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Research article

Consensus control of multi-agent systems with delays

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Abstract: This paper concerns the consensus problem of linear time-invariant multi-agent systems (MASs) with multiple state delays and communicate delays. Consensus control is widely applied in spacecraft formation, sensor networks, robotic manipulators, autonomous vehicles, and others. By introducing a linear transformation, the consensus problem of the delayed MAS under an undirected network was converted into a robust asymptotic stability problem associated with the eigenvalues of the normalized Laplacian matrix of the network. By means of the argument principle and optimization technologies, a numerical controller design method was presented for the delayed MAS to reach consensus. The effectiveness of the proposed approach was illustrated by some numerical examples. The proposed approach may be applied to multi-agent systems with distributed delays.

Keywords: multi-agent systems; optimization; argument principle; time delay

1. Introduction

Consensus control of MASs is one of the basic problems in the control community, having attracted considerable attention. The basic idea of the consensus control problem is to design a distributed protocol for each agent by using the information of the agent and its neighbors such that all the agents in the network asymptotically reach a common value [1]. In recent years, consensus control has been widely applied in spacecraft formation, sensor networks, robotic manipulators, autonomous vehicles, and others (see for instance, [2] and the references therein).

In practice, MASs are often accompanied with time delays that arise in sensor response or information transmission, which might cause negative effects on the consensus performance or, even worse, make the system fail to achieve consensus [3]. Therefore, the consensus problem of MASs with time delays is a challenging topic that is worthy of investigating.

In recent years, many research works involving consensus problem of linear time-invariant MASs in the presence of time delays have been developed. These works have been conducted under different perspectives. For example, based on the frequency domain analysis for MASs with communication delays, a large number of works have dedicated to seeking the consensus margin for system parameters such that the consensus can be achieved within this range [4–14]. Among these, the delay margin problem was investigated in [4–9], while the gain margin problem was studied in [10–12]. Using Lyapunov functional and linear matrix inequality technologies, sufficient conditions for the consensus of delayed MASs were derived in [15–19]. By means of the algebraic Riccati equation, a delay-dependent consensus protocol was proposed by [20]. In [21], approaches including Lyapunov theorems and the Nyquist stability criterion were used to study consensus algorithms for MASs with both communication and input delays. Using z-transformation and Routh criterion, a necessary and sufficient condition for the consensus of discrete-time MASs with communication delay was provided by [22]. Based on the impulsive observer and piecewise Lyapunov functional, sufficient conditions in terms of linear matrix inequalities for consensus of MASs with single delay were derived in [23]. Besides, leader-follower consensus problems of MASs with time delays were investigated in [24] and [25]. In [26], the consensus problems of MASs with time-varying delays were considered. In [27], the consensus problems of MASs with the random input delay were discussed.

Overall, the existing works have been focused on communication delays in the information transmission or a single delay in system interior.

In practical MASs, multiple time delays might arise in system states caused by mechanical transmission, sensor sensing, hydraulic process, and others. These state delays affect the system performance and complicate the problem analysis. For general linear MASs with multiple state delays, it is still challenging work to design a consensus protocol. Motivated by the above statements, this paper investigates the consensus problem of general continuous-time linear MASs with multiple state delays and communication delays. We consider a multi-agent networks with N agents intercommunicated by a fixed undirected networks graph; the dynamics of agent i is given by

$$\dot{x}_i(t) = \sum_{d=0}^{m} A_d x_i(t-\tau_d) + Bu_i(t), \quad y_i(t) = Cx_i(t), \tag{1.1}$$

where $x_i \in \mathbb{R}^p$ and $u_i \in \mathbb{R}^q$ are system states and the control input of agent i, respectively, $A_d \in \mathbb{R}^{p\times p}$ for $d = 0, 1, ..., m$ are system matrices, and $B \in \mathbb{R}^{p\times q}$ is input matrix, $0 = \tau_0 < \tau_1 < ... < \tau_m$ are time delays.

Let us give the output feedback control protocol

$$u_i(t) = \sum_{d=0}^{m} \left(\frac{K_d}{d_i} \sum_{j\in\mathcal{N}_i} a_{ij} \left[y_i(t-\tau_d-\tau_c) - y_j(t-\tau_d-\tau_c) \right] \right), \tag{1.2}$$

where $\tau_c > 0$ represents the communication delay, a_{ij} is the entries of the network adjacency matrix, $K_d \in \mathbb{R}^{q\times p}$ for $d = 0, 1, ..., m$ are feedback gain matrices to be determined.

Considerations on the structure of protocol (1.2) are illustrated as two aspects: On the one hand, the agent i receives a set of available historical information with respect to its neighbors; on the other hand, for general linear systems with multiple delays, the more information of delayed states being used, the more possible it is to stabilize it.

The consensus problem in this paper is: for the delayed MAS (1.1) with the protocol structure (1.2), determining the gain matrices $K_0, ..., K_m$ such that $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$ for $i, j = 1, 2, ..., N$.

Along the line of [28], we investigate consensus problem of the delayed MAS (1.1) with the protocol structure (1.2). By introducing a linear transformation, we will show that the consensus of N agents is equivalent to the simultaneous asymptotic stability of $N-1$ delayed subsystems associated with the eigenvalues of normalized Laplacian matrix of the network. We prove that a sufficient condition for the consensus of the delayed MASs is that a linear delay system with an uncertain parameter is robust stable, where the uncertain parameter covers the interval bounded by the maximum and minimum eigenvalues of the normalized Laplacian matrix. Then, using the argument principle and the properties of state transition matrix, we derive a sufficient condition for the consensus of the delayed MAS. Based on the obtained condition, we construct an optimization scheme to numerically solve the feedback gain matrices.

The main contributions of this paper are summarized as follows:

1) Compared with the existing works, this paper considers multiple state delays in linear MASs and designs a distributed consensus protocol with neighbor historical output information.

2) By the argument principle and optimization technologies, a sufficient condition is derived for the consensus of the MAS with multiple state delays. Based on this, an optimization-based method for calculating the consensus gain matrices is presented.

The rest of this paper is organized as follows. Section 2 introduces the basic material on graph theory and the stability criteria based on the argument principle needed in this paper. The main results are given in Section 3. In Section 4, simulation examples are presented to illustrate the accuracy of the theoretical results. Section 5 is a brief conclusion.

