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Research article

Regularity criterion of three dimensional magneto-micropolar fluid equations with fractional dissipation

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Abstract: In this paper, we investigate the regularity criterion of weak solutions to three-dimensional magneto-micropolar fluid equations with fractional dissipation. A regularity criterion is established via the third component of the velocity fields, the micro-rotational velocity fields, and the magnetic fields.

Keywords: magneto-micropolar fluid equations; fractional dissipation; regularity criterion; weak solutions

1. Introduction

In this paper, we consider the three-dimensional magneto-micropolar fluid equations with fractional dissipation

$$\begin{cases} \partial_t u + \mu(-\Delta)^{\alpha} u - \chi\Delta u + u\cdot\nabla u - b\cdot\nabla b + \nabla p - 2\chi\nabla\times v = 0, \\ \partial_t v + \eta(-\Delta)^{\beta} v - \kappa\nabla\nabla\cdot v + 4\chi v + u\cdot\nabla v - 2\chi\nabla\times u = 0, \\ \partial_t b + \lambda(-\Delta)^{\gamma} b + u\cdot\nabla b - b\cdot\nabla u = 0, \\ \nabla\cdot u = 0, \nabla\cdot b = 0, \end{cases} \tag{1.1}$$

with an initial value

$$t = 0 : u = u_0(x), \quad v = v_0(x), \quad b = b_0(x), \quad x\in\mathbb{R}^3. \tag{1.2}$$

Here $u = u(x,t)$, $v = v(x,t)$, $b = b(x,t)\in\mathbb{R}^3$, and $p = p(x,t)\in\mathbb{R}$ are the velocity, micro-rotational velocity, magnetic fields, and scalar pressure, respectively. μ, χ, and $\frac{1}{\lambda}$ represent the kinematic viscosity, vortex viscosity, and magnetic Reynolds number, respectively. η and κ are angular viscosities. α, β and γ are the parameters of the fractional dissipations corresponding to the velocity, micro-rotational velocity and magnetic field, respectively. The fractional Laplace operator $(-\Delta)^{\alpha}$ is defined through the Fourier transform as

$$\widehat{(-\Delta)^{\alpha} f}(\xi) = \widehat{\Lambda^{2\alpha} f} = |\xi|^{2\alpha}\hat{f}(\xi).$$

The incompressible magneto-micropolar fluid equations have made analytic studies a great challenge but offer new opportunities due to their distinctive mathematical features. Regularity criteria for weak solutions are established by Fan and Zhong [1] in pointwise multipliers for $1 \leq \alpha = \beta = \gamma \leq \frac{5}{4}$. Local and global well-posedness have been established in [2–4], respectively. For $\alpha = \beta = \gamma = 1$, we refer to [5–7] for the existence of strong solutions and weak solutions, respectively. In the study field of the magneto-micropolar fluid equations, regularity criteria for weak solutions and blow-up criteria for smooth solutions are very important topics. The readers may refer to regularity criteria of weak solutions in Morrey-Campanato space [8], in Lorentz space [9], Besov space [10], Triebel-Lizorkin space [11] and other regularity criteria for weak solutions [12–15], and [16, 17] for blow-up criteria of smooth solutions in different function spaces, respectively. Serrin-type regularity criteria for weak solutions via the velocity fields and the gradient of the velocity field were established in Yuan [13], respectively. We may refer to [18–20] for global well-posedness. On the other hand, the global regularity of weak solutions to (1.1) with partial viscosities becomes more complex. In the case of 2D, we may refer to [21–25], and in the case of 3D, we may refer to [26, 27].

If $\nu = 0$ and $\chi = 0$, then (1.1) reduces to MHD equations with fractional dissipation. The MHD equations govern the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas, liquid metals, and salt water. We only recall regularity criteria for our purpose. If $\alpha, \beta > \frac{5}{4}$, some regularity criteria have been established by Wu [28, 29], which are given in terms of the velocity u. If $1 \leq \alpha = \beta \leq \frac{3}{2}$, Zhou [30] obtained the Serrin-type criteria $u \in L^p_T L^q_x$ with $\frac{2\alpha}{p} + \frac{3}{q} \leq 2\alpha - 1$ and $\frac{3}{2\alpha-1} < q \leq \infty$. Later, Yuan [14] extended the above function space L^q to $B^s_{q,\infty}$. Recently, the regularity criterion involving $u_3, b \in L^\omega_T L^q_x$ is given in [31]. We also refer to [32, 33] for well-posedness and [34] for blow up criterion of smooth solutions.

Motivated by the Serrin-type regularity criterion of weak solutions to Navier-Stokes equations [35, 36] and MHD equations [30, 31]. The main purpose is to investigate the regularity criterion of weak solutions to the systems (1.1) and (1.2) in this paper and establish the Serrin-type regularity criterion of weak solutions involving partial components. We state our main result as follows:

Theorem 1.1. *Let $1 \leq \alpha = \beta = \gamma \leq \frac{3}{2}$ and $\chi, \kappa \geq 0$. Assume that $(u_0, v_0, b_0) \in H^1(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Furthermore, if*

$$u_3, v, b \in L^\varrho(0, T; L^q(\mathbb{R}^3)),$$

with

$$\frac{2\alpha}{\varrho} + \frac{3}{q} \leq \frac{3}{4}(2\alpha - 1) + \frac{3(1-\epsilon)}{4q}, \quad \frac{3+\epsilon}{2\alpha - 1} < q \leq \infty, \quad 0 < \epsilon \leq \frac{1}{3}, \tag{1.3}$$

then the solution (u, v, b) to the systems (1.1) *and* (1.2) *remains smooth on* $[0, T]$.

Remark 1.2. Since the concrete values of the constants μ, η, and λ play no role in our proof, for this reason, we shall assume them to be all equal to one throughout this paper. For convenience of description, we define horizontal derivatives $\nabla_h := (\partial_1, \partial_2)$.

Remark 1.3. When $\nu = 0$ and $\chi = 0$, the conclusion in Theorem 1.1 is reduced to the one in [31].

Remark 1.4. Compared with [31], the main difficulty in this paper comes from the nonlinear term $u \cdot \nabla v$. In order to overcome the difficulty caused by the nonlinear term, owing to the energy functional (see (2.2)), we first use integrating by parts and $\nabla \cdot u = 0$ to transform it into a control of the horizontal derivative, and then use Hölder's inequality, multiplicative Sobolev inequality, the Gagliardo-Nirenberg inequality, and Young's inequality to control the nonlinear term.

