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Research article

Global boundedness in a Keller-Segel system with nonlinear indirect signal consumption mechanism

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Abstract: In this paper, we study a quasilinear chemotaxis model with a nonlinear indirect consumption mechanism

$$\begin{cases} v_{1t} = \nabla \cdot (\psi(v_1)\nabla v_1 - \chi\phi(v_1)\nabla v_2) + \lambda_1 v_1 - \lambda_2 v_1^\beta, & x \in \Omega,\ t > 0, \\ v_{2t} = \Delta v_2 - w^\theta v_2, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + v_1^\alpha, & x \in \Omega,\ t > 0, \end{cases}$$

in a smooth and bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with homogeneous Neumann boundary conditions, where $\chi,\ \lambda_1,\ \lambda_2,\ \theta > 0,\ 0 < \alpha \leq \frac{1}{\theta},\ \beta \geq 2,\ \psi$, and ϕ are nonlinear functions that satisfy $\psi(s) \geq a_0(s+1)^{r_1}$ and $0 \leq \phi(s) \leq b_0 s(s+1)^{r_2}$ for all $s \geq 0$ with $a_0, b_0 > 0$ and $r_1, r_2 \in \mathbb{R}$. It has been proven that if $r_1 > 2r_2 + 1$, then the problem admits a global and bounded classical solution for some appropriate nonnegative initial data.

Keywords: chemotaxis system; global boundedness; nonlinear indirect signal

1. Introduction

As we all know, Keller and Segel [1] first proposed the classical chemotaxis model (hereafter called K-S model), which has been widely applied in biology and medicine. The model can be given by the following:

$$\begin{cases} v_{1t} = \Delta v_1 - \chi\nabla \cdot (v_1 \nabla v_2) + f(v_1), & x \in \Omega,\ t > 0, \\ \tau v_{2t} = \Delta v_2 - v_2 + v_1, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where v_1 is the cell density, v_2 is the concentration of the chemical signal, and $f(v_1)$ is the logistic source function. For the case of $\tau = 1$ and $f(v_1) = 0$, it has been proven that the classical solutions to system (1.1) always remain globally bounded when $n = 1$ [2]. A critical mass phenomenon of system (1.1) has been shown in a two-dimensional space. Namely, if the initial data v_{10} satisfies $\|v_{10}\|_{L^1(\Omega)} < \frac{4\pi}{\chi}$, then the solution (v_1, v_2) is globally bounded [3]. Alternatively, if the initial data v_{10} satisfies $\|v_{10}\|_{L^1(\Omega)} > \frac{4\pi}{\chi}$, then the solution (v_1, v_2) is unbounded in finite or infinite time, provided Ω is simply connected [4, 5]. In particular, for a framework of radially symmetric solutions in a planar disk, the solutions blow up in finite time if $\|v_{10}\|_{L^1(\Omega)} > \frac{8\pi}{\chi}$ [6]. When $f(v_1) = 0$, Liu and Tao [7] changed $\tau v_{2t} = \Delta v_2 - v_2 + v_1$ to $v_{2t} = \Delta v_2 - v_2 + g(v_1)$ with $0 \leq g(v_1) \leq K v_1^{\alpha}$ for $K, \alpha > 0$, and obtained the global well-posedness of model (1.1) provided that $0 < \alpha < \frac{2}{n}$. Later on, the equation $\tau v_{2t} = \Delta v_2 - v_2 + v_1$ was changed to $0 = \Delta v_2 - \varpi(t) + g(v_1)$ with $\varpi(t) = \frac{1}{|\Omega|} \int_\Omega g(v_1(\cdot, t))$ for $g(v_1) = v_1^{\alpha}$. Winkler [8] deduced that for any v_{10}, the model (1.1) is globally and classical solvable if $\alpha < \frac{2}{n}$; conversely, if $\alpha > \frac{2}{n}$, then the solutions are unbounded in a finite-time for any $\int_\Omega v_{10} = m > 0$. For $\tau = 0$, when $f(v_1) \leq v_1(c - dv_1)$ with $c, d > 0$, Tello and Winkler [9] deduced the global well-posedness of model (1.1) provided that $d > \frac{n-2}{n}\chi$. Afterwards, when $f(v_1) = cv_1 - dv_1^{\epsilon}$ with $\epsilon > 1,\ c \geq 0,\ d > 0$, Winkler [10] defined a concept of very weak solutions and observed that these solutions are globally bounded under some conditions. For more results on (1.1), the readers can refer to [11–14].