Notations: Let $\mathbb{R}^p$ be the p-dimensional Euclidean space, $\mathbb{R}^{p\times q}$ represents the $p\times q$ real matrix, $\|x\|$ denotes Euclidean norm of matrix $x\in\mathbb{R}^{p\times q}$, I_p denotes $p\times p$ identity matrix, $\mathbf{1}_p$ denotes p-dimensional column vector with entries all being 1, $\mathbf{0}_{p\times q}$ denotes $p\times q$ matrix with all the elements being zero. The notation diag$\{\cdot\}$ denotes a block-diagonal matrix. The notation $\otimes$ denotes the Kronecker product. The notations ker(W) and im(W) denote the kernel and image of matrix W, respectively. The notation $\lceil a \rfloor$ denotes the smallest integer that is greater than or equal to $a\in\mathbb{R}$.

2. Preliminaries

Let a simple graph (no self-loops or multiple edges) $\mathcal{G} = \{\mathcal{V},\mathcal{E},\mathcal{A}\}$ denote the undirected communication topology between MASs with the set of vertices $\mathcal{V}=\{1,2,...,N\}$ and the set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. The ith vertex represents agent i, and the edge (i,j) denotes the communication channel between agent i and agent j. The set $\mathcal{E}\subset\{(i,j): i,j\in\mathcal{V}\}$ is the edge set. The set of neighbors of the ith agent is denoted by $\mathcal{N}_i=\{j\in\mathcal{V}|(i,j)\in\mathcal{E}\}$. $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij}=a_{ji}>0$ if $(i,j)\in\mathcal{E}$ and $i\neq j$; $a_{ij}=0$ otherwise. The degree of the ith vertex is denoted by $\sum_{j=1}^N a_{ij}$ and the degree matrix $\mathcal{D}=\text{diag}\{d_1,d_2,...,d_N\}$. The Laplacian matrix L of $\mathcal{G}$ is defined by $L=\mathcal{D}-\mathcal{A}$. Note that $\mathcal{L}\mathbf{1}_N=\mathbf{0}_N$. Let $\mathcal{L}:=\mathcal{D}^{-1}L$ be the normalized Laplacian matrix associated with $\mathcal{G}$ by replacing the original weight a_{ij} with a new weight $\frac{a_{ij}}{d_i}$. For an undirected graph, $\mathcal{L}$ is a symmetric, positive semi-definite matrix, and all its eigenvalues are nonnegative. The eigenvalues of $\mathcal{L}$ can be arranged as $0=\lambda_1(\mathcal{L})<\lambda_2(\mathcal{L})\leq ...\leq\lambda_N(\mathcal{L})$. Throughout this paper, we consider connected and undirected network graph.

For preparation, we review some results in [28] that provide stability criteria for linear time-invariant delay systems described as

$$\dot{x}(t) = \sum_{d=0}^{m} A_d x_i(t - \tau_d). \tag{2.1}$$

Lemma 2.1. *Consider the linear delay system (2.1) with characteristic equation*

$$p(s) = \det\left(sI_p - \sum\nolimits_{d=0}^{m} A_d e^{-\tau_d s}\right) = 0,$$

let s be an unstable characteristic root of $p(s)$, then $|s| \leq r = \sum_{d=0}^{m} \|A_d\|$. Furthermore, system (2.1) is asymptotically stable if and only if

$$p(s) \neq 0 \text{ for } s \in l, \text{ and } \Delta_l \arg p(s) = 0,$$

where $l = l_1 \cup l_2$ is a closed half-circumference with radius r, $l_1 = \{s : s = \mathbf{i}w, -r \leq w \leq r, \mathbf{i}^2 = -1\}$ and $l_2 = \{s : |s| = r, -\pi/2 \leq \arg s \leq \pi/2\}$, the notation $\arg p(s)$ stands for the argument of complex function $p(s)$, and $\Delta_l \arg p(s)$ stands for the change of the argument as the complex variable s traverses the contour l once in the positive direction.

The explanation for Lemma 2.1 is as follows. First, we obtain that all the unstable characteristic roots of $p(s)$ are contained in a region Ω, which is enclosed by the half-circumference l (see Figure 1). Then, by means of the argument principle, the number of the unstable roots located in Ω is equal to $\Delta_l \arg p(s)/2\pi$. Hence, by calculating $\Delta_l \arg p(s)$, we can check the stability of system (2.1). The following Algorithm 1 is to check the stability of a linear delay system described by (2.1).

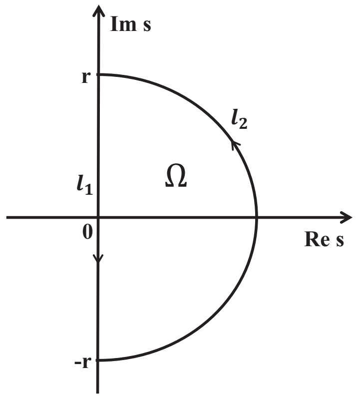

Figure 1. Explanation for Lemma 2.1.

3. Main results

In this section, we first prove that delayed MAS (1.1) can achieve consensus if and only if $N-1$ delayed systems associated with the eigenvalues of the Laplacian matrix of the network are

Algorithm 1 An algorithm to check the stability of linear delay system

1: Calculate the upper bound r of the unstable roots and get the half-circumference l.
2: Take a sufficiently large integer n_l to discretize l as uniformly as possible in the clockwise direction, and record these nodes as $\{s_k\}_{k=1}^{n_l}$.
3: For $k = 1, ..., n_l$, calculate complex value $p(s_k)$, check whether $p(s_k) = 0$ by evaluating its magnitude satisfies $|p(s_k)| \leq \delta_1$ with the preassigned tolerance δ_1. If it holds, i.e., $p(s_k) = 0$, then the system is not asymptotically stable and stop the algorithm. Otherwise, we continue.
4: Compute $\Delta_l \arg p(s)$ along the ordered node $\{s_k\}_{k=1}^{n_l}$ by checking $\left|\Delta_l \arg p(s)\right| \leq \delta_2$ with the preassigned tolerance δ_2. If $\Delta_l \arg p(s) = 0$, the system is asymptotically stable, otherwise is not stable.

simultaneously asymptotically stable. With the protocol (1.2), the overall closed-loop dynamics of the delayed MAS (1.1) can be written as

$$\dot{x}(t) = \sum_{d=0}^{m} (I_N \otimes A_d)\, x(t-\tau_d) + \sum_{d=0}^{m} (\mathcal{L} \otimes BK_d C)\, x(t-\tau_d-\tau_c), \tag{3.1}$$

where $x(t) = [x_1(t)^T, x_2(t)^T, ..., x_N(t)^T]^T \in \mathbb{R}^{Np}$ represents the aggregate state of the delayed MAS.

