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Research article

Global solution to the complex short pulse equation

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Abstract: This paper deals with global well-posedness of the solution to the complex short pulse equation. We first use regularized technology and the approximation argument to prove the local existence and uniqueness of this equation. Then, based on conserved quantities and energy analysis, we show that the solution can be extended globally in time for suitably small initial data.

Keywords: complex short pulse equation; conserved quantities; regularized solution; energy method; uniqueness

1. Introduction

The short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \tag{1.1}$$

was derived by Schäfer and Wayne [1] as a nonlinear model to describe the propagation of ultra-short optical pulses in isotropic optical fibers. Here, $u = u(t, x)$ is a real-valued function, representing the magnitude of the electric field, and the subscripts denote partial derivatives with respect to t and x. It is an integrable differential equation and has attracted much attention in the past two decades. Various solutions to this equation have been obtained, including, its periodic and solitary wave solutions in [2]; loop and pulse solutions in [3]; two-loop soliton solutions in [4]; and multiloop solutions, multibreather, and periodic solutions in [5, 6]. Concerning the Cauchy problem of (1.1), local well-posedness of solution was obtained in [1, 7], where in [7] the global existence of the solution for small initial data in H^2 was also established, and modified scattering behavior was proved in [8–10] under different conditions on initial data.

A general model (the generalized Ostrovsky equation) related to the short pulse equation is

$$u_{xt} = u + (u^p)_{xx} \tag{1.2}$$

where $p \geq 2$ is an integer. The case of $p = 2$ is usually referred as the Ostrovsky-Hunter equation [11] and short-wave equation [12]. For $p \geq 4$, the global well-posedness and scattering was proved by

Stefanov et al. [13] and Hayashi et al. [14]. Apart from the above generalization of the short pulse equation (1.1), there are several other different versions, such as, the higher-order nonlinearity corrections in [15], the vector short pulse equations in [16, 17], and the multi-component short pulse model [18, 19].

In this paper we study the complex short pulse equation

$$q_{xt} + q + \frac{1}{2}(|q|^2 q_x)_x = 0 \tag{1.3}$$

where $q(t, x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$, is a complex-valued function. Equation (1.3) was produced from the negative order Wadati-Konno-Ichikawa (WKI) hierarchy in [20, 21], where the Lax pair for the whole WKI hierarchy and algebraic structure with r-matrix were discussed. See [18, 22] for the derivation of this equation and [23–25] for the symmetry methods to solve the equations. Zhaqilao et al. [19] studied multi-soliton solutions and the Cauchy problem for (1.3). As far as we know, the global existence theory for this equation has not been established, and the goal of this paper is to obtain the global well-posedness result.

To apply the partial differential equation theories for (1.3), we rewrite it into a first order equation with respect to t. Hence, we integrate Eq (1.3) to get

$$q_t + \partial_x^{-1} q + \frac{1}{2}|q|^2 q_x = 0, \tag{1.4}$$

where $\partial_x^{-1} q$ is defined through the Fourier transform, namely,

$$\mathcal{F}(\partial_x^{-1} f)(\xi) = \frac{1}{i\xi}\hat{f}(\xi).$$

From now on, we mainly focus on Eq (1.4).

Throughout the paper, L^p ($p \geq 1$) is the usual Lebesgue space, H^k denotes the inhomogeneous Sobolev spaces equipped with the norm

$$\|f\|_{H^k} := \|(1 + \xi^2)^{k/2} \hat{f}(\xi)\|_{L^2},$$

and $\dot{H}^{-1}$ is the homogeneous Sobolev space with

$$\|f\|_{\dot{H}^{-1}} := \||\xi|^{-1} \hat{f}(\xi)\|_{L^2}.$$

The main results of the paper are the following two theorems.

Theorem 1.1. *Assume that the initial data $q_0 \in H^2 \cap \dot{H}^{-1}$. Then, there exists a time $T > 0$, depending only on the norms of initial data, such that the complex short pulse equation* (1.4) *has a unique solution $q \in C([0, T]; H^2 \cap \dot{H}^{-1})$ satisfying $q(0) = q_0$. Moreover, if T^* is the time that the solution can not be continued to $T = T^*$, then either $T^* = \infty$ or $\|q(t, x)\|_{H^2}$ tends to infinite as $t \to T^*$.*

We remark that the sine-Gordon transformation method studying the Eq (1.1) in [7] doesn't work for the complex Eq (1.3). Here, we use regularized technology to compensate the loss of derivative for the nonlinear term and take approximation argument to obtain the local existence result for the complex short pulse equation.

To state the global result, we set

$$\mathcal{H}_1(0) = \int_{\mathbb{R}} |q_0|^2 dx,$$
$$\mathcal{H}_2(0) = \int_{\mathbb{R}} (\sqrt{1+|q_{0x}|^2} - 1) dx,$$
$$\mathcal{H}_3(0) = \int_{\mathbb{R}} \Big[\frac{|q_{0xx}|^2}{(1+|q_{0x}|^2)^{\frac{5}{2}}} - \frac{1}{4} \frac{(q_{0x}\overline{q_{0xx}} - \overline{q_{0x}} q_{0xx})^2}{(1+|q_{0x}|^2)^{\frac{5}{2}}} \Big] dx,$$

and we will show that they are all conserved as long as the solution exists (see Lemma 4.1 in Section 4). We note that the second term in the quantity $\mathcal{H}_3(0)$ is new and nonzero in our complex case. The conserved quantity $\mathcal{H}_3$ is derived by delicate analysis and computations. We now state the global existence of solution to (1.3). Due to the work of Liu et al. [26, 27] where wave-break phenomena is demonstrated, the global result obtained here requires the smallness assumption on the initial data.

Theorem 1.2. *Let $q_0 \in H^2 \cap \dot{H}^{-1}$ with $\mathcal{H}_1(0), \mathcal{H}_2(0), \mathcal{H}_3(0)$ small. Then, the complex short pulse equation* (1.3) *admits a unique global solution $q \in C([0, +\infty); H^2 \cap \dot{H}^{-1})$ satisfying $q(0) = q_0$.*

To prove Theorem 1.2, the crucial step is to obtain the H^2 bound for the solution q from the above three conserved quantities. This aim is achieved by combining change of variable, interpolation inequalities, and the continuous lemma.

This paper is organized as follows. In Section 2, we use the regularized operator to construct a regularized equation for (1.4) and prove the global well-posedness of smooth solution to this equation. Giving an *a priori* estimate for the regularized equation and taking the limit argument, Theorem 1.1 is proved in Section 3. Finally, in Section 4, we present the proof of Theorem 1.2.

2. A regularization of the complex short pulse equation

In this section, we will prove the existence of solution for a regularized problem of the complex short pulse equation. To this end, we introduce the regularized operator $J_\epsilon = (I - \epsilon\partial_{xx})^{-1}$ and consider the following regularized equation

$$q_t^\epsilon + \partial_x^{-1} q^\epsilon + \frac{1}{2} J_\epsilon [|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)] = 0. \tag{2.1}$$

Formally, when $\epsilon \to 0$, Eq (2.1) converges to (1.4). Rigorous justification of this convergence behavior actually gives the proof of Theorem 1.1. Here, we show the global well-poseness for the regularized equation (2.1).

Using the approach of Fourier transform, we can derive the following properties for the operator J_ϵ.