Considering the volume filling effect [15], the self-diffusion functions and chemotactic sensitivity functions may have nonlinear forms of the cell density. The general model can be written as follows:

$$\begin{cases} v_{1t} = \nabla \cdot (\psi(v_1)\nabla v_1 - \phi(v_1)\nabla v_2) + f(v_1), & x \in \Omega,\ t > 0, \\ \tau v_{2t} = \Delta v_2 - v_2 + v_1, & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

Here, $\psi(v_1)$ and $\phi(v_1)$ are nonlinear functions. When $\tau = 1$ and $f(v_1) = 0$, for any $\int_\Omega v_{10} = M > 0$, Winkler [16] derived that the solution (v_1, v_2) is unbounded in either finite or infinite time if $\frac{\phi(v_1)}{\psi(v_1)} \geq c v_1^{\alpha}$ with $\alpha > \frac{2}{n}$, $n \geq 2$ and some constant $c > 0$ for all $v_1 > 1$. Later on, Tao and Winkler [17] deduced the global well-posedness of model (1.5) provided that $\frac{\phi(v_1)}{\psi(v_1)} \leq c v_1^{\alpha}$ with $\alpha < \frac{2}{n}$, $n \geq 1$ and some constant $c > 0$ for all $v_1 > 1$. Furthermore, in a high-dimensional space where $n \geq 5$, Lin et al. [18] changed the equation $\tau v_{2t} = \Delta v_2 - v_2 + v_1$ to $0 = \Delta v_2 - \varpi(t) + v_1$ with $\varpi(t) = \frac{1}{|\Omega|} \int_\Omega v_1(x, t) dx$, and showed that the solution (v_1, v_2) is unbounded in a finite time.

Next, we introduce the chemotaxis model that involves an indirect signal mechanism. The model can be given by the following:

$$\begin{cases} v_{1t} = \nabla \cdot (\psi(v_1)\nabla v_1 - \phi(v_1)\nabla v_2) + f(v_1), & x \in \Omega,\ t > 0, \\ \tau v_{2t} = \Delta v_2 - v_2 + w, & x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w - w + v_1, & x \in \Omega,\ t > 0. \end{cases} \tag{1.3}$$

For $\tau = 1$, when $\psi(v_1) = 1, \phi(v_1) = v_1$ and $f(v_1) = \lambda(v_1 - v_1^{\alpha})$, the conclusion in [19] implied that the system is globally classical solvable if $\alpha > \frac{n}{4} + \frac{1}{2}$ with $n \geq 2$. Furthermore, the authors in [20–22] extended the boundedness result to a quasilinear system. Ren [23] derived the global well-posedness of system (1.3) and provided the qualitative analysis of such solutions. For $\tau = 0$, when $\psi(s) \geq c(s + 1)^{\theta}$ and $|\phi(s)| \leq ds(s + 1)^{\kappa - 1}$ with $s \geq 0,\ c, d > 0$ and $\theta, \kappa \in R$, Li and Li [24] obtained that the model (1.3) is globally classical solvable. Meanwhile, they also provided the qualitative analysis of such solutions. More results of the system with an indirect signal mechanism can be found in [25–28].

Considering that the cell or bacteria populations have a tendency to move towards a degraded nutrient, the authors obtain another well-known chemotaxis-consumption system:

$$\begin{cases} v_{1t} = \Delta v_1 - \chi \nabla \cdot (v_1 \nabla v_2), & x \in \Omega,\ t > 0, \\ v_{2t} = \Delta v_2 - v_1 v_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

where v_1 denotes the cell density, and v_2 denotes the concentration of oxygen. If $0 < \chi \leq \frac{1}{6(n+1)\|v_{20}\|_{L^\infty(\Omega)}}$ with $n \geq 2$, then the results of [29] showed that the system (1.4) is globally classical solvable. Thereafter, Zhang and Li [30] deduced the global well-posedness of model (1.4) provided that $n \leq 2$ or $0 < \chi \leq \frac{1}{6(n+1)\|v_{20}\|_{L^\infty(\Omega)}}$, $n \geq 3$. In addition, for a sufficiently large v_{10} and v_{20}, Tao and Winkler [31] showed that the defined weak solutions globally exist when $n = 3$. Meanwhile, they also analyzed the qualitative properties of these weak solutions.

