 \frac{\beta x}{\sigma q}\left(\frac{p'-r}{p}\right)|v^n|^2 dx - \int_0^1 \frac{\beta x q'}{\sigma q^2}|v^n|^2 dx.$$

By using the above equations in (3.18), we obtain

$$\begin{aligned}
&\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2 dx + \int_0^1 \eta|v_x^n|^2 dx + \int_0^1 \frac{\beta}{\sigma q}|v^n|^2 dx = \int_0^1 \frac{x}{\sigma}\left(\frac{p'-r}{p}\right)|\zeta_n v^n|^2 dx + \int_0^1 \frac{xr}{p}\eta|v_x^n|^2 dx \\
&+ \int_0^1 \frac{\beta x}{\sigma q}\left(\frac{p'-r}{p}\right)|v^n|^2 dx + \int_0^1 \frac{\beta x q'}{\sigma q^2}|v^n|^2 dx + 2\Re\left(\int_0^1 \left(f_n^2 + i\zeta_n f_n^1\right)\frac{x}{\sigma}\overline{v_x^n}dx\right) + \frac{1}{\sigma(1)}|\zeta_n v^n(1)|^2 \\
&- \lim_{x\to 0}\frac{x}{\sigma}|\zeta_n v^n|^2 + \eta(1)|v_x^n(1)|^2 - \lim_{x\to 0} x\eta|v_x^n|^2 + \frac{\beta}{\sigma(1)q(1)}|v^n(1)|^2 - \lim_{x\to 0}\frac{\beta x}{\sigma q}|v^n|^2.
\end{aligned} \tag{3.19}$$

Now, multiplying (3.8) by $\frac{K_p}{2\sigma}\overline{v^n}$ and integrating over $(0,1)$ leads to

$$\begin{aligned}&\frac{K_p}{2}\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx - \frac{K_p}{2}\int_0^1 \eta|v_x^n|^2dx + \frac{K_p}{2}\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx = -\frac{K_p}{2}(\eta v_x^n)(1)\overline{v^n}(1)\\&+\frac{K_p}{2}\lim_{x\to 0}(\eta v_x^n)\overline{v^n} - \frac{K_p}{2}\Re\left(\int_0^1 \frac{1}{\sigma}f_n^2\overline{v^n}dx\right) - \frac{K_p}{2}\Re\left(i\int_0^1 \frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}dx\right).\end{aligned} \tag{3.20}$$

Adding (3.19) and (3.20) and using Lemma 2.4, we obtain (3.17).

Step 4. The aim of this step is to show that the solution (v^n, z^n, φ^n) of (3.7) satisfies the following:

$$\int_0^1 |\zeta_n v^n|dx \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad \int_0^1 |\eta v_x^n|^2dx - \int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \xrightarrow[n\to\infty]{} 0. \tag{3.21}$$

From (3.17), we have

$$\begin{aligned}&\left(1+\frac{K_p}{2}\right)\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx + \left(1-\frac{K_p}{2}\right)\int_0^1 \eta|v_x^n|^2dx + \left(1+\frac{K_p}{2}\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx\\&\leq \mathcal{P}_0\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx + \mathcal{R}_0\int_0^1 \eta|v_x^n|^2dx + (\mathcal{P}_0+\mathcal{Q}_0)\int_0^1 \frac{|\beta|}{\sigma q}|v^n|^2dx + \mathcal{L}_n,\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}\mathcal{L}_n = &\left|2\Re\left(\int_0^1\left(f_n^2 + i\zeta_n f_n^1\right)\frac{x}{\sigma}\overline{v_x^n}dx\right)\right| + \left|\frac{K_p}{2}\Re\left(i\int_0^1 \frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}dx\right)\right| + \left|\frac{K_p}{2}(\eta v_x^n)(1)\overline{v^n}(1)\right|\\&+\left|\frac{K_p}{2}\Re\left(\int_0^1 \frac{1}{\sigma}f_n^2\overline{v^n}dx\right)\right| + \frac{1}{\sigma(1)}|\zeta_n v^n(1)|^2 + \eta(1)|v_x^n(1)|^2 + \frac{\beta}{\sigma(1)q(1)}|v^n(1)|^2.\end{aligned}$$

Next, we need to estimate the terms of $\mathcal{L}_n$. By applying the Cauchy–Schwarz inequality and considering the fact that $\frac{x}{\sqrt{p}}$ is non-decreasing on the interval $(0,1]$, $\|F\|_{\mathcal{H}} \xrightarrow[n\to\infty]{} 0$ and $\zeta_n \xrightarrow[n\to\infty]{} \delta$, we obtain:

$$\left|2\Re\left(\int_0^1 f_n^2\frac{x}{\sigma}\overline{v_x^n}dx\right)\right| \leq \frac{2}{\sqrt{p(1)}}\left(\int_0^1 \frac{1}{\sigma}|f_n^2|^2\right)^{\frac{1}{2}}\left(\int_0^1 \eta|v_x^n|^2\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0,$$

$$\left|2\Re\left(\int_0^1 i\zeta_n f_n^1\frac{x}{\sigma}\overline{v_x^n}dx\right)\right| \leq 2|\zeta_n|\sqrt{\frac{C'_H}{p(1)}}\left(\int_0^1 \eta|(f_n^1)_x|^2\right)^{\frac{1}{2}}\left(\int_0^1 \eta|v_x^n|^2\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0,$$

$$\left|\frac{K_p}{2}\Re\left(i\int_0^1 \frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}dx\right)\right| \leq \frac{K_pC'_H|\zeta_n|}{2}\left(\int_0^1 \eta|(f_n^1)_x|^2\right)^{\frac{1}{2}}\left(\int_0^1 \eta|v_x^n|^2\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0,$$

$$\left|\frac{K_p}{2}\Re\left(\int_0^1 \frac{1}{\sigma}f_n^2\overline{v^n}dx\right)\right| \leq \frac{K_p}{2}\sqrt{\frac{C'_H}{p(1)}}\left(\int_0^1 \frac{1}{\sigma}|f_n^2|^2\right)^{\frac{1}{2}}\left(\int_0^1 \eta|v_x^n|^2\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0.$$

