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The normalizer problem for finite groups having normal 2-complements

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Abstract: Assume that H is a finite group that has a normal 2-complement. Under some conditions, it is proven that the normalizer property holds for H. In particular, if there is a nilpotent subgroup of index 2 in H, then H has the normalizer property. The result of Li, Sehgal and Parmenter, stating that the normalizer property holds for finite groups that have an abelian subgroup of index 2 is generalized.

Keywords: the normalizer property; Coleman automorphisms; class-preserving automorphisms; normal 2-complements

1. Introduction

Throughout this paper, H denotes a finite group and q is a prime. Denote the integral group ring of H over $\mathbb{Z}$ by $\mathbb{Z}H$. The unit group of $\mathbb{Z}H$ is denoted by $\mathcal{U}(\mathbb{Z}H)$ denote and $\mathcal{Z}(\mathcal{U}(\mathbb{Z}H))$ to denote the center of $\mathcal{U}(\mathbb{Z}H)$. The normalizer problem asks whether $N_{\mathcal{U}(\mathbb{Z}H)}(H) = H\,\mathcal{Z}(\mathcal{U}(\mathbb{Z}H))$, where $\mathrm{N}_{\mathcal{U}(\mathbb{Z}H)}(H)$ is the normalizer of H in $\mathcal{U}(\mathbb{Z}H)$ (see [1], Problem 43). If the equation holds, then it is said that H has the normalizer property. Historically, it has been proven that the normalizer property holds for finite nilpotent groups in Coleman [2]. Jackowski and Marciniak [3] proved that the finite group having a normal Sylow 2-subgroup has the normalizer property. In particular, groups of odd order have the normalizer property. A few years later, Mazur [4–6] found that the well-known isomorphism problem of the integral group ring and the normalizer problem are closely connected. Based on Mazur's observation, among other things, a first counterexample to the normalizer problem was found by Hertweck [7], and later a first counterexample to the isomorphism problem. However, investigating which groups have the normalizer property is still an interesting problem. Recently, a number of related works on this subject were published, see [8–12].

To explain our results in detail, as in [10], some kinds of automorphisms of finite groups are defined in the following:

Let $\theta \in \mathrm{Aut}(H)$. We call θ a class-preserving automorphism if θ maps every element of H to its conjugate. Denote by $\mathrm{Aut_c}(H)$ the class-preserving automorphism group of H.

Let $\theta \in \mathrm{Aut}(H)$. We call θ a Coleman automorphism if, for any $P \in \mathrm{Syl}(H)$, the restriction of θ to P coincides with that of some inner automorphism of H. Denote by $\mathrm{Aut_{Col}}(H)$ the Coleman automorphism group of H.

$\mathrm{Aut}_{\mathbb{Z}}(H) = \{\theta_u \in \mathrm{Aut}(H) \mid x^{\theta_u} = u^{-1}xu, u \in \mathrm{N}_{\mathcal{U}(\mathbb{Z}H)}(H), x \in H\} \leq \mathrm{Aut}(H)$.

Write

$$\mathrm{Out_c}(H) = \mathrm{Aut_c}(H)/\mathrm{Inn}(H),$$
$$\mathrm{Out_{Col}}(H) = \mathrm{Aut_{Col}}(H)/\mathrm{Inn}(H),$$
$$\mathrm{Out}_{\mathbb{Z}}(H) = \mathrm{Aut}_{\mathbb{Z}}(H)/\mathrm{Inn}(H).$$

Jackowski and Marciniak [3] showed that $N_{\mathcal{U}(\mathbb{Z}H)}(H) = H\,\mathcal{Z}(\mathcal{U}(\mathbb{Z}H))$ if and only if $\mathrm{Out}_{\mathbb{Z}}(H) = 1$. In addition, $\mathrm{Out}_{\mathbb{Z}}(H) \leq \mathrm{Out}_c(H) \cap \mathrm{Out_{Col}}(H)$, and $\mathrm{Out}_{\mathbb{Z}}(H)$ is an elementary abelian 2-group (see [1]). Thus, $\mathrm{Out}_{\mathbb{Z}}(H) = 1$ is equivalent to the order of $\mathrm{Out_c}(H) \cap \mathrm{Out_{Col}}(H)$ being an odd number, i.e., H has the normalizer property if and only if the order of $\mathrm{Out_c}(H) \cap \mathrm{Out_{Col}}(H)$ is an odd number.

