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Research article

On derivations of Leibniz algebras

Kailash C. Misra[1], Sutida Patlertsin[2], Suchada Pongprasert[2,*] and Thitarie Rungratgasame[2]

[1] Department of Mathematics, North Carolina State University, Raleigh, North Carolina, USA
[2] Department of Mathematics, Faculty of Science, Srinakharinwirot University, Bangkok, Thailand

* **Correspondence:** Email: suchadapo@g.swu.ac.th.

Abstract: Leibniz algebras are non-antisymmetric generalizations of Lie algebras. In this paper, we investigate the properties of complete Leibniz algebras under certain conditions on their extensions. Additionally, we explore the properties of derivations and direct sums of Leibniz algebras, proving several results analogous to those in Lie algebras.

Keywords: Leibniz algebra; completeness; extension; holomorph; derivation; direct sum; decomposition

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1. Introduction

A vector space $\mathbf{A}$ over a field $\mathbb{F}$ equipped with a bilinear product $[\ ,\ ] : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ is said to be a Leibniz algebra (specifically, a left Leibniz algebra) if it satisfies the Leibniz identity. Leibniz algebras were first studied by Bloh [1] in 1965 and later popularized by Loday [2] in 1993. They are generalizations of Lie algebras; a Leibniz algebra $\mathbf{A}$ is a Lie algebra if and only if $\mathrm{Leib}(\mathbf{A}) = \{0\}$, where $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}_{\mathbb{F}}\{[a, a] \mid a \in \mathbf{A}\}$.

In [3], Jacobson introduced the notion of complete Lie algebras $\mathbf{L}$ as those with a trivial center and whose derivations are inner derivations. He proved that all semisimple Lie algebras are complete, but nilpotent Lie algebras are not. In 1994, Meng [4] showed that a Lie algebra $\mathbf{L}$ is complete if and only if the holomorph $\mathrm{hol}(\mathbf{L}) = \mathbf{L} \oplus \mathrm{Der}(\mathbf{L})$ is a direct sum of $\mathbf{L}$ and the centralizer of $\mathbf{L}$ in the holomorph. Meng also demonstrated that if a Lie algebra $\mathbf{L}$ with a trivial center is decomposed as $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$, where $\mathbf{L}_1$ and $\mathbf{L}_2$ are ideals, then $\mathrm{Der}(\mathbf{L}) = \mathrm{Der}(\mathbf{L}_1) \oplus \mathrm{Der}(\mathbf{L}_2)$.

In 2013, Ancochea Bermúdez and Campoamor-Stursberg [5] defined a complete Leibniz algebra as one with a trivial center and such that all derivations are left multiplications, analogous to complete Lie algebras. However, as noted in [6], with this definition, there exist semisimple Leibniz algebras that are not complete. To address this, Boyle et al. [6] redefined a derivation $\delta \in \mathrm{Der}(\mathbf{A})$ to be inner if $\mathrm{im}(\delta - L_a) \subseteq \mathrm{Leib}(\mathbf{A})$ for some $a \in \mathbf{A}$, where $L_a : \mathbf{A} \to \mathbf{A}$ is the left multiplication operator defined by

$L_a(b) = [a, b]$ for all $b \in \mathbf{A}$. They then defined a Leibniz algebra $\mathbf{A}$ to be complete if all its derivations are inner and the center of $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is trivial. Using this definition, they showed that all semisimple Leibniz algebras are complete, whereas nilpotent Leibniz algebras are not. They also proved that if $\mathbf{A}$ is a complete non-Lie Leibniz algebra, then the holomorph of $\mathbf{A}$ is not the direct sum of $\mathbf{A}$ and the centralizer of $\mathbf{A}$ in the holomorph.

The organization of this paper is as follows: In Section 2, we recall relevant definitions and results on Leibniz algebras needed for later sections. In Section 3, we investigate the properties of complete Leibniz algebras as defined in [6] and their extensions. In particular, we give a new characterization of complete Leibniz algebras (see Theorem 3.7). In the last section, we study the properties of derivations and ideals with respect to direct sum decompositions of Leibniz algebras. Specifically, we prove that if the Leibniz algebra $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$, then, unlike the Lie algebra case, $\mathrm{Der}(\mathbf{A}) \neq \mathrm{Der}(\mathbf{A}_1) \oplus \mathrm{Der}(\mathbf{A}_2)$ (see Proposition 4.3). We also prove that $\mathbf{A}$ is complete if and only if $\mathbf{A}_1$ and $\mathbf{A}_2$ are complete (see Theorem 4.7). Throughout this paper, all algebras are assumed to be finite-dimensional over an algebraically closed field $\mathbb{F}$ with characteristic zero.

2. Preliminaries

A *(left) Leibniz algebra* $\mathbf{A}$ is a vector space over a field $\mathbb{F}$ equipped with a bilinear product $[\ ,\ ] : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ satisfing the Leibniz identity $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$ for all $a, b, c \in \mathbf{A}$. A subspace I of $\mathbf{A}$ is an *ideal* if $[I, \mathbf{A}] \subseteq I$ and $[\mathbf{A}, I] \subseteq I$. The subspace $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{[a, a] \mid a \in \mathbf{A}\}$ is an abelian ideal of $\mathbf{A}$, and $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is a Lie algebra. Indeed, the Leibniz algebra $\mathbf{A}$ is a Lie algebra if and only if $\mathrm{Leib}(\mathbf{A}) = \{0\}$. A linear transformation $\delta : \mathbf{A} \to \mathbf{A}$ is called a *derivation* of $\mathbf{A}$ if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathbf{A}$. The set of all derivations of $\mathbf{A}$ is denoted by $\mathrm{Der}(\mathbf{A})$. For $a \in \mathbf{A}$, the *left multiplication operator* $L_a : \mathbf{A} \to \mathbf{A}$ defined by $L_a(b) = [a, b]$ for all $b \in \mathbf{A}$ is a derivation, but the *right multiplication operator* $R_a : \mathbf{A} \to \mathbf{A}$ defined by $R_a(b) = [b, a]$ for all $b \in \mathbf{A}$ is not a derivation. We denote $L(\mathbf{A}) = \{L_a \mid a \in \mathbf{A}\} \subseteq \mathrm{Der}(\mathbf{A})$, which forms a Lie algebra under the commutator bracket. Note that $L(\mathbf{A}) = \mathrm{ad}(\mathbf{A})$ when $\mathbf{A}$ is a Lie algebra. Throughout this paper, a Leibniz algebra refers to a left Leibniz algebra.