Thus, using the above limits with the limits in (3.15), we deduce that

$$\mathcal{L}_n \xrightarrow[n\to\infty]{} 0. \tag{3.23}$$

Now, let us consider two cases based on the value of β:

If $\beta > 0$: from (3.22), we obtain that

$$\left(1+\frac{K_p}{2}-\mathcal{P}_0\right)\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx+\left(1-\frac{K_p}{2}-\mathcal{R}_0\right)\int_0^1 \eta|v_x^n|^2dx$$
$$+\left(1+\frac{K_p}{2}-\mathcal{P}_0-\mathcal{Q}_0\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \leq \mathcal{L}_n.$$

By using Hypothesis 2 and the result in (3.23) in the above equation, we obtain that

$$\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx \xrightarrow[n\to\infty]{} 0,\ \int_0^1 \eta|v_x^n|^2dx \xrightarrow[n\to\infty]{} 0,\ \int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \xrightarrow[n\to\infty]{} 0. \tag{3.24}$$

From Eq $(3.7)_1$, we have

$$\int_0^1 \frac{1}{\sigma}|z^n|^2dx \leq 2\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx+2\int_0^1 \frac{1}{\sigma}|f_n^1|^2dx \xrightarrow[n\to\infty]{} 0. \tag{3.25}$$

Therefore, the limits stated in (3.21) hold true in this case.

If $\beta < 0$: from (3.22), we obtain

$$\begin{aligned}&\left(1+\frac{K_p}{2}-\mathcal{P}_0\right)\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx+\left(1-\frac{K_p}{2}-\mathcal{R}_0\right)\int_0^1 \eta|v_x^n|^2dx\\&+\left(1+\frac{K_p}{2}+\mathcal{P}_0+\mathcal{Q}_0\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \leq \mathcal{L}_n.\end{aligned} \tag{3.26}$$

Hardy's inequality in Proposition 2.1 and the fact that $\beta < 0$ lead to

$$\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \geq \beta C_H\int_0^1 \eta|v_x^n|^2dx.$$

Using the above inequality, the fact that $K_p + K_q \leq 2$, and Hypothesis 2, we obtain

$$\left(1+\frac{K_p}{2}+\mathcal{P}_0+\mathcal{Q}_0\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \geq \left(1+\frac{K_p}{2}+\mathcal{P}_0+K_q\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \geq 4\beta C_H\int_0^1 \eta|v_x^n|^2dx.$$

Using the above inequality and Hypothesis 2 in (3.26), we obtain

$$\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad \int_0^1 \eta|v_x^n|^2dx \xrightarrow[n\to\infty]{} 0. \tag{3.27}$$

Using $(3.7)_1$ and the first limit in the preceding equation, we establish the first limit in (3.21). Further, using the norm equivalence from Corollary 2.1, we get the second limit in (3.21). Finally, using (3.9) and (3.21), we deduce that $\|\Upsilon^n\|_{\mathcal{H}}$ tends to 0 as n approaches infinity. This contradicts the assertion that $\|\Upsilon^n\|_{\mathcal{H}} = 1$ in (3.5). Hence, $i\mathbb{R}^* \subseteq \rho(\mathcal{A})$ holds true, completing the proof of Proposition 3.1. □

Proof of Theorem 3.1. By Proposition 3.1, we have $i\mathbb{R} \subseteq \rho(\mathcal{A})$ if $\tau > 0$ and consequently $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ and $i\mathbb{R}^* \subseteq \rho(\mathcal{A})$ if $\tau = 0$ and consequently $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Therefore, according to Arendt–Batty's Theorem, we obtain the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ of contractions is strongly stable, and the proof is complete.

4. Polynomial stability

This section is devoted to studying the polynomial stability of the system (S) when $\tau > 0$ and when p is weakly or strongly degenerate. The main results are presented in the following theorem.

Theorem 4.1. *Assume that $\tau > 0$ and Hypothesis 2 holds. Then, the C_0- semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is polynomially stable, i.e., there exists a constant $C > 0$ such that for every $\Upsilon_0 \in D(\mathcal{A})$, we have*

$$\mathcal{E}(t) \leq \frac{C}{t^{\frac{2}{1-\alpha}}} \|\Upsilon_0\|^2_{D(\mathcal{A})}, \quad t > 0, \ \forall\, \Upsilon \in D(\mathcal{A}). \tag{4.1}$$

According to the Theorem of Borichev and Tomilov [50] (see also [51,52]), in order to prove Theorem 4.1, we need to prove that the following two conditions hold:

$$i\mathbb{R} \subseteq \rho(\mathcal{A}), \tag{P1}$$

and

$$\limsup_{|\zeta|\to\infty} \frac{1}{|\zeta|^{1-\alpha}} \left\|(i\zeta I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{P2}$$

are satisfied.