2. Proof of main result

In this section, our main purpose is to complete the proof of Theorem 1.1. To this end, we introduce the following lemma:

Lemma 2.1. *([37]) The multiplicative Sobolev inequality*

$$\|\nabla u\|_{L^{3q}} \leq C\|\partial_1 \nabla u\|_{L^2}^{\frac{1}{3}}\|\partial_2 \nabla u\|_{L^2}^{\frac{1}{3}}\|\partial_3 \nabla u\|_{L^q}^{\frac{1}{3}}, \ \ 1 \leq q < \infty, \tag{2.1}$$

holds.

In what follows, we prove Theorem 1.1.

Proof. Let

$$\begin{aligned} E(t) :=& \|\nabla_h u(t)\|_{L^2}^2 + \|\nabla_h v(t)\|_{L^2}^2 + \|\nabla_h b(t)\|_{L^2}^2 + \int_0^t (\|\nabla_h \Lambda^\alpha u(\tau)\|_{L^2}^2 + \\ & \|\nabla_h \Lambda^\alpha v(\tau)\|_{L^2}^2 + \|\nabla_h \Lambda^\alpha b(\tau)\|_{L^2}^2) d\tau + \kappa \int_0^t \|\nabla_h \nabla \cdot v(\tau)\|_{L^2}^2 d\tau. \end{aligned} \tag{2.2}$$

The proof is divided into two cases: $\frac{3+\epsilon}{2\alpha-1} < q < \infty$ and $q = \infty$. We first consider the case $\frac{3+\epsilon}{2\alpha-1} < q < \infty$.

Taking the inner product of the first three equations of (1.1) with (u, v, b), and adding them up, using integrating by parts, the divergence-free condition, and Cauchy inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \|\Lambda^\alpha u(t)\|_{L^2}^2 + \|\Lambda^\alpha v(t)\|_{L^2}^2 \\ &+ \|\Lambda^\alpha b(t)\|_{L^2}^2 + \kappa\|\nabla \cdot v(t)\|_{L^2}^2 \leq 0. \end{aligned}$$

Integrating the above inequality with respect to t and then obtaining

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t (\|\Lambda^\alpha u(\tau)\|_{L^2}^2 + \|\Lambda^\alpha v(\tau)\|_{L^2}^2 \\ &+ \|\Lambda^\alpha b(\tau)\|_{L^2}^2 + \kappa\|\nabla \cdot v(\tau)\|_{L^2}^2) d\tau \leq \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \end{aligned}$$

By multiplying the first three equations of (1.1) by $\Delta_h u$, $\Delta_h v$, and $\Delta_h b$, respectively, and adding them up, using integrating by parts and the divergence-free condition, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla_h u(t)\|_{L^2}^2 + \|\nabla_h v(t)\|_{L^2}^2 + \|\nabla_h b(t)\|_{L^2}^2) + \|\nabla_h \Lambda^\alpha u(t)\|_{L^2}^2 + \|\nabla_h \Lambda^\alpha v(t)\|_{L^2}^2 \\ &+ \|\nabla_h \Lambda^\alpha b(t)\|_{L^2}^2 + \kappa\|\nabla_h \nabla \cdot v(t)\|_{L^2}^2 + \chi\|\nabla_h \nabla u(t)\|_{L^2}^2 + 4\chi\|\nabla_h v\|_{L^2}^2 \\ :=& \sum_{i=1}^{6} I_i, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta_h u \, dx, \\ I_2 &= -\int_{\mathbb{R}^3} (b \cdot \nabla b) \cdot \Delta_h u \, dx, \\ I_3 &= \int_{\mathbb{R}^3} (u \cdot \nabla b) \cdot \Delta_h b \, dx, \\ I_4 &= -\int_{\mathbb{R}^3} (b \cdot \nabla u) \cdot \Delta_h b \, dx, \\ I_5 &= \int_{\mathbb{R}^3} (u \cdot \nabla v) \cdot \Delta_h v \, dx, \\ I_6 &= -2\chi \int_{\mathbb{R}^3} [(\nabla \times v) \cdot \Delta_h u + (\nabla \times u) \cdot \Delta_h v] \, dx. \end{aligned}$$

Thanks to integration by parts and Cauchy's inequality, we arrive at

$$\begin{aligned} I_6 &= 4\chi \int_{\mathbb{R}^3} \nabla_h (\nabla \times u) \cdot \nabla_h v \, dx \\ &\leq \chi \|\nabla_h (\nabla \times u)\|_{L^2}^2 + 4\chi \|\nabla_h v\|_{L^2}^2 \\ &= \chi \|\nabla_h \nabla u\|_{L^2}^2 + 4\chi \|\nabla_h v\|_{L^2}^2. \end{aligned} \tag{2.4}$$

For I_1, we divide it into the following three items: $I_{1i} (i = 1, 2, 3)$ as

$$\begin{aligned} I_1 &= \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} u_j \partial_j u_k \Delta_h u_k \, dx + \sum_{j=1}^{3} \int_{\mathbb{R}^3} u_j \partial_j u_3 \Delta_h u_3 \, dx + \\ &\quad \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_3 \partial_3 u_k \Delta_h u_k \, dx \\ &:= I_{11} + I_{12} + I_{13}. \end{aligned} \tag{2.5}$$

The divergence-free condition and integration by parts entail that

$$\begin{aligned} I_{11} &= \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} u_j \partial_j u_k \partial_{ii}^2 u_k \, dx \\ &= -\sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} \partial_i u_j \partial_j u_k \partial_i u_k \, dx + \frac{1}{2} \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} \partial_j u_j |\partial_i u_k|^2 \, dx \\ &= -\sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} \partial_i u_j \partial_j u_k \partial_i u_k \, dx - \frac{1}{2} \sum_{i,k=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 |\partial_i u_k|^2 \, dx \\ &= -\int_{\mathbb{R}^3} \partial_1 u_1 \partial_1 u_1 \partial_1 u_1 \, dx - \int_{\mathbb{R}^3} \partial_1 u_1 \partial_1 u_2 \partial_1 u_2 \, dx - \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_1 u_1 \, dx \\ &\quad - \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_2 \partial_1 u_2 \, dx - \int_{\mathbb{R}^3} \partial_2 u_1 \partial_1 u_1 \partial_2 u_1 \, dx - \int_{\mathbb{R}^3} \partial_2 u_1 \partial_1 u_2 \partial_2 u_2 \, dx \end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\partial_2u_2\partial_2u_1\partial_2u_1\,dx-\int_{\mathbb{R}^3}\partial_2u_2\partial_2u_2\partial_2u_2\,dx-\frac{1}{2}\sum_{i,k=1}^{2}\int_{\mathbb{R}^3}\partial_3u_3|\partial_iu_k|^2\,dx\\
=\;&-\int_{\mathbb{R}^3}\partial_1u_1\partial_1u_1\partial_1u_1\,dx-\int_{\mathbb{R}^3}\partial_2u_2\partial_2u_2\partial_2u_2\,dx+\int_{\mathbb{R}^3}\partial_3u_3\partial_2u_1\partial_2u_1\,dx\\
&+\int_{\mathbb{R}^3}\partial_3u_3\partial_1u_2\partial_1u_2\,dx+\int_{\mathbb{R}^3}\partial_3u_3\partial_2u_1\partial_1u_2\,dx-\frac{1}{2}\sum_{i,k=1}^{2}\int_{\mathbb{R}^3}\partial_3u_3|\partial_iu_k|^2\,dx\\
=\;&\frac{1}{2}\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}\partial_3u_3\partial_ku_j\partial_ku_j\,dx-\int_{\mathbb{R}^3}\partial_3u_3\partial_1u_1\partial_2u_2\,dx+\int_{\mathbb{R}^3}\partial_3u_3\partial_2u_1\partial_1u_2\,dx\\
=\;&-\sum_{j,k=1}^{2}\int_{\mathbb{R}^3}u_3\partial_{3k}^2u_j\partial_ku_j\,dx+\int_{\mathbb{R}^3}u_3(\partial_{32}^2u_2\partial_1u_1+\partial_{31}^2u_1\partial_2u_2)\,dx\\
&-\int_{\mathbb{R}^3}u_3(\partial_{32}^2u_1\partial_1u_2+\partial_{31}^2u_2\partial_2u_1)\,dx,
\end{aligned}
\tag{2.6}
$$