Lemma 2.1. *The following two statements hold for J_ϵ.*
(i) Suppose $f(x) \in H^k(\mathbb{R})$ with $k \geq 0$. Then,

$$\|J_\epsilon f(x)\|_{H^k} \leq \|f(x)\|_{H^k}, \tag{2.2}$$
$$\|J_\epsilon f(x)\|_{H^{k+2}} \leq C(\epsilon) \|f(x)\|_{H^k}, \tag{2.3}$$
$$\|J_\epsilon f(x)\|_{L^\infty} + \|J_\epsilon f'(x)\|_{L^\infty} \leq C(\epsilon) \|f(x)\|_{L^2} \tag{2.4}$$

where $C(\epsilon)$ denotes a constant depending on ϵ.

(ii) If $u(x) \in L^2(\mathbb{R})$, $v(x) \in L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} (J_\epsilon u) \cdot v dx = \int_{\mathbb{R}} u \cdot (J_\epsilon v) dx. \tag{2.5}$$

Proof. Using the definition of the H^{k+2}-norm and the property of the Fourier transform, for any given $\epsilon > 0$, one gets

$$\begin{aligned} \|J_\epsilon f\|_{H^{k+2}} &= \|(1+\xi^2)^{\frac{k+2}{2}} \widehat{J_\epsilon f}(\xi)\|_{L^2} \\ &= \Big(\int_{\mathbb{R}} (1+\xi^2)^{k+2} \frac{1}{(1+\epsilon\xi^2)^2} |\widehat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}} \\ &\le C(\epsilon) \Big(\int_{\mathbb{R}} (1+\xi^2)^k |\widehat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}} \\ &= C(\epsilon) \|f\|_{H^k}. \end{aligned}$$

Hence, the estimate (2.3) holds. The proof of (2.2) is similar. For (2.4), we have

$$\|J_\epsilon f(x)\|_{L^\infty} + \|J_\epsilon f'(x)\|_{L^\infty} \le C\|J_\epsilon f(x)\|_{H^2} \le C(\epsilon)\|f(x)\|_{L^2}.$$

The equality (2.5) follows by using the fundamental property of the Fourier transform. □

Taking the basic L^2 energy estimate, we can obtain following conservation law.

Lemma 2.2. *Suppose that $q^\epsilon \in C([0,T]; H^2 \cap \dot{H}^{-1})$ is a solution of* (2.1)*. Then, we have*

$$\int_{\mathbb{R}} |q^\epsilon(t,x)|^2 dx = const, \quad t \in [0,T].$$

Proof. Note that the above assumption implies $\partial_x^{-1} q^\epsilon \in C([0,T]; H^1)$, so $\partial_x^{-1} q^\epsilon$ decays to zero as $|x| \to \infty$. Multiplying (2.1) by $2\overline{q^\epsilon}$ and integrating the real part over $\mathbb{R}$, we obtain

$$\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 = -\mathrm{Re} \int_{\mathbb{R}} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)] \cdot \overline{q^\epsilon} dx.$$

By Lemma 2.1 (*ii*), we have

$$\begin{aligned} \mathrm{Re} \int_{\mathbb{R}} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)] \cdot \overline{q^\epsilon} dx &= \mathrm{Re} \int_{\mathbb{R}} |J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon) \cdot (J_\epsilon \overline{q^\epsilon}) dx \\ &= \frac{1}{4} \int_{\mathbb{R}} \partial_x (|J_\epsilon q^\epsilon|^4) dx \\ &= 0. \end{aligned}$$

So the L^2 conserved quantity is proved. □

From Lemma 2.2 and (2.4), we see that in the linear level the role of J_ϵ gives L^∞ estimate of $J_\epsilon q^\epsilon$ which is crucial in the global extension argument of Theorem 2.1. Moreover, as we will see later, in the nonlinear level, the appearance of J_ϵ absorbs the derivative in the nonlinear term which makes the estimate for such terms easier.

Now, we give the global existence result for the regularized problem (2.1).

Theorem 2.1. *For any given $\epsilon > 0$ and $q_0^\epsilon \in H^k \cap \dot{H}^{-1}$ with $k \geq 2$ as an integer, the Eq* (2.1) *admits a unique solution*

$$q^\epsilon \in C([0, +\infty); H^k \cap \dot{H}^{-1})$$

satisfying $q^\epsilon(0) = q_0^\epsilon$.

Proof. Our proof is based on the contraction mapping principle and continuation principle of an autonomous ODE on a Banach space. We split the proof into two steps.

1) First, we prove the local well-posedness of the Eq (2.1). The fundamental solution of the linear problem for (2.1)

$$\begin{cases} Q_t^\epsilon + \partial_x^{-1} Q^\epsilon = 0, \\ Q^\epsilon(0) = Q_0^\epsilon \end{cases}$$

is

$$Q^\epsilon(t) = e^{-t\partial_x^{-1}} Q_0^\epsilon.$$

Note that the solution operator $e^{-t\partial_x^{-1}}$ is a norm-preserving map from H^k to H^k in the sense of

$$\|Q^\epsilon(t)\|_{H^k} = \|e^{-t\partial_x^{-1}} Q_0^\epsilon\|_{H^k} = \|Q_0^\epsilon\|_{H^k}. \tag{2.6}$$

By Duhamel's principle, we obtain the integral equation for the nonlinear problem (2.1) satisfying $q^\epsilon(0) = q_0^\epsilon$:

$$q^\epsilon = Q^\epsilon - \frac{1}{2} \int_0^t e^{-(t-s)\partial_x^{-1}} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)](s) ds,$$

from which we notice that $Q_0^\epsilon = q_0^\epsilon$.

Define the operator Φ by

$$\Phi(q^\epsilon) = Q^\epsilon - \frac{1}{2} \int_0^t e^{-(t-s)\partial_x^{-1}} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)](s) ds.$$

To prove the local well-posedness result, we need to show Φ maps H^k into H^k and Φ is locally Lipschitz continuous in H^k.

By the triangle inequality, the norm preserving property (2.6) and Lemma 2.1 (i), we deduce

$$\begin{aligned} \|\Phi(q^\epsilon)\|_{H^k} &\leq \|Q^\epsilon\|_{H^k} + \frac{1}{2} \int_0^t \left\| e^{-(t-s)\partial_x^{-1}} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)] \right\|_{H^k} ds \\ &= \|q_0^\epsilon\|_{H^k} + \frac{1}{2} \int_0^t \left\| J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)] \right\|_{H^k} ds \\ &\leq \|q_0^\epsilon\|_{H^k} + C(\epsilon) \int_0^t \|q^\epsilon\|_{H^k}^3 ds, \end{aligned}$$

where, for the last inequality, we have used the Banach algebra property (see, e.g., [28])

$$\|fg\|_{H^s} \leq C_s \|f\|_{H^s} \|g\|_{H^s}, \ s > \frac{1}{2}. \tag{2.7}$$

Therefore, this shows that the map Φ is a closed map of H^k to itself. Moreover, a similar analysis can be used to prove that Φ is Lipschitz with respect to q^ϵ and it is contractive if $T_\epsilon = T(\epsilon, q_0^\epsilon) > 0$ is sufficiently small.