Noting that when $\tau > 0$, Proposition 3.1 implies $i\mathbb{R} \subseteq \rho(\mathcal{A})$, we infer, according to the Borichev–Tomilov Theorem, that polynomial stability depends on the fulfillment of conditions (P1) and (P2). Since condition (P1) has already been established (as seen in Proposition 3.1), we must now establish condition (P2). To do so, we will proceed by contradiction. Let's assume that (P2) is false. Consequently, there exists a sequence $\{(\zeta_n, \Upsilon^n = (z^n, v^n, \varphi^n)^\top)\} \subset \mathbb{R}^* \times D(\mathcal{A})$ such that

$$|\zeta_n| \to +\infty \quad \text{and} \quad \|\Upsilon^n\|_{\mathcal{H}} = \|(v^n, z^n, \varphi^n)\|_{\mathcal{H}} = 1, \tag{4.2}$$

and

$$(\zeta_n)^{1-\alpha} (i\zeta_n I - \mathcal{A})\Upsilon^n = \mathcal{F}_n = (f_n^1, f_n^2, f_n^3)^\top \to 0 \quad \text{in} \quad \mathcal{H}. \tag{4.3}$$

Detailing the above equation yields:

$$\begin{cases} i\zeta_n v^n - z^n = \dfrac{f_n^1}{\zeta_n^{1-\alpha}}, \\ i\zeta_n z^n - \sigma(\eta v_x^n)_x - \dfrac{\beta}{q} v = \dfrac{f_n^2}{\zeta_n^{1-\alpha}}, \\ (|\xi|^2 + \tau + i\zeta_n)\varphi^n(\xi) - \mu(\xi) z^n(1) = \dfrac{f_n^3(\xi)}{\zeta_n^{1-\alpha}}, \quad \forall\, \xi \in \mathbb{R}. \end{cases} \tag{4.4}$$

Combining $(4.4)_1$ and $(4.4)_2$ yields

$$\zeta_n^2 v^n + \sigma(\eta v_x^n)_x + \frac{\beta}{q} v^n = -\frac{1}{\zeta_n^{1-\alpha}} (f_n^2 + i\zeta_n f_n^1). \tag{4.5}$$

Here, we will verify condition (P2) by seeking a contradiction with $\|\Upsilon^n\|_{\mathcal{H}} = 1$ through the demonstration of $\|\Upsilon^n\|_{\mathcal{H}} = o(1)$. To maintain clarity, we divide the proof into several lemmas.

Lemma 4.1. *Assume Hypothesis 2 and $\tau > 0$. Then, the solution (v^n, z^n, φ^n) of (4.3) satisfies the following:*

$$\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2 d\xi = \frac{o(1)}{\zeta_n^{1-\alpha}}, \quad |z^n(1)| = o(1), \quad \textit{and} \quad \int_{\mathbb{R}}|\varphi^n(\xi)|^2 d\xi = \frac{o(1)}{\zeta_n^{1-\alpha}}. \tag{4.6}$$

Proof. First, taking the inner product of (3.6) with Υ^n in $\mathcal{H}$ and using the fact that $\|\mathcal{F}_n\|_{\mathcal{H}} = o(1)$ and $\|\Upsilon^n\|_{\mathcal{H}} = 1$, we derive

$$\gamma\int_{\mathbb{R}}(|\xi|^2+\tau)|\varphi^n(\xi)|^2 d\xi = \Re(\langle(i\zeta_n I - \mathcal{A})\Upsilon^n, \Upsilon^n\rangle_{\mathcal{H}}) = -\Re(\langle\mathcal{A}\Upsilon^n, \Upsilon^n\rangle_{\mathcal{H}}) \le \frac{1}{|\zeta_n|^{1-\alpha}}\|\mathcal{F}\|_{\mathcal{H}}\|\Upsilon^n\|_{\mathcal{H}} = \frac{o(1)}{\zeta_n^{1-\alpha}}.$$

In fact,

$$\begin{aligned}
\Re\left(\langle\mathcal{A}\Upsilon^n, \Upsilon^n\rangle_{\mathcal{H}}\right) =& -\Re\left(\int_0^1 \eta v_x^n\overline{z_x^n}dx\right) + \Re\left((\eta v_x^n)(1)\overline{z^n(1)}\right) + \Re\left(\int_0^1 \frac{\beta}{\sigma q}v^n\overline{z^n}dx\right) + \Re\left(\int_0^1 \eta z_x^n\overline{v_x^n}dx\right)\\
&- \Re\left(\int_0^1 \frac{\beta}{\sigma q}z^n\overline{v^n}dx\right) - \gamma\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2 d\xi + \Re\left(\gamma z^n(1)\int_{\mathbb{R}}\mu(\xi)\overline{\varphi^n(\xi)}d\xi\right)\\
=& -\Re\left(\gamma\overline{z^n(1)}\int_{\mathbb{R}}\mu(\xi)\varphi^(\xi)d\xi\right) - \gamma\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2 d\xi + \Re\left(\gamma z^n(1)\int_{\mathbb{R}}\mu(\xi)\overline{\varphi^n(\xi)}d\xi\right)\\
=& -\gamma\int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2 d\xi
\end{aligned}$$

Now, from $(4.4)_3$, we have that

$$\mu(\xi)|z^n(1)| \le (|\zeta_n| + |\xi|^2 + \tau)|\varphi^n(\xi)| + \frac{|f^3(\xi)|}{\zeta_n^{1-\alpha}}.$$

Multiplying the above equation by $\frac{|\xi|}{(|\zeta_n|+|\xi|^2+\tau)^2}$ and integrating over $\mathbb{R}$ leads to

$$\mathtt{N}_3(\zeta,\tau,\alpha)|z^n(1)| \le \mathtt{N}_4(\zeta,\tau)\left(\int_{\mathbb{R}}|\xi\varphi^n(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \mathtt{N}_5(\zeta,\tau)\left(\int_{\mathbb{R}}|f_3^n(\xi)|^2 d\xi\right)^{\frac{1}{2}},$$

where $\mathtt{N}_3(\zeta,\tau,\alpha)$, $\mathtt{N}_4(\zeta,\tau)$, and $\mathtt{N}_5(\zeta,\tau)$ are defined in Lemma 2.2.