and

$$
I_{12}=-\sum_{j=1}^{3}\sum_{l=1}^{2}\int_{\mathbb{R}^3}\partial_lu_j\partial_ju_3\partial_lu_3\,dx=\sum_{j=1}^{3}\sum_{l=1}^{2}\int_{\mathbb{R}^3}\partial_lu_ju_3\partial_{jl}^2u_3\,dx.
\tag{2.7}
$$

Therefore, we obtain

$$
|I_1|\le C\int_{\mathbb{R}^3}|u_3||\nabla u||\nabla_h\nabla u|\,dx.
\tag{2.8}
$$

From Hölder's inequality, Lemma 2.1, the Gagliardo-Nirenberg inequality, and Young's inequality, it follows that

$$
\begin{aligned}
|I_1|\;&\le\;C\int_{\mathbb{R}^3}|u_3||\nabla u||\nabla_h\nabla u|\,dx\\
&\le\;C\|u_3\|_{L^q}\|\nabla u\|_{L^{\theta_1}}\|\nabla_h\nabla u\|_{L^{\theta_2}}\\
&\le\;C\|u_3\|_{L^q}\|\nabla_h\nabla u\|_{L^2}^{\frac{2}{3}}\|\Delta u\|_{L^{\frac{\theta_1}{3}}}^{\frac{1}{3}}\|\nabla_h\nabla u\|_{L^{\theta_2}}\\
&\le\;C\|u_3\|_{L^q}\|\nabla_hu\|_{L^2}^{\frac{2s_1}{3}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{2(1-s_1)}{3}}\|\nabla u\|_{L^2}^{\frac{s_2}{3}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1-s_2}{3}}\|\nabla_hu\|_{L^2}^{s_3}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{1-s_3}\\
&\le\;C\|u_3\|_{L^q}\|\nabla u\|_{L^2}^{\frac{2s_1}{3}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{2(1-s_1)}{3}}\|\nabla u\|_{L^2}^{\frac{s_2}{3}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1-s_2}{3}}\|\nabla u\|_{L^2}^{s_3}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{1-s_3}\\
&\le\;C\|u_3\|_{L^q}\|\nabla u\|_{L^2}^{\frac{2s_1}{3}+\frac{s_2}{3}+s_3}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1-s_2}{3}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{2(1-s_1)}{3}+1-s_3}\\
&\le\;C[\|u_3\|_{L^q}\|\nabla u\|_{L^2}^{\frac{2s_1}{3}+\frac{s_2}{3}+s_3}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1-s_2}{3}}]^{m'}+\frac{1}{6}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{(\frac{2(1-s_1)}{3}+1-s_3)m},
\end{aligned}
\tag{2.9}
$$

where the constants $1 < \theta_1, \theta_2, m, m' < \infty$ and $0 \le s_1, s_2, s_3 \le 1$ satisfy

$$\begin{cases} \frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{q} = 1, \\ 2 - \frac{3}{2} = (1 - \frac{3}{2})s_1 + (1 + \alpha - \frac{3}{2})(1 - s_1), \\ 2 - \frac{3}{\theta_1/3} = (1 - \frac{3}{2})s_2 + (1 + \alpha - \frac{3}{2})(1 - s_2), \\ 2 - \frac{3}{\theta_2} = (1 - \frac{3}{2})s_3 + (1 + \alpha - \frac{3}{2})(1 - s_3), \\ \frac{1}{m} + \frac{1}{m'} = 1, \\ (\frac{2(1 - s_1)}{3} + 1 - s_3)m = 2. \end{cases} \tag{2.10}$$

Noting that $1 \le \alpha \le \frac{3}{2}$ and $\frac{3+\epsilon}{2\alpha-1} < q \le \infty$, one solution to (2.10) can be written as

$$\begin{cases} \theta_1 = \frac{18q}{5q - 18\epsilon}, \\ \theta_2 = \frac{18q}{13q - 18(1-\epsilon)}, \\ s_1 = 1 - \frac{1}{\alpha}, \\ s_2 = 1 - \frac{9\epsilon}{\alpha q}, \\ s_3 = 1 - \frac{1}{3\alpha} - \frac{3(1-\epsilon)}{\alpha q}, \\ m = \frac{2\alpha q}{q + 3(1-\epsilon)}, \\ m' = \frac{2\alpha q}{(2\alpha - 1)q - 3(1-\epsilon)}. \end{cases} \tag{2.11}$$

To bound I_3, we decompose it into three pieces as

$$\begin{aligned} I_3 &= \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} u_j \partial_j b_k \Delta_h b_k \, dx + \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_j \partial_j b_3 \Delta_h b_3 \, dx + \sum_{k=1}^{3} \int_{\mathbb{R}^3} u_3 \partial_3 b_k \Delta_h b_k \, dx \\ &:= I_{31} + I_{32} + I_{33}. \end{aligned} \tag{2.12}$$

By using integrating by parts (see [31]), we have

$$\begin{aligned} I_{31} &= \sum_{j,k,l=1}^{2} \int_{\mathbb{R}^3} [\partial_{ll}^2 u_j \partial_j b_k b_k + \partial_l u_j \partial_{lj}^2 b_k b_k] \, dx - \\ &\quad \frac{1}{2} \sum_{j,k,l=1}^{2} \int_{\mathbb{R}^3} [\partial_{lj}^2 u_j \partial_l b_k b_k + \partial_j u_j \partial_{ll}^2 b_k b_k] \, dx. \end{aligned} \tag{2.13}$$