Hence, by the contraction mapping principle, we know that there exists a unique solution $q^\epsilon(t,x) \in C([0,T_\epsilon);H^k)$ for some $T_\epsilon > 0$. As $q_0^\epsilon \in \dot{H}^{-1}$, we can also show $q^\epsilon(t,x) \in C([0,T_\epsilon);H^{-1})$. Indeed, differentiating equation (2.1) gives

$$q_{tt}^\epsilon + \partial_x^{-1} q_t^\epsilon + \frac{1}{2} J_\epsilon[|J_\epsilon q^\epsilon|_t^2 J_\epsilon q_x^\epsilon] + \frac{1}{2} J_\epsilon[|J_\epsilon q^\epsilon|^2 J_\epsilon q_{xt}^\epsilon] = 0,$$

which is equivalent to

$$\begin{aligned} q_{tt}^\epsilon + \partial_x^{-1} q_t^\epsilon =& -\frac{1}{2} J_\epsilon[(J_\epsilon q_t^\epsilon J_\epsilon \overline{q^\epsilon} + J_\epsilon q^\epsilon J_\epsilon \overline{q_t^\epsilon}) J_\epsilon q_x^\epsilon] \\ & -\frac{1}{2} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q^\epsilon + \frac{1}{2} J_\epsilon^2 (|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon)_x]. \end{aligned}$$

Thus, we have

$$\begin{aligned} q_t^\epsilon =& e^{-t\partial_x^{-1}} q_t^\epsilon(0) - \frac{1}{2}\int_0^t e^{-(t-s)\partial_x^{-1}} J_\epsilon[(J_\epsilon q_t^\epsilon J_\epsilon \overline{q^\epsilon} + J_\epsilon q^\epsilon J_\epsilon \overline{q_t^\epsilon}) J_\epsilon q_x^\epsilon] ds \\ & - \frac{1}{2}\int_0^t e^{-(t-s)\partial_x^{-1}} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q^\epsilon + \frac{1}{2} J_\epsilon^2 (|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon)_x] ds \end{aligned}$$

which implies

$$\begin{aligned} \sup_{t\in[0,\tau]} \|q_t^\epsilon(t,x)\|_{H^{k-2}} \le& C\|q_0^\epsilon\|_{\dot{H}^{-1}\cap H^k} + C\tau \sup_{t\in[0,T_\epsilon]} \|q^\epsilon(t,x)\|_{H^k}^2 \sup_{t\in[0,\tau]} \|q_t^\epsilon(t,x)\|_{H^{k-2}} \\ & + C\tau \sup_{t\in[0,T_\epsilon]} \|q^\epsilon(t,x)\|_{H^k}^3. \end{aligned}$$

This shows that $q_t^\epsilon(t,x) \in H^{k-2}$ for

$$\tau \le \tau_0 := [2C \sup_{t\in[0,T_\epsilon]} \|q^\epsilon(t,x)\|_{H^k}^2]^{-1}.$$

Moreover, with similar arguments, we can easily obtain

$$\|q_t^\epsilon(t_2,x) - q_t^\epsilon(t_1,x)\|_{H^{k-2}} \le C|t_2 - t_1| \sup_{t\in[0,T_\epsilon]} \|q^\epsilon(t,x)\|_{H^k}^3, \quad t_1, t_2 \in [0,\tau_0],$$

so $q_t^\epsilon \in C([0,\tau_0];H^{k-2})$. Since τ_0 depends only on the H^k energy norm of q^ϵ, a bootstrap argument shows that the H^{k-2}-norm continuity of q_t^ϵ holds in the whole interval $[0,T_\epsilon)$. This result together with Eq (2.1) also imply that

$$q^\epsilon \in C([0,T_\epsilon);\dot{H}^{-1}).$$

2) Second, we will show that $T_\epsilon = +\infty$. Assume the maximal existence time $T_\epsilon < +\infty$, from the continuation principle, it suffices for us to obtain an *a priori* bound for $\|q^\epsilon(\cdot,t)\|_{H^k}$ in the time interval $[0,T_\epsilon)$. From (2.1), we have

$$q_{xt}^\epsilon + q^\epsilon + \frac{1}{2} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)]_x = 0. \tag{2.8}$$

Multiplying (2.8) by $2\overline{q_x^\epsilon}$ and integrating the real part over $\mathbb{R}$, we obtain

$$\frac{d}{dt}\|q_x^\epsilon\|_{L^2}^2 + \mathrm{Re}\int_{\mathbb{R}} J_\epsilon[|J_\epsilon q^\epsilon|^2 (J_\epsilon q_x^\epsilon)]_x \cdot \overline{q_x^\epsilon} dx = 0. \tag{2.9}$$

Applying Cauchy-Schwarz inequality and Lemma 2.1 (i), the nonlinear term is estimated by

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}} J_\epsilon[|J_\epsilon q^\epsilon|^2(J_\epsilon q_x^\epsilon)]_x \cdot \overline{q_x^\epsilon}dx &\le \left\|J_\epsilon[|J_\epsilon q^\epsilon|^2(J_\epsilon q_x^\epsilon)]_x\right\|_{L^2}\|\overline{q_x^\epsilon}\|_{L^2}\\ &\le C(\epsilon)\left\||J_\epsilon q^\epsilon|^2(J_\epsilon q_x^\epsilon)\right\|_{L^2}\|\overline{q_x^\epsilon}\|_{L^2}\\ &\le C(\epsilon)\left\||J_\epsilon q^\epsilon|^2\right\|_{L^\infty}\|q_x^\epsilon\|_{L^2}^2\\ &\le C(\epsilon)\|q_x^\epsilon\|_{L^2}^2,\end{aligned}$$

where, for the last inequality we have used Lemma 2.2 and (2.4). As a result, we conclude that

$$\frac{d}{dt}\|q_x^\epsilon\|_{L^2}^2 \le C(\epsilon)\|q_x^\epsilon\|_{L^2}^2,$$

which, by Gronwall's inequality, gives us

$$\|q_x^\epsilon(t,x)\|_{L^2} \le C(\epsilon, T_\epsilon),\ t \in [0, T_\epsilon). \tag{2.10}$$

Taking the partial derivative ∂_x to Eq (2.8) and multiplying both sides by $2\overline{q_{xx}^\epsilon}$, we have

$$\frac{d}{dt}\|q_{xx}^\epsilon\|_{L^2}^2 = I_1 + I_2 + I_3 \tag{2.11}$$

with

$$\begin{aligned}I_1 &:= -\,\mathrm{Re}\int_{\mathbb{R}} |J_\epsilon q^\epsilon|_{xx}^2 J_\epsilon q_x^\epsilon \cdot J_\epsilon\overline{q_{xx}^\epsilon}dx,\\ I_2 &:= -\,2\mathrm{Re}\int_{\mathbb{R}} |J_\epsilon q^\epsilon|_x^2 J_\epsilon q_{xx}^\epsilon \cdot J_\epsilon\overline{q_{xx}^\epsilon}dx,\\ I_3 &:= -\,\mathrm{Re}\int_{\mathbb{R}} |J_\epsilon q^\epsilon|^2 J_\epsilon q_{xxx}^\epsilon \cdot J_\epsilon\overline{q_{xx}^\epsilon}dx.\end{aligned}$$

We use Hölder's inequality and (2.4) to obtain

$$\begin{aligned}|I_1| + |I_2| &\le C\|J_\epsilon q_{xx}^\epsilon\|_{L^2}^2\|J_\epsilon q^\epsilon\|_{L^\infty}\|J_\epsilon q_x^\epsilon\|_{L^\infty} + C\|J_\epsilon q_{xx}^\epsilon\|_{L^2}\|J_\epsilon q_x^\epsilon\|_{L^2}\|J_\epsilon q_x^\epsilon\|_{L^\infty}^2\\ &\le C(\epsilon)\|J_\epsilon q_{xx}^\epsilon\|_{L^2}^2 + C(\epsilon).\end{aligned}$$

For the term I_3, we first integrate it by part, then we have

$$|I_3| \le C\|J_\epsilon q_{xx}^\epsilon\|_{L^2}^2\|J_\epsilon q^\epsilon\|_{L^\infty}\|J_\epsilon q_x^\epsilon\|_{L^\infty} \le C(\epsilon)\|J_\epsilon q_{xx}^\epsilon\|_{L^2}^2.$$

Hence, by (2.11) and Gronwall's inequality, we get

$$\|q_{xx}^\epsilon(t,x)\|_{L^2} \le C(\epsilon, T_\epsilon),\ t \in [0, T_\epsilon), \tag{2.12}$$

which yields the boundedness of the H^2 norm for q^ϵ.