The matrix $\mathcal{L}$ is a real symmetric matrix and there exists an orthogonal matrix $U \in \mathbb{R}^{N\times N}$ such that $U^T \mathcal{L} U = \Lambda := \mathrm{diag}\{0, \lambda_2(\mathcal{L}), \lambda_3(\mathcal{L}), ..., \lambda_\mathrm{N}(\mathcal{L})\}$, where $0 < \lambda_2(\mathcal{L}) \leq \lambda_3(\mathcal{L}) \leq ... \leq \lambda_N(\mathcal{L})$. By using the linear transformation $\tilde{x}(t) = (U^T \otimes I_p)x(t)$, the overall closed-loop dynamics (3.1) becomes

$$\dot{\tilde{x}}(t) = \sum_{d=0}^{m} (I_N \otimes A_d)\, \tilde{x}(t-\tau_d) + \sum_{d=0}^{m} (\Lambda \otimes BK_d C)\, \tilde{x}(t-\tau_d-\tau_c). \tag{3.2}$$

Theorem 3.1. *Under an undirected and connected network graph, the delayed MAS (1.1) with the protocol (1.2) achieves consensus if and only if the following $N-1$ delayed subsystems are simultaneously asymptotically stable:*

$$\dot{\tilde{x}}_i(t) = \sum_{d=0}^{m} A_d \tilde{x}_i(t-\tau_d) + \lambda_i(\mathcal{L}) \sum_{d=0}^{m} BK_d C \tilde{x}_i(t-\tau_d-\tau_c), \quad i = 2, 3, ..., N. \tag{3.3}$$

Proof. Since $\ker(\mathcal{L}) = \mathrm{im}(\mathbf{1}_N)$, $x_i(t) - x_j(t) \to 0$ if and only if $x(t) \to \mathrm{im}(\mathbf{1}_N \otimes I_p) = \ker(\mathcal{L} \otimes I_p)$. This holds if and only if $(\mathcal{L} \otimes I_p)x(t) \to 0$, i.e., $(U\Lambda \otimes I_p)\tilde{x}(t) \to 0$. Since U is nonsingular, this holds if and only if $(\Lambda \otimes I_p)\tilde{x}(t) \to 0$, which is equivalent to $\tilde{x}_i(t) \to 0$ for $i = 2, 3, ..., N$. □

In order to seek the gain matrices $K_0, ..., K_m$ such that each delayed subsystem of (3.3) is asymptotically stable, we first consider the stability of the following linear delay system with uncertain parameters:

$$\dot{\eta}(t) = \sum_{d=0}^{m} A_d \eta(t-\tau_d) + (\alpha + \Delta\alpha) \sum_{d=0}^{m} BK_d C \eta(t-\tau_d-\tau_c), \tag{3.4}$$

where

$$\alpha := 0.5(\lambda_2(\mathcal{L}) + \lambda_N(\mathcal{L})) \quad \text{and} \quad |\Delta\alpha| \leq \overline{\Delta\alpha} := 0.5(\lambda_N(\mathcal{L}) - \lambda_2(\mathcal{L})). \tag{3.5}$$

Remark 3.2. *In system (3.4), the range of continuous parameter $\alpha + \Delta\alpha$ covers the closed interval $[\lambda_2(\mathcal{L}), \lambda_N(\mathcal{L})]$. It is obvious that if system (3.4) is robust asymptotically stable, then delayed subsystems (3.3) are simultaneously asymptotically stable. Therefore, if there are gain matrices $K_0, ..., K_m$ such that system (3.4) is robust asymptotically stable, then delayed MAS (1.1) can achieve consensus by using these gain matrices.*

In the rest of this section, we will mainly discuss the robust stability problem of system (3.4); its nominal model is

$$\dot{\eta}(t) = \sum_{d=0}^{m} A_d \eta(t - \tau_d) + \alpha \sum_{d=0}^{m} BK_d C\eta(t - \tau_d - \tau_c). \tag{3.6}$$

Before giving the main results results in this paper, the following lemmas and definitions are useful.

Lemma 3.3. *([28]) The nominal system (3.6) is asymptotically stable if and only if*

$$\int_0^\infty \|F(t)\|^2 dt \leq \beta, \tag{3.7}$$

where β is a finite constant, and $F(t)$ represents the state transition matrix of system (3.6), which is the solution of the matrix differential equation

$$\begin{aligned} \dot{F}(t) &= \sum_{d=0}^{m} A_d F(t - \tau_d) + \alpha \sum_{d=0}^{m} BK_d CF(t - \tau_d - \tau_c), \\ F(0) &= I_p, \quad F(t) = \mathbf{0}_{p\times p} \text{ for } t < 0. \end{aligned} \tag{3.8}$$

Lemma 3.4. *Denote the characteristic polynomial of system (3.4) by*

$$\tilde{P}(s) = \det\left(sI_p - \sum_{d=0}^{m} A_d e^{-\tau_d s} - (\alpha + \Delta\alpha) \sum_{d=0}^{m} BK_d C e^{-\tau_d s - \tau_c s} \right), \tag{3.9}$$

let s be an unstable characteristic root, i.e., $\tilde{P}(s) = 0$, then

$$|s| \leq r = \sum_{d=0}^{m} (\|A_d\| + \lambda_N(\mathcal{L})\|B\|\|K_d\|\|C\|). \tag{3.10}$$

