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Research article

Normalized ground states for a doubly nonlinear Schrödinger equation on periodic metric graphs

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Abstract: We investigate the existence of ground states for a class of Schrödinger equations with both a standard power nonlinearity and delta nonlinearity concentrated at finite vertices of the periodic metric graphs G. Using variational methods, if $\alpha > 0$ and the standard nonlinearity power is L^2-subcritical, we establish the existence of ground states for every mass and every periodic graph. If $\alpha < 0$ and the standard nonlinearity power is L^2-critical, we show that two types of topological structures on G will prevent the existence of ground states. Furthermore, for graphs that do not satisfy these two types of topological structures, ground states exist when the given mass belongs to an appropriate range and the parameter $|\alpha|$ is small enough.

Keywords: ground states; point-wise nonlinearity; periodic metric graph

1. Introduction and main results

In this paper, we address the existence of ground states for the following NLS energy functionals

$$F_{\alpha,V}(u,G) = \frac{1}{2}\int_G |u'|^2 dx - \frac{1}{p}\int_G |u|^p dx - \frac{\alpha}{q}\sum_{\mathrm{v}\in V} |u(\mathrm{v})|^q, \tag{1.1}$$

with the mass constraint

$$\int_G |u|^2 dx = \mu, \tag{1.2}$$

where G is a general periodic metric graph, $2 < p \le 6$, $2 < q < 4$, $\alpha \in \mathbb{R}\setminus\{0\}$, $V \subset V(G)$ is a subset of all the vertices of G, and the number of vertices in V is finite, i.e., $\#V < +\infty$.

A metric graph $G = (V(G), E(G))$ is defined as a connected network composed of edges $E(G)$ (may be multiple edges and self-loops) and vertices $V(G)$ (endpoints of the edges). Every edge e may be bounded or unbounded. A bounded edge e is usually defined as a finite interval $e := [0, l_e]$, with $l_e < +\infty$ denoting the length of e. Any unbounded edge can be referred to as a half-line $\mathbb{R}^+ = [0, +\infty)$.

The periodic graph G is made up of an infinite number of duplicates of a given compact graph $\mathcal{K}$, which is called the periodicity cell of G (see Section 2 for a rigorous definition of the $Z-$periodic graph). It is readily seen that every edge of a periodic graph G is bounded. For the present paper, we consider this type of periodic graph, the periodicity cell of which reproduces itself only in one direction, for example, the ladder-type graph in Figure 1(a) that has been investigated in [1]. We highlight that this structure of the periodic graph enjoys a Z-symmetry (see [2] and [3] for more details). As for the periodic graphs, the periodicity cell of which reproduces itself along more than one direction (Figure 1(b)), we suggest reading [4] for more information.

From the perspective of the topological structure of the graphs in the present paper, without loss of generality, we can assume that every vertex of G has a degree that is not equal to 2. In this way, it is natural to exclude the special case of $G = \mathbb{R}$.

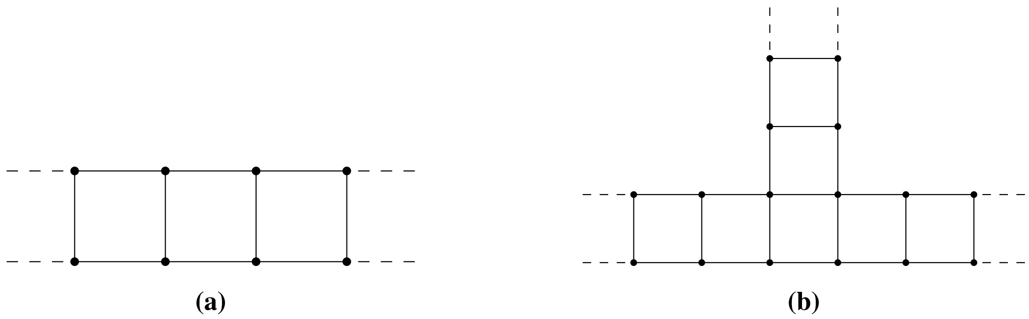

Figure 1. (a) the ladder-type graph; (b) a graph in which the periodicity cell replicates itself along two directions.