The *left center* of $\mathbf{A}$ is $Z^l(\mathbf{A}) = \{x \in \mathbf{A} \mid [x, a] = 0 \text{ for all } a \in \mathbf{A}\}$. The *right center* of $\mathbf{A}$ is $Z^r(\mathbf{A}) = \{x \in \mathbf{A} \mid [a, x] = 0 \text{ for all } a \in \mathbf{A}\}$. The *center* of $\mathbf{A}$ is $Z(\mathbf{A}) = Z^l(\mathbf{A}) \cap Z^r(\mathbf{A})$, which is an ideal of $\mathbf{A}$. We have the chain of ideals $\{\mathbf{A}^j\}_{j \geq 1}$ defined by $\mathbf{A}^1 = \mathbf{A}$ and $\mathbf{A}^j = [\mathbf{A}, \mathbf{A}^{j-1}]$ for $j \geq 2$. The Leibniz algebra $\mathbf{A}$ is *nilpotent* if $\mathbf{A}^j = 0$ for some positive integer j. An ideal I of $\mathbf{A}$ is a *characteristic ideal* if $\delta(I) \subseteq I$ for all $\delta \in \mathrm{Der}(\mathbf{A})$. As shown in [6], $\mathrm{Leib}(\mathbf{A})$ is a characteristic ideal. For $\delta \in \mathrm{Der}(\mathbf{A})$, we have $\delta(\mathrm{Leib}(\mathbf{A})) \subseteq \mathrm{Leib}(\mathbf{A})$, hence δ naturally induces a derivation $\overline{\delta} \in \mathrm{Der}(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$ where $\overline{\delta}(x + \mathrm{Leib}(\mathbf{A})) = \delta(x) + \mathrm{Leib}(\mathbf{A})$ for all $x + \mathrm{Leib}(\mathbf{A}) \in \mathbf{A}/\mathrm{Leib}(\mathbf{A})$.

3. Complete Leibniz algebras

In this section, we analyze the properties of complete Leibniz algebras. We begin by recalling the definition of complete Leibniz algebras given in [6].

Definition 3.1. [6] A Leibniz algebra $\mathbf{A}$ is *complete* if

(i) $Z\left(\mathbf{A}/\mathrm{Leib}(\mathbf{A})\right) = 0$, and

(ii) Any derivation $\delta \in \mathrm{Der}(\mathbf{A})$ is inner, i.e., $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$ for some $x \in \mathbf{A}$.

Recall that any Lie algebra is a Leibniz algebra, but not conversely. Let $\mathbf{L}$ be a Lie algebra; hence, it is also a Leibniz algebra. If $\mathbf{L}$ is complete as a Leibniz algebra, then it is likewise complete as a Lie algebra since $\mathrm{Leib}(\mathbf{L}) = \{0\}$, and for $x \in \mathbf{L}$, $\mathrm{ad}_x = L_x$.

Example 3.2. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{x, y, z\}$ *with the nonzero multiplications given by* $[x, z] = \alpha z, \alpha \in \mathbb{F}\backslash\{0\}, [x, y] = y$ *and* $[y, x] = -y$*. Then, we have* $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$ *and* $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \{0\}$*. As shown in [7], the derivation algebra* $\mathrm{Der}(\mathbf{A})$ *is given by* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{\delta_1, \delta_2, \delta_3\}$ *where*

$$\begin{array}{lll} \delta_1(x) = y, & \delta_1(y) = 0, & \delta_1(z) = 0, \\ \delta_2(x) = 0, & \delta_2(y) = y, & \delta_2(z) = 0, \\ \delta_3(x) = 0, & \delta_3(y) = 0, & \delta_3(z) = z. \end{array}$$

Since $\mathrm{im}(\delta_1 - L_{-y}) \subseteq \mathrm{Leib}(\mathbf{A})$*,* $\mathrm{im}(\delta_2 - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$*, and* $\mathrm{im}(\delta_3 - L_0) \subseteq \mathrm{Leib}(\mathbf{A})$*, by Definition 3.1, we conclude that* $\mathbf{A}$ *is complete.*

Example 3.3. *Consider the Leibniz algebra* $\mathbf{A}_n = \mathrm{span}\{e_1, e_2, \ldots, e_n, e\}$ *with the nonzero multiplications given by* $[e_1, e_i] = e_{i+1}$ *for* $i = 1, \ldots, n-1$*,* $[e_1, e] = e_1$*, and* $[e, e_i] = -ie_i$ *for* $i = 1, \ldots, n$*. Clearly, we have* $\mathrm{Leib}(\mathbf{A}_n) = \mathrm{span}\{e_2, e_3, \ldots, e_n\}$*, implying that* $\mathbf{A}_n/\mathrm{Leib}(\mathbf{A}_n) = \mathrm{span}\{e_1 + \mathrm{Leib}(\mathbf{A}_n), e + \mathrm{Leib}(\mathbf{A}_n)\}$ *and* $Z(\mathbf{A}_n/\mathrm{Leib}(\mathbf{A}_n)) = \{0\}$*. In [5], it is proved that* $\mathrm{Der}(\mathbf{A}_n) = L(\mathbf{A}_n)$*, hence* $\mathbf{A}_n$ *is a complete Leibniz algebra.*