Proof. Since s is an unstable root, Re s ≥ 0, $|e^{-\tau_d s}| \leq 1$, $|e^{-\tau_d s - \tau_c s}| \leq 1$, $\forall \tau_d, \tau_c \geq 0$. Define

$$W(s) := \sum_{d=0}^{m} A_d e^{-\tau_d s} + (\alpha + \Delta\alpha) \sum_{d=0}^{m} BK_d C e^{-\tau_d s - \tau_c s},$$

then $\tilde{P}(s) = \det\left(sI_p - W(s)\right) = 0$, this implies that s is an eigenvalue of matrix $W(s)$, and it can be deduced that

$$\begin{aligned} |s| = |\lambda_k(W(s))| \leq \|W(s)\| &= \| \sum_{d=0}^{m} A_d e^{-\tau_d s} + (\alpha + \Delta\alpha) \sum_{d=0}^{m} BK_d C e^{-\tau_d s - \tau_c s} \| \\ &\leq \sum_{d=0}^{m} \|A_d\| \, |e^{-\tau_d s}| + (\alpha + \Delta\alpha) \sum_{d=0}^{m} \|BK_d C\| \, |e^{-\tau_d s - \tau_c s}| \end{aligned}$$

$$\leq \sum_{d=0}^{m} (\|A_d\| + \lambda_N(\mathcal{L})\|BK_dC\|) \leq r.$$

□

Definition 3.5. *A semicircular region* Ω *is defined as* $\Omega = \{s : \text{Re } s \geq 0, |s| \leq r\}$*, where* r *is given by (3.24). The boundary of region* Ω *is denoted as* l*.*

Next, we present the main results for the robust stability of system (3.4).

Lemma 3.6. *System (3.4) is asymptotically stable if and only if*

$$\tilde{P}(s) \neq 0, \quad \text{for} \;\; s \in l \tag{3.11}$$

and

$$\triangle_l \arg \tilde{P}(s) = 0. \tag{3.12}$$

where $\tilde{P}(s)$ *is defined by (3.9) and* l *is given by Definition (3.5).*

Proof. The results can be derived by applying Lemma 2.1 to system (3.4). □

Theorem 3.7. *System (3.4) is robust asymptotically stable, if*

1) The nominal system (3.6) is asymptotically stable;

2) The condition

$$\sup_{s\in l} \|R^{-1}(s)\| \sum_{d=0}^{m} \|BK_dC\| < \frac{1}{\Delta\alpha} \tag{3.13}$$

holds, where

$$R(s) := sI_p - \sum_{d=0}^{m} A_d e^{-\tau_d s} - \alpha \sum_{d=0}^{m} BK_dCe^{-\tau_d s - \tau_c s}, \tag{3.14}$$

and l *is given by Definition 3.5.*

Proof. Since the nominal system (3.6) is asymptotically stable, $R(s)^{-1}$ exists for $s \in l$. Let $P(s) = \det(R(s))$ be the characteristic polynomial of the nominal system (3.6), let $D(s) = \Delta\alpha \sum_{d=0}^{m} BK_dCe^{-\tau_d s - \tau_c s}$. For $s \in l$, we have

$$\begin{aligned}
\tilde{P}(s) &= \det\left(sI_p - \sum_{d=0}^{m} A_d e^{-\tau_d s} - (\alpha + \Delta\alpha) \sum_{d=0}^{m} BK_dCe^{-\tau_d s - \tau_c s}\right) \\
&= \det\left(R(s) - D(s)\right) \\
&= \det\left(R(s)\right) \det\left(I_p - R^{-1}(s)D(s)\right) \\
&= P(s) \det\left(I_p - R^{-1}(s)D(s)\right).
\end{aligned}$$

Thus,

$$\tilde{P}(s) = P(s) \det\left(I_p - R^{-1}(s)D(s)\right). \tag{3.15}$$

According to Lemma 3.4 and Definition 3.5, all the unstable characteristic roots of $\tilde{P}(s)$ lie in region Ω. It is sufficient to check whether $\tilde{P}(s)$ has roots in Ω for the stability of system (3.4). Using (3.15), for $s \in l$,

$$\tilde{P}(s) = P(s) \prod_{k=1}^{p} \Big[1 - \lambda_k(R^{-1}(s)D(s))\Big]. \tag{3.16}$$

It can be known that the eigenvalue $\lambda_k(R^{-1}(s)D(s))$ is continuous for $s \in l$. Since the nominal system (3.6) is asymptotically stable, $P(s) \neq 0$ for $s \in l$. With the condition (3.13), for $s \in l$ and $k = 1, ..., p$, we have

$$\begin{aligned}
|\lambda_k(R^{-1}(s)D(s))| &\leq \|R^{-1}(s)D(s)\| \leq \|R^{-1}(s)\| \, \|D(s)\| \\
&= \|R^{-1}(s)\| \, \|\Delta\alpha \sum_{d=0}^{m} BK_d C e^{-\tau_d s - \tau_c s}\| \\
&\leq \|R^{-1}(s)\| \, |\Delta\alpha| \sum_{d=0}^{m} \|BK_d C\| \, |e^{-\tau_d s - \tau_c s}| \\
&\leq \sup_{s \in l} \|R^{-1}(s)\| \overline{\Delta\alpha} \sum_{d=0}^{m} \|BK_d C\| < 1,
\end{aligned}$$

i.e.,

$$|\lambda_k(R^{-1}(s)D(s))| < 1. \tag{3.17}$$

By means of (3.16) and (3.17), we can deduce that for $s \in l$,

$$\tilde{P}(s) \neq 0. \tag{3.18}$$

Furthermore, using (3.16) and (3.17), we have that for $s \in l$,

$$\Delta_l \arg \tilde{P}(s) = \Delta_l \arg P(s) + \sum_{k=1}^{p} \Delta_l \arg \Big[1 - \lambda_k(R^{-1}(s)D(s))\Big] = 0 + \sum_{k=1}^{p} 0 = 0, \tag{3.19}$$

where $\Delta_l \arg P(s) = 0$ since system (3.6) is asymptotically stable.

Using (3.18), (3.19), and Lemma 3.6, it can be deduced that system (3.4) is robust asymptotically stable, and the proof is completed. □

As is illustrated in the foregoing, if system (3.4) is robust asymptotically stable, it is sufficient to obtain that the delayed MAS (1.1) achieves consensus. Combining Lemma 3.3 and Theorem 3.7, we have the following corollary for the consensus of the delayed MAS (1.1).