Based on the model (1.4), some researchers have considered the model that involves an indirect signal consumption:

$$\begin{cases} v_{1t} = \Delta v_1 - \chi \nabla \cdot (v_1 \nabla v_2), & x \in \Omega,\ t > 0, \\ v_{2t} = \Delta v_2 - v_1 v_2, & x \in \Omega,\ t > 0, \\ w_t = -\delta w + v_1, & x \in \Omega,\ t > 0, \end{cases} \tag{1.5}$$

where w represents the indirect signaling substance produced by cells for degrading oxygen. Fuest [32] obtained the global well-posedness of model (1.5) provided that $n \leq 2$ or $\|v_{20}\|_{L^\infty(\Omega)} \leq \frac{1}{3n}$, and studied the convergence rate of the solution. Subsequently, the authors in [33] extended the boundedness conclusion of model (1.5) using conditions $n \geq 3$ and $0 < \|v_{20}\|_{L^\infty(\Omega)} \leq \frac{\pi}{\sqrt{n}}$. For more results on model (1.5), the readers can refer to [34–39].

Inspired by the work mentioned above, we find that there are few papers on the quasilinear chemotaxis model that involve the nonlinear indirect consumption mechanism. In view of the complexity of the biological environment, this signal mechanism may be more realistic. In this manuscript, we are interested in the following system:

$$\begin{cases} v_{1t} = \nabla \cdot (\psi(v_1) \nabla v_1 - \chi \phi(v_1) \nabla v_2) + \lambda_1 v_1 - \lambda_2 v_1^\beta, & x \in \Omega,\ t > 0, \\ v_{2t} = \Delta v_2 - w^\theta v_2, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + v_1^\alpha, & x \in \Omega,\ t > 0, \\ \frac{\partial v_1}{\partial \nu} = \frac{\partial v_2}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ v_1(x,0) = v_{10}(x), v_2(x,0) = v_{20}(x), & x \in \Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded and smooth domain, ν denotes the outward unit normal vector on $\partial\Omega$, and $\chi,\ \lambda_1,\ \lambda_2,\ \theta > 0,\ 0 < \alpha \leq \frac{1}{\theta},\ \beta \geq 2$. Here, v_1 is the cell density, v_2 is the concentration of oxygen, and w is the indirect chemical signal produced by v_1 to degrade v_2. The diffusion functions $\psi,\ \phi \in C^2[0, \infty)$ are assumed to satisfy

$$\psi(s) \geq a_0 (s+1)^{r_1} \text{ and } 0 \leq \phi(s) \leq b_0 s (s+1)^{r_2}, \tag{1.7}$$

for all $s \geq 0$ with $a_0, b_0 > 0$ and $r_1, r_2 \in \mathbb{R}$. In addition, the initial data v_{10} and v_{20} fulfill the following:

$$v_{10}, v_{20} \in W^{1,\infty}(\Omega) \text{ with } v_{10}, v_{20} \geq 0, \not\equiv 0 \text{ in } \Omega. \tag{1.8}$$

Theorem 1.1. *Assume that* $\chi,\ \lambda_1,\ \lambda_2,\ \theta > 0,\ 0 < \alpha \le \frac{1}{\theta}$, *and* $\beta \ge 2$, *and that* $\Omega \subset \mathbb{R}^n (n \ge 1)$ *is a smooth bounded domain. Let* $\psi,\ \phi \in C^2[0,\infty)$ *satisfy* (1.7). *Suppose that the initial data* v_{10} *and* v_{20} *fulfill* (1.8). *It has been proven that if* $r_1 > 2r_2 + 1$, *then the problem* (1.6) *has a nonnegative classical solution*

$$(v_1, v_2, w) \in \left(C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))\right)^2 \times C^{2,0}(\bar{\Omega} \times (0,\infty)),$$

which is globally bounded in the sense that

$$\|v_1(\cdot,t)\|_{L^\infty(\Omega)} + \|v_2(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C,$$

for all $t > 0$, *with* $C > 0$.

Remark 1.2. *Our main ideas are as follows. First, we obtain the* L^∞ *bound for* v_2 *by the maximum principle of the parabolic equation. Next, we establish an estimate for the functional* $y(t) := \frac{1}{p}\int_\Omega (v_1+1)^p + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p}$ *for any* $p > 1$ *and* $t > 0$. *Finally, we can derive the global solvability of model* (1.6).