Applying Young's inequality to the above equation, we obtain

$$|z^n(1)|^2 \le \frac{1}{c_1(|\zeta_n|+\tau)^{\alpha-1}}\int_{\mathbb{R}}|\xi\varphi^n(\xi)|^2 d\xi dx + \frac{\sqrt{\pi}}{4}\frac{1}{c_1\zeta_n^{2-2\alpha}(|\zeta_n|+\tau)^{\alpha}}\int_{\mathbb{R}}|f_3^n(\xi)|^2 d\xi.$$

By applying the first estimate in (4.6), together with the fact that $\|\mathcal{F}^n\| = o(1)$, we derive the second estimate in (4.6). Now, regarding the final estimate in (4.6), in the case where $\tau > 0$, it becomes evident that:

$$\int_{\mathbb{R}}|\varphi^n(\xi)|^2 d\xi \le \int_{\mathbb{R}}(\xi^2+\tau)|\varphi^n(\xi)|^2 d\xi = \frac{o(1)}{|\zeta_n|^{1-\alpha}}.$$

□

Lemma 4.2. *Assume Hypothesis 2. Then, the solution (v^n, z^n, φ^n) of (4.3) satisfies the following:*

$$\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2 dx = o(1), \int_0^1 \eta|v_x^n|^2 dx - \int_0^1 \frac{\beta}{\sigma q}|v^n|^2 dx = o(1). \tag{4.7}$$

Proof. By multiplying (4.5) by $\dfrac{-2x}{\sigma}\overline{v}_x$, integrating over $(0,1)$, and employing a similar computation, as in Step 3. in Proposition 3.1, we obtain that the following equation holds:

$$\begin{aligned}
&\left(1+\frac{K_p}{2}\right)\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx+\left(1-\frac{K_p}{2}\right)\int_0^1 \eta|v_x^n|^2dx+\left(1+\frac{K_p}{2}\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx\\
&=\int_0^1 \frac{x}{\sigma}\left(\frac{p'-r}{p}\right)|\zeta_n v^n|^2dx+\int_0^1 \frac{xr}{p}\eta|v_x^n|^2dx+\int_0^1 \frac{\beta x}{\sigma q}\left(\frac{p'-r}{p}\right)|v^n|^2dx\\
&+\int_0^1 \frac{\beta x q'}{\sigma q^2}|v^n|^2dx+2\Re\left(\frac{1}{\zeta_n^{1-\alpha}}\int_0^1\left(f_n^2+i\zeta_n f_n^1\right)\frac{x}{\sigma}\overline{v_x^n}dx\right)-\frac{K_p}{2}\Re\left(\frac{i}{\zeta_n^{1-\alpha}}\int_0^1\frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}dx\right)\\
&+\frac{1}{\sigma(1)}|\zeta_n v^n(1)|^2+\eta(1)|v_x^n(1)|^2+\frac{\beta}{\sigma(1)q(1)}|v^n(1)|^2-\frac{K_p}{2}\Re\left(\frac{1}{\zeta_n^{1-\alpha}}\int_0^1\frac{1}{\sigma}f_n^2\overline{v^n}dx\right)-\frac{K_p}{2}(\eta v_x^n)(1)\overline{v^n}(1).
\end{aligned}\tag{4.8}$$

By integrating by parts, the term $\Re\left(\dfrac{2i}{\zeta_n^{1-\alpha}}\displaystyle\int_0^1 \zeta_n f_n^1\frac{x}{\sigma}\overline{v_x^n}dx\right)$ in (4.8), using the fact that $\left(\frac{x}{\sigma}\right)'=\frac{1}{\sigma}-\frac{1}{\sigma}\left(\frac{p'-r}{p}\right)$ along with Lemma 2.4, we derive:

$$\begin{aligned}
&\left(1+\frac{K_p}{2}\right)\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx+\left(1-\frac{K_p}{2}\right)\int_0^1 \eta|v_x^n|^2dx+\left(1+\frac{K_p}{2}\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx\\
&\le \mathcal{P}_0\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx+\mathcal{R}_0\int_0^1 \eta|v_x^n|^2dx+(\mathcal{P}_0+\mathcal{Q}_0)\int_0^1\frac{|\beta|}{\sigma q}|v^n|^2dx+\mathcal{M}_n,
\end{aligned}\tag{4.9}$$