Corollary 3.8. *Under an undirected and connected network graph represented by a Laplacian matrix $\mathcal{L}$, the delayed MAS (1.1) with the protocol (1.2) achieves consensus if there are gain matrices $K_0, K_1, ..., K_m$ such that*

1) $\int_0^\infty \|F(t)\|^2 dt \leq \beta$, *and*

2) $\sup_{s \in l} \|R^{-1}(s)\| \sum_{d=0}^{m} \|BK_d C\| < \frac{1}{\overline{\Delta\alpha}}$.

where $F(t)$ is subject to (3.8), $R(s)$ is defined in (3.14), and l is given by Definition 3.5.

For the sake of convenience, we introduce a decision variable $\vec{K} = [K_0, K_1, ..., K_m]$ to stand for all the gain matrices to be determined. Since $F(t)$ and $R(s)$ are functions of $\vec{K}$, let us re-denote them as $F(\vec{K}, t)$ and $R(\vec{K}, s)$, respectively. Our goal is to calculate $\vec{K}$ such that the two conditions in Corollary 3.8 are satisfied. Based on this, we construct an optimization problem that minimizes the objective function

$$\min_{\vec{K}} \; J(\vec{K}) = \int_0^T \|F(\vec{K}, t)\|_F^{\;2} dt \tag{3.20}$$

subject to equality constraints

$$\begin{cases} \dot{F}(\vec{K}, t) = \sum\limits_{d=0}^{m} A_d F(\vec{K}, t - \tau_d) + \alpha \sum\limits_{d=0}^{m} BK_d F(\vec{K}, t - \tau_d - \tau_c), \\ F(\vec{K}, t) = \mathbf{0}_{p\times p} \;\text{ for }\; t < 0 \;\text{ and }\; F(\vec{K}, 0) = I_p, \end{cases} \tag{3.21}$$

and inequality constraints

$$\sup_{s\in l} \|R^{-1}(\vec{K}, s)\|_F \sum\nolimits_{d=0}^{m} \|BK_d C\|_F < \frac{1}{\overline{\Delta\alpha}}, \tag{3.22}$$

$$\sum_{d=0}^{m} \|K_d\|_F \leq b, \tag{3.23}$$

where $R(\vec{K}, s)$ is given by (3.14), α and $\overline{\Delta\alpha}$ are given by (3.5), T is a sufficient large positive constant, b is a positive constant, and the symbol $\|\cdot\|_F$ represents the Frobenius norm.

Remark 3.9. *For solving the optimization problem, T is taken as a finite. Assume that there exists a feasible solution $\vec{K}_f$ such that the nominal system (3.6) is asymptotically stable, then we can find a constant T_0 and a sufficient small value ϵ such that*

$$\left| \int_0^T \|F(\vec{K}_f, t)\|_F^{\;2} - \int_0^\infty \|F(\vec{K}_f, t)\|_F^{\;2} \right| \leq \epsilon,$$

for $T \geq T_0$. In addition, if $\|F(\vec{K}_f, t)\|_F$ decreases faster (the transient process is very short), T_0 may not be vary large. Hence, for a sufficiently large constant T, an optimal solution $\vec{K}^*$ that minimizes the integral $\int_0^T \|F(\vec{K}, t)\|_F^{\;2} dt$ can compress the integral $\int_0^\infty \|F(\vec{K}, t)\|_F^{\;2} dt$ to the finite.

Remark 3.10. *The constraint (3.23) is necessary, since in engineering practice the cost of the control law should be constrained. On the other hand, this constraint is also used to determine the radius of the semicircle l. According to Lemma 3.4, we have*

$$r = \sum_{d=0}^{m} \|A_d\|_F + b\lambda_N(\mathcal{L})\|B\|_F\|C\|_F. \tag{3.24}$$

In order to obtain a numerical solution on $\vec{K}$, it is necessary to discrete the above optimization problem. Assume that the numerical solution gives a sequence of approximated values $\{F_1(\vec{K}), F_2(\vec{K}), ..., F_M(\vec{K})\}$ of $\{F(\vec{K}, t_1), F(\vec{K}, t_2), ..., F(\vec{K}, t_M)\}$ of Eq (3.8) on certain equidistant step-values $\{t_n = nh\}$ with the step-size $h = T/M$, where M is a positive integer. Let $v_d = \lceil \tau_d h^{-1} \rfloor$, $\delta_d = v_d - \tau_d h^{-1}$,

$0 \leq \delta_d < 1$, $n_d = \lceil(\tau_d + \tau_c)h^{-1}\rfloor$, $\varepsilon_d = n_d - (\tau_d + \tau_c)h^{-1}$, $0 \leq \varepsilon_d < 1$, for $d = 0, ..., m$. Besides, take a sufficiently small step-size h_l to discretize l as uniformly as possible in the clockwise direction, record these nodes as $\{s_k\}_{k=1}^{n_l}$, and denote $N_l = \{1, 2, ..., n_l\}$. In this paper, these discrete points on l are taken as $\left\{s_k = \mathbf{i}w : w = -r + kh_l,\ k = 0, 1, ..., \frac{2r}{h_l}\right\} \cup \left\{s_k = re^{j\theta} : \theta = -\frac{\pi}{2} + kh_l,\ k = 0, 1, ..., \frac{\pi}{h_l}\right\}$,

Thus, combining the Euler scheme with the linear interpolation, a numerical solvable version for the optimization problem with the objective function (3.20) subject to the constraints (3.21) and (3.22) is obtained as follows.