With the description of G as above, we define a function $u : G \to \mathbb{R}$ as a family of functions $\{u_e\}_{e\in E(G)}$, where every u_e is defined on the bounded interval $[0, l_e]$. Lebesgue spaces L^r $(1 \le r < +\infty)$ on G can be defined in a natural way, with the norm

$$\|u\|^r_{L^r(G)} := \sum_{e\in E(G)} \|u_e\|^r_{L^r(I_e)}.$$

The Sobolev space $H^1(G)$ is defined as the set of those functions $u : G \to \mathbb{R}$ such that $u = (u_e)_{e\in E(G)}$ is continuous on G and $u_e \in H^1(I_e)$, with the natural norm

$$\|u\|^2_{H^1(G)} := \|u'\|^2_{L^2(G)} + \|u\|^2_{L^2(G)}.$$

According to the mass constraint (1.2), for every mass $\mu > 0$, we introduce the corresponding space

$$H^1_\mu(G) := \left\{u \in H^1(G) : \int_G |u|^2 dx = \mu\right\}.$$

Thus, by ground states of mass μ we mean the global minimizers of the energy functional (1.1) in the space $H^1_\mu(G)$, i.e., the solutions of the following minimization problem

$$\mathcal{F}_{\alpha,V}(\mu, G) := \inf_{u\in H^1_\mu(G)} F_{\alpha,V}(u, G). \tag{1.3}$$

Our aim is to clarify, under what conditions, there exists a function $u \in H_\mu^1(G)$ such that $F_{\alpha,V}(u,G) = \mathcal{F}_{\alpha,V}(\mu,G)$. Without loss of generality, it is readily seen that we only need to consider the nonnegative, real−valued functions.

Ground states satisfy, for a suitable Lagrange multiplier $\omega \in \mathbb{R}$, the stationary NLS equation

$$-u'' + \omega u = |u|^{p-2}\, u, \tag{1.4}$$

on each edge of G, with a standard Kirchhoff boundary condition (see [5] for a discussion)

$$\sum_{e \succ \mathrm{v}} \frac{\mathrm{d}u_e}{\mathrm{d}x}(\mathrm{v}) = 0, \quad \text{for any } \mathrm{v} \in V(G) \setminus V, \tag{1.5}$$

and the non-Kirchhoff condition (usually referred to as delta interaction)

$$\sum_{e \succ \mathrm{v}} \frac{\mathrm{d}u_e}{\mathrm{d}x}(\mathrm{v}) = -\alpha\, |u(\mathrm{v})|^{q-2}\, u(\mathrm{v}), \quad \text{for any } \mathrm{v} \in V. \tag{1.6}$$

The condition in (1.6) can be interpreted as an effect of point-like defects or impurities in the propagation medium (see [6, 7] for more about the non-Kirchhoff vertex conditions on graphs).

In the past few years, the study of the nonlinear Schrödinger equations on metric graphs mainly focuses on three cases: on compact metric graphs (see, for instance, [8–12]), on noncompact metric graphs with at least one unbounded edge (see [5, 13–16]), and on noncompact metric graphs with no unbounded edges (see [17–19]). In particular, we refer interested readers to [20–23] and the references there for more about the present model on noncompact metric graphs with half-lines.

Before stating our main results, we wish to emphasize again that every periodic metric graph G discussed in the present paper enjoys a Z-symmetry, which means that each periodicity cell $\mathcal{K}$ connects only two of all the others.

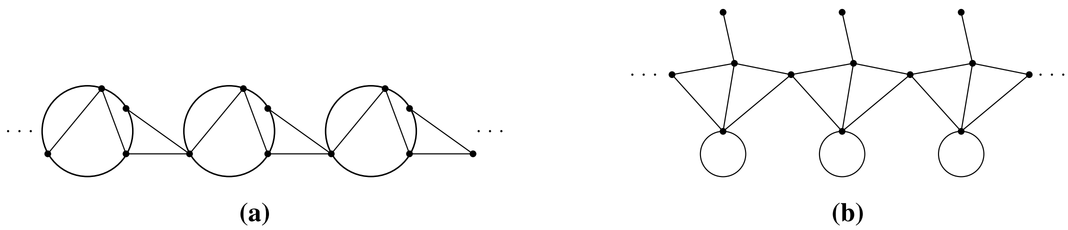

Figure 2. (*a*) a graph satisfying assumption (A_1); (*b*) a graph with a terminal point.

In this paper, we mainly discuss the following two cases:

$$\text{(Case 1)} \quad \alpha > 0,\ 2 < p < 6, \ \text{ and } 2 < q < 4;$$

$$\text{(Case 2)} \quad \alpha < 0,\ p = 6, \ \text{and } 2 < q < 4.$$

In case 1, we present the first conclusion.

Theorem 1.1. *Let G be a periodic metric graph. $2 < p < 6$ and $2 < q < 4$. Then, for every $\alpha > 0$, we have*

$$\mathcal{F}_{\alpha,V}(\mu,G) \in (-\infty, 0)\,, \quad \forall\, \mu > 0, \tag{1.7}$$

and ground states of mass μ always exist.