where

$$\begin{aligned}
\mathcal{M}_n=&\left|\frac{2}{\zeta_n^{1-\alpha}}\Re\left(\int_0^1 f_n^2\frac{x}{\sigma}\overline{v_x^n}dx\right)\right|+\left|\Re\left(\frac{2i}{\zeta_n^{1-\alpha}}\int_0^1 (f_n^1)_x\frac{x}{\sigma}\zeta_n\overline{v^n}dx\right)\right|+\left|\Re\left(\frac{2i}{\zeta_n^{1-\alpha}}\int_0^1\frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}dx\right)\right|\\
&+\left|\Re\left(\frac{2i}{\zeta_n^{1-\alpha}}\int_0^1\frac{x}{\sigma}\left(\frac{p'-r}{p}\right)f_n^1\zeta_n\overline{v^n}dx\right)\right|+\left|\frac{K_p}{2\zeta_n^{1-\alpha}}\Re\left(i\int_0^1\frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}dx\right)\right|+\left|\frac{K_p}{2\zeta_n^{1-\alpha}}\Re\left(\int_0^1\frac{1}{\sigma}f_n^2\overline{v^n}dx\right)\right|\\
&+\frac{1}{\sigma(1)}|\zeta_n v^n(1)|^2+\eta(1)|v_x^n(1)|^2+\frac{\beta}{\sigma(1)q(1)}|v^n(1)|^2+\left|\Re\left(\frac{2i}{\zeta_n^{1-\alpha}}f_n^1(1)\frac{1}{\sigma(1)}\zeta_n v_n(1)\right)\right|+\left|\frac{K_p}{2}(\eta v_x^n)(1)\overline{v^n}(1)\right|.
\end{aligned}$$

Our aim now is to show that $\mathcal{M}_n=o(1)$. First, using $(4.4)_1$ and Hardy's inequality, we obtain

$$\int_0^1\frac{1}{\sigma}|\zeta_n v^n|^2dx\le 2\int_0^1\frac{1}{\sigma}|z^n|^2dx+2C_H\int_0^1\eta|(f_n^1)_x|^2dx\le 2\max(1,C_H)\left(\|\Upsilon\|_{\mathcal{H}}^2+\|\mathcal{F}\|_{\mathcal{H}}^2\right).\tag{4.10}$$

To estimate the terms in $\mathcal{M}_n$, we employ the Cauchy–Schwarz inequality, together with the fact that $\mathcal{F}_n=o(1)$ and $\frac{x}{\sqrt{p}}$ is non-decreasing on $(0,1]$. Utilizing (4.10), we arrive at:

$$\left|\frac{2}{\zeta_n^{1-\alpha}}\Re\left(\int_0^1 f_n^2\frac{x}{\sigma}\overline{v_x^n}dx\right)\right|\le\frac{2}{\zeta_n^{1-\alpha}\sqrt{p(1)}}\left(\int_0^1\frac{1}{\sigma}|f_n^2|^2dx\right)^{\frac12}\left(\int_0^1\eta|v_x^n|^2dx\right)^{\frac12}=\frac{o(1)}{\zeta_n^{1-\alpha}},\tag{4.11}$$

$$\left|\frac{2}{\zeta_n^{1-\alpha}}\Re\left(\int_0^1 i(f_n^1)_x\frac{x}{\sigma}\zeta_n\overline{v^n}dx\right)\right|\le\frac{2}{\sqrt{p(1)}\zeta_n^{1-\alpha}}\left(\int_0^1\eta|(f_n^1)_x|^2dx\right)^{\frac12}\left(\int_0^1\frac{1}{\sigma}|\zeta_n v^n|^2dx\right)^{\frac12}=\frac{o(1)}{\zeta_n^{1-\alpha}},\tag{4.12}$$

$$\left|\Re\left(\frac{2i}{\zeta_n^{1-\alpha}}\int_0^1\left(\frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}+\frac{x}{\sigma}\left(\frac{p'-r}{p}\right)f_n^1\zeta_n\overline{v^n}+\frac{K_p}{4}\int_0^1\frac{1}{\sigma}f_n^1\zeta_n\overline{v^n}\right)dx\right)\right|$$
$$\leq\frac{2}{\zeta_n^{1-\alpha}}\left(1+\mathcal{P}_0+\frac{K_p}{4}\right)\left(\int_0^1\eta|(f_n^1)_x|^2dx\right)^{\frac{1}{2}}\left(\int_0^1\frac{1}{\sigma}|\zeta_n v^n|^2dx\right)^{\frac{1}{2}}=\frac{o(1)}{\zeta_n^{1-\alpha}},\tag{4.13}$$

$$\left|\frac{K_p}{2\zeta_n^{1-\alpha}}\Re\left(\int_0^1\frac{1}{\sigma}f_n^2\overline{v^n}dx\right)\right|\leq\frac{K_p}{2\zeta_n^{1-\alpha}}\left(\int_0^1\frac{1}{\sigma}|f_n^2|^2dx\right)^{\frac{1}{2}}\left(\int_0^1\frac{1}{\sigma}|v^n|^2dx\right)^{\frac{1}{2}}=\frac{o(1)}{\zeta_n^{1-\alpha}},\tag{4.14}$$

Now, we have that from $(4.4)_1$:

$$|\zeta_n v^n(1)|\leq|z^n(1)|+\frac{1}{\zeta_n^{1-\alpha}}|f_n^1(1)|.$$

Using the fact that $|f_n^1(1)|\leq\sqrt{\max_{x\in[0,1]}\eta^{-1}}\|F\|_{\mathcal{H}}=o(1)$ and using the second estimation in (4.6) in the above inequality, we obtain

$$|\zeta_n v^n(1)|=o(1).\tag{4.15}$$

Also, we can deduce that

$$\left|\Re\left(\frac{2i}{\zeta_n^{1-\alpha}}f_n^1(1)\frac{1}{\sigma(1)}\zeta_n v_n(1)\right)\right|=\frac{o(1)}{\zeta_n^{1-\alpha}}.\tag{4.16}$$