Proposition 3.4. *Let* $\mathbf{A}$ *be a Leibniz algebra, and* $\mathrm{Leib}(\mathbf{A}) = \mathbf{A}^2 \subsetneq \mathbf{A}$*. Then* $\mathbf{A}$ *is not complete.*

Proof. Assume that $\mathrm{Leib}(\mathbf{A}) = \mathbf{A}^2 \subsetneq \mathbf{A}$. Then there exists $0 \neq x \in \mathbf{A}\backslash\mathbf{A}^2$ such that $[x + \mathrm{Leib}(\mathbf{A}), y + \mathrm{Leib}(\mathbf{A})] = [x, y] + \mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A})$ for any $y \in \mathbf{A}$. Hence, $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) \neq \{0\}$, which implies $\mathbf{A}$ is not complete. □

Example 3.5. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{x, y, z\}$ *with the nonzero multiplications given by* $[x, z] = z$*. Then* $\mathbf{A}^2 = \mathrm{span}\{z\} = \mathrm{Leib}(\mathbf{A})$*, since* $z = [x + z, x + z]$*, and* $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \mathbf{A}/\mathrm{Leib}(\mathbf{A})$*. Hence,* $\mathbf{A}$ *is not complete.*

It is shown in [6] that if $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$ is complete as a Lie algebra, then $\mathbf{A}$ is complete as a Leibniz algebra. The following example demonstrates that the existence of outer derivations of $\mathbf{A}$ implies the existence of outer derivations of $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$.

Example 3.6. *Consider the Leibniz algebra* $\mathbf{A} = \mathrm{span}\{x, y, z\}$ *with the nonzero multiplications given by* $[x, x] = z$ *and* $[y, y] = z$*. As shown in [8], the derivations* δ_1, δ_2 *defined by*

$$\begin{array}{lll} \delta_1(x) = x, & \delta_1(y) = y, & \delta_1(z) = 2z, \\ \delta_2(x) = y, & \delta_2(y) = -x, & \delta_2(z) = 0 \end{array}$$

are outer derivations. Clearly, $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{z\}$*. Define* $\overline{\delta_1}, \overline{\delta_2}$ *by*

$$\begin{array}{ll} \overline{\delta_1}(x + \mathrm{Leib}(\mathbf{A})) = x + \mathrm{Leib}(\mathbf{A}), & \overline{\delta_1}(y + \mathrm{Leib}(\mathbf{A})) = y + \mathrm{Leib}(\mathbf{A}), \\ \overline{\delta_2}(x + \mathrm{Leib}(\mathbf{A})) = y + \mathrm{Leib}(\mathbf{A}), & \overline{\delta_2}(y + \mathrm{Leib}(\mathbf{A})) = -x + \mathrm{Leib}(\mathbf{A}). \end{array}$$

Then $\overline{\delta_1}$ *and* $\overline{\delta_2}$ *are outer derivations of* $\mathbf{A}/\mathrm{Leib}(\mathbf{A})$*.*

Following the definition analogous to the holomorph of a Lie algebra, the *holomorph* of the Leibniz algebra $\mathbf{A}$ is defined as $\mathrm{hol}(\mathbf{A}) := \mathbf{A} \oplus \mathrm{Der}(\mathbf{A})$ where the multiplication is given by $[x + \delta_1, y + \delta_2] = [x, y] + \delta_1(y) + [L_x, \delta_2] + [\delta_1, \delta_2]$ for all $x, y \in \mathbf{A}$ and $\delta_1, \delta_2 \in \mathrm{Der}(\mathbf{A})$ (see [6]). For two subspaces $M \subseteq N$ of $\mathrm{hol}(\mathbf{A})$, the *left centralizer* of M in N is defined as $Z^l_N(M) = \{x \in N \mid [x, M] = 0\}$. The following theorem, analogous to a result in Lie algebra theory [4], is applied to Leibniz algebras. We say that a Leibniz algebra $\mathbf{B}$ containing the Leibniz algebra $\mathbf{A}$ is a *special extension* of $\mathbf{A}$ if $\mathbf{A}$ is an ideal of $\mathbf{B}$.

Theorem 3.7. *Let $\mathbf{A}$ be a Leibniz algebra. Then the following conditions are equivalent:*