Remark 1.3. *Theorem 1.1 shows that self-diffusion and logical source are advantageous for the boundedness of the solutions. In this manuscript, due to the indirect signal substance w that consumes oxygen, the aggregation of cells or bacterial is almost impossible when self-diffusion is stronger than cross-diffusion, namely* $r_1 > 2r_2 + 1$. *We can control the logical source to ensure the global boundedness of the solution for model* (1.6). *Thus, we can study the effects of the logistic source, the diffusion functions, and the nonlinear consumption mechanism on the boundedness of the solutions.*

2. Preliminaries

In this section, we first state a lemma on the local existence of classical solutions. The proof can be proven by the fixed point theory. The readers can refer to [40, 41] for more details.

Lemma 2.1. *Let the assumptions in Theorem 1.1 hold. Then, there exists* $T_{\max} \in (0,\infty]$ *such that the problem* (1.6) *has a nonnegative classical solution* (v_1, v_2, w) *that satisfies the following:*

$$(v_1, v_2, w) \in \left(C^0(\bar{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max}))\right)^2 \times C^{2,0}(\bar{\Omega} \times (0,T_{\max})).$$

Furthermore, if $T_{\max} < \infty$, *then*

$$\limsup_{t \nearrow T_{\max}} \left(\|v_1(\cdot,t)\|_{L^\infty(\Omega)} + \|v_2(\cdot,t)\|_{W^{1,\infty}(\Omega)}\right) = \infty.$$

Lemma 2.2. *(cf. [42]) Let* $\Omega \subset \mathbb{R}^n (n \ge 1)$ *be a smooth bounded domain. For any* $s \ge 1$ *and* $\epsilon > 0$, *one can obtain*

$$\int_{\partial\Omega} |\nabla z|^{2s-2} \frac{\partial |\nabla z|^2}{\partial \nu} \le \epsilon \int_\Omega |\nabla z|^{2s-2} |D^2 z|^2 + C_\epsilon \int_\Omega |\nabla z|^{2s},$$

for all $z \in C^2(\bar{\Omega})$ *fulfilling* $\frac{\partial z}{\partial \nu}\big|_{\partial\Omega} = 0$, *with* $C_\epsilon = C(\epsilon, s, \Omega) > 0$.

Lemma 2.3. *(cf. [43]) Let* $\Omega \subset \mathbb{R}^n (n \ge 1)$ *be a bounded and smooth domain. For* $s \ge 1$, *we have*

$$\int_\Omega |\nabla z|^{2s+2} \le 2(4s^2 + n)\|z\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla z|^{2s-2} |D^2 z|^2,$$

for all $z \in C^2(\bar{\Omega})$ *fulfilling* $\frac{\partial z}{\partial \nu}\big|_{\partial\Omega} = 0$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded and smooth domain. For any $z \in C^2(\Omega)$, one has the following:*

$$(\Delta z)^2 \leq n|D^2 z|^2,$$

where $D^2 z$ represents the Hessian matrix of z and $|D^2 z|^2 = \sum_{i,j=1}^n z_{x_i x_j}^2$.

Proof. The proof can be found in [41, Lemma 3.1].

Lemma 2.5. *(cf. [44, 45]) Let $a_1, a_2 > 0$. The non-negative functions $f \in C([0,T)) \cap C^1((0,T))$ and $y \in L^1_{loc}([0,T))$ fulfill*

$$f'(t) + a_1 f(t) \leq y(t), \quad t \in (0,T),$$

and

$$\int_t^{t+\tau} y(s)ds \leq a_2, \quad t \in (0, T-\tau),$$

where $\tau = \min\{1, \frac{T}{2}\}$ and $T \in (0, \infty]$. Then, one deduces the following:

$$f(t) \leq f(0) + 2a_2 + \frac{a_2}{a_1}, \quad t \in (0,T).$$

3. Global boundedness of the solutions

In this section, we provide some useful Lemmas to prove Theorem 1.1.