In this paper, the normalizer problem of finite groups with normal 2-complements is checked. Mazur [5] conjectured that finite groups having abelian Sylow 2-subgroups have the normalizer property. He proved that the conjecture holds if Sylow 2-subgroups have order 2. Later, this result was generalized by Hertweck [11], who proved that H has the normalizer property, provided that H has a normal 2-complement and H has a cycli Sylow 2-subgroup or an abelian of exponent at most 4. Marciniak and Roggenkamp [13] showed that the normalizer property holds for metabelian groups having abelian Sylow 2-subgroups. They also constructed a metabelian group $H = (\mathrm{C}_2^4 \times \mathrm{C}_3) \rtimes \mathrm{C}_2^3$ such that the order of $\mathrm{Out_c}(H) \cap \mathrm{Out_{Col}}(H)$ is an even number. This example illustrates that if the Sylow 2-subgroup of H is not abelian, then in general it is not the case that the order of $\mathrm{Out}_c(H) \cap \mathrm{Out_{Col}}(H)$ is an odd number. Our main results are the following:

Theorem 1.1. Let M be a subgroup of H such that $|H : M| = 2$. If $M = O_2(M) \times O_{2'}(M)$, then $\mathrm{Out}_{\mathbb{Z}}(H) = 1$, that is, H has the normalizer property.

Theorem 1.2. Let $H = O_{2'}(H) \rtimes P$ be a semidirect product of a nilpotent normal 2-complement $O_{2'}(H)$ by an abelian Sylow 2-subgroup P. Then the order of $\mathrm{Out_c}(H) \cap \mathrm{Out_{Col}}(H)$ is an odd number. In particular, H has the normalizer property.

Theorem 1.3. Let $H = O_{2'}(H) \rtimes Q$ be a semidirect product of a normal 2-complement $O_{2'}(H)$ by a Hamilton Sylow 2-subgroup Q. Then $\mathrm{Out}_{\mathbb{Z}}(H) = 1$.

Let $M \leq H$ or $M \trianglelefteq H$, and $\sigma \in \mathrm{Aut}(H)$. If σ fixes M or H/M, which will be denoted by $\sigma|_M$ or $\sigma|_{H/M}$, respectively. For a $h \in H$, we use $\mathrm{conj}(h)$ to denote the inner automorphism induced by h. For any $p \| H|$, denote by $O_p(H)$ the largest normal p-subgroup of H and $O_{p'}(H)$ the largest normal p'-subgroup of H. The other notation is standard; refer to [10, 14].

2. Preliminaries

Lemma 2.1. [11] Assume that H has cyclic Sylow 2-subgroups. Then the order of $\mathrm{Out_c}(H) \cap \mathrm{Out_{Col}}(H)$ is an odd number.

Lemma 2.2. [11] Let K be a normal 2-complement of H. Suppose that $P \in \mathrm{Syl}_2(H)$, and $D \trianglelefteq P$ such that $\exp(P/D) \leqslant 4$. If P/D is abelian and $\theta \in \mathrm{Aut}_{\mathbb{Z}}(H)$, then there is $\tau \in \mathrm{Inn}(H)$ such that $\theta\tau|_{KD} \in \mathrm{Aut_{Col}}(KD)$.

Lemma 2.3. [10] Suppose that $M \trianglelefteq H$. Suppose that q does not divide $|H/M|$. Then we have the following:
(1) If $\theta \in \mathrm{Aut_c}(H)$ is a q-element, then $\theta|_M \in \mathrm{Aut_c}(M)$;
(2) If $\theta \in \mathrm{Aut_{Col}}(H)$ is a q-element, then $\theta|_M \in \mathrm{Aut_{Col}}(M)$;
(3) If $\mathrm{Out_c}(M)$ or $\mathrm{Out_{Col}}(M)$ is a q'-group, then $\mathrm{Out_c}(H)$ or $\mathrm{Out_{Col}}(H)$ is also a q'-group.