$$\min_{\vec{K}} J_n(\vec{K}) = \sum_{n=0}^{M} \|F_n(\vec{K})\|_F^{\,2} h \tag{3.25}$$

subject to

$$\begin{aligned}
&F_{n+1}(\vec{K}) = F_n(\vec{K}) + h\left[\sum_{d=0}^{m} A_d F_{n-\nu_d+\delta_d}(\vec{K}) + \alpha \sum_{d=0}^{m} BK_d F_{n-n_d+\varepsilon_d}(\vec{K})\right],\\
&F_{n-\nu_d+\delta_d}(\vec{K}) = (1-\delta_d)F_{n-\nu_d}(\vec{K}) + \delta_d F_{n-\nu_d+1}(\vec{K}) \ \text{ for } \ n - \nu_d + \delta_d > 0,\\
&F_{n-n_d+\varepsilon_d}(\vec{K}) = (1-\varepsilon_d)F_{n-n_d}(\vec{K}) + \varepsilon_d F_{n-n_d+1}(\vec{K}) \ \text{ for } \ n - n_d + \varepsilon_d > 0,\\
&F_{n-\nu_d+\delta_d}(\vec{K}) = \mathbf{0}_{p\times p} \ \text{ for } \ n - \nu_d + \delta_d < 0,\ \ F_{n-\nu_d+\delta_d}(\vec{K}) = I_p \ \text{ for } \ n - \nu_d + \delta_d = 0,\\
&F_{n-n_d+\varepsilon_d}(\vec{K}) = \mathbf{0}_{p\times p} \ \text{ for } \ n - n_d + \varepsilon_d < 0,\ \ F_{n-n_d+\varepsilon_d}(\vec{K}) = I_p \ \text{ for } \ n - n_d + \varepsilon_d = 0.
\end{aligned} \tag{3.26}$$

$$\sup_{k\in N_l} \|R^{-1}(\vec{K}, s_k)\|_F \sum_{d=0}^{m} \|BK_d C\|_F < \frac{1}{\Delta\alpha}, \tag{3.27}$$

and

$$\sum_{d=0}^{m} \|K_d\|_F \leq b. \tag{3.28}$$

Thus, the optimization problem with the objective function (3.20) subject to the constraints (3.21) and (3.22) is reduced to one with the objective function (3.25) subject to the constraints (3.26) and (3.27). This is a nonlinear minimization problem with the quadratic equality constraints and a non-convex inequality constraint.

Remark 3.11. *The numerical method that is used for the discretization of the optimization problem is not limited to the Euler method; other numerical methods such as Runge-Kutta or Multi-step methods can also be applied.*

Remark 3.12. *For the inequality constraint (3.27), it should be noticed that $R^{-1}(\vec{K}, s)$ is well defined only when the nominal system (3.6) is asymptotically stable. Hence, an initial value of $\vec{K}$ that stabilizes system (3.6) is needed for the iteration of the optimization algorithm. Such an initial value can be obtained by minimizing the objective function (3.25) subject to the constraints (3.26).*

Remark 3.13. *We note that the constraints may describe a non-convex feasibility region, and thus multiple local solutions can arise [29]. Conventional numerical techniques are efficient at locating an optimum but they do not necessarily locate the global optimum unless some a priori knowledge is available. It is necessary for checking whether the optimal solution $\vec{K}^*$ stabilizes the nominal system (3.6).*

Algorithm 2 Computing consensus gain matrices for delayed MAS

1: Given a starting point $\vec{K}^0$, a sufficient large constant T, a weight constant ρ.
2: Solve the optimization problem with objective function (3.25) subject to constraints (3.26) and obtain $\vec{K}_I = \{K_0^I, K_1^I, ..., K_m^I\}$.
3: For the $\vec{K}_I$ obtained in Step 2, use Algorithm 1 to check whether system (3.6) is asymptotically stable. If $\vec{K}_I$ stabilizes the system, then go to Step 4. Otherwise, increase T or choose another $\vec{K}^0$ and go to Step 2.
4: Let $\vec{K}_I$ be the starting point, solve the optimization problem with objective function (3.25) subject to constraints (3.26) and (3.27) and obtain $\vec{K}^* = \{K_0^*, K_1^*, ..., K_m^*\}$.
5: For the $\vec{K}^*$ obtained in step 4, use Algorithm 1 to check whether system (3.6) is asymptotically stable. If $\vec{K}^*$ stabilizes the system, then stop the algorithm. Otherwise, increase T or choose another $\vec{K}^0$ and go to Step 2.

In view of the above statements, we propose an algorithm to numerically compute the consensus gain matrices $\vec{K}$ as follows.

Remark 3.14. *Our results have a close relationship with the second smallest eigenvalue $\lambda_2(\mathcal{L})$ and the largest eigenvalue $\lambda_N(\mathcal{L})$ of the normalized Laplacian matrix $\mathcal{L}$. Denote the lower bound of $\lambda_2(\mathcal{L})$ by b_l and the upper bound of $\lambda_N(\mathcal{L})$ by b_u. From Remark 3.2, we know that as long as Theorem 3.7 holds for*

$$\alpha = 0.5(b_l + b_u) \ \ and \ \ \Delta\alpha = 0.5(b_u - b_l),$$

then delayed MAS (1.1) is consensusable. It is easy to obtain the upper bound of $\lambda_N(\mathcal{L})$ that

$$\lambda_N(\mathcal{L}) \leq \|\mathcal{L}\|_\infty = 2.$$

For the lower bound of $\lambda_2(\mathcal{L})$, we have the estimation as follows.

Lemma 3.15. *([30]) Let $\mathcal{G}'$ be an undirected and connected simple graph with N vertices, let $e(\mathcal{G}')$ denote the edge connectivity of $\mathcal{G}'$, i.e., the minimal number of edges whose removal would result in losing connectivity of the graph $\mathcal{G}'$, and let $\lambda_2(\mathcal{L})$ denote the second smallest eigenvalue associated with the Laplacian matrix of $\mathcal{G}'$. Then*

$$\lambda_2(\mathcal{L}) \geq 2e(\mathcal{G}')(1 - \cos\frac{\pi}{N}) \geq 2(1 - \cos\frac{\pi}{N}).$$

Remark 3.16. *The proposed algorithm based on Theorem 3.1 is less conservative than the existing methods based on LMI. Theorem 3.1 is a sufficient and necessary condition for asymptotic stability of system* (3.3)*. However, the results based on LMI are only sufficient conditions for asymptotic stability of system* (3.3)*.*

4. Numerical examples

In this section, we provide two examples to illustrate the main results. For both examples, we consider a network graph with the Laplacian matrix and the normalized Laplacian matrix as follows.

$$L = \begin{bmatrix} 5 & -1 & -2 & -2 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} 1 & -0.2 & -0.4 & -0.4 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

where the eigenvalues of $\mathcal{L}$ are $\lambda_1(\mathcal{L}) = 0, \lambda_2(\mathcal{L}) = \lambda_3(\mathcal{L}) = 1, \lambda_4(\mathcal{L}) = 2$.