Lemma 3.1. *Let $\beta > 1$, then, there exist $M,\ M_1,\ M_2 > 0$ such that*

$$\|v_2(\cdot,t)\|_{L^\infty(\Omega)} \leq M \ \textit{for all } t \in (0, T_{\max}), \tag{3.1}$$

and

$$\int_\Omega v_1 \leq M_1 \ \textit{for all } t \in (0, T_{\max}). \tag{3.2}$$

Proof. By the parabolic comparison principle for $v_{2t} = \Delta v_2 - w_1^\theta v_2$, we can derive (3.1). Invoking the integration for the first equation of (1.6), one has the following:

$$\frac{d}{dt}\int_\Omega v_1 = \lambda_1 \int_\Omega v_1 - \lambda_2 \int_\Omega v_1^\beta \ \text{ for all } t \in (0, T_{\max}). \tag{3.3}$$

Invoking the Hölder inequality, we obtain the following:

$$\frac{d}{dt}\int_\Omega v_1 \leq \lambda_1 \int_\Omega v_1 - \frac{\lambda_2}{|\Omega|^{\beta-1}}\left(\int_\Omega v_1\right)^\beta. \tag{3.4}$$

We can apply the comparison principle to deduce the following:

$$\int_\Omega v_1 \leq \max\left\{\int_\Omega v_{10}, \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{\beta-1}} |\Omega|\right\} = M_1. \tag{3.5}$$

Thereupon, we complete the proof.

Lemma 3.2. *For any* $\gamma > 1$, *we have the following:*

$$\int_\Omega w^\gamma \le C_0 \int_\Omega v_1^{\alpha\gamma} \text{ for all } t \in (0, T_{\max}), \tag{3.6}$$

where $C_0 = \frac{2^\gamma}{1+\gamma} > 0$.

Proof. For $\gamma > 1$, multiplying equation $0 = \Delta w - w + v_1^\alpha$ by $w^{\gamma-1}$, one obtain the following:

$$\begin{aligned} 0 &= -(\gamma - 1)\int_\Omega w^{\gamma-2}|\nabla w|^2 - \int_\Omega w^\gamma + \int_\Omega v_1^\alpha w^{\gamma-1} \\ &\le \int_\Omega v_1^\alpha w^{\gamma-1} - \int_\Omega w^\gamma \text{ for all } t \in (0, T_{\max}). \end{aligned} \tag{3.7}$$

By Young's inequality, it is easy to deduce the following:

$$\int_\Omega v_1^\alpha w^{\gamma-1} \le \frac{\gamma - 1}{2\gamma}\int_\Omega w^\gamma + 2^{\gamma-1} \cdot \frac{1}{\gamma}\int_\Omega v_1^{\alpha\gamma}. \tag{3.8}$$

Thus, we arrive at (3.6) by combining (3.7) with (3.8).

Lemma 3.3. *Let the assumptions in Lemma 2.1 hold. For any* $p > \max\{1, \frac{1}{\theta} - 1\}$, *there exists* $C > 0$ *such that*

$$\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v_2|^{2p} + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} + \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2 v_2|^2 \le C\int_\Omega v_1^{\theta\alpha(p+1)} + C, \tag{3.9}$$

for all $t \in (0, T_{\max})$.

Proof. Using the equation $v_{2t} = \Delta v_2 - w_1^\theta v_2$, we obtain the following:

$$\begin{aligned} \nabla v_2 \cdot \nabla v_{2t} &= \nabla v_2 \cdot \nabla \Delta v_2 - \nabla v_2 \cdot \nabla\left(w^\theta v_2\right) \\ &= \frac{1}{2}\Delta|\nabla v_2|^2 - |D^2 v_2|^2 - \nabla v_2 \cdot \nabla\left(w^\theta v_2\right), \end{aligned} \tag{3.10}$$

where we used the equality $\nabla v_2 \cdot \nabla\Delta v_2 = \frac{1}{2}\Delta|\nabla v_2|^2 - |D^2 v_2|^2$. Testing (3.10) by $|\nabla v_2|^{2p-2}$ and integrating by parts, we derive the following:

$$\begin{aligned} &\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v_2|^{2p} + \int_\Omega |\nabla v_2|^{2p-2}|D^2 v_2|^2 + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} \\ &= \frac{1}{2}\int_\Omega |\nabla v_2|^{2p-2}\Delta|\nabla v_2|^2 + \int_\Omega |\nabla v_2|^{2p} - \int_\Omega |\nabla v_2|^{2p-2}\nabla v_2 \cdot \nabla\left(w^\theta v_2\right) \\ &= I_1 + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} + I_2. \end{aligned} \tag{3.11}$$