(i) $\mathbf{A}$ *is complete.*
(ii) Any special extension $\mathbf{B}$ of $\mathbf{A}$ can be written as $\mathbf{B} = \mathbf{A} + X$, where $\mathbf{A} \cap X = \mathrm{Leib}(\mathbf{A})$ and $X = \{x \in \mathbf{B} \mid [x, \mathbf{A}] \subseteq \mathrm{Leib}(\mathbf{A})\}$.
(iii) $\mathrm{hol}(\mathbf{A}) = \mathbf{A} + (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I)$ *and* $\mathbf{A} \cap (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I) = \mathrm{Leib}(\mathbf{A})$, *where* $I = \{\delta \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(\delta) \subseteq \mathrm{Leib}(\mathbf{A})\}$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathbf{A}$ is complete. Let $\mathbf{B}$ be a special extension of $\mathbf{A}$. It is clear that $\mathrm{Leib}(\mathbf{A}) \subseteq \mathbf{A} \cap X$. To show that $\mathbf{A} \cap X \subseteq \mathrm{Leib}(\mathbf{A})$, we let $x \in \mathbf{A} \cap X$. Then for all $a \in \mathbf{A}$, $[x + \mathrm{Leib}(\mathbf{A}), a + \mathrm{Leib}(\mathbf{A})] = [x, a] + \mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A})$ and hence $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \{0\}$ which implies $x \in \mathrm{Leib}(\mathbf{A})$. Therefore, $\mathbf{A} \cap X = \mathrm{Leib}(\mathbf{A})$. Let $x \in \mathbf{B}$. Since $\mathbf{A}$ is an ideal of $\mathbf{B}$, $\mathrm{ad}_x|_{\mathbf{A}} \in \mathrm{Der}(\mathbf{A})$. Thus, there exists $b \in \mathbf{A}$ such that $\mathrm{im}(\mathrm{ad}_x|_{\mathbf{A}} - L_b) \subseteq \mathrm{Leib}(\mathbf{A})$. So, we have $[x - b, A] \subseteq \mathrm{Leib}(\mathbf{A})$ and $x - b \in X$. Hence, $x \in \mathbf{A} + X$, which implies $\mathbf{B} \subseteq \mathbf{A} + X$. Since the reverse inclusion is clear, we have $\mathbf{B} = \mathbf{A} + X$.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Since $\mathbf{A}$ is an ideal of $\mathrm{hol}(\mathbf{A})$, we have $\mathrm{hol}(\mathbf{A})$ is a special extension of $\mathbf{A}$. Then $\mathrm{hol}(\mathbf{A}) = \mathbf{A} + X$ where $X = \{x + \delta \in \mathrm{hol}(\mathbf{A}) \mid [x + \delta, \mathbf{A}] = [x, \mathbf{A}] + \delta(\mathbf{A}) \subseteq \mathrm{Leib}(\mathbf{A})\}$. Set $I = \{\delta \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(\delta) \subseteq \mathrm{Leib}(\mathbf{A})\}$. To show that $\mathbf{A} + (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I) \subseteq \mathbf{A} + X$, let $a \in \mathbf{A} + (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I)$. By [6], $Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) = \{c - L_c \mid c \in \mathbf{A}\}$, hence $a = b + c - L_c + \delta$ for some $b, c \in \mathbf{A}$ and $\delta \in I$. For all $d \in \mathbf{A}$, we have $[c - L_c + \delta, d] = [c, d] - L_c(d) + \delta(d) = \delta(d) \in \mathrm{Leib}(\mathbf{A})$. Hence, $c - L_c + \delta \in X$, which implies $a = b + c - L_c + \delta \in \mathbf{A} + X$. To prove the converse, let $a \in \mathbf{A} + X = \mathrm{hol}(\mathbf{A}) = \mathbf{A} \oplus \mathrm{Der}(\mathbf{A})$. Then $a = b + c + \delta_1 = d + \delta_2$ for some $b, d \in \mathbf{A}, c + \delta_1 \in X$, and $\delta_2 \in \mathrm{Der}(\mathbf{A})$. Hence, $\delta_1 = \delta_2$ and $b + c = d$, which implies $\mathrm{im}(L_c + \delta_2) \subseteq \mathrm{Leib}(\mathbf{A})$. Thus, $a = d + \delta_2 = d - c + c - L_c + L_c + \delta_2 \in \mathbf{A} + (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I)$. This implies $\mathbf{A} + X \subseteq \mathbf{A} + (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I)$. Therefore, we have that $\mathrm{hol}(\mathbf{A}) = \mathbf{A} + X = \mathbf{A} + (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I)$. Also, $\mathbf{A} \cap (Z^l_{\mathrm{hol}(\mathbf{A})}(\mathbf{A}) \oplus I) = \mathbf{A} \cap X = \mathrm{Leib}(\mathbf{A})$.

(iii) $\Rightarrow$ (i) follows from [6]. $\square$

4. Decomposition of Leibniz algebras

We assume throughout this section that the Leibniz algebra $\mathbf{A}$ is the direct sum of two ideals, i.e., $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are ideals of $\mathbf{A}$. For $\delta \in \mathrm{Der}(\mathbf{A}_1)$, we can extend δ to be a derivation on $\mathbf{A}$ by defining $\delta(a_1 + a_2) = \delta(a_1)$ for any $a_1 \in \mathbf{A}_1$ and $a_2 \in \mathbf{A}_2$. Similarly, for $\delta \in \mathrm{Der}(\mathbf{A}_2)$, we can extend δ to be a derivation on $\mathbf{A}$ by defining $\delta(a_1 + a_2) = \delta(a_2)$ for any $a_1 \in \mathbf{A}_1$ and $a_2 \in \mathbf{A}_2$. Hence, we can tacitly take into account that $\delta \in \mathrm{Der}(\mathbf{A}_i)$ as a derivation on $\mathbf{A}$ and $\mathrm{Der}(\mathbf{A}_i) \subseteq \mathrm{Der}(\mathbf{A})$ for $i = 1, 2$.