For solving the optimization problems in Algorithm 2, we use the interior point method, which is included by the OPTI Toolbox of the MATLAB.

Example 4.1. *Consider the dynamics of agent i given by*

$$\dot{x}_i(t) = A_0 x_i(t) + A_1 x_i(t-\tau_1) + A_2 x_i(t-\tau_2) + Bu_i(t), \ \ y_i(t) = Cx_i(t),$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},$$

$$\tau_1 = 1, \tau_2 = 2, \tau_c = 0.5.$$

Let the controller structure be

$$u_i(t) = \sum_{d=0}^{2} \left(\frac{K_d}{d_i} \sum_{j \in \mathcal{N}_i} a_{ij} \left[y_i(t-\tau_d-\tau_c) - y_j(t-\tau_d-\tau_c) \right] \right)$$

with the constraint

$$\|K_0\|_F + \|K_1\|_F + \|K_2\|_F \leq 10.$$

By calculating, we obtain the radius $r = 11.20$. Let the step-size be $h = h_l = 0.01$ and the initial value be $\vec{K}^0 = 0_{1\times3}$.

For $T = 10$, using Algorithm 2 we obtain the gains

$$K_0 = -1.1654, \ K_1 = -0.0321, \ K_2 = -0.0192.$$

For $T = 20$, we obtain the gains

$$K_0 = -1.2054, \ K_1 = -0.0024, \ K_2 = -0.0089.$$

Using Lemma 2.1, we check that the subsystems (3.3) with the above two groups of gains are both asymptotically stable, i.e., the delayed MAS (1.1) achieves consensus. By taking the initial values as $x_1(t) = [50, 20]^T$, $x_2(t) = [-50, -20]^T$, $x_3(t) = [100\cos(t), 60e^t]^T$, $x_4(t) = [-100 + \sin(t), -60e^t]^T$ for $t \in [-2.5, 0]$, we draw the trajectories of the delayed MAS (1.1) with the gain matrices $K_0 = -1.2054$, $K_1 = -0.0024$, $K_2 = -0.0089$ in Figures 2 and 3. From the figures, we can see that the consensus is achieved within 10 seconds.

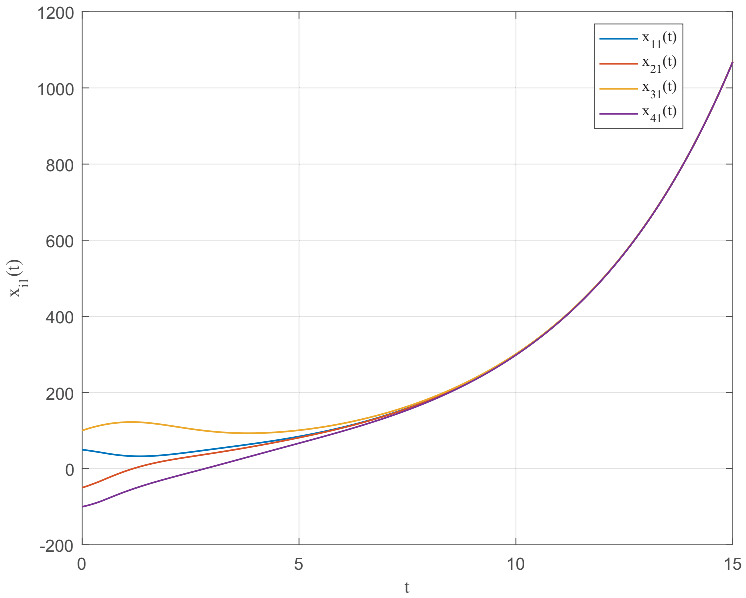

Figure 2. Example 1: trajectories of $x_{i1}(t)$, $i = 1, 2, 3, 4$.

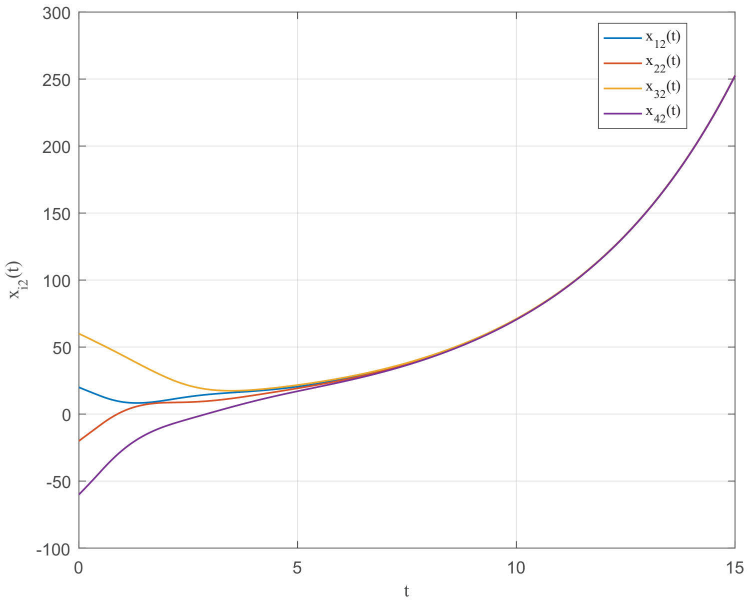

Figure 3. Example 1: trajectories of $x_{i2}(t)$, $i = 1, 2, 3, 4$.

Let the controller structure be

$$u_i(t) = \frac{K_0}{d_i} \sum_{j \in N_i} a_{ij} \left[y_i(t - \tau_c) - y_j(t - \tau_c) \right],$$

where $||K_0||_F \leq 10$.

For $T = 20$, we obtain the gain $K_0 = -1.2166$. Using Lemma 2.1, we check that the delayed MAS (1.1) can also achieve consensus with this gain.