Lemma 2.4. Let $M \leq H$, and let $\eta \in \mathrm{Aut}(H)$ be a q-element. If $\eta|_M = \mathrm{conj}(h)|_M$ for some $h \in H$, then there exists a $\delta \in \mathrm{Inn}(H)$ with the property that $\eta\delta|_M = id|_M$ and $o(\eta\delta)$ is still a power of the prime q.

Proof. Set $o(\eta) = q^i$ for some positive integer i. Then $\eta\mathrm{conj}(h^{-1})|_M = id|_M$. Let j be a positive integer satisfying $(\eta\mathrm{conj}(h^{-1}))^j$ is the q-part of $\eta\mathrm{conj}(h^{-1})$, where $(j, q) = 1$. Thus, there are integers s and t such that $sj + tq^i = 1$. Obviously, the order of $(\eta\mathrm{conj}(h^{-1}))^{sj}$ is a power of q, and $(\eta\mathrm{conj}(h^{-1}))^{sj}|_M = id|_M$. Since $\mathrm{Inn}(H) \trianglelefteq \mathrm{Aut}(H)$, then there is a $\delta \in \mathrm{Inn}(H)$ such that $(\eta\mathrm{conj}(h^{-1}))^{sj} = \eta^{sj}\delta = \eta^{1-tq^i}\delta = \eta\delta$. We are done.

Lemma 2.5. Let $\theta \in \mathrm{Aut}(H)$ be a q-element. Suppose that $M \leq H$, such that θ fixes M. If $\theta|_M = \mathrm{conj}(h)|_M$ for some $h \in H$, then we may find a q-element $t \in H$ satisfying $\theta|_M = \mathrm{conj}(t)|_M$.

Proof. Set $o(\theta) = q^i$, $o(h) = q^j t$, and $(q, t) = 1$, where i, j, and t are positive integers. Set $k := max\{i, j\}$. Since $(q^k, t) = 1$, then there are integers u and v satisfying $uq^k + vt = 1$. Write $t = h^{vt}$. Consequently, t is a q-element. For any $m \in M$, we have $m = m^{\theta^{uq^k}} = m^{h^{uq^k}}$, it follows that $m^\theta = m^h = m^{h^{uq^k+vt}} = (m^{h^{uq^k}})^{h^{vt}} = m^{h^{vt}} = m^t$, i.e., $\theta|_M = \mathrm{conj}(t)|_M$. We are done.

Lemma 2.6. [15] Let $\eta \in \mathrm{Aut}(H)$ be a q-element. Suppose that $M \trianglelefteq H$, such that $\eta|_M = id|_M$, and that $\eta|_{H/M} = id|_{H/M}$. Then $\eta|_{H/O_q(Z(M))} = id|_{H/O_q(Z(M))}$. Further, if $\eta|_R = id|_R$ for some $R \in \mathrm{Syl_q}(H)$, then $\eta \in \mathrm{Inn}(H)$.

Lemma 2.7. [12] Assume that Q is a q-group and $\bar{Q} = Q/\Phi(Q)$, where $\Phi(Q)$ is the Frattini subgroup of Q. Suppose that B is an abelian q'-group such that B acts on Q. Then one can find some $x \in Q$ satisfying $C_B(Q) = C_B(x) = C_B(\bar{Q}) = C_B(\bar{x})$.

Lemma 2.8. [16] Let $\pi(H)$ and $\pi(\mathrm{Aut_{Col}}(H))$ be the sets of prime divisors of $|H|$ and $|\mathrm{Aut_{Col}}(H)|$, respectively. Then $\pi(\mathrm{Aut_{Col}}(H)) \subseteq \pi(H)$.

Lemma 2.9. Let $\theta \in \mathrm{Aut}(H)$, and $M \trianglelefteq H$. Then we have the following:
(1) If $\theta \in \mathrm{Aut_c}(H)$, then θ fixes M, and $\theta|_{H/M} \in \mathrm{Aut_c}(H/M)$;
(2) If $\theta \in \mathrm{Aut_{Col}}(H)$, then θ fixes M, and $\theta|_{H/M} \in \mathrm{Aut_{Col}}(H/M)$.

Proof. These proofs are obvious, so we omit them.