Similarly, we have

$$I_{32} = \sum_{j,l=1}^{2}\int_{\mathbb{R}^3}[\partial_{ll}^2 u_j\partial_j b_3 b_3 + \partial_l u_j\partial_{lj}^2 b_3 b_3]\,dx - \frac{1}{2}\sum_{j,k,l=1}^{2}\int_{\mathbb{R}^3}[\partial_{lj}^2 u_j\partial_l b_3 b_3 + \partial_j u_j\partial_{ll}^2 b_3 b_3]\,dx, \tag{2.14}$$

and

$$I_{33} = \sum_{k=1}^{3}\sum_{l=1}^{2}\int_{\mathbb{R}^3}[\partial_{3l}^2 u_3\partial_l b_k b_k + \partial_l u_3\partial_{3l}^2 b_k b_k]\,dx + \frac{1}{2}\sum_{k=1}^{3}\sum_{j,l=1}^{2}\int_{\mathbb{R}^3}[\partial_{lj}^2 u_j\partial_l b_k b_k + \partial_j u_j\partial_{ll}^2 b_k b_k]\,dx. \tag{2.15}$$

Collecting (2.13)–(2.15), it is easy to derive that

$$|I_3| \le C\int_{\mathbb{R}^3}|b|(|\nabla u| + |\nabla b|)(|\nabla_h\nabla u| + |\nabla_h\nabla b|)\,dx. \tag{2.16}$$

Furthermore, we have

$$|I_2 + I_3 + I_4| \le C\int_{\mathbb{R}^3}|b|(|\nabla u| + |\nabla b|)(|\nabla_h\nabla u| + |\nabla_h\nabla b|)\,dx. \tag{2.17}$$

Similar to (2.13), it follows from Hölder's inequality, Lemma 2.1, Gagliardo-Nirenberg inequality, and Young's inequality that

$$\begin{aligned}
&|I_2 + I_3 + I_4| \\
\le\ & C\int_{\mathbb{R}^3}|b|(|\nabla u| + |\nabla b|)(|\nabla_h\nabla u| + |\nabla_h\nabla b|)\,dx \\
\le\ & C\|b\|_{L^q}\||\nabla u| + |\nabla b|\|_{L^{\theta_1}}\||\nabla_h\nabla u| + |\nabla_h\nabla b|\|_{L^{\theta_2}} \\
\le\ & C\|b\|_{L^q}(\|\nabla_h\nabla u\|_{L^2}^{\frac{2}{3}}\|\Delta u\|_{L^{\frac{\theta_1}{3}}}^{\frac{1}{3}} + \|\nabla_h\nabla b\|_{L^2}^{\frac{2}{3}}\|\Delta b\|_{L^{\frac{\theta_1}{3}}}^{\frac{1}{3}})\cdot \\
& (\|\nabla_h\nabla u\|_{L^{\theta_2}} + \|\nabla_h\nabla b\|_{L^{\theta_2}}) \\
\le\ & C\|b\|_{L^q}\Big(\|\nabla u\|_{L^2}^{\frac{2s_1}{3}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{2(1-s_1)}{3}}\|\nabla u\|_{L^2}^{\frac{s_2}{3}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1-s_2}{3}} + \\
& \|\nabla b\|_{L^2}^{\frac{2s_1}{3}}\|\nabla_h\Lambda^\alpha b\|_{L^2}^{\frac{2(1-s_1)}{3}}\|\nabla b\|_{L^2}^{\frac{s_2}{3}}\|\Lambda^{\alpha+1}b\|_{L^2}^{\frac{1-s_2}{3}}\Big)\cdot \\
& (\|\nabla u\|_{L^2}^{s_3}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{1-s_3} + \|\nabla b\|_{L^2}^{s_3}\|\nabla_h\Lambda^\alpha b\|_{L^2}^{1-s_3}) \\
\le\ & C\|b\|_{L^q}(\|\nabla u\|_{L^2}^{\frac{2s_1}{3}} + \|\nabla b\|_{L^2}^{\frac{2s_1}{3}})(\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{2(1-s_1)}{3}} + \|\nabla_h\Lambda^\alpha b\|_{L^2}^{\frac{2(1-s_1)}{3}})\cdot \\
& (\|\nabla u\|_{L^2}^{\frac{s_2}{3}} + \|\nabla b\|_{L^2}^{\frac{s_2}{3}})(\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1-s_2}{3}} + \|\Lambda^{\alpha+1}b\|_{L^2}^{\frac{1-s_2}{3}})\cdot \\
& (\|\nabla u\|_{L^2}^{s_3} + \|\nabla b\|_{L^2}^{s_3})(\|\nabla_h\Lambda^\alpha u\|_{L^2}^{1-s_3} + \|\nabla_h\Lambda^\alpha b\|_{L^2}^{1-s_3}) \\
\le\ & C\|b\|_{L^q}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{2s_1}{3}+\frac{s_2}{3}+s_3}(\|\Lambda^{\alpha+1}u\|_{L^2} + \|\Lambda^{\alpha+1}b\|_{L^2})^{\frac{1-s_2}{3}}\cdot
\end{aligned} \tag{2.18}$$

$$
\begin{aligned}
&(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^{\frac{2(1-s_1)}{3}+1-s_3}\\
\leq\ & C\big[\|b\|_{L^q}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{2s_1}{3}+\frac{s_2}{3}+s_3}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2})^{\frac{1-s_2}{3}}\big]^{m'}+\\
&\frac{1}{6}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^{\left(\frac{2(1-s_1)}{3}+1-s_3\right)m},
\end{aligned}
$$

where the constants $1<\theta_1,\theta_2,m,m'<\infty$ and $0\leq s_1,s_2,s_3\leq 1$ satisfy (2.10).

Similar to I_3, we bound I_5 as

$$
|I_5|\leq C\int_{\mathbb{R}^3}|v|(|\nabla u|+|\nabla v|)(|\nabla_h\nabla u|+|\nabla_h\nabla v|)\,dx. \tag{2.19}
$$

Using the same steps as (2.18), we obtain

$$
\begin{aligned}
|I_5|\ \leq\ & C\int_{\mathbb{R}^3}|v|(|\nabla u|+|\nabla v|)(|\nabla_h\nabla u|+|\nabla_h\nabla v|)\,dx\\
\leq\ & C\big[\|v\|_{L^q}(\|\nabla u\|_{L^2}+\|\nabla v\|_{L^2})^{\frac{2s_1}{3}+\frac{s_2}{3}+s_3}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}v\|_{L^2})^{\frac{1-s_2}{3}}\big]^{m'}+\\
&\frac{1}{6}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha v\|_{L^2})^{\left(\frac{2(1-s_1)}{3}+1-s_3\right)m},
\end{aligned}
$$

where the constants $1<\theta_1,\theta_2,m,m'<\infty$ and $0\leq s_1,s_2,s_3\leq 1$ satisfy (2.10).