Using Lemma 2.4 and (3.1), one has the following:

$$\int_\Omega |\nabla v_2|^{2p+2} \le C_1 \int_\Omega |\nabla v_2|^{2p-2}|D^2 v_2|^2 \text{ for all } t \in (0, T_{\max}), \tag{3.12}$$

where $C_1 = 2(4p^2 + n)M^2$. In virtue of Lemma 2.2, Young's inequality, and (3.12), an integration by parts produces the following:

$$
\begin{aligned}
I_1 + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} &= \frac{1}{2}\int_\Omega |\nabla v_2|^{2p-2}\Delta|\nabla v_2|^2 + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} \\
&= \frac{1}{2}\int_{\partial\Omega} |\nabla v_2|^{2p-2}\frac{\partial|\nabla v_2|^2}{\partial\nu} - \frac{1}{2}\int_\Omega \nabla|\nabla v_2|^{2p-2}\cdot\nabla|\nabla v_2|^2 + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} \\
&\le \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 + C_2\int_\Omega |\nabla v_2|^{2p} - \frac{p-1}{2}\int_\Omega |\nabla v_2|^{2p-4}\left|\nabla|\nabla v_2|^2\right|^2 \\
&\le \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 + \frac{1}{4C_1}\int_\Omega |\nabla v_2|^{2p+2} + C_3 \\
&\le \frac{1}{2}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 + C_3 \ \text{ for all } t\in(0,T_{\max}),
\end{aligned} \tag{3.13}
$$

with $C_2, C_3 > 0$. Due to $|\Delta v_2| \le \sqrt{n}|D^2v_2|$, we can conclude from (3.1) and the integration by parts that

$$
\begin{aligned}
I_2 &= -\int_\Omega |\nabla v_2|^{2p-2}\nabla v_2\cdot\nabla\left(w^\theta v_2\right) = \int_\Omega w^\theta v_2\nabla\cdot\left(\nabla v_2|\nabla v_2|^{2p-2}\right) \\
&\le \int_\Omega w^\theta v_2\left(\Delta v_2|\nabla v_2|^{2p-2} + (2p-2)|\nabla v_2|^{2p-2}|D^2v_2|\right) \\
&\le \int_\Omega \left(\sqrt{n} + 2(p-2)\right)Mw^\theta|\nabla v_2|^{2p-2}|D^2v_2| \\
&= C_4\int_\Omega w^\theta|\nabla v_2|^{2p-2}|D^2v_2| \ \text{ for all } t\in(0,T_{\max}),
\end{aligned} \tag{3.14}
$$

with $C_4 = (\sqrt{n} + 2(p-2))M > 0$. Due to $p > \max\{1, \frac{1}{\theta} - 1\}$, we have $\theta(p+1) > 1$. With applications of Young's inequality, (3.12), and Lemma 3.2, we obtain the following from (3.14):

$$
\begin{aligned}
C_4\int_\Omega w^\theta|\nabla v_2|^{2p-2}|D^2v_2| &\le \frac{1}{8}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 + C_5\int_\Omega w^{2\theta}|\nabla v_2|^{2p-2} \\
&\le \frac{1}{8}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 + \frac{1}{8C_1}\int_\Omega |\nabla v_2|^{2p+2} + C_6\int_\Omega w^{\theta(p+1)} \\
&\le \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 + C_7\int_\Omega w^{\theta(p+1)} \\
&\le \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 + C_8\int_\Omega v_1^{\theta\alpha(p+1)},
\end{aligned} \tag{3.15}
$$

with $C_5, C_6, C_7, C_8 > 0$. Substituting (3.13) and (3.15) into (3.11), we derive the following:

$$
\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v_2|^{2p} + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} + \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2v_2|^2 \le C_8\int_\Omega v_1^{\theta\alpha(p+1)} + C_3, \tag{3.16}
$$

for all $t \in (0, T_{\max})$. Thereupon, we complete the proof.

Lemma 3.4. *Let the assumptions in Lemma 2.1 hold. If* $r_1 > 2r_2 + 1$, *then for any* $p > 1$, *we obtain the following:*

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (v_1+1)^p + \frac{1}{p}\int_\Omega (v_1+1)^p$$
$$\leq \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2 v_2|^2 + (C+\lambda_1+\frac{1}{p})\int_\Omega (v_1+1)^p - \lambda_2\int_\Omega v_1^{p+\beta-1} + C, \tag{3.17}$$

for all $t \in (0, T_{\max})$, *with* $C > 0$.