Lemma 2.10. [8] Let $v \in \mathrm{N}_{\mathcal{U}(\mathbb{Z}H)}(H)$, $M \trianglelefteq H$, and let Q be a q-subgroup of H. Assume that $v^\eta = Mh \in H/M$ for some $h \in H$, where $\eta : \mathbb{Z}H \to \mathbb{Z}(H/M)$ is the natural homomorphism. Then one can find some $m \in M$ such that $v^{-1}yv = (mh)^{-1}y(mh)$ for all $y \in Q$.

3. Proof of The Theorems

Theorem 3.1. Let M be a subgroup of H satisfying $|H : M| = 2$. If $M = O_2(M) \times O_{2'}(M)$, then $\mathrm{Out}_{\mathbb{Z}}(H) = 1$, that is, H has the normalizer property.

Proof. Let $\rho \in \mathrm{Aut}_{\mathbb{Z}}(H)$, and we will show that $\rho \in \mathrm{Inn}(H)$. If $O_2(M) = 1$, then Sylow 2-subgroups of H are cyclic. By Mazur's result, then $\mathrm{Out}_{\mathbb{Z}}(H) = 1$. Hereafter, we suppose that $O_2(M) \neq 1$. Let $Q \in \mathrm{Syl}_2(H)$. Then $O_2(M) = Q \cap M$. Since by assumption $|H : M| = 2$, it follows that $O_{2'}(M)$ is a normal 2-complement Q in H. By Lemma 2.2, then $\rho\gamma|_M \in \mathrm{Aut}_{\mathrm{Col}}(M)$ for some $\gamma \in \mathrm{Inn}(H)$. Write $\beta = \rho\gamma$. Since $\mathrm{Out}_{\mathbb{Z}}(H)$ is an elementary abelian 2-group, we may suppose that $o(\beta)$ is a power of 2. So is $\beta|_M$. Note that $M = O_2(M) \times O_{2'}(M)$, where $O_2(M) \in \mathrm{Syl}_2(M)$, which implies that $\beta|_M \in \mathrm{Inn}(M)$ by Lemma 2.3(3). Thus, $\beta|_M = \mathrm{conj}(x)|_M$ for some $x \in M$. By Lemma 2.4, we obtain that $\beta\delta|_M = id|_M$ for some $\delta \in \mathrm{Inn}(H)$, and $o(\beta\delta)$ is still a power of 2. By replacing β with $\beta\delta$, we may suppose that

$$\beta|_M = id|_M. \tag{3.1}$$

Since H/M is cyclic, we obtain that

$$\beta|_{H/M} = id_{H/M}. \tag{3.2}$$

It is clear that β acts on the set $\mathrm{Syl}_2(H)$. Since $|\mathrm{Syl}_2(H)|$ is an odd number and β is of 2-power order, one can find an $R \in \mathrm{Syl}_2(H)$ such that $R^\beta = R$. Note that $\mathrm{Aut}_{\mathbb{Z}}(H) \le \mathrm{Aut}_{\mathrm{Col}}(H)$, so $\beta|_R = \mathrm{conj}(g)|_R$ for some $g \in H$. By Lemma 2.5, we may suppose that g is a 2-element. It follows that $R^\beta = R = R^g$, i.e., $g \in N_H(R)$, which implies that $g \in R$. Note that $O_2(M) = M \cap R$. Then we have $\beta|_{O_2(M)} = \mathrm{conj}(g)|_{O_2(M)}$. However, $\beta|_{O_2(M)} = id|_{O_2(M)}$. Consequently, $x^g = x$ for all $x \in O_2(M)$, that is, $g \in C_R(O_2(M))$.

Case 1. Assume that g is not in $O_2(M)$. Since $|R/O_2(M)| = |R/M \cap R| = |MR/M| = |H/M| = 2$, we obtain that $O_2(M)$ is a maximal subgroup of R. Thus, $R = < O_2(M), g >$. Since $g \in C_R(O_2(M))$, it implies that $g \in Z(R)$, and thus

$$\beta|_R = \mathrm{conj}(g)|_R = id|_R. \tag{3.3}$$

Hence, by Lemma 2.6, Eqs (3.1)–(3.3) yield that $\beta \in \mathrm{Inn}(H)$, i.e., $\rho \in \mathrm{Inn}(H)$, as desired.