Applying a similar argument as above, we can actually obtain

$$\|q^\epsilon(t,x)\|_{H^k} \le C(\epsilon, T_\epsilon),\quad t \in [0, T_\epsilon),$$

which implies $T_\epsilon = +\infty$. Hence, the proof of Theorem 2.1 is finished. □

We remark that the constant $C(\epsilon)$ that appears in the above proof tends to ∞ as $\epsilon \to 0$.

3. Local existence and uniqueness

We now present the proof of Theorem 1.1.
Proof of Theorem 1.1. To prove the local existence result of Theorem 1.1, we should drive an *a priori* estimate for the solution q^ϵ of the regularized equation. Note that the constant C obtained in (2.10) and (2.12) depends on ϵ, which are not sufficient for our argument. We first regularize the initial data by

$$q_0^\epsilon(x) := (q_0 * \rho_\epsilon)(x),$$

where $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$ and $\rho(x)$ is a radial function satisfying

$$\rho(x) \in C_0^\infty(\mathbb{R}),\ \rho(x) \geq 0,\ \int_{\mathbb{R}} \rho(x)dx = 1.$$

Clearly, we have $q_0^\epsilon(x) \in H^m(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ for all $m \geq 0$, and

$$\lim_{\epsilon\to 0} \|q_0^\epsilon - q_0\|_{H^2\cap \dot{H}^{-1}} = 0. \tag{3.1}$$

See [29, Section 3.5] for the proof for (3.1). By Theorem 2.1, Eq (2.1) has a unique solution $q^\epsilon \in C([0,+\infty); H^m \cap \dot{H}^{-1})$ equipped with the initial data $q_0^\epsilon(x)$, where the integer m can be taken large enough to proceed all the following differential calculations.

From Lemma 2.2, we have

$$\|q^\epsilon(t,x)\|_{L^2} = \|q_0^\epsilon(x)\|_{L^2}. \tag{3.2}$$

Next, we will estimate L^2 norm of q_x^ϵ. By Lemma 2.1 (*ii*), the triangle inequality, we obtain from the equality (2.9),

$$\begin{aligned}\frac{d}{dt}\|q_x^\epsilon\|_{L^2}^2 &\leq \int_{\mathbb{R}} \left| |J_\epsilon q^\epsilon|_x^2 (J_\epsilon q_x^\epsilon)\cdot J_\epsilon \overline{q_x^\epsilon}\right| dx - \mathrm{Re}\int_{\mathbb{R}} |J_\epsilon q^\epsilon|^2 (J_\epsilon q_{xx}^\epsilon)\cdot J_\epsilon \overline{q_x^\epsilon} dx\\ &=: \widetilde{I_1} + \widetilde{I_2}.\end{aligned}$$

Using Hölder's inequality, Lemma 2.1 (*i*), and noting that $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, the first term $\widetilde{I_1}$ is estimated by

$$\begin{aligned}\widetilde{I_1} &\leq 2\left\|J_\epsilon q_x^\epsilon \cdot J_\epsilon \overline{q^\epsilon}\right\|_{L^2} \cdot \|J_\epsilon q_x^\epsilon\|_{L^\infty} \cdot \|J_\epsilon \overline{q_x^\epsilon}\|_{L^2}\\ &\leq 2\|J_\epsilon q_x^\epsilon\|_{L^2} \cdot \|J_\epsilon \overline{q^\epsilon}\|_{L^\infty} \cdot \|J_\epsilon q_x^\epsilon\|_{L^\infty} \cdot \|J_\epsilon \overline{q_x^\epsilon}\|_{L^2}\\ &\leq C\|q_x^\epsilon\|_{L^2} \cdot \|J_\epsilon \overline{q^\epsilon}\|_{H^1} \cdot \|J_\epsilon q_x^\epsilon\|_{H^1} \cdot \|\overline{q_x^\epsilon}\|_{L^2}\\ &\leq C\|q^\epsilon\|_{H^2}^4.\end{aligned}$$

For the term $\widetilde{I_2}$, estimating in the same way, there holds

$$\widetilde{I_2} = \frac{1}{2}\int_{\mathbb{R}} |J_\epsilon q^\epsilon|_x^2 \cdot |J_\epsilon q_x^\epsilon|^2 dx \leq C\|q^\epsilon\|_{H^2}^4.$$

So, we get from these two estimates that

$$\frac{d}{dt}\|q_x^\epsilon\|_{L^2}^2 \leq C\|q^\epsilon\|_{H^2}^4. \tag{3.3}$$

Integrating with respect to t on both sides of (3.3), we obtain that

$$\|q_x^\epsilon(t)\|_{L^2}^2 \le C\int_0^t \|q^\epsilon\|_{H^2}^4 dt + \|q_x^\epsilon(0)\|_{L^2}^2. \tag{3.4}$$

Now we will estimate the L^2 norm of q_{xx}^ϵ. For differentiating equation (2.8) with respect to x, we get

$$q_{xxt}^\epsilon + q_x^\epsilon + \frac{1}{2}J_\epsilon[|J_\epsilon q^\epsilon|^2(J_\epsilon q_x^\epsilon)]_{xx} = 0. \tag{3.5}$$

Multiplying (3.5) by $2\overline{q_{xx}^\epsilon}$ and taking the real part of the result, we have

$$\frac{d}{dt}\|q_{xx}^\epsilon\|_{L^2}^2 + \mathrm{Re}\int_{\mathbb{R}} J_\epsilon[|J_\epsilon q^\epsilon|^2(J_\epsilon q_x^\epsilon)]_{xx}\cdot\overline{q_{xx}^\epsilon}dx = 0.$$

Using Lemma 2.1 (*ii*) and the triangle inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\|q_{xx}^\epsilon\|_{L^2}^2 \le& \int_{\mathbb{R}}\left||J_\epsilon q^\epsilon|_{xx}^2(J_\epsilon q_x^\epsilon)\cdot J_\epsilon\overline{q_{xx}^\epsilon}\right|dx + 2\int_{\mathbb{R}}\left||J_\epsilon q^\epsilon|_x^2(J_\epsilon q_{xx}^\epsilon)\cdot J_\epsilon\overline{q_{xx}^\epsilon}\right|dx\\ &-\mathrm{Re}\int_{\mathbb{R}}|J_\epsilon q^\epsilon|^2(J_\epsilon q_{xxx}^\epsilon)\cdot J_\epsilon\overline{q_{xx}^\epsilon}dx\\ =:& \widetilde{I_3}+\widetilde{I_4}+\widetilde{I_5}.\end{aligned}$$