Now, from the boundary conditions, we have

$$|(\eta v_x^n)(1)|^2=\left|-\gamma\int_{\mathbb{R}}\mu(\xi)\varphi(\xi)d\xi\right|^2\leq\gamma^2\left|\int_{\mathbb{R}}\frac{|\xi|^{\frac{2\alpha-1}{2}}}{\sqrt{\xi^2+1}}\sqrt{\xi^2+1}\varphi(\xi)d\xi\right|^2\leq\gamma^2\mathtt{M}_1^2\int_{\mathbb{R}}(\xi^2+1)|\varphi(\xi)|^2d\xi=\frac{o(1)}{\zeta_n^{1-\alpha}}.\tag{4.17}$$

For the last term in $\mathcal{M}_n$, we have

$$\left|\frac{K_p}{2}(\eta v_x^n)(1)\overline{v^n}(1)\right|\leq\frac{o(1)}{\zeta_n^{\frac{3-\alpha}{2}}}\tag{4.18}$$

Thus, using (4.11)–(4.18), we deduce that $\mathcal{M}_n=o(1)$.

Next, we proceed by distinguishing two cases according to the sign of β:

Case 1. If $\beta>0$:
From (4.9) and the fact that $\mathcal{M}_n=o(1)$, we have

$$\left(1+\frac{K_p}{2}-\mathcal{P}_0\right)\int_0^1\frac{1}{\sigma}|\zeta_n v^n|^2dx+\left(1-\frac{K_p}{2}-\mathcal{R}_0\right)\int_0^1\eta|v_x^n|^2dx$$
$$+\left(1+\frac{K_p}{2}-\mathcal{P}_0-\mathcal{Q}_0\right)\int_0^1\frac{\beta}{\sigma q}|v^n|^2dx=o(1).$$

By using Hypothesis 2, we deduce that

$$\int_0^1\frac{1}{\sigma}|\zeta_n v^n|^2dx=o(1),\quad\int_0^1\eta|v_x^n|^2dx=o(1),\quad\text{and}\quad\int_0^1\frac{\beta}{\sigma q}|v^n|^2dx=o(1).\tag{4.19}$$

From Eq $(4.4)_1$, we have

$$\int_0^1\frac{1}{\sigma}|z^n|^2dx\leq2\int_0^1\frac{1}{\sigma}|\zeta_n v^n|^2dx+\frac{2}{\zeta^{1-\alpha}}\int_0^1\frac{1}{\sigma}|f_n^1|^2dx=o(1).\tag{4.20}$$

Therefore, we proved (4.7) in this case.
Case 2. If $\beta < 0$: From (4.9), we obtain

$$\begin{aligned}&\left(1+\frac{K_p}{2}-\mathcal{P}_0\right)\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx+\left(1-\frac{K_p}{2}-\mathcal{R}_0\right)\int_0^1 \eta|v_x^n|^2dx\\&+\left(1+\frac{K_p}{2}+\mathcal{P}_0+Q_0\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx=o(1).\end{aligned} \tag{4.21}$$

Utilizing Hardy's inequality from Proposition 2.1 and the condition $\beta < 0$, along with the condition $K_p + K_q \leq 2$ and Hypothesis 2, we obtain:

$$\left(1+\frac{K_p}{2}+\mathcal{P}_0+Q_0\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \geq \left(1+\frac{K_p}{2}+\mathcal{P}_0+K_q\right)\int_0^1 \frac{\beta}{\sigma q}|v^n|^2dx \geq 4\beta C_H\int_0^1 \eta|v_x^n|^2dx.$$

Applying the aforementioned inequality along with Hypotheses 2 to (4.21), we obtain

$$\int_0^1 \frac{1}{\sigma}|\zeta_n v^n|^2dx = o(1) \quad \text{and} \quad \int_0^1 \eta|v_x^n|^2dx = o(1).$$

Employing $(4.4)_1$ and the preceding estimation, we establish the first estimate in (4.7). Also, using the norm equivalence provided by Corollary 2.1, we derive the second estimate in (4.7). □

Proof of Theorem 4.1: Finally, using Lemmas 4.1 and 4.2, we conclude that $\|\Upsilon^n\|_{\mathcal{H}} = o(1)$. This contradicts the assertion $\|\Upsilon^n\|_{\mathcal{H}} = 1$ in (4.3). Conequently, (P2) holds true, and thus the proof is complete.

Remark 1.