Case 2. Assume that $g \in O_2(M)$. Recall that $\beta|_R = \mathrm{conj}(g)|_R$. Write $\tau = \beta\mathrm{conj}(g^{-1})$. Then we have

$$\tau|_R = id_R. \tag{3.4}$$

Since $M = O_2(M) \times O_{2'}(M)$, $g \in C_R(O_2(M))$ and Eq (3.1), we have

$$\tau|_M = id_M. \tag{3.5}$$

In addition, by Eq (3.2) and $g \in O_2(M) \le M$, we obtain that

$$\tau|_{H/M} = id_{H/M}. \tag{3.6}$$

Let m be the $2'$-component of the order of τ. Then τ^m is of 2-power order and satisfies the following conditions: $\tau^m|_R = id|_R, \tau^m|_M = id_M, \tau^m|_{H/M} = id_{H/M}$. Thus, by Lemma 2.6, $\tau^m \in \mathrm{Inn}(H)$. Since $\mathrm{Out}_{\mathbb{Z}}(H)$ is an elementary abelian 2-group, this implies that $\tau \in \mathrm{Inn}(H)$, i.e., $\beta\mathrm{conj}(g^{-1}) \in \mathrm{Inn}(H)$. Hence $\rho \in \mathrm{Inn}(H)$. We are done.

The following results are immediate from Theorem 3.1, which generalizes the finite version of a result due to Li et al. ([17], Theorem 2).

Corollary 3.2. Let $F(H)$ be the Fitting subgroup of H satisfying $|H : F(H)| = 2$. Then $\mathrm{Out}_{\mathbb{Z}}(H) = 1$, that is, H has the normalizer property.

Theorem 3.3. Let $H = O_{2'}(H) \rtimes P$ be a semidirect product of a nilpotent normal 2-complement $O_{2'}(H)$ by an abelian Sylow 2-subgroup P. Then the order of $\mathrm{Out}_c(H) \cap \mathrm{Out}_{\mathrm{Col}}(H)$ is an odd number. In particular, H has the normalizer property.

Proof. Let $\rho \in \mathrm{Aut}_c(H) \cap \mathrm{Aut}_{\mathrm{Col}}(H)$ be of 2-power order. Our goal is to prove that $\rho \in \mathrm{Inn}(H)$. Since $H = O_{2'}(H) \rtimes P$ is a semidirect product. First, we show that $\rho\mathrm{conj}(h^{-1})|_{O_{2'}(H)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(H))$ for some $h \in H$. Let $\pi(O_{2'}(H)) = \{q_1, q_2, \cdots, q_r\}$ and let $Q_i \in \mathrm{Syl}_{q_i}(O_{2'}(H))$, where $i = 1, 2, \cdots, r$. Then $O_{2'}(H) = Q_1 \times Q_2 \times \cdots \times Q_r$. Since $\rho \in \mathrm{Aut}_{\mathrm{Col}}(H)$, we obtain that $\rho|_{Q_i} = \mathrm{conj}(t_i)|_{Q_i}$ for some $t_i \in H$, where $i = 1, 2, \cdots, r$. In Lemma 2.5, we have

$$\rho|_{Q_i} = \mathrm{conj}(h_i)|_{Q_i}, \tag{3.7}$$

where $h_i \in H$ is a 2-element. Since $Q_i \trianglelefteq H$, thus P acts on Q_i. By Lemma 2.7, $C_P(Q_i) = C_P(b_i) = C_P(\bar{b}_i) = C_P(\bar{Q}_i)$ for some $b_i \in Q_i$. Set $b = b_1 b_2 \cdots b_r$. Since $\rho \in \mathrm{Aut}_c(H)$, then there is a $h \in H$ such that $b^\rho = b^h$, i.e., $(h^{-1}b_1h)\cdots(h^{-1}b_rh) = (h_1^{-1}b_1h_1)\cdots(h_r^{-1}b_rh_r)$. From this, we obtain $h^{-1}b_ih = h_i^{-1}b_ih_i$. It follows that

$$[h_ih^{-1}, b_i] = 1. \tag{3.8}$$

Since $H = O_{2'}(H) \rtimes P$, let $h_ih^{-1} = mk$, where $m \in O_{2'}(H)$ and $k \in P$. Next, we will show that $k \in C_P(Q_i)$. Since H acts on $\bar{Q}_i$, by Eq (3.8), we obtain that

$$[h_ih^{-1}, \bar{b}_i] = 1. \tag{3.9}$$

On the other hand,

$$[h_ih^{-1}, \bar{b}_i] = [mk, \bar{b}_i] = [m, \bar{b}_i]^k[k, \bar{b}_i]. \tag{3.10}$$