Proof. Testing the first equation of problem (1.6) by $(v_1+1)^{p-1}$, one can obtain the following:

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (v_1+1)^p + \frac{1}{p}\int_\Omega (v_1+1)^p = -(p-1)\int_\Omega (v_1+1)^{p-2}\psi(v_1)|\nabla v_1|^2 + \frac{1}{p}\int_\Omega (v_1+1)^p$$
$$+\chi(p-1)\int_\Omega (v_1+1)^{p-2}\phi(v_1)\nabla v_1\cdot\nabla v_2$$
$$+\lambda_1\int_\Omega v_1(v_1+1)^{p-1} - \lambda_2\int_\Omega v_1^\beta(v_1+1)^{p-1}, \tag{3.18}$$

for all $t \in (0, T_{\max})$. In view of (1.7), the first term on the right-hand side of (3.18) can be estimated as follows:

$$-(p-1)\int_\Omega (v_1+1)^{p-2}\psi(v_1)|\nabla v_1|^2 \leq -(p-1)a_0\int_\Omega (v_1+1)^{p+r_1-2}|\nabla v_1|^2. \tag{3.19}$$

For the second term on the right-hand side of (3.18), we can see that

$$\chi(p-1)\int_\Omega (v_1+1)^{p-2}\phi(v_1)\nabla v_1\cdot\nabla v_2 \leq \chi(p-1)b_0\int_\Omega v_1(v_1+1)^{p+r_2-2}\nabla v_1\cdot\nabla v_2. \tag{3.20}$$

We can obtain the following from Young's inequality:

$$\chi(p-1)b_0\int_\Omega v_1(v_1+1)^{p+r_2-2}\nabla v_1\cdot\nabla v_2$$
$$\leq \chi(p-1)b_0\int_\Omega (v_1+1)^{p+r_2-1}\nabla v_1\cdot\nabla v_2$$
$$\leq (p-1)a_0\int_\Omega (v_1+1)^{p+r_1-2}|\nabla v_1|^2 + C_1\int_\Omega (v_1+1)^{p+2r_2-r_1}|\nabla v_2|^2, \tag{3.21}$$

with $C_1 > 0$. Utilizing Young's inequality and (3.12), one has the following:

$$C_1\int_\Omega (v_1+1)^{p+2r_2-r_1}|\nabla v_2|^2 \leq \frac{1}{8(4p^2+n)M^2}\int_\Omega |\nabla v_2|^{2(p+1)} + C_2\int_\Omega (v_1+1)^{\frac{(p+1)(p+2r_2-r_1)}{p}}$$
$$\leq \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2 v_2|^2 + C_2\int_\Omega (v_1+1)^{\frac{(p+1)(p+2r_2-r_1)}{p}}, \tag{3.22}$$

where $C_2 > 0$. Due to $r_1 > 2r_2+1$, for any $p > 1 > \frac{r_1-2r_2}{2r_2-r_1+1}$, we can obtain $\frac{(p+1)(p+2r_2-r_1)}{p} < p$. Applying Young's inequality, we obtain the following:

$$C_2\int_\Omega (v_1+1)^{\frac{(p+1)(p+2r_2-r_1)}{p}} \leq C_3\int_\Omega (v_1+1)^p + C_3, \tag{3.23}$$

where $C_3 > 0$. Hence, substituting (3.19)–(3.23) into (3.18), one obtains the following:

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (v_1+1)^p + \frac{1}{p}\int_\Omega (v_1+1)^p \\ &\le \frac{1}{4}\int_\Omega |\nabla v_2|^{2p-2}|D^2 v_2|^2 + (C_3+\lambda_1+\frac{1}{p})\int_\Omega (v_1+1)^p - \lambda_2\int_\Omega v_1^{p+\beta-1} + C_4,\end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$, where $C_4 > 0$.

Lemma 3.5. *Let the assumptions in Lemma 2.1 hold. If* $r_1 > 2r_2 + 1$, *then for any* $p > \max\{1, \frac{1}{\theta} - 1\}$, *we obtain the following:*

$$\int_\Omega (v_1+1)^p + \int_\Omega |\nabla v_2|^{2p} \le C, \tag{3.25}$$

where $C > 0$.