Combining (2.3), (2.4), (2.9), (2.18), and (2.20), we arrive at

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla_h u(t)\|_{L^2}^2+\|\nabla_h v(t)\|_{L^2}^2+\|\nabla_h b(t)\|_{L^2}^2)+\|\nabla_h\Lambda^\alpha u(t)\|_{L^2}^2+\\
&\|\nabla_h\Lambda^\alpha v(t)\|_{L^2}^2+\|\nabla_h\Lambda^\alpha b(t)\|_{L^2}^2+\kappa\|\nabla_h\nabla\cdot v(t)\|_{L^2}^2\\
\leq\ & C\|u_3\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3(1-\epsilon)}}\|\nabla u\|_{L^2}^{\frac{2\left((2\alpha-1)q-3\right)}{(2\alpha-1)q-3(1-\epsilon)}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{6\epsilon}{(2\alpha-1)q-3(1-\epsilon)}}+\\
&\|b\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3(1-\epsilon)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{2\left((2\alpha-1)q-3\right)}{(2\alpha-1)q-3(1-\epsilon)}}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2})^{\frac{6\epsilon}{(2\alpha-1)q-3(1-\epsilon)}}+\\
&\|v\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3(1-\epsilon)}}(\|\nabla u\|_{L^2}+\|\nabla v\|_{L^2})^{\frac{2\left((2\alpha-1)q-3\right)}{(2\alpha-1)q-3(1-\epsilon)}}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}v\|_{L^2})^{\frac{6\epsilon}{(2\alpha-1)q-3(1-\epsilon)}}\\
\leq\ & C(\|u_3\|_{L^q}+\|b\|_{L^q}+\|v\|_{L^q})^{\frac{2\alpha q}{(2\alpha-1)q-3(1-\epsilon)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+\\
&\|\nabla v\|_{L^2})^{\frac{2\left((2\alpha-1)q-3\right)}{(2\alpha-1)q-3(1-\epsilon)}}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2}+\|\Lambda^{\alpha+1}v\|_{L^2})^{\frac{6\epsilon}{(2\alpha-1)q-3(1-\epsilon)}}.
\end{aligned}
\tag{2.20}
$$

Set

$$
\begin{aligned}
\Theta_1&=\frac{2\alpha q}{(2\alpha-1)q-3(1-\epsilon)},\\
\Theta_2&=\frac{2((2\alpha-1)q-3)}{(2\alpha-1)q-3(1-\epsilon)},\\
\Theta_3&=\frac{6\epsilon}{(2\alpha-1)q-3(1-\epsilon)}.
\end{aligned}
\tag{2.21}
$$

Integrating (2.20) with respect to t, we obtain

$$
\begin{aligned}
E(t)\leq CJ_0+C\int_0^t&\left(\|u_3\|_{L^q}+\|b\|_{L^q}+\|v\|_{L^q}\right)^{\Theta_1}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+\\
&\|\nabla v\|_{L^2})^{\Theta_2}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2}+\|\Lambda^{\alpha+1}v\|_{L^2})^{\Theta_3}d\tau,
\end{aligned}
\tag{2.22}
$$

where $J_0 = \|\nabla u(0)\|_{L^2}^2 + \|\nabla v(0)\|_{L^2}^2 + \|\nabla b(0)\|_{L^2}^2$.

By taking the inner product of the first three equations of (1.1) with $(-\Delta u, -\Delta v, -\Delta b)$ and integrating by parts, the divergence-free condition, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2) + \|\Lambda^{\alpha+1} u(t)\|_{L^2}^2 + \|\Lambda^{\alpha+1} v(t)\|_{L^2}^2 + \\ &\|\Lambda^{\alpha+1} b(t)\|_{L^2}^2 + \kappa\|\nabla\nabla\cdot v(t)\|_{L^2}^2 + \chi\|\nabla\nabla u(t)\|_{L^2}^2 + 4\chi\|\nabla v(t)\|_{L^2}^2 \\ := & \sum_{i=1}^{6} J_i,\end{aligned} \tag{2.23}$$

where

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\Delta u\, dx, \\ J_2 &= -\int_{\mathbb{R}^3} (b\cdot\nabla b)\cdot\Delta u\, dx, \\ J_3 &= \int_{\mathbb{R}^3} (u\cdot\nabla b)\cdot\Delta b\, dx, \\ J_4 &= -\int_{\mathbb{R}^3} (b\cdot\nabla u)\cdot\Delta b\, dx, \\ J_5 &= \int_{\mathbb{R}^3} (u\cdot\nabla v)\cdot\Delta v\, dx, \\ J_6 &= -2\chi\int_{\mathbb{R}^3} [(\nabla\times v)\cdot\Delta u + (\nabla\times u)\cdot\Delta v]\, dx. \end{aligned}$$

By integration by parts and Cauchy's inequality, we arrive at

$$\begin{aligned} J_6 &= 4\chi\int_{\mathbb{R}^3} \nabla(\nabla\times u)\cdot\nabla v\, dx \\ &\le \chi\|\nabla(\nabla\times u)\|_{L^2}^2 + 4\chi\|\nabla v\|_{L^2}^2 \\ &= \chi\|\nabla\nabla u\|_{L^2}^2 + 4\chi\|\nabla v\|_{L^2}^2. \end{aligned} \tag{2.24}$$

For J_1, we divide it into the following three items: $J_{1i}(i = 1, 2, 3)$

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^3} u_3\partial_3 u\cdot\Delta_h u\, dx + \sum_{j=1}^{2}\int_{\mathbb{R}^3} u_j\partial_j u\cdot\Delta u\, dx + \int_{\mathbb{R}^3} u_3\partial_3 u\cdot\partial_{33}^2 u\, dx \\ &:= J_{11} + J_{12} + J_{13}. \end{aligned} \tag{2.25}$$