Compared to the results on noncompact metric graphs with half-lines, Theorem 1.1 has exhibited a similarity with the real line $G = \mathbb{R}$ in [24], where a unique positive ground state exists for any $\mu > 0$. On the other hand, for general noncompact metric graphs with half-lines, Theorem 1.1 has unveiled a significant difference with the ones in [23], where the existence of ground states depends not only on μ, p, q but also on the topological and metric features of the graphs.

In case 2, with the critical exponent $p = 6$, in order to state our results, we introduce the definition of a critical mass in [13], which is denoted as

$$\mu_G := \sqrt{\frac{3}{K_G}},$$

with K_G denoting the sharp constant of the following inequality:

$$\|u\|^6_{L^6(G)} \leq K_G \|u\|^4_{L^2(G)} \|u'\|^2_{L^2(G)}, \ \forall\, u \in H^1(G). \tag{1.8}$$

When $G = \mathbb{R}$ or $\mathbb{R}^+$, there are results

$$\mu_{\mathbb{R}} = \frac{\sqrt{3}\pi}{2} = 2\mu_{\mathbb{R}^+}.$$

It is well known that, for every noncompact metric graph G (see [13]),

$$\mu_{\mathbb{R}^+} \leq \mu_G \leq \mu_{\mathbb{R}}.$$

In what follows, we introduce two types of topological assumptions on periodic metric graphs, and they play a significant role in the nonexistence of ground states:

(A_1) : Every point x in G can serve as the origin for two disjoint edges extending to infinity,

(A_2) : G has a terminal edge.

Here, assumption (A_1) is an equivalent deformation of the assumption $\left(\mathrm{H}_{per}\right)$ in [6] (see Figure 2(a)), and another version of this assumption can be traced back to [5] on graphs with at least a half-line. A terminal edge in assumption (A_2) denotes an edge that ends with a vertex of degree 1 (see Figure 2(b)). Obviously, these two types of graphs are mutually exclusive.

As for the periodic metric graph G at present, we have (see Proposition 4.1 in [2])

$$\mu_G = \begin{cases} \mu_{\mathbb{R}}, & \text{if } G \text{ satisfies assumption } (\mathrm{A}_1), \\ \mu_{\mathbb{R}^+}, & \text{if } G \text{ satisfies assumption } (\mathrm{A}_2). \end{cases} \tag{1.9}$$

Now we state the results of case 2 as follow.

Theorem 1.2. *Let G be a periodic metric graph.* $p = 6$ *and* $2 < q < 4$. *If G satisfies assumption* (A_1) *or* (A_2)*, then for every* $\alpha < 0$*, we have*

$$\mathcal{F}_{\alpha,V}(\mu, G) = \begin{cases} 0, & \text{for } \mu \leq \mu_G, \\ -\infty, & \text{for } \mu > \mu_G, \end{cases} \tag{1.10}$$

and ground states do not exist for every μ.

Theorem 1.3. *Let G be a periodic metric graph. $p = 6$ and $2 < q < 4$. If G satisfies neither assumption,* (A_1) *nor* (A_2)*, then for every $\alpha < 0$, we have*

$$\mathcal{F}_{\alpha,V}(\mu, G) = \begin{cases} 0, & \text{for } \mu \leq \mu_G, \\ -\infty, & \text{for } \mu > \mu_{\mathbb{R}}, \end{cases} \tag{1.11}$$

and ground states do not exist when $\mu \leq \mu_G$. Moreover, if $\mu_G < \mu_{\mathbb{R}}$, then for every $\mu \in (\mu_G, \mu_{\mathbb{R}}]$, we can obtain a value $\tilde{\alpha} < 0$ (possibly equal to $-\infty$) depending on $\mu, q, V$and the periodic graph G such that

$$\mathcal{F}_{\alpha,V}(\mu, G) \in (-\infty, 0)\,,\ \forall\, \alpha \in (\tilde{\alpha}, 0)\,, \tag{1.12}$$

and ground states exist when $\mu \in (\mu_G, \mu_{\mathbb{R}}]$ and $\alpha \in (\tilde{\alpha}, 0)$.

Theorems 1.2 and 1.3 provide a comprehensive discussion about the existence of ground states based on the topological features of the periodic metric graph G. It is obvious that the topological assumption (A_1) or (A_2) will prevent the existence of ground states.

For the periodic metric graphs satisfying neither assumption (A_1) nor (A_2), the precondition $\mu_G < \mu_{\mathbb{R}}$ in Theorem 1