Since $\bar{Q}_i$ is abelian and $O_{2'}(H)$ is nilpotent, it follows that $[m, \bar{b}_i] = 1$. By (3.9) and (3.10), we obtain that $[k, \bar{b}_i] = 1$, that is, $k \in C_P(\bar{b}_i) = C_P(Q_i)$. Thus, by (3.7), we imply that $\rho\mathrm{conj}(h^{-1})|_{Q_i} = \mathrm{conj}(m)|_{Q_i}$. This shows that $\rho\mathrm{conj}(h^{-1})|_{O_{2'}(H)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(H))$. Since $O_{2'}(H)$ is of odd order, by Lemma 2.8, $\rho\mathrm{conj}(h^{-1})|_{O_{2'}(H)} = id|_{O_{2'}(H)}$, that is, $\rho|_{O_{2'}(H)} = \mathrm{conj}(h)|_{O_{2'}(H)}$. In Lemma 2.4, let us set

$$\rho|_{O_{2'}(H)} = id|_{O_{2'}(H)}. \tag{3.11}$$

Since $H/O_{2'}(H)$ is an abelian 2-group. Then

$$\rho|_{H/O_{2'}(H)} = id|_{H/O_{2'}(H)}. \tag{3.12}$$

By Lemma 2.6, Eqs (3.11) and (3.12) yield that $\rho|_{H/O_2(Z(O_{2'}(H)))} = id|_{H/O_2(Z(O_{2'}(H)))}$. Note that $|O_{2'}(H)|$ is an odd number. We obtain that $O_2(Z(O_{2'}(H))) = 1$, i.e., $\rho = id$. Hence $\rho \in \mathrm{Inn}(H)$. We are done.

A group Q is called a Hamilton 2-group if $Q = Q_8 \times E$, where Q_8 is a quaternion group of order 8 and E is an elementary abelian 2-group. Higman ([1], Theorem 2.7) proved that the units of $\mathbb{Z}Q$ are trivial.

Theorem 3.4. Let $H = O_{2'}(H) \rtimes Q$ be a semidirect product of a normal 2-complement $O_{2'}(H)$ by a Hamilton Sylow 2-subgroup Q. Then $\mathrm{Out}_{\mathbb{Z}}(H) = 1$.

Proof. Let $\rho \in \mathrm{Aut}_{\mathbb{Z}}(H)$, and we will show that $\rho \in \mathrm{Inn}(H)$. By the definition of $\mathrm{Aut}_{\mathbb{Z}}(H)$, then there is a $v \in \mathrm{N}_{\mathcal{U}(\mathbb{Z}H)}(H)$ such that $h^\rho = v^{-1}hv$ for all $h \in H$. Let $\epsilon : \mathbb{Z}H \to \mathbb{Z}\,(\sum_{h\in H} r_h h \mapsto \sum_{h\in H} r_h)$ be the augmentation map for $\mathbb{Z}H$ to $\mathbb{Z}$, where $r_h \in \mathbb{Z}$ for each $h \in H$. Then we have $\epsilon(v) = 1$ or -1 since v is a unit of $\mathbb{Z}H$. Note that $\rho = \mathrm{conj}(v) = \mathrm{conj}(-v)$. Thus, we may suppose that $\epsilon(v) = 1$.

Consider the quotient $H/O_{2'}(H)$. We set $\bar{h} := hO_{2'}(H)$ for any $h \in H$ and set $\bar{H} := H/O_{2'}(H)$. Denote by

$$\theta : \mathbb{Z}H \to \mathbb{Z}\bar{H} \quad (\sum_{g\in H} r_g g \mapsto \sum_{g\in H} r_g \bar{g}), \tag{3.13}$$

the natural homomorphism for $\mathbb{Z}H$ to $\mathbb{Z}\bar{H}$.