By Hölder's inequality, Lemma 2.1 (*i*), the embedding relation $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and the Banach algebra property (2.7), the term $\widetilde{I_3}+\widetilde{I_4}$ can be estimated by

$$\begin{aligned}\widetilde{I_3}+\widetilde{I_4} \le& \left\||J_\epsilon q^\epsilon|_{xx}^2(J_\epsilon q_x^\epsilon)\right\|_{L^2}\cdot\|J_\epsilon\overline{q_{xx}^\epsilon}\|_{L^2} + 2\left\||J_\epsilon q^\epsilon|_x^2(J_\epsilon q_{xx}^\epsilon)\right\|_{L^2}\cdot\|J_\epsilon\overline{q_{xx}^\epsilon}\|_{L^2}\\ \le& \left\||J_\epsilon q^\epsilon|_{xx}^2\right\|_{L^2}\cdot\|J_\epsilon q_x^\epsilon\|_{L^\infty}\cdot\|q_{xx}^\epsilon\|_{L^2} + 2\left\||J_\epsilon q^\epsilon|_x^2\right\|_{L^\infty}\cdot\|J_\epsilon q_{xx}^\epsilon\|_{L^2}\cdot\|\overline{q_{xx}^\epsilon}\|_{L^2}\\ \le& C\|q^\epsilon\|_{H^2}^4.\end{aligned}$$

For the term $\widetilde{I_5}$, we integrate it by parts to get

$$\begin{aligned}\widetilde{I_5} =& -\frac{1}{2}\int_{\mathbb{R}}|J_\epsilon q^\epsilon|^2\cdot|J_\epsilon q_{xx}^\epsilon|_x^2dx\\ =& \frac{1}{2}\int_{\mathbb{R}}|J_\epsilon q^\epsilon|_x^2\cdot|J_\epsilon q_{xx}^\epsilon|^2dx\\ \le& C\|q^\epsilon\|_{H^2}^4.\end{aligned}$$

Therefore, we concude that

$$\frac{d}{dt}\|q_{xx}^\epsilon\|_{L^2}^2 \le C\|q^\epsilon\|_{H^2}^4. \tag{3.6}$$

Integrating with respect to t on both sides of (3.6), we obtain

$$\|q_{xx}^\epsilon(t)\|_{L^2}^2 \le C\int_0^t\|q^\epsilon\|_{H^2}^4dt + \|q_{xx}^\epsilon(0)\|_{L^2}^2. \tag{3.7}$$

Combining (3.2), (3.4), and (3.7), we get

$$\|q^\epsilon\|_{H^2}^2 \le C\int_0^t\|q^\epsilon\|_{H^2}^4dt + \|q_0^\epsilon\|_{H^2}^2 = C\int_0^t\|q^\epsilon\|_{H^2}^4dt + C_1,$$

where we have used (3.1) in the last step. Note that the constant C is independent of ϵ, and by (3.1), one has

$$C_1 = \|q_0^\epsilon\|_{H^2}^2 \to \|q_0\|_{H^2}^2,\ \epsilon \to 0. \tag{3.8}$$

Let

$$\varphi^\epsilon(t) = C\int_0^t \|q^\epsilon\|_{H^2}^4 dt + C_1.$$

Then, there holds

$$\frac{d\varphi^\epsilon(t)}{dt} \le C[\varphi^\epsilon(t)]^2$$

which gives us

$$\|q^\epsilon\|_{H^2}^2 \le \varphi^\epsilon(t) \le \frac{C_1}{1 - CC_1 t}. \tag{3.9}$$

Hence, using (3.8), we see that there exists a time $T > 0$ $(T = T(\|q_0\|_{H^2}))$ such that

$$\|q^\epsilon\|_{H^2} \le C,\ \ \forall\, t \in [0, T]. \tag{3.10}$$

This shows that the regularized solution q^ϵ is uniformly bounded in H^2, and the argument used in the first step of Theorem 2.1 also yields the uniform bound of q_t^ϵ in L^2.

Moreover, we can show that the solution family q^ϵ (also q_t^ϵ) forms a Cauchy sequence in $C([0, T]; L^2)$. In fact, for $\epsilon > \epsilon' > 0$, taking energy estimate of the equation

$$\begin{aligned}(q^\epsilon - q^{\epsilon'})_t &+ \partial_x^{-1}(q^\epsilon - q^{\epsilon'}) \\ &+ \frac{1}{2}(J_\epsilon - J_{\epsilon'})[|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon] + \frac{1}{2}J_{\epsilon'}[|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon - |J_{\epsilon'} q^{\epsilon'}|^2 J_{\epsilon'} q_x^{\epsilon'}] = 0\end{aligned} \tag{3.11}$$

at L^2 level yields that

$$\frac{d}{dt}\|q^\epsilon - q^{\epsilon'}\|_{L^2}^2 = K_1 + K_2,$$

where

$$\begin{aligned}K_1 &:= -\operatorname{Re}\int_{\mathbb{R}} (J_\epsilon - J_{\epsilon'})[|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon] \cdot (\overline{q^\epsilon} - \overline{q^{\epsilon'}})dx, \\ K_2 &:= -\operatorname{Re}\int_{\mathbb{R}} J_{\epsilon'}[|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon - |J_{\epsilon'} q^{\epsilon'}|^2 J_{\epsilon'} q_x^{\epsilon'}] \cdot (\overline{q^\epsilon} - \overline{q^{\epsilon'}})dx.\end{aligned}$$

Using Plancherel's theorem and (3.10), we have

$$\begin{aligned}\|(J_\epsilon - J_{\epsilon'})[|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon]\|_{L^2} &= \|\frac{(\epsilon - \epsilon')\xi^2}{(1 + \epsilon\xi^2)(1 + \epsilon'\xi^2)}\mathcal{F}[|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon](\xi)\|_{L^2} \\ &= \sqrt{\epsilon - \epsilon'}\|\frac{\sqrt{\epsilon - \epsilon'}\xi}{(1 + \epsilon\xi^2)(1 + \epsilon'\xi^2)}\mathcal{F}[(|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon)_x](\xi)\|_{L^2} \\ &\le \sqrt{\epsilon}\|[|J_\epsilon q^\epsilon|^2 J_\epsilon q_x^\epsilon]_x\|_{L^2} \\ &\le C\sqrt{\epsilon}.\end{aligned}$$

Therefore,

$$|K_1| \le C\sqrt{\epsilon}\|q^\epsilon - q^{\epsilon'}\|_{L^2}.$$

To estimate K_2, we rewrite it as

$$K_2 = K_{21} + K_{22} + K_{23}$$

with

$$K_{21} := -\operatorname{Re}\int_{\mathbb{R}} [|J_\epsilon q^\epsilon|^2 (J_\epsilon - J_{\epsilon'}) q_x^\epsilon] \cdot J_{\epsilon'}(\overline{q^\epsilon} - \overline{q^{\epsilon'}})dx,$$
$$K_{22} := -\operatorname{Re}\int_{\mathbb{R}} [|J_\epsilon q^\epsilon|^2 J_{\epsilon'}(q_x^\epsilon - q_x^{\epsilon'})] \cdot J_{\epsilon'}(\overline{q^\epsilon} - \overline{q^{\epsilon'}})dx,$$
$$K_{23} := -\operatorname{Re}\int_{\mathbb{R}} [(|J_\epsilon q^\epsilon|^2 - |J_{\epsilon'} q^{\epsilon'}|^2) J_{\epsilon'} q_x^{\epsilon'}] \cdot J_{\epsilon'}(\overline{q^\epsilon} - \overline{q^{\epsilon'}})dx.$$

Using the uniform bound (3.10) and the same treatment for $J_\epsilon - J_{\epsilon'}$ as above, we can get

$$|K_{21}| \le C\sqrt{\epsilon}\|q^\epsilon - q^{\epsilon'}\|_{L^2}.$$

Similarly, the term K_{23} is estimated by

$$|K_{23}| \le C\sqrt{\epsilon}\|q^\epsilon - q^{\epsilon'}\|_{L^2} + C\|q^\epsilon - q^{\epsilon'}\|_{L^2}^2.$$

Then, integrating by part gives us

$$|K_{22}| \le C\|q^\epsilon - q^{\epsilon'}\|_{L^2}^2.$$

Combining these estimates gives us

$$\|q^\epsilon - q^{\epsilon'}\|_{C([0,T];L^2)} \to 0, \ \epsilon, \ \epsilon' \to 0.$$

By interpolation, we also have

$$\|q^\epsilon - q^{\epsilon'}\|_{C([0,T];H^{k'})} \to 0, \ \epsilon, \ \epsilon' \to 0. \tag{3.12}$$

for any $0 \le k' < 2$. A similar strategy can be applied to show the Cauchy property of the sequence q_t^ϵ in $C([0,T];L^2)$, that is

$$\|q_t^\epsilon - q_t^{\epsilon'}\|_{C([0,T];L^2)} \to 0, \ \epsilon, \ \epsilon' \to 0. \tag{3.13}$$

and further details are omitted.