Integrating by parts and using the divergence-free condition yields

$$J_{11} = \frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{2}\int_{\mathbb{R}^3} \partial_3 u_3\partial_l u_k\partial_l u_k\, dx - \sum_{k=1}^{3}\sum_{l=1}^{2}\int_{\mathbb{R}^3} \partial_l u_3\partial_3 u_k\partial_l u_k\, dx, \tag{2.26}$$

$$J_{12} = \frac{1}{2}\sum_{j=1}^{3}\sum_{k,l=1}^{3}\int_{\mathbb{R}^3} \partial_j u_j\partial_l u_k\partial_l u_k\, dx - \sum_{j=1}^{2}\sum_{k,l=1}^{3}\int_{\mathbb{R}^3} \partial_l u_j\partial_j u_k\partial_l u_k\, dx, \tag{2.27}$$

and

$$J_{13} = \frac{1}{2}\sum_{k=1}^{3}\int_{\mathbb{R}^3}(\partial_1 u_1 + \partial_2 u_2)\partial_3 u_k \partial_3 u_k \, dx. \tag{2.28}$$

Therefore, we have

$$|J_1| \le C\int_{\mathbb{R}^3} |\nabla_h u||\nabla u|^2 \, dx. \tag{2.29}$$

From Hölder's inequality and Lemma 2.1, it follows that

$$\begin{aligned} |J_1| &\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^4}^2 \\ &\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\Lambda^\alpha u\|_{L^6}^{\frac{3}{2\alpha}} \\ &\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h \Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1} u\|_{L^2}^{\frac{1}{2\alpha}}. \end{aligned} \tag{2.30}$$

By using integrating by parts and the divergence-free condition, we have

$$\begin{aligned} J_3 &= -\sum_{j,k,l=1}^{3}\int_{\mathbb{R}^3}\partial_l(u_j\partial_j b_k)\partial_l b_k \, dx \\ &= -\sum_{j,k,l=1}^{3}\int_{\mathbb{R}^3}(\partial_l u_j \partial_j b_k \partial_l b_k + u_j\partial_{lj}^2 b_k \partial_l b_k) \, dx \\ &= \sum_{j,k,l=1}^{3}\int_{\mathbb{R}^3} b_k\partial_l(\partial_l u_j \partial_j b_k) \, dx \\ &= \sum_{j,k,l=1}^{3}\int_{\mathbb{R}^3}(b_k\partial_{ll}^2 u_j\partial_j b_k + b_k\partial_l u_j \partial_{jl}^2 b_k) \, dx. \end{aligned} \tag{2.31}$$

Then we arrive at

$$|J_3| \le C\int_{\mathbb{R}^3}|b|(|\nabla u| + |\nabla b|)(|\Delta u| + |\Delta b|) \, dx. \tag{2.32}$$

Furthermore, we have

$$|J_2 + J_3 + J_4| \le C\int_{\mathbb{R}^3}|b|(|\nabla u| + |\nabla b|)(\Delta u| + |\Delta b|) \, dx. \tag{2.33}$$

It follows from the same procedure (2.18) that

$$\begin{aligned} &|J_2 + J_3 + J_4| \\ \le\ & C\int_{\mathbb{R}^3}|b|(|\nabla u| + |\nabla b|)(|\Delta u| + |\Delta b|) \, dx \\ \le\ & C\|b\|_{L^q}\||\nabla u| + |\nabla b|\|_{L^{\theta_1}}\||\Delta u| + |\Delta b|\|_{L^{\theta_2}} \\ \le\ & C\|b\|_{L^q}(\|\Delta u\|_{L^2}^{\frac{2}{3}}\|\Delta u\|_{L^{\frac{\theta_1}{3}}}^{\frac{1}{3}} + \|\Delta b\|_{L^2}^{\frac{2}{3}}\|\Delta b\|_{L^{\frac{\theta_1}{3}}}^{\frac{1}{3}})(\|\Delta u\|_{L^{\theta_2}} + \|\Delta b\|_{L^{\theta_2}}) \\ \le\ & C\|b\|_{L^q}\Big(\|\nabla u\|_{L^2}^{\frac{2s_1}{3}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{2(1-s_1)}{3}}\|\nabla u\|_{L^2}^{\frac{s_2}{3}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1-s_2}{3}} + \\ & \|\nabla b\|_{L^2}^{\frac{2s_1}{3}}\|\Lambda^{\alpha+1}b\|_{L^2}^{\frac{2(1-s_1)}{3}}\|\nabla b\|_{L^2}^{\frac{s_2}{3}}\|\Lambda^{\alpha+1}b\|_{L^2}^{\frac{1-s_2}{3}}\Big) \times \end{aligned} \tag{2.34}$$

$$
\begin{aligned}
&(\|\nabla u\|_{L^2}^{s_3}\|\Lambda^{\alpha+1}u\|_{L^2}^{1-s_3}+\|\nabla b\|_{L^2}^{s_3}\|\Lambda^{\alpha+1}b\|_{L^2}^{1-s_3})\\
\le\ & C\|b\|_{L^q}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{2s_1}{3}+\frac{s_2}{3}+s_3}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2})^{\frac{2(1-s_1)}{3}+\frac{1-s_2}{3}+1-s_3}\\
\le\ & C\|b\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\frac{1}{8}(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2),
\end{aligned}
$$

where the constants $1<\theta_1,\theta_2,m,m'<\infty$ and $0\le s_1,s_2,s_3\le 1$ satisfy (2.10).

Similar to J_3, we bound J_5 as

$$
|J_5|\le C\int_{\mathbb{R}^3}|v|(|\nabla u|+|\nabla v|)(|\Delta u|+|\Delta v|)\,dx. \tag{2.35}
$$

The same procedure leads to (2.34) yields

$$
\begin{aligned}
|J_5| &\le C\int_{\mathbb{R}^3}|v|(|\nabla u|+|\nabla v|)(|\Delta u|+|\Delta v|)\,dx\\
&\le C\|v\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}}(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2)+\frac{1}{8}(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}v\|_{L^2}^2).
\end{aligned}
$$

Combining (2.23), (2.24), (2.30), (2.34), and (2.36), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2)+\frac{3}{4}(\|\Lambda^{\alpha+1}u(t)\|_{L^2}^2+\|\Lambda^{\alpha+1}v(t)\|_{L^2}^2)\\
&+\frac{3}{4}\|\Lambda^{\alpha+1}b(t)\|_{L^2}^2+\kappa\|\nabla\nabla\cdot v(t)\|_{L^2}^2\\
\le\ & C(\|b\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}}+\|v\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla v\|_{L^2}^2)+\\
&C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1}{2\alpha}}.
\end{aligned} \tag{2.36}
$$