Observe that if $L_a = L_{a_1+a_2} \in L(\mathbf{A})$ for some $a_1 \in \mathbf{A}_1, a_2 \in \mathbf{A}_2$, then for any $b = b_1 + b_2 \in \mathbf{A}$ where $b_1 \in \mathbf{A}_1, b_2 \in \mathbf{A}_2$, we have $L_a(b) = L_{a_1}(b_1) + L_{a_2}(b_2)$ since $[a_i, b_j] \in [\mathbf{A}_i, \mathbf{A}_j] \in \mathbf{A}_i \cap \mathbf{A}_j = \{0\}$ for $i \neq j$. This implies that $L_a \in L(\mathbf{A}_1) + L(\mathbf{A}_2)$. So $L(\mathbf{A}) \subseteq L(\mathbf{A}_1) + L(\mathbf{A}_2)$. It is clear that $L(\mathbf{A}_1) + L(\mathbf{A}_2) \subseteq L(\mathbf{A})$ and $L(\mathbf{A}_1) \cap L(\mathbf{A}_2) = \{0\}$; hence, $L(\mathbf{A}) = L(\mathbf{A}_1) \oplus L(\mathbf{A}_2)$. Moreover, if $a = a_1 + a_2, b = b_1 + b_2 \in \mathbf{A}$, where $a_1, b_1 \in \mathbf{A}_1$ and $a_2, b_2 \in \mathbf{A}_2$, then $[a, b] = [a_1, b_1] + [a_2, b_2] \in \mathbf{A}_1^2 + \mathbf{A}_2^2$. Thus, $\mathbf{A}^2 \subseteq \mathbf{A}_1^2 + \mathbf{A}_2^2$. Clearly, $\mathbf{A}_1^2 + \mathbf{A}_2^2 \subseteq \mathbf{A}^2$ and $\mathbf{A}_1^2 \cap \mathbf{A}_2^2 = \{0\}$, hence, $\mathbf{A}^2 = \mathbf{A}_1^2 \oplus \mathbf{A}_2^2$.

In [4], Meng proved that for a Lie algebra $\mathbf{L}$, if $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$, where $\mathbf{L}_1$ and $\mathbf{L}_2$ are ideals of $\mathbf{L}$, then $Z(\mathbf{L}) = Z(\mathbf{L}_1) \oplus Z(\mathbf{L}_2)$. Moreover, $\mathrm{Der}(\mathbf{L}) = \mathrm{Der}(\mathbf{L}_1) \oplus \mathrm{Der}(\mathbf{L}_2)$ if $Z(\mathbf{L}) = \{0\}$. In the following propositions, we obtain some analogous results for Leibniz algebras.

Proposition 4.1. *Let* $\mathbf{A}$ *be a Leibniz algebra such that* $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$*, where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are ideals of* $\mathbf{A}$*. Then*

(i) $\mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A}_1) \oplus \mathrm{Leib}(\mathbf{A}_2)$,
(ii) $Z(\mathbf{A}) = Z(\mathbf{A}_1) \oplus Z(\mathbf{A}_2)$.

Proof. (i) If $a = a_1 + a_2 \in \mathrm{Leib}(\mathbf{A})$ where $a_1 \in \mathbf{A}_1$ and $a_2 \in \mathbf{A}_2$, then $[a, a] = [a_1, a_1] + [a_2, a_2] \in \mathrm{Leib}(\mathbf{A}_1) + \mathrm{Leib}(\mathbf{A}_2)$. Hence, $\mathrm{Leib}(\mathbf{A}) \subseteq \mathrm{Leib}(\mathbf{A}_1) + \mathrm{Leib}(\mathbf{A}_2)$. Since the inverse inclusion is clear, $\mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A}_1) + \mathrm{Leib}(\mathbf{A}_2)$. Additionally, we have $\mathrm{Leib}(\mathbf{A}_1) \cap \mathrm{Leib}(\mathbf{A}_2) \subseteq \mathbf{A}_1 \cap \mathbf{A}_2 = \{0\}$, which implies $\mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A}_1) \oplus \mathrm{Leib}(\mathbf{A}_2)$. (ii) Clearly, $Z(\mathbf{A}_1) \cap Z(\mathbf{A}_2) = \{0\}$. Let $a_1 \in Z(\mathbf{A}_1), a_2 \in Z(\mathbf{A}_2)$, and $b = b_1 + b_2 \in \mathbf{A}$, where $b_1 \in \mathbf{A}_1$ and $b_2 \in \mathbf{A}_2$. Then we have $[a_1 + a_2, b] = [a_1, b_1] + [a_2, b_2] = 0$. Similarly, $[b, a_1 + a_2] = [b_1, a_1] + [b_2, a_2] = 0$. Thus, $a_1 + a_2 \in Z(\mathbf{A})$. This implies $Z(\mathbf{A}_1) \oplus Z(\mathbf{A}_2) \subseteq Z(\mathbf{A})$. If $a = a_1 + a_2 \in Z(\mathbf{A})$ where $a_1 \in \mathbf{A}_1$ and $a_2 \in \mathbf{A}_2$, then for any $b_1 \in \mathbf{A}_1$, we have $[a_1, b_1] = [a - a_2, b_1] = [a, b_1] - [a_2, b_1] = 0$ and $[b_1, a_1] = [b_1, a - a_2] = [b_1, a] - [b_1, a_2] = 0$. This implies $a_1 \in Z(\mathbf{A}_1)$. Similarly, we have $a_2 \in Z(\mathbf{A}_2)$. Hence, $a = a_1 + a_2 \in Z(\mathbf{A}_1) \oplus Z(\mathbf{A}_2)$, which completes the proof. □

By the above proposition,

$$Z\left(\mathbf{A}/\mathrm{Leib}(\mathbf{A})\right) = Z\left(\frac{\mathbf{A}_1 \oplus \mathbf{A}_2}{\mathrm{Leib}(\mathbf{A}_1) \oplus \mathrm{Leib}(\mathbf{A}_2)}\right) \cong Z\left(\mathbf{A}_1/\mathrm{Leib}(\mathbf{A}_1)\right) \oplus Z\left(\mathbf{A}_2/\mathrm{Leib}(\mathbf{A}_2)\right).$$

Hence, the following result holds:

Corollary 4.2. *Let* $\mathbf{A}$ *be a Leibniz algebra such that* $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$*, where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are ideals of* $\mathbf{A}$*. Then* $Z\left(\mathbf{A}/\mathrm{Leib}(\mathbf{A})\right) = \{0\}$ *if and only if* $Z\left(\mathbf{A}_i/\mathrm{Leib}(\mathbf{A}_i)\right) = \{0\}$ *for all* $i = 1, 2$.