Now we can prove the existence part of Theorem 1.1. Indeed, from (3.12), (3.13), and the equation (2.1), applying the standard limit argument, we see that there exists $q \in C([0,T];H^{k'} \cap \dot{H}^{-1}) \cap C_W([0,T];H^2)$ satisfying Eq (1.4) with $k' < 2$, here $C_W([0,T];H^2)$ denotes the continuity on $[0,T]$ with values in the weak topology of H^2. Furthermore, we can show that q is also continuous in the strong topology of H^2. To see this result, we rewrite (3.9) in the form

$$\|q^\epsilon(t)\|_{H^2}^2 - \|q_0^\epsilon\|_{H^2}^2 \le \frac{CC_1 t}{1 - CC_1 t}.$$

By (3.8) and the weak convergence property of $q^\epsilon(t)$ in H^2, we obtain

$$\|q(t)\|_{H^2}^2 - \|q_0^\epsilon\|_{H^2}^2 \le \frac{CC_1 t}{1 - CC_1 t},$$

which implies that

$$\limsup_{t\to0+}\|q(t)\|_{H^2}\le\|q_0\|_{H^2}.$$

On the other hand, the fact $q\in C_W([0,T];H^2)$ gives

$$\|q_0\|_{H^2}\le\liminf_{t\to0+}\|q(t)\|_{H^2}.$$

Hence, the strong continuity of q at $t=0$ is proved. This argument also yields the continuity of q in H^2 at any time. Then, we have $q\in C([0,T];H^2)$.

Finally, it remains to prove the uniqueness. In fact, if q and $\tilde q$ both satisfy (1.4) with the same initial data, then $q-\tilde q$ satisfies

$$(q-\tilde q)_t+\partial_x^{-1}(q-\tilde q)+\frac{1}{2}(|q|^2-|\tilde q|^2)q_x+\frac{1}{2}|\tilde q|^2(q-\tilde q)_x=0. \tag{3.14}$$

Multiplying (3.14) by $2\overline{q-\tilde q}$ and integrating the real part over $\mathbb{R}$, we get

$$\frac{d}{dt}\|q-\tilde q\|_{L^2}^2=-\mathrm{Re}\int_{\mathbb{R}}(|q|^2-|\tilde q|^2)q_x\overline{q-\tilde q}dx-\mathrm{Re}\int_{\mathbb{R}}|\tilde q|^2(q-\tilde q)_x\overline{q-\tilde q}dx.$$

Then, applying integration by parts, Hölder's inequality, and the Banach algebra property (2.7), we obtain

$$\begin{aligned}\frac{d}{dt}\|q-\tilde q\|_{L^2}^2&\le\int_{\mathbb{R}}|(q(\overline{q}-\overline{\tilde q})+\overline{\tilde q}(q-\tilde q))q_x(\overline{q}-\overline{\tilde q})|dx+\frac{1}{2}\int_{\mathbb{R}}|\tilde q|_x^2|q-\tilde q|^2dx\\&\le2\|q_x\|_{L^\infty}\|q\|_{L^\infty}\|q-\tilde q\|_{L^2}^2+\frac{1}{2}\||\tilde q|_x^2\|_{L^\infty}\|q-\tilde q\|_{L^2}^2\\&\le C\|q-\tilde q\|_{L^2}^2,\end{aligned}$$

where we have used the fact that $q,\tilde q\in H^2$ in the last inequality. Since q and $\tilde q$ satisfy the same initial data, the uniqueness follows from Gronwall's inequality.

□

4. Global well-posedness

In this section, we will prove the global well-posedness of the complex short pulse equation (1.3), namely, Theorem 1.2. The proof is based only on the energy analysis. To prove this result, we need to control the H^2 norm of $q(t)$ by a t-independent constant. This constant will be found from the values of the conserved quantities of (1.4).

Lemma 4.1. *Let $q(t,x)\in C([0,T);H^2\cap\dot H^{-1})$ be the solution of* (1.4) *obtained in Theorem 2.1. Then, the following quantities are conserved on* $[0,T)$*:*

$$\begin{aligned}\mathcal{H}_1(t):&=\int_{\mathbb{R}}|q|^2dx,\\\mathcal{H}_2(t):&=\int_{\mathbb{R}}(\sqrt{1+|q_x|^2}-1)dx=\int_{\mathbb{R}}\frac{|q_x|^2}{1+\sqrt{1+|q_x|^2}}dx,\\\mathcal{H}_3(t):&=\int_{\mathbb{R}}\Big[\frac{|q_{xx}|^2}{(1+|q_x|^2)^{\frac52}}-\frac{1}{4}\frac{(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac52}}\Big]dx.\end{aligned}$$

Proof. For the sake of simplicity, q is assumed smooth in this proof and rigorous limit argument is omitted. Multiplying (1.4) by $2\overline{q}$, and taking the real part, we derive the first balance equation

$$\partial_t(|q|^2) = -\partial_x(|\partial_x^{-1}q|^2 + \frac{1}{4}|q|^4), \tag{4.1}$$

where $\partial_x^{-1}q = -q_t - \frac{1}{2}|q|^2 q_x$.

Multiplying (1.3) by $\frac{2\overline{q_x}}{\sqrt{1+|q_x|^2}}$, and taking the real part, we derive the second balance equation

$$\partial_t(\sqrt{1+|q_x|^2} - 1) = -\frac{1}{2}\partial_x(|q|^2\sqrt{1+|q_x|^2}). \tag{4.2}$$

Integrating (4.1) and (4.2) over x in $\mathbb{R}$, we obtain the conversations of $\mathcal{H}_1(t)$ and $\mathcal{H}_2(t)$.