Integrating (2.36) over the interval $(0,t)$ and using Hölder's inequality, it was deduced that

$$
\begin{aligned}
&\frac{1}{2}(\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2)+\frac{3}{4}\int_0^t(\|\Lambda^{\alpha+1}u(\tau)\|_{L^2}^2+\\
&\|\Lambda^{\alpha+1}v(\tau)\|_{L^2}^2+\|\Lambda^{\alpha+1}b(\tau)\|_{L^2}^2)d\tau+\int_0^t\kappa\|\nabla\nabla\cdot v(\tau)\|_{L^2}^2d\tau\\
\le\ & C+C\int_0^t(\|b\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}}+\|v\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla v\|_{L^2}^2)d\tau+\\
&C\int_0^t\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1}{2\alpha}}d\tau\\
\le\ & C+C\int_0^t(\|b\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}}+\|v\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla v\|_{L^2}^2)d\tau+\\
&C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}\int_0^t\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1}{2\alpha}}d\tau.
\end{aligned} \tag{2.37}
$$

From Young's inequality, it follows that

$$
\begin{aligned}
& C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}\int_0^t\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1}{2\alpha}}d\tau\\
\le\ & C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}[\int_0^t\|\nabla u\|_{L^2}^2d\tau]^{1-\frac{3}{4\alpha}}[\int_0^t\|\nabla_h\Lambda^\alpha u\|_{L^2}^2d\tau]^{\frac{1}{2\alpha}}[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4\alpha}}\\
\le\ & C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}[\int_0^t\|u\|_{L^2}^{\frac{2\alpha}{1+\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{2}{1+\alpha}}d\tau]^{1-\frac{3}{4\alpha}}[\int_0^t\|\nabla_h\Lambda^\alpha u\|_{L^2}^2d\tau]^{\frac{1}{2\alpha}}[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4\alpha}}\\
\le\ & C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}[\int_0^t\|\nabla_h\Lambda^\alpha u\|_{L^2}^2d\tau]^{\frac{1}{2\alpha}}[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4\alpha}+\frac{4\alpha-3}{4\alpha(1+\alpha)}}\\
\le\ & C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}[(\int_0^t\|\nabla_h\Lambda^\alpha u\|_{L^2}^2d\tau)^{\frac{1}{2}}+1][(\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau)^{\frac{1}{4}}+1] \qquad (2.38)\\
\le\ & CE(t)[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+\\
& CE(t)+C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}\\
\le\ & CE(t)[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+C(\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}^2+1)[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+\\
& CE(t)+C\sup_{0\le\tau\le t}\|\nabla_h u\|_{L^2}^2+C\\
\le\ & CE(t)[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+C[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+CE(t)+C.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& \frac{1}{2}(\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2)+\frac{3}{4}\int_0^t(\|\Lambda^{\alpha+1}u(\tau)\|_{L^2}^2+\\
& \|\Lambda^{\alpha+1}v(\tau)\|_{L^2}^2+\|\Lambda^{\alpha+1}b(\tau)\|_{L^2}^2)d\tau+\int_0^t\kappa\|\nabla\nabla\cdot v(\tau)\|_{L^2}^2d\tau\\
\le\ & C+C\int_0^t(\|b\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}}+\|v\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla v\|_{L^2}^2)d\tau+ \qquad (2.39)\\
& CE(t)[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+C[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2d\tau]^{\frac{1}{4}}+CE(t)+C.
\end{aligned}
$$

By using Hölder's inequality, Young's inequality, and (2.22), we deduce that

$$
\begin{aligned}
CE(t)\ \le\ & C+C\int_0^t(\|u_3\|_{L^q}+\|b\|_{L^q}+\|v\|_{L^q})^{\Theta_1}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+\\
& \|\nabla v\|_{L^2})^{\Theta_2}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2}+\|\Lambda^{\alpha+1}v\|_{L^2})^{\Theta_3}d\tau\\
\le\ & C+C[\int_0^t(\|u_3\|_{L^q}+\|b\|_{L^q}+\|v\|_{L^q})^{\frac{2\alpha q}{(2\alpha-1)q-3}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+ \qquad (2.40)\\
& \|\nabla v\|_{L^2})^2d\tau]^{\Theta_2}[\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2}+\|\Lambda^{\alpha+1}v\|_{L^2})^2d\tau]^{\frac{1}{2}\Theta_3}
\end{aligned}
$$

$$\leq \ C + C\int_0^t (\|u_3\|_{L^q} + \|b\|_{L^q} + \|v\|_{L^q})^{\frac{2\alpha q}{(2\alpha-1)q-3}}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla v\|_{L^2})^2 d\tau + \frac{1}{16}\int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2} + \|\Lambda^{\alpha+1}b\|_{L^2} + \|\Lambda^{\alpha+1}v\|_{L^2})^2 d\tau.$$

Similarly, it follows from (2.22) and Hölder's inequality and Young's inequality that

$$\begin{aligned}
& CE(t)[\int_0^t \|\Lambda^{\alpha+1}u\|_{L^2}^2 d\tau]^{\frac{1}{4}} \\
\leq \ & C[\int_0^t \|\Lambda^{\alpha+1}u\|_{L^2}^2 d\tau]^{\frac{1}{4}} + C[\int_0^t \|\Lambda^{\alpha+1}u\|_{L^2}^2 d\tau]^{\frac{1}{4}} \int_0^t (\|u_3\|_{L^q} + \|b\|_{L^q} + \\
& \|v\|_{L^q})^{\Theta_1}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla v\|_{L^2})^{\frac{\Theta_2}{2}}(\|\Lambda^{\alpha+1}u\|_{L^2} + \|\Lambda^{\alpha+1}b\|_{L^2} + \|\Lambda^{\alpha+1}v\|_{L^2})^{\Theta_3} d\tau \\
\leq \ & C[\int_0^t \|\Lambda^{\alpha+1}u\|_{L^2}^2 d\tau]^{\frac{1}{4}} + C[\int_0^t \|\Lambda^{\alpha+1}u\|_{L^2}^2 d\tau]^{\frac{1}{4}}[\int_0^t (\|u_3\|_{L^q} + \|b\|_{L^q} + \\
& \|v\|_{L^q})^{\frac{2\alpha q}{(2\alpha-1)q-3}}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla v\|_{L^2})^2 d\tau]^{\frac{\Theta_2}{2}}[\int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2} + \\
& \|\Lambda^{\alpha+1}b\|_{L^2} + \|\Lambda^{\alpha+1}v\|_{L^2})^2 d\tau]^{\frac{\Theta_3}{2}} \qquad (2.41) \\
\leq \ & C[\int_0^t \|\Lambda^{\alpha+1}u\|_{L^2}^2 d\tau]^{\frac{1}{4}} + C[\int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2} + \|\Lambda^{\alpha+1}b\|_{L^2} + \|\Lambda^{\alpha+1}v\|_{L^2})^2 d\tau]^{\frac{2\Theta_3+1}{4}} \cdot \\
& [\int_0^t (\|u_3\|_{L^q} + \|b\|_{L^q} + \|v\|_{L^q})^{\frac{2\alpha q}{(2\alpha-1)q-3}}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla v\|_{L^2})^2 d\tau]^{\Theta_2} \\
\leq \ & C[\int_0^t \|\Lambda^{\alpha+1}u\|_{L^2}^2 d\tau]^{\frac{1}{4}} + C[\int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2} + \|\Lambda^{\alpha+1}b\|_{L^2} + \|\Lambda^{\alpha+1}v\|_{L^2})^2 d\tau]^{\frac{2\Theta_3+1}{4}} \cdot \\
& [\int_0^t (\|u_3\|_{L^q} + \|b\|_{L^q} + \|v\|_{L^q})^{\Theta_4}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla v\|_{L^2})^2 d\tau]^{\frac{3(2\alpha-1)q+3(1-\epsilon)-12}{4[(2\alpha-1)q-3(1-\epsilon)]}} \\
\leq \ & C + C\int_0^t (\|u_3\|_{L^q} + \|b\|_{L^q} + \|v\|_{L^q})^{\Theta_4}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2} + \\
& \|\nabla v\|_{L^2})^2 d\tau + \frac{1}{16}\int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2} + \|\Lambda^{\alpha+1}b\|_{L^2} + \|\Lambda^{\alpha+1}v\|_{L^2})^2 d\tau,
\end{aligned}$$