- *This work generalizes the results in [34]. Specifically, by considering $\alpha \to 1$ in our work, we obtain the system studied in [34].*
- *We conjecture that the energy decay rate $t^{1-\alpha}$ is optimal. However, the conditions assumed to achieve stability have not been proven to be optimal, leaving this an open problem.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. W. Feller, The parabolic differential equations and the associated semi-groups of transformations, *Ann. Math.*, **55** (1952), 468–519. https://doi.org/10.2307/1969644
2. P. Cannarsa, G. Fragnelli, D. Rocchetti, Null controllability of degenerate parabolic operators with drift, *Networks Heterogen. Media*, **2** (2007), 695–715. https://doi.org/10.1137/04062062X
3. G. Fragnelli, Interior degenerate/singular parabolic equations in nondivergence form: well-posedness and carleman estimates, *J. Differ. Equations*, **260** (2016), 1314–1371. https://doi.org/10.1016/j.jde.2015.09.019
4. M. Badii, J. I. Díaz, Time periodic solutions for a diffusive energy balance model in climatology, *J. Math. Anal. Appl.*, **233** (1999), 713–729. https://doi.org/10.1006/jmaa.1999.6335
5. I. H. Biswas, A. K. Majee, G. Vallet, On the cauchy problem of a degenerate parabolic-hyperbolic pde with lévy noise, *Adv. Nonlinear Anal.*, **8** (2019), 809–844. https://doi.org/10.1515/anona-2017-0113
6. A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Cloaking devices, electromagnetic wormholes, and transformation optics, *SIAM Rev.*, **51** (2009), 3–33. https://doi.org/10.1137/080716827
7. O. Nikan, Z. Avazzadeh, J. A. T. Machado, Numerical simulation of a degenerate parabolic problem occurring in the spatial diffusion of biological population, *Chaos, Solitons Fractals*, **151** (2021), 111220. https://doi.org/10.1016/j.chaos.2021.111220
8. F. Alabau-Boussouira, P. Cannarsa, G. Leugering, Control and stabilization of degenerate wave equations, *SIAM J. Control Optim.*, **55** (2017), 2052–2087. https://doi.org/10.1137/15M1020538
9. M. Gueye, Exact boundary controllability of 1-d parabolic and hyperbolic degenerate equations, *SIAM J. Control Optim.*, **52** (2014), 2037–2054. https://doi.org/10.1137/120901374
10. F. Chouaou, C. Aichi, A. Benaissa, Decay estimates for a degenerate wave equation with a dynamic fractional feedback acting on the degenerate boundary, *Filomat*, **35** (2021), 3219–3239. https://doi.org/10.2298/FIL2110219C
11. I. Boutaayamou, G. Fragnelli, D. Mugnai, Boundary controllability for a degenerate wave equation in nondivergence form with drift, *SIAM J. Control Optim.*, **61** (2023), 1934–1954. https://doi.org/10.1137/22M151491X
12. M. Akil, G. Fragnelli, I. Issa, Stability for degenerate wave equations with drift under simultaneous degenerate damping, preprint, arXiv:2308.08645. https://doi.org/10.48550/arXiv.2308.08645
13. M. Akil, G. Fragnelli, I. Issa, Energy decay rate of a transmission system governed by degenerate wave equation with drift and under heat conduction with memory effect, preprint, arXiv:2311.16296. https://doi.org/10.48550/arXiv.2311.16296
14. B. Allal, A. Hajjaj, J. Salhi, A. Sbai, Boundary controllability for a coupled system of degenerate/singular parabolic equations, *Evol. Equations Control Theory*, **11** (2022), 1579–1604. https://doi.org/10.3934/eect.2021055
15. I. Boutaayamou, G. Fragnelli, A degenerate population system: Carleman estimates and controllability, *Nonlinear Anal.*, **195** (2020), 111742. https://doi.org/10.1016/j.na.2019.111742

16. G. Fragnelli, Carleman estimates and null controllability for a degenerate population model, *J. Math. Pures Appl.*, **115** (2018), 74–126. https://doi.org/10.1016/j.matpur.2018.01.003

17. I. Boutaayamou, G. Fragnelli, L. Maniar, Carleman estimates for parabolic equations with interior degeneracy and neumann boundary conditions, *J. Anal. Math.*, **135** (2018), 1–35. https://doi.org/10.1007/s11854-018-0030-2

18. G. Fragnelli, Null controllability for a degenerate population model in divergence form via carleman estimates, *Adv. Nonlinear Anal.*, **9** (2019), 1102–1129. https://doi.org/10.1515/anona-2020-0034

19. G. Fragnelli, M. Yamamoto, Carleman estimates and controllability for a degenerate structured population model, *Appl. Math. Optim.*, **84** (2020), 999–1044. https://doi.org/10.1007/s00245-020-09669-0

20. C. L. Epstein, R. Mazzeo, *Degenerate Diffusion Operators Arising in Population Biology*, Princeton University Press, 2013. https://doi.org/10.1365/s13291-015-0131-0

21. J. Vancostenoble, E. Zuazua, Null controllability for the heat equation with singular inverse-square potentials, *J. Funct. Anal.*, **254** (2008), 1864–1902. https://doi.org/10.1016/j.jfa.2007.12.015

22. P. S. Hagan, D. E. Woodward, Equivalent black volatilities, *Appl. Math. Finance*, **6** (1999), 147–157. https://doi.org/10.1080/135048699334500

23. F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability, *J. Evol. Equations*, **6** (2006), 161–204. https://doi.org/10.1007/s00028-006-0222-6

24. P. Cannarsa, P. Martinez, J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, *SIAM J. Control Optim.*, **47** (2008), 1–19. https://doi.org/10.1137/04062062X

25. G. Fragnelli, D. Mugnai, Carleman estimates and observability inequalities for parabolic equations with interior degeneracy, *Adv. Nonlinear Anal.*, **2** (2013). https://doi.org/10.1515/anona-2013-0015

26. G. Fragnelli, D. Mugnai, *Carleman Estimates, Observability Inequalities and Null Controllability for Interior Degenerate Non Smooth Parabolic Equations*, American Mathematical Society, **242** (2016). https://doi.org/10.1090/MEMO/1146