Let $I_{\mathbf{A}} = \{a \in \mathbf{A} \mid \mathrm{im}(L_a) \subseteq \mathrm{Leib}(\mathbf{A})\}$. Then $I_{\mathbf{A}}$ is an ideal of $\mathbf{A}$ [9]. Observe that $I_{\mathbf{A}_1} \cap I_{\mathbf{A}_2} = \{0\}$. If $a = a_1 + a_2 \in I_{\mathbf{A}}$ where $a_1 \in \mathbf{A}_1$ and $a_2 \in \mathbf{A}_2$, then $L_{a_1}(\mathbf{A}) + L_{a_2}(\mathbf{A}) = L_{a_1+a_2}(\mathbf{A}) = L_a(\mathbf{A}) \subseteq \mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A}_1) \oplus \mathrm{Leib}(\mathbf{A}_2)$ which implies that $L_{a_i}(\mathbf{A}) \subseteq \mathrm{Leib}(\mathbf{A}_i)$ for $i = 1, 2$. Hence, $a = a_1 + a_2 \in I_{\mathbf{A}_1} \oplus I_{\mathbf{A}_2}$. Since the reverse inclusion is clear, we have $I_{\mathbf{A}} = I_{\mathbf{A}_1} \oplus I_{\mathbf{A}_2}$.

Proposition 4.3. *Let* $\mathbf{A}$ *be a Leibniz algebra such that* $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ *where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are ideals of* $\mathbf{A}$*. Then*

$$\mathrm{Der}(\mathbf{A}) = (\mathrm{Der}(\mathbf{A}_1) + I_1) \oplus (\mathrm{Der}(\mathbf{A}_2) + I_2)$$

where $I_1 = \{\delta \in \mathrm{Der}(\mathbf{A}) \mid \delta(\mathbf{A}_2) = \{0\}, \delta(\mathbf{A}_1) \subseteq \mathbf{A}_2 \cap Z(\mathbf{A})\}$ *and* $I_2 = \{\delta \in \mathrm{Der}(\mathbf{A}) \mid \delta(\mathbf{A}_1) = \{0\}, \delta(\mathbf{A}_2) \subseteq \mathbf{A}_1 \cap Z(\mathbf{A})\}$.

Proof. First we observe that if $\delta \in (\mathrm{Der}(\mathbf{A}_1) + I_1) \cap (\mathrm{Der}(\mathbf{A}_2) + I_2)$, then $\delta \in \mathrm{Der}(\mathbf{A}_i) + I_i, i = 1, 2$. So $\delta(\mathbf{A}) = \delta(\mathbf{A}_1) + \delta(\mathbf{A}_2) \subseteq \mathbf{A}_1 \cap \mathbf{A}_2 = \{0\}$, which implies $\delta = 0$. To show $\mathrm{Der}(\mathbf{A}) \subseteq (\mathrm{Der}(\mathbf{A}_1) + I_1) \oplus (\mathrm{Der}(\mathbf{A}_2) + I_2)$, let $0 \neq \delta \in \mathrm{Der}(\mathbf{A})$. Suppose there exists $a \in \mathbf{A}_1$ such that $0 \neq \delta(a) \in \mathbf{A}_2$. Then we have that for all $a_1 \in \mathbf{A}_1$, $[\delta(a), a_1] = 0 = [a_1, \delta(a)]$ and for all $a_2 \in \mathbf{A}_2$, $[\delta(a), a_2] = \delta([a, a_2]) - [a, \delta(a_2)] =$

$-[a, \delta(a_2)] \in \mathbf{A}_1 \cap \mathbf{A}_2 = \{0\}$ and $[a_2, \delta(a)] = \delta([a_2, a]) - [\delta(a_2), a] = -[\delta(a_2), a] \in \mathbf{A}_1 \cap \mathbf{A}_2 = \{0\}$. Thus, $\delta(a) \in Z(\mathbf{A})$, which implies that $\delta(a) \in Z(\mathbf{A}) \cap \mathbf{A}_2$. Similarly, if there exists $a \in \mathbf{A}_2$ such that $0 \neq \delta(a) \in \mathbf{A}_1$, then $\delta(a) \in Z(\mathbf{A}) \cap \mathbf{A}_1$.

We define $\delta_{11}, \delta_{12}, \delta_{21}$, and δ_{22} as follows: for $a = a_1 + a_2 \in \mathbf{A}$, where $a_1 \in \mathbf{A}_1$ and $a_2 \in \mathbf{A}_2$,

$$\delta_{11}(a) = \delta|_{\mathbf{A}_1}(a_1), \qquad \delta_{12}(a) = (\delta - \delta|_{\mathbf{A}_1})(a_1),$$
$$\delta_{21}(a) = (\delta - \delta|_{\mathbf{A}_2})(a_2), \qquad \delta_{22}(a) = \delta|_{\mathbf{A}_2}(a_2).$$

Clearly, $\delta = \delta_{11} + \delta_{12} + \delta_{21} + \delta_{22}$ and $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22} \in \mathrm{Der}(\mathbf{A})$. In particular, $\delta_{11} \in \mathrm{Der}(\mathbf{A}_1), \delta_{12} \in I_1, \delta_{21} \in I_2$ and $\delta_{22} \in \mathrm{Der}(\mathbf{A}_2)$. This implies that $\mathrm{Der}(\mathbf{A}) \subseteq (\mathrm{Der}(\mathbf{A}_1) + I_1) \oplus (\mathrm{Der}(\mathbf{A}_2) + I_2)$. Since the reverse inclusion is clear, we have $\mathrm{Der}(\mathbf{A}) = (\mathrm{Der}(\mathbf{A}_1) + I_1) \oplus (\mathrm{Der}(\mathbf{A}_2) + I_2)$. □

Corollary 4.4. *Let* $\mathbf{A}$ *be a Leibniz algebra such that* $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$*, where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are ideals of* $\mathbf{A}$*. Then* $\mathrm{Der}(\mathbf{A}) = \mathrm{Der}(\mathbf{A}_1) \oplus \mathrm{Der}(\mathbf{A}_2)$ *if one of the following conditions holds:*

(i) $Z(\mathbf{A}) = \{0\}$.
(ii) $\mathbf{A}_i^2 = \mathbf{A}_i$ *for all* $i = 1, 2$.