Differentiating equation (1.3) with respect to x, we get

$$q_{xxt} + q_x + \frac{1}{2}(|q|^2_{xx}q_x + 2|q|^2_x q_{xx} + |q|^2 q_{xxx}) = 0. \tag{4.3}$$

Multiplying (4.3) by $\frac{2\overline{q_{xx}}}{(1+|q_x|^2)^{\frac{5}{2}}}$, taking the real part, and noting that

$$|q|^2_{xx} = \overline{q}q_{xx} + 2|q_x|^2 + q\overline{q_{xx}},$$

we obtain

$$\partial_t\Big[\frac{|q_{xx}|^2}{(1+|q_x|^2)^{\frac{5}{2}}}\Big] + \frac{5|q_{xx}|^2\mathrm{Re}(\overline{q_x}\cdot q_{xt})}{(1+|q_x|^2)^{\frac{7}{2}}} + \frac{|q_x|^2_x(\overline{q}q_{xx} + 2|q_x|^2 + q\overline{q_{xx}})}{2(1+|q_x|^2)^{\frac{5}{2}}}$$
$$+\frac{|q_x|^2_x}{(1+|q_x|^2)^{\frac{5}{2}}} + \frac{2|q|^2_x|q_{xx}|^2}{(1+|q_x|^2)^{\frac{5}{2}}} + \frac{|q|^2|q_{xx}|^2_x}{2(1+|q_x|^2)^{\frac{5}{2}}} = 0.$$

From Eq (1.3), we have

$$\frac{5|q_{xx}|^2\mathrm{Re}(\overline{q_x}\cdot q_{xt})}{(1+|q_x|^2)^{\frac{7}{2}}} = -\frac{5|q|^2_x|q_{xx}|^2}{2(1+|q_x|^2)^{\frac{5}{2}}} - \frac{5|q|^2|q_x|^2_x|q_{xx}|^2}{4(1+|q_x|^2)^{\frac{7}{2}}}.$$

Thus, we deduce

$$\partial_t\Big[\frac{|q_{xx}|^2}{(1+|q_x|^2)^{\frac{5}{2}}}\Big] = \frac{|q|^2_x|q_{xx}|^2}{2(1+|q_x|^2)^{\frac{5}{2}}} + \frac{5|q|^2|q_x|^2_x|q_{xx}|^2}{4(1+|q_x|^2)^{\frac{7}{2}}} - \frac{|q_x|^2_x}{(1+|q_x|^2)^{\frac{3}{2}}}$$
$$-\frac{|q_x|^2_x(\overline{q}q_{xx} + q\overline{q_{xx}})}{2(1+|q_x|^2)^{\frac{5}{2}}} - \frac{|q|^2|q_{xx}|^2_x}{2(1+|q_x|^2)^{\frac{5}{2}}}. \tag{4.4}$$

By a direct computation, we get

$$\partial_t\Big[-\frac{1}{4}\frac{(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac{5}{2}}}\Big] = -\frac{1}{2}\frac{(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})}{(1+|q_x|^2)^{\frac{5}{2}}}(q_{xt}\overline{q_{xx}} - \overline{q_{xt}}q_{xx} + q_x\overline{q_{xxt}} - \overline{q_x}q_{xxt})$$

$$+\frac{5}{8}\frac{(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})^2|q_x|_t^2}{(1+|q_x|^2)^{\frac{7}{2}}}.$$

From Eqs (1.3) and (4.3), we have

$$q_{xt}\overline{q_{xx}}-\overline{q_{xt}}q_{xx}=\overline{q}q_{xx}-q\overline{q_{xx}}+\frac{1}{2}|q|_x^2(\overline{q_x}q_{xx}-q_x\overline{q_{xx}}),$$

$$q_x\overline{q_{xxt}}-\overline{q_x}q_{xxt}=|q|_x^2(\overline{q_x}q_{xx}-q_x\overline{q_{xx}})+\frac{1}{2}|q|^2(\overline{q_x}q_{xxx}-q_x\overline{q_{xxx}}),$$

and

$$|q_x|_t^2=-|q|_x^2(1+|q_x|^2)-\frac{1}{2}|q|^2|q_x|_x^2.$$

Note that

$$(\overline{q}q_x-q\overline{q_x})_x=\overline{q}q_{xx}-q\overline{q_{xx}}\,,\quad(\overline{q_x}q_{xx}-q_x\overline{q_{xx}})_x=\overline{q_x}q_{xxx}-q_x\overline{q_{xxx}}.$$

So, we deduce

$$\partial_t\Big[-\frac{1}{4}\frac{(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac{5}{2}}}\Big]=\frac{1}{8}\frac{|q|_x^2(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac{5}{2}}}-\frac{5}{16}\frac{|q|^2|q_x|_x^2(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac{7}{2}}}$$
$$-\frac{1}{2}\frac{(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})(\overline{q}q_x-q\overline{q_x})_x}{(1+|q_x|^2)^{\frac{5}{2}}}+\frac{1}{4}\frac{|q|^2(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})_x}{(1+|q_x|^2)^{\frac{5}{2}}}. \tag{4.5}$$

Combining (4.4) and (4.5), we conclude

$$\partial_t\Big[\frac{|q_{xx}|^2}{(1+|q_x|^2)^{\frac{5}{2}}}-\frac{1}{4}\frac{(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac{5}{2}}}\Big]=\partial_x\Big[\frac{2}{(1+|q_x|^2)^{\frac{1}{2}}}$$
$$-\frac{|q|^2|q_{xx}|^2}{2(1+|q_x|^2)^{\frac{5}{2}}}+\frac{1}{8}\frac{|q|^2(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac{5}{2}}}\Big], \tag{4.6}$$

where, we have used the following equality

$$-\frac{1}{2}\frac{(q_x\overline{q_{xx}}-\overline{q_x}q_{xx})(\overline{q}q_x-q\overline{q_x})_x}{(1+|q_x|^2)^{\frac{5}{2}}}=-\frac{|q|_x^2|q_{xx}|^2}{(1+|q_x|^2)^{\frac{5}{2}}}+\frac{1}{2}\frac{|q_x|_x^2(\overline{q}q_{xx}+q\overline{q_{xx}})}{(1+|q_x|^2)^{\frac{5}{2}}}.$$

Integrating (4.6) over x in $\mathbb{R}$, we obtain the conversation of $\mathcal{H}_3(t)$. □

To control the H^2 norm of $q(t)$, we also need the following two results, which can be found in [30] and [31, Lemma 2.1.3], respectively.

Lemma 4.2. *Let u belongs to L^b in $\mathbb{R}^n$ and its derivatives of order m, $D^m u$, belongs to L^r, $1\le b,r\le\infty$. For the derivatives $D^j u$, $0\le j<m$, the following inequalities hold*

$$\|D^j u\|_{L^a}\le C\|D^m u\|_{L^r}^p\|u\|_{L^b}^{1-p},$$

where

$$\frac{1}{a}=\frac{j}{n}+p(\frac{1}{r}-\frac{m}{n})+(1-p)\frac{1}{b},$$

for all p in the interval $\frac{j}{m}\le p\le 1$ (The constant depending only on n,m,j,b,r,p).

Lemma 4.3. *Let $f(x)$ be a nonnegative continuous function on $\mathbb{R}^+$ satisfying*

$$f(x) \leq a + bf^{\kappa}(x), \quad a, b > 0,\ \kappa > 1.$$

If a and b further satisfy

$$a^{\kappa-1} b < \frac{(\kappa-1)^{\kappa-1}}{\kappa^{\kappa}}$$

and $f(0) \leq a$, then $f(x)$ is bounded on $\mathbb{R}^+$.

Proof of Theorem 1.2. As shown in Theorem 2.1, the $\dot{H}^{-1}$ norm of $q(t,x)$ is essentially controlled by its H^2 energy norm. To estimate $\|q(t)\|_{H^2}$, noting that $\mathcal{H}_1(t) = \|q(t)\|_{L^2}^2$ is a constant, it suffices for us to estimate $\|q_x(t)\|_{H^1}$. To this end, we introduce a variable

$$Q(x) = \frac{q_x}{\sqrt{1+|q_x|^2}},$$

that is,

$$q_x = \frac{Q}{\sqrt{1-|Q|^2}}.$$

When $|Q| < 1$, we can expand q_x in the Taylor series

$$q_x = Q \cdot \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n! \cdot 2^n} (|Q|^2)^n.$$

By the Banach algebra property (2.7) and the triangle inequality, we have

$$\begin{aligned} \|q_x\|_{H^1} &\leq \|Q\|_{H^1} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n! \cdot 2^n} \left\||Q|^2\right\|_{H^1}^n \\ &= \frac{\|Q\|_{H^1}}{\sqrt{1 - \left\||Q|^2\right\|_{H^1}}} \\ &\leq \frac{\|Q\|_{H^1}}{\sqrt{1 - \left\|Q\right\|_{H^1}^2}}, \end{aligned} \tag{4.7}$$

where the equality in (4.7) holds under the condition $\|Q\|_{H^1} < 1$.