27. M. Fotouhi, L. Salimi, Controllability results for a class of one dimensional degenerate/singular parabolic equations, *Commun. Pure Appl. Anal.*, **12** (2013), 1415–1430. https://doi.org/10.3934/cpaa.2013.12.1415

28. G. Fragnelli, D. Mugnai, Singular parabolic equations with interior degeneracy and non smooth coefficients: The neumann case, *Discrete Contin. Dyn. Syst. Ser. S*, **13** (2020), 1495–1511. https://doi.org/10.3934/dcdss.2020084

29. Z. J. Han, Z. Liu, J. Wang, Sharper and finer energy decay rate for an elastic string with localized kelvin-voigt damping, *Discrete Contin. Dyn. Syst. Ser. S*, **15** (2022), 1455–1467. https://doi.org/10.3934/dcdss.2022031

30. Z. J. Han, Z. Liu, Q. Zhang, Sharp stability of a string with local degenerate kelvin–voigt damping, *ZAMM*, **102** (2022). https://doi.org/10.1002/zamm.202100602

31. Z. J. Han, Z. Liu, K. i Yu, Stabilization for wave equation with localized kelvin–voigt damping on cuboidal domain: A degenerate case, *SIAM J. Control Optim.*, **62** (2024), 441–465. https://doi.org/10.1137/22M153210X

32. B. Allal, A. Moumni, J. Salhi, Boundary controllability for a degenerate and singular wave equation, *Math. Methods Appl. Sci.* , **45** (2022), 11526–11544. https://doi.org/10.1002/mma.8464

33. G. Fragnelli, D. Mugnai, A. Sbai, Boundary controllability for degenerate/singular hyperbolic equations in nondivergence form with drift, preprint, arXiv:2402.18247. https://doi.org/10.48550/arXiv.2402.18247

34. G. Fragnelli, D. Mugnai, A. Sbai, Stabilization for degenerate equations with drift and small singular term, preprint, arXiv:2403.17802. https://doi.org/10.48550/arXiv.2403.17802

35. M. Akil, G. Fragnelli, I. Issa, Stability of degenerate wave equation with a singular potential and local damping, (2024), hal-04539408f, In press.

36. D. Matignon, Asymptotic stability of webster-lokshin equation, *Math. Control Relat. Fields*, **4** (2014). https://doi.org/10.3934/mcrf.2014.4.481

37. R. L. Bagley, P. J. Torvik, Fractional calculus—a different approach to the analysis of viscoelastically damped structures, *AIAA J.*, **21** (1983), 741–748. https://doi.org/10.2514/3.8142

38. M. Mainardi, E. Bonetti, The application of real-order derivatives in linear viscoelasticity, in *Progress and Trends in Rheology II*, Steinkopff, Heidelberg, (1988), 64–67. https://doi.org/10.1007/978-3-642-49337-9_11

39. P. Torvik, R. Bagley, On the appearance of the fractional derivative in the behavior of real materials, *J. Appl. Mech.*, **51** (1984). https://doi.org/10.1115/1.3167615

40. M. Akil, Y. Chitour, M. Ghader, A. Wehbe. Stability and exact controllability of a timoshenko system with only one fractional damping on the boundary, *Asymptotic Anal.*, **119** (2020), 221–280. https://doi.org/10.3233/ASY-191574

41. M. Akil, I. Issa, A. Wehbe, Energy decay of some boundary coupled systems involving wave euler-bernoulli beam with one locally singular fractional kelvin-voigt damping, *Math. Control Relat. Fields*, **13** (2023), 330–381. https://doi.org/10.3934/mcrf.2021059

42. M. Akil, A. Wehbe, Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric conditions, *Math. Control Relat. Fields*, **9** (2019), 97–116. https://doi.org/10.3934/mcrf.2019005

43. A. Benaissa, S. Gaouar, Asymptotic stability for the lamé system with fractional boundary damping, *Comput. Math. Appl.*, **77** (2019), 1331–1346. https://doi.org/10.1016/j.camwa.2018.11.011

44. B. Mbodje, Wave energy decay under fractional derivative controls, *IMA J. Math. Control Inf.*, **23** (2006), 237–257. https://doi.org/10.1093/imamci/dni056

45. A. Pazy, Semigroups of linear operators and applications to partial differential equations, *Appl. Math. Sci.*, **44** (1983). https://doi.org/10.1007/978-1-4612-5561-1

46. G. Fragnelli, D. Mugnai, Linear stabilization for a degenerate wave equation in non divergence form with drift, preprint arXiv:2212.05264. https://doi.org/10.48550/arXiv.2212.05264

47. W. Arendt, C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Am. Math. Soc.*, **306** (1988), 837–852. https://doi.org/10.2307/2000826

48. M. Akil, Stability of piezoelectric beam with magnetic effect under (coleman or pipkin)–gurtin thermal law, *Z. Angew. Math. Phys.* , **73** (2022). https://doi.org/10.1007/s00033-022-01867-w

49. Z. Liu, S. Zheng, *Semigroups Associated with Dissipative Systems*, CRC Press, 1999.

50. A. Borichev, Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.*, **347** (2010), 455–478. https://doi.org/10.1007/s00208-009-0439-0

51. Z. Liu, B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, *Z. Angew. Math. Phys.*, **56** (2005), 630–644. https://doi.org/10.1007/s00033-004-3073-4

52. C. J. K. Batty, T. Duyckaerts, Non-uniform stability for bounded semi-groups on Banach spaces, *J. Evol. Equations*, **8** (2008), 765–780. https://doi.org/10.1007/s00028-008-0424-1

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