Since $v \in \mathrm{N}_{\mathcal{U}(\mathbb{Z}H)}(H)$, then $v^\theta \in \mathrm{N}_{\mathcal{U}(\mathbb{Z}\bar{H})}(\bar{H})$. By Lemma 2.9, ρ fixes $O_{2'}(H)$ and ρ induces an automorphism of $H/O_{2'}(H)$. Since $h^\rho = v^{-1}hv$ for any $h \in H$, then

$$(hO_{2'}(H))^{\rho|_{H/O_{2'}(H)}} = \overline{v^{-1}hv} = (v^{-1}hv)^\theta = (v^\theta)^{-1}\bar{h}v^\theta. \tag{3.14}$$

Thus, $\rho|_{H/O_{2'}(H)} \in \mathrm{Aut}_{\mathbb{Z}}(H/O_{2'}(H))$. Since $H/O_{2'}(H)$ is a Hamilton 2-group, by Higman's result, $\mathbb{Z}(H/O_{2'}(H))$ has only trivial units. Hence, one can find an element $\bar{g} = gO_{2'}(H) \in H/O_{2'}(H)$ satisfying $v^\theta = \bar{g}$. By Eq (3.14), we obtain that

$$\rho|_{H/O_{2'}(H)} = \mathrm{conj}(g)|_{H/O_{2'}(H)}. \tag{3.15}$$

Since $\rho\mathrm{conj}(g^{-1}) \in \mathrm{Aut}_{\mathbb{Z}}(H) \subseteq \mathrm{Out}_{\mathrm{c}}(H) \cap \mathrm{Out}_{\mathrm{Col}}(H)$. By Lemma 2.9, $\rho\mathrm{conj}(g^{-1})|_{O_{2'}(H)} \in \mathrm{Aut}(O_{2'}(H))$. Next, we show that $\rho\mathrm{conj}(g^{-1})|_{O_{2'}(H)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(H))$. Let $p \in \pi(O_{2'}(H))$ and $P \in \mathrm{Syl}_p(O_{2'}(H))$. By Lemma 2.10, then $\rho\mathrm{conj}(g^{-1})|_P = \mathrm{conj}(n)|_P$ for some $n \in O_{2'}(H)$. Consequently, this shows that $\rho\mathrm{conj}(g^{-1})|_{O_{2'}(H)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(H))$. Write $\gamma := \rho\mathrm{conj}((ng)^{-1})$. By Eq (3.15), we obtain that

$$\gamma|_{H/O_{2'}(H)} = id|_{H/O_{2'}(H)}. \tag{3.16}$$

Since $\mathrm{Out}_{\mathbb{Z}}(H)$ is an elementary abelian 2-group, we may suppose that $o(\gamma)$ is a power of 2. So is $\gamma|_{O_{2'}(H)}$. Note that $\gamma|_{O_{2'}(H)} = \rho\mathrm{conj}(g^{-1})\mathrm{conj}(n^{-1})|_{O_{2'}(H)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(H))$. By Lemma 2.8 and the fact that the order of $\gamma|_{O_{2'}(H)}$ is a power of 2, we deduce that

$$\gamma|_{O_{2'}(H)} = id|_{O_{2'}(H)}. \tag{3.17}$$

Now, by Lemma 2.6, Eqs (3.16) and (3.17) one can see that $\gamma|_{H/O_2(Z(O_{2'}(H)))} = id|_{H/O_2(Z(O_{2'}(H)))}$. Since $O_{2'}(H)$ has odd order, we have $\gamma = id$. Thus, $\rho\mathrm{conj}((ng)^{-1}) = id$. Hence $\rho \in \mathrm{Inn}(H)$. We are done.

4. Conclusions

In conclusion, we investigate the normalizer problem of finite groups with normal 2-complements. We have proven that H has the normalizer property, if H is a semidirect product of a nilpotent normal 2-complement by an abelian Sylow 2-subgroup or H is a semidirect product of a normal 2-complement by a Hamilton Sylow 2-subgroup. Additionally, we have proven that the normalizer property holds for finite groups with a nilpotent subgroup of index 2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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