Example 4.2. *Consider the dynamics of agent i given by*

$$\dot{x}_i(t) = A_0 x_i(t) + A_1 x_i(t-\tau_1) + A_2 x_i(t-\tau_2) + Bu_i(t), \quad y_i(t) = Cx_i(t),$$

where

$$A_0 = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -0.8 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & -0.1 & 0 & 0.2 \\ 0 & -0.2 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.3 \\ 0 & -0.3 & 0.1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.1 \\ -0.1 \\ -0.1 \\ 0.1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0.1 & 0 & 0 \\ 0.2 & 0 & 0.1 & 0.1 \end{bmatrix}, \tau_1 = 0.2, \tau_2 = 0.3, \tau_c = 0.1.$$

Let the controller structure be

$$u_i(t) = \sum_{d=0}^{2} \left(\frac{K_d}{d_i} \sum_{j \in \mathcal{N}_i} a_{ij} \left[y_i(t-\tau_d-\tau_c) - y_j(t-\tau_d-\tau_c) \right] \right)$$

with the constraint

$$\|K_0\|_F + \|K_1\|_F + \|K_2\|_F \leq 10.$$

By calculating, we obtain the radius $r = 5.62$. Let the step-size $h = k_l = 0.01$ and the initial value be $\vec{K}^0 = 0_{1\times 6}$.

For $T = 10$, using Algorithm 2, we obtain the gain matrices

$$K_0 = \begin{bmatrix} 0.2246 & -1.8883 \end{bmatrix}, K_1 = \begin{bmatrix} -0.0851 & -1.3535 \end{bmatrix}, K_2 = \begin{bmatrix} -0.1969 & -1.3399 \end{bmatrix}.$$

For $T = 20$, we obtain the gain matrices

$$K_0 = \begin{bmatrix} 0.2228 & -1.8868 \end{bmatrix}, K_1 = \begin{bmatrix} -0.0870 & -1.3536 \end{bmatrix}, K_2 = \begin{bmatrix} -0.1992 & -1.3411 \end{bmatrix}.$$

Using Lemma 2.1 we check that the delayed MAS (1.1) achieves consensus. By taking the initial values as $x_1(t) = [5, 5\cos(t), 12e^t, 13(t+1)]^T$, $x_2(t) = [-5, -5+\sin(t), -12e^t, -15(t+1)]^T$, $x_3(t) = [10, 10(t+1), 20\cos(t), 20e^t]^T$, $x_4(t) = [-10, -10(t+1)^2, -20, -20+t]^T$ for $t \in [-2.5, 0]$, we draw the differences of corresponding components between $x_i(t)$ and $x_1(t)$ with the second group of gain matrices above in Figure 4. From the figure, we can see that the consensus is achieved within 15 seconds.

Let the controller structure be

$$u_i(t) = \frac{K_0}{d_i} \sum_{j \in N_i} a_{ij} \left[y_i(t-\tau_c) - y_j(t-\tau_c) \right],$$

where $\|K_0\|_F \leq 10$.

For $T = 20$, we obtain the gain matrix $K_0 = \begin{bmatrix} -0.5255 & -3.1183 \end{bmatrix}$. Using Lemma 2.1, we check that the delayed MAS (1.1) achieve consensus.

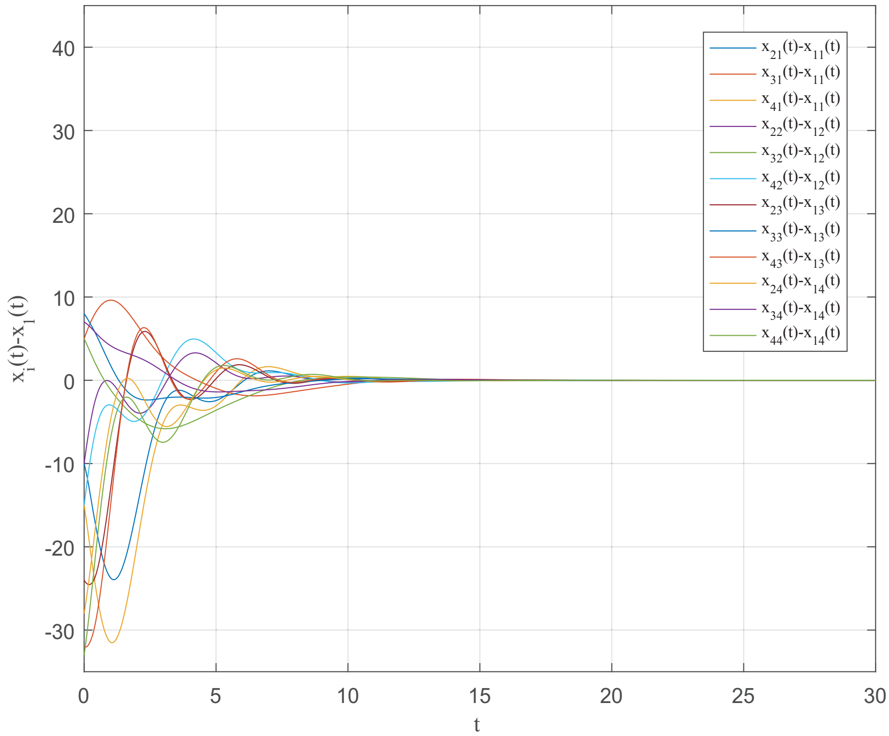

Figure 4. Example 2: differences of corresponding components between $x_i(t)$ and $x_1(t)$, $i = 2, 3, 4$.

Remark 4.3. *In principle, the greater T is, the better effect of $\vec{K}$ is for the consensus problem. However, a great value of T needs more computational effort. In the implementation of solving the optimization problem with objective function (3.25) subject to constraints (3.26) and (3.27), when T is large enough, the gain matrices change a little. The examples show the argument.*

Remark 4.4. *The examples show that, under the controller structure with less feedback term (e.g., $\vec{K} = K_0$), the delayed MAS can also achieve consensus as long as there exists a feasible solution for $\vec{K}$ such that the Corollary 3.8 holds. Hence, we can flexibly try different controller structures and solve the gain matrices by Algorithm 2.*

5. Conclusions

This paper investigated the consensus problem of linear MASs with multiple state delays and communication delay under an undirected network graph. The consensus problem of N agents was converted into the simultaneously asymptotically stable problem of $N - 1$ delayed subsystems associated with the eigenvalues of normalized Laplacian matrix by employing a linear transformation. We presented a sufficient condition for the consensus of delayed MASs in the form of the integral performance and inequalities, which were derived by integral stability criteria for linear delay systems and argument principle. Based on the obtained condition, we formulated an optimization-based framework to numerically solve the consensus gain matrices. The numerical examples demonstrated that the proposed Algorithm is efficient for the consensus of the delayed MASs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflicts of interest.

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