where $\Theta_4 = \frac{8\alpha q}{3(2\alpha-1)q+3(1-\epsilon)-12}$.

We substitute (2.40) and (2.41) into (2.39) and then use Young's inequality to obtain

$$\begin{aligned}
& \frac{1}{2}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2) + \frac{3}{4}\int_0^t (\|\Lambda^{\alpha+1}u(\tau)\|_{L^2}^2 + \\
& \|\Lambda^{\alpha+1}v(\tau)\|_{L^2}^2 + \|\Lambda^{\alpha+1}b(\tau)\|_{L^2}^2)d\tau + \int_0^t \kappa\|\nabla\nabla\cdot v(\tau)\|_{L^2}^2 d\tau \\
\leq \ & C + C\int_0^t (\|b\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}} + \|v\|_{L^q}^{\frac{2\alpha q}{(2\alpha-1)q-3}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla v\|_{L^2}^2)d\tau + \\
& C\int_0^t (\|u_3\|_{L^q} + \|b\|_{L^q} + \|v\|_{L^q})^{\Theta_4}(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla v\|_{L^2})^2 d\tau + \qquad (2.42)
\end{aligned}$$

$$C\int_0^t (\|u_3\|_{L^q}+\|b\|_{L^q}+\|v\|_{L^q})^{\frac{2\alpha q}{(2\alpha-1)q-3}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+\|\nabla v\|_{L^2})^2 d\tau +$$
$$\frac{1}{8}[\int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2}+\|\Lambda^{\alpha+1}v\|_{L^2})^2 d\tau]$$
$$\leq C+C\int_0^t (\|u_3\|_{L^q}^{\Theta_4}+\|b\|_{L^q}^{\Theta_4}+\|v\|_{L^q}^{\Theta_4})(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+\|\nabla v\|_{L^2})^2 d\tau +$$
$$\frac{1}{4}\int_0^t (\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2+\|\Lambda^{\alpha+1}v\|_{L^2}^2)d\tau.$$

Then we have

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\int_0^t (\|\Lambda^{\alpha+1}u(\tau)\|_{L^2}^2+ \\
&\|\Lambda^{\alpha+1}v(\tau)\|_{L^2}^2+\|\Lambda^{\alpha+1}b(\tau)\|_{L^2}^2)d\tau+\int_0^t \kappa\|\nabla\nabla\cdot v(\tau)\|_{L^2}^2 d\tau \\
\leq\ & C+C\int_0^t (\|u_3\|_{L^q}^{\Theta_4}+\|b\|_{L^q}^{\Theta_4}+\|v\|_{L^q}^{\Theta_4})(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+\|\nabla v\|_{L^2})^2 d\tau.
\end{aligned} \tag{2.43}$$

Thanks to Gronwall's inequality and condition (1.3), we obtain

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\int_0^t (\|\Lambda^{\alpha+1}u(\tau)\|_{L^2}^2+ \\
&\|\Lambda^{\alpha+1}v(\tau)\|_{L^2}^2+\|\Lambda^{\alpha+1}b(\tau)\|_{L^2}^2)d\tau+\int_0^t \kappa\|\nabla\nabla\cdot v(\tau)\|_{L^2}^2 d\tau \\
\leq\ & C\exp[C\int_0^t (\|u_3\|_{L^q}^{\Theta_4}+\|b\|_{L^q}^{\Theta_4}+\|v\|_{L^q}^{\Theta_4})d\tau]<\infty.
\end{aligned} \tag{2.44}$$

Finally, we consider the case $q=\infty$. By repeating the above procedure, we derive that

$$E(t)\leq CJ_0+C\int_0^t (\|u_3\|_{L^\infty}+\|b\|_{L^\infty}+\|v\|_{L^\infty})^{\frac{2\alpha}{2\alpha-1}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}+\|\nabla v\|_{L^2})^2 d\tau.$$

Thanks to Gronwall's inequality and condition (1.3), we obtain

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\int_0^t (\|\Lambda^{\alpha+1}u(\tau)\|_{L^2}^2+ \\
&\|\Lambda^{\alpha+1}v(\tau)\|_{L^2}^2+\|\Lambda^{\alpha+1}b(\tau)\|_{L^2}^2)d\tau+\int_0^t \kappa\|\nabla\nabla\cdot v(\tau)\|_{L^2}^2 d\tau \\
\leq\ & C\exp[C\int_0^t (\|u_3\|_{L^\infty}^{\frac{8\alpha}{3(2\alpha-1)}}+\|b\|_{L^\infty}^{\frac{8\alpha}{3(2\alpha-1)}}+\|v\|_{L^\infty}^{\frac{8\alpha}{3(2\alpha-1)}})d\tau]<\infty.
\end{aligned} \tag{2.45}$$

By the above steps, we establish a higher-order a priori estimate of the solutions, and then we obtain that the higher-order norm of the solutions is bounded, thus proving the smoothness of the solutions. This completes the proof of Theorem 1.1.

3. Conclusions

In this paper, the regularity criterion of the weak solution of the three-dimensional magnetic micropolar fluid equation when $1 \le \alpha = \beta = \gamma \le \frac{3}{2}$ is studied. However, the regularity of the weak solution of the magnetic micropolar fluid equation when $1 \le \alpha, \beta, \gamma \le \frac{3}{2}$ on $\mathbb{R}^3$ is still an open problem, and it is hoped that the method in this paper can provide inspiration for the solution of this problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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