Proof. We can combine Lemma 3.3 with Lemma 3.4 to infer the following:

$$\begin{aligned}&\frac{d}{dt}(\frac{1}{p}\int_\Omega (v_1+1)^p + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p}) + \frac{1}{p}\int_\Omega (v_1+1)^p + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} \\ &\le C_1\int_\Omega v_1^{\theta\alpha(p+1)} + (C_1+\lambda_1+\frac{1}{p})\int_\Omega (v_1+1)^p - \lambda_2\int_\Omega v_1^{p+\beta-1} + C_1,\end{aligned} \tag{3.26}$$

where $C_1 > 0$. Due to $0 < \alpha \le \frac{1}{\theta}$ and $\beta \ge 2$, we can obtain $\theta\alpha(p+1) \le p+1 \le p+\beta-1$. Using Young's inequality, we can obtain the following:

$$C_1\int_\Omega v_1^{\theta\alpha(p+1)} \le \frac{\lambda_2}{2}\int_\Omega v_1^{p+\beta-1} + C_2, \tag{3.27}$$

where $C_2 > 0$. By the inequality $(w+s)^\kappa \le 2^\kappa(w^\kappa + s^\kappa)$ with $w, s > 0$ and $\kappa > 1$, we deduce the following:

$$(C_1+\lambda_1+\frac{1}{p})\int_\Omega (v_1+1)^p \le \frac{\lambda_2}{2}\int_\Omega v_1^{p+\beta-1} + C_3, \tag{3.28}$$

where $C_3 > 0$, where we have applied Young's inequality. Thus, we obtain the following:

$$\frac{d}{dt}(\frac{1}{p}\int_\Omega (v_1+1)^p + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p}) + \frac{1}{p}\int_\Omega (v_1+1)^p + \frac{1}{2p}\int_\Omega |\nabla v_2|^{2p} \le C_4, \tag{3.29}$$

where $C_4 > 0$. Therefore, we can obtain (3.25) by Lemma 2.5. Thereupon, we complete the proof.

The proof of Theorem 1.1. Recalling Lemma 3.5, for any $p > \max\{1, \frac{1}{\theta} - 1\}$, and applying the L^p−estimates of elliptic equation, there exists $C_1 > 0$ such that

$$\sup_{t\in(0,T_{\max})} \|w(\cdot,t)\|_{W^{2,\frac{p}{\alpha}}(\Omega)} \le C_1 \text{ for all } t \in (0, T_{\max}). \tag{3.30}$$

The Sobolev imbedding theorem enables us to obtain the following:

$$\sup_{t\in(0,T_{\max})} \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_2 \text{ for all } t \in (0, T_{\max}), \tag{3.31}$$

with $C_2 > 0$. Besides, using the well-known heat semigroup theory to the second equation in system (1.6), we can find $C_3 > 0$ such that

$$\|v_2(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_3 \text{ for all } t \in (0, T_{\max}). \tag{3.32}$$

Therefore, using the Moser-iteration [17], we can find $C_4 > 0$ such that

$$\|v_1(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4 \text{ for all } t \in (0, T_{\max}). \tag{3.33}$$

Based on (3.31)–(3.33), we can find $C_5 > 0$ that fulfills the following:

$$\|v_1(\cdot, t)\|_{L^\infty(\Omega)} + \|v_2(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_5, \tag{3.34}$$

for all $t \in (0, T_{\max})$. According to Lemma 2.1, we obtain $T_{\max} = \infty$. Thereupon, we complete the proof of Theorem 1.1.

4. Conclusions and outlook

In this manuscript, based on the model established in [35], we further considered that self-diffusion and cross-diffusion are nonlinear functions, as well as the mechanism of nonlinear generation and consumption of the indirect signal substance w. We mainly studied the effects of diffusion functions, the logical source, and the nonlinear consumption mechanism on the boundedness of solutions, which enriches the existing results of chemotaxis consumption systems. Compared with previous results [29, 32], the novelty of this manuscript is that our boundedness conditions are more generalized and do not depend on spatial dimension or the sizes of $\|v_{20}\|_{L^\infty(\Omega)}$ established in [29, 32], which may be more in line with the real biological environment. In addition, we will further explore interesting problems related to system (1.6) in our future work, such as the qualitative analysis of system (1.6), the global classical solvability for full parabolic of system (1.6), and so on.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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