Proof. (i) Assume that $Z(\mathbf{A}) = \{0\}$. Then we have $I_1 = \{0\} = I_2$ which implies $\mathrm{Der}(\mathbf{A}) = \mathrm{Der}(\mathbf{A}_1) \oplus \mathrm{Der}(\mathbf{A}_2)$. (ii) Assume that $\mathbf{A}_i^2 = \mathbf{A}_i$ for all $i = 1, 2$. Let $\delta \in \mathrm{Der}(\mathbf{A})$. Then we have $\delta(\mathbf{A}_i) = \delta(\mathbf{A}_i^2) = \delta([\mathbf{A}_i, \mathbf{A}_i]) = [\delta(\mathbf{A}_i), \mathbf{A}_i] + [\mathbf{A}_i, \delta(\mathbf{A}_i)] \subseteq \mathbf{A}_i$ for all $i = 1, 2$. Thus, we have $I_i = \{0\}$ for all $i = 1, 2$ and hence $\mathrm{Der}(\mathbf{A}) = \mathrm{Der}(\mathbf{A}_1) \oplus \mathrm{Der}(\mathbf{A}_2)$. □

The following example shows that the direct sum $\mathrm{Der}(\mathbf{A}_1) \oplus \mathrm{Der}(\mathbf{A}_2)$ may not be $\mathrm{Der}(\mathbf{A})$ when $\mathbf{A}_i^2 \neq \mathbf{A}_i$ and $\mathbf{A}_j \cap Z(\mathbf{A}) \neq \{0\}$ for $i \neq j$.

Example 4.5. *Consider the Leibniz algebra* $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$*, where* $\mathbf{A}_1 = \mathrm{span}\{x, y, z\}$ *and* $\mathbf{A}_2 = \mathrm{span}\{a, b, c\}$ *with the nonzero multiplications in* $\mathbf{A}$ *given by* $[x, z] = \alpha z, \alpha \in \mathbb{F}\backslash\{0\}$, $[x, y] = y, [y, x] = -y, [a, a] = c, [a, b] = b$ *and* $[b, a] = -b$*. Then* $\mathbf{A}_1^2 = \mathrm{span}\{y, z\} \neq \mathbf{A}_1$ *and* $\mathbf{A}_2 \cap Z(\mathbf{A}) = \mathrm{span}\{c\} \neq \{0\}$*. We have that* $\mathrm{Der}(\mathbf{A}) = \mathrm{span}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$ *where*

$$\begin{array}{llllll}
\delta_1(x) = y, & \delta_1(y) = 0, & \delta_1(z) = 0, & \delta_1(a) = 0, & \delta_1(b) = 0, & \delta_1(c) = 0,\\
\delta_2(x) = 0, & \delta_2(y) = y, & \delta_2(z) = 0, & \delta_2(a) = 0, & \delta_2(b) = 0, & \delta_2(c) = 0,\\
\delta_3(x) = 0, & \delta_3(y) = 0, & \delta_3(z) = z, & \delta_3(a) = 0, & \delta_3(b) = 0, & \delta_3(c) = 0,\\
\delta_4(x) = 0, & \delta_4(y) = 0, & \delta_4(z) = 0, & \delta_4(a) = b, & \delta_4(b) = 0, & \delta_4(c) = 0,\\
\delta_5(x) = 0, & \delta_5(y) = 0, & \delta_5(z) = 0, & \delta_5(a) = c, & \delta_5(b) = 0, & \delta_5(c) = 0,\\
\delta_6(x) = 0, & \delta_6(y) = 0, & \delta_6(z) = 0, & \delta_6(a) = 0, & \delta_6(b) = b, & \delta_6(c) = 0,\\
\delta_7(x) = c, & \delta_7(y) = 0, & \delta_7(z) = 0, & \delta_7(a) = 0, & \delta_7(b) = 0, & \delta_7(c) = 0.
\end{array}$$

However, $\mathrm{Der}(\mathbf{A}_1) = \mathrm{span}\{\delta_1, \delta_2, \delta_3\}$, $\mathrm{Der}(\mathbf{A}_2) = \mathrm{span}\{\delta_4, \delta_5, \delta_6\}$ *and* $\delta_7 \in I_1$.

Proposition 4.6. *Let* $\mathbf{A}$ *be a Leibniz algebra such that* $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$*, where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are ideals of* $\mathbf{A}$*. Let* M *be a subalgebra of* $\mathbf{A}$*, and* $\mathbf{A}_1 \subseteq M$*. Then*

$$M = \mathbf{A}_1 \oplus (\mathbf{A}_2 \cap M)$$

and M *is an ideal of* $\mathbf{A}$ *if and only if* $\mathbf{A}_2 \cap M$ *is an ideal of* $\mathbf{A}_2$.