Next, we shall give an upper bound for $\|Q\|_{H^1}$. On one hand,

$$\begin{aligned} \|Q\|_{L^2}^2 &= \int_{\mathbb{R}} |Q|^2 dx \\ &= \int_{\mathbb{R}} \frac{|q_x|^2}{1+|q_x|^2} dx \\ &= \int_{\mathbb{R}} \frac{|q_x|^2}{1+\sqrt{1+|q_x|^2}} \frac{1+\sqrt{1+|q_x|^2}}{1+|q_x|^2} dx \end{aligned}$$

$$\begin{aligned}
&\leq 2\int_{\mathbb{R}} \frac{|q_x|^2}{1+\sqrt{1+|q_x|^2}}dx \\
&= 2\mathcal{H}_2.
\end{aligned} \tag{4.8}$$

On the other hand, noting that

$$|q_x|_x^2 = \overline{q_x}q_{xx} + q_x\overline{q_{xx}},$$

and

$$(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})^2 = (|q_x|_x^2)^2 - 4|q_x|^2|q_{xx}|^2,$$

by a direct computation, we get

$$\begin{aligned}
\|Q_x\|_{L^2}^2 &= \int_{\mathbb{R}} |Q_x|^2 dx \\
&= \int_{\mathbb{R}} \Big|\partial_x\big(\frac{q_x}{\sqrt{1+|q_x|^2}}\big)\Big|^2 dx \\
&= \int_{\mathbb{R}} \Big|\frac{q_{xx}}{\sqrt{1+|q_x|^2}} - \frac{q_x|q_x|_x^2}{2(\sqrt{1+|q_x|^2})^3}\Big|^2 dx \\
&= \int_{\mathbb{R}} \Big[\frac{|q_{xx}|^2}{1+|q_x|^2} - \frac{|q_x|_x^2(\overline{q_x}q_{xx} + q_x\overline{q_{xx}})}{2(1+|q_x|^2)^2} + \frac{(|q_x|_x^2)^2|q_x|^2}{4(1+|q_x|^2)^3}\Big]dx \\
&= \int_{\mathbb{R}} \Big\{\frac{|q_{xx}|^2}{(1+|q_x|^2)^3} - \frac{(2+|q_x|^2)[(|q_x|_x^2)^2 - 4|q_x|^2|q_{xx}|^2]}{4(1+|q_x|^2)^3}\Big\}dx \\
&= \int_{\mathbb{R}} \Big[\frac{|q_{xx}|^2}{(1+|q_x|^2)^3} - \frac{(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})^2}{4(1+|q_x|^2)^3} - \frac{(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})^2}{4(1+|q_x|^2)^2}\Big]dx.
\end{aligned}$$

Then,

$$\begin{aligned}
\|Q_x\|_{L^2}^2 &\leq \int_{\mathbb{R}} \Big[\frac{|q_{xx}|^2}{(1+|q_x|^2)^3} - \frac{(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})^2}{4(1+|q_x|^2)^3}\Big] \cdot \sqrt{1+|q_x|^2}dx \\
&\quad - \int_{\mathbb{R}} \frac{(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})^2}{4(1+|q_x|^2)^2}dx \\
&= \mathcal{H}_3 - \frac{1}{4}\int_{\mathbb{R}} \frac{(q_x\overline{q_{xx}} - \overline{q_x}q_{xx})^2}{(1+|q_x|^2)^{\frac{5}{2}}} \cdot (1+|q_x|^2)^{\frac{1}{2}}dx \\
&\leq \mathcal{H}_3 + \mathcal{H}_3 \cdot \Big\|\sqrt{1+|q_x|^2}\Big\|_{L^\infty} \\
&\leq \mathcal{H}_3 + \mathcal{H}_3 \cdot (1 + \|q_x\|_{L^\infty}).
\end{aligned} \tag{4.9}$$

By Lemma 4.2, we have

$$\begin{aligned}
\|q_x\|_{L^\infty} &\leq C\|q\|_{L^2}^{\frac{1}{4}}\|q_{xx}\|_{L^2}^{\frac{3}{4}} \\
&\leq C \cdot \mathcal{H}_1^{\frac{1}{8}} \cdot \Big\|\frac{q_{xx}}{(1+|q_x|^2)^{\frac{5}{4}}} \cdot (1+|q_x|^2)^{\frac{5}{4}}\Big\|_{L^2}^{\frac{3}{4}} \\
&\leq C \cdot \mathcal{H}_1^{\frac{1}{8}} \cdot \Big\|\frac{q_{xx}}{(1+|q_x|^2)^{\frac{5}{4}}}\Big\|_{L^2}^{\frac{3}{4}} \cdot \Big\|(1+|q_x|^2)^{\frac{5}{4}}\Big\|_{L^\infty}^{\frac{3}{4}}
\end{aligned}$$

$$\leq C \cdot \mathcal{H}_1^{\frac{1}{8}} \cdot \mathcal{H}_3^{\frac{3}{8}}(1 + \|q_x\|_{L^\infty}^{\frac{5}{2}}).$$

Note that the condition of Lemma 4.3 holds with the smallness assumption on $\mathcal{H}_1$ and $\mathcal{H}_3$. Hence, applying Lemma 4.3 implies that $\|q_x\|_{L^\infty}$ has an upper bound $C(\mathcal{H}_1, \mathcal{H}_3)$, provided that $\mathcal{H}_1(0)$ and $\mathcal{H}_3(0)$ are sufficiently small. So from (4.8)–(4.9), we see that the norm

$$\|Q\|_{H^1} = \sqrt{\|Q\|_{L^2}^2 + \|Q_x\|_{L^2}^2}$$

is bounded from above by a constant depending on $\mathcal{H}_1(0)$, $\mathcal{H}_2(0)$ and $\mathcal{H}_3(0)$, that is,

$$\|Q\|_{H^1} \leq \sqrt{2\mathcal{H}_2 + 2\mathcal{H}_3 + \mathcal{H}_3 \cdot C(\mathcal{H}_1, \mathcal{H}_3)} =: C(\mathcal{H}_1(0), \mathcal{H}_2(0), \mathcal{H}_3(0)). \tag{4.10}$$

Combining (4.7) and (4.10), the following holds

$$\|q_x\|_{H^1} \leq \frac{C(\mathcal{H}_1(0), \mathcal{H}_2(0), \mathcal{H}_3(0))}{\sqrt{1 - (C(\mathcal{H}_1(0), \mathcal{H}_2(0), \mathcal{H}_3(0)))^2}},$$

which results in the t-independent bound of the norm $\|q\|_{H^2}$

$$\|q\|_{H^2} \leq \Big(\mathcal{H}_1 + \frac{(C(\mathcal{H}_1(0), \mathcal{H}_2(0), \mathcal{H}_3(0)))^2}{1 - (C(\mathcal{H}_1(0), \mathcal{H}_2(0), \mathcal{H}_3(0)))^2}\Big)^{\frac{1}{2}}.$$

This bound allows us to apply bootstrap argument, and obtain global existence of solution to the Eq (1.4). This completes the proof of Theorem 1.2. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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