Proof. It is clear that $M = \mathbf{A} \cap M = (\mathbf{A}_1 \oplus \mathbf{A}_2) \cap M = \mathbf{A}_1 \oplus (\mathbf{A}_2 \cap M)$ since $\mathbf{A}_1 \subseteq M$. If M is an ideal of $\mathbf{A}$, then $\mathbf{A}_2 \cap M$ is an ideal of $\mathbf{A}$, and hence an ideal of $\mathbf{A}_2$. Conversely, assume that $\mathbf{A}_2 \cap M$ is an ideal of $\mathbf{A}_2$. Let $m = m_1 + m_2 \in M$ and $a = a_1 + a_2 \in \mathbf{A}$, where $m_1 \in \mathbf{A}_1 \subseteq M, m_2 \in \mathbf{A}_2 \cap M, a_1 \in \mathbf{A}_1$, and $a_2 \in \mathbf{A}_2$. Then $[m, a] = [m_1, a_1] + [m_2, a_2] \in M$ since $[m_1, a_2] = 0 = [m_2, a_1]$ and $\mathbf{A}_2 \cap M$ is an ideal of $\mathbf{A}_2$. Similarly, $[a, m] \in M$. Hence, M is an ideal of $\mathbf{A}$. □

The following theorem is the Leibniz algebra analog of the Lie algebra result proved in [4].

Theorem 4.7. *Let* $\mathbf{A}$ *be a Leibniz algebra such that* $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$*, where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are ideals of* $\mathbf{A}$*. Then* $\mathbf{A}$ *is a complete Leibniz algebra if and only if* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are complete.*

Proof. Assume that $\mathbf{A}$ is complete. Then $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \{0\}$. By Corollary 4.2, we have $Z(\mathbf{A}_i/\mathrm{Leib}(\mathbf{A}_i)) = \{0\}$, for $i = 1, 2$. Let $\delta_1 \in \mathrm{Der}(\mathbf{A}_1)$ and $\delta_2 \in \mathrm{Der}(\mathbf{A}_2)$. Since all derivations of $\mathbf{A}$ are inner, for each $i = 1, 2$, we have that there exists $a_i = a_{i1} + a_{i2} \in \mathbf{A}$ where $a_{i1} \in \mathbf{A}_1$ and $a_{i2} \in \mathbf{A}_2$ such that $\mathrm{im}(\delta_i - L_{a_i}) \subseteq \mathrm{Leib}(\mathbf{A})$. For all $b_1 \in \mathbf{A}_1$ and $b_2 \in \mathbf{A}_2$, we have that $L_{a_{12}}(b_1) = 0 = L_{a_{21}}(b_2)$ which implies $\delta_1(b_1) - L_{a_{11}}(b_1) + \delta_2(b_2) - L_{a_{22}}(b_2) = \delta_1(b_1) - L_{a_1}(b_1) + \delta_2(b_2) - L_{a_2}(b_2) \in \mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A}_1) \oplus \mathrm{Leib}(\mathbf{A}_2)$. It follows that $\delta_1(b_1) - L_{a_{11}}(b_1) \in \mathrm{Leib}(\mathbf{A}_1)$ and $\delta_2(b_2) - L_{a_{22}}(b_2) \in \mathrm{Leib}(\mathbf{A}_2)$. Hence, δ_1 and δ_2 are inner. This proves that $\mathbf{A}_1$ and $\mathbf{A}_2$ are complete. Conversely, assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ are complete. Then $Z(\mathbf{A}_i/\mathrm{Leib}(\mathbf{A}_i)) = \{0\}$ for i = 1, 2. By Corollary 4.2, $Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A})) = \{0\}$. Let $\delta \in \mathrm{Der}(\mathbf{A})$. Then, by Proposition 4.3, $\delta = (\delta_{11} + \delta_{12}) + (\delta_{21} + \delta_{22})$ where $\delta_{11} \in \mathrm{Der}(\mathbf{A}_1), \delta_{12} \in I_1, \delta_{21} \in I_2, \delta_{22} \in \mathrm{Der}(\mathbf{A}_2)$. Since δ_{11} and δ_{22} are inner, there exist $a_1 \in \mathbf{A}_1$ and $a_2 \in \mathbf{A}_2$ such that $\delta_{11}(b_1) - L_{a_1}(b_1) \in \mathrm{Leib}(\mathbf{A}_1)$ and $\delta_{22}(b_2) - L_{a_2}(b_2) \in \mathrm{Leib}(\mathbf{A}_2)$ for all $b_1 \in \mathbf{A}_1, b_2 \in \mathbf{A}_2$. Since $\delta_{12}(b_1), \delta_{21}(b_2) \in Z(\mathbf{A})$, we have that for all $x \in \mathbf{A}$, $[\delta_{12}(b_1) + \mathrm{Leib}(\mathbf{A}), a + \mathrm{Leib}(\mathbf{A})] = [\delta_{12}(b), a] + \mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A})$ and $[\delta_{21}(b_2) + \mathrm{Leib}(\mathbf{A}), a + \mathrm{Leib}(\mathbf{A})] = [\delta_{21}(b), a] + \mathrm{Leib}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A})$. This implies $\delta_{12}(b_1) + \mathrm{Leib}(\mathbf{A}), \delta_{21}(b_2) + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A}/\mathrm{Leib}(\mathbf{A}))$, hence, $\delta_{12}(b_1), \delta_{21}(b_2) \in \mathrm{Leib}(\mathbf{A})$. Let $a = a_1 + a_2$. Then, for all $b = b_1 + b_2$ where $b_1 \in \mathbf{A}_1$ and $b_2 \in \mathbf{A}_2$, we have that $\delta(b) - L_a(b) = (\delta_{11}(b_1) - L_{a_1}(b_1)) + (\delta_{22}(b_2) - L_{a_2}(b_2)) + \delta_{12}(b_1) + \delta_{12}(b_2) + \delta_{21}(b_1) + \delta_{21}(b_2) \in \mathrm{Leib}(\mathbf{A})$. This implies δ is inner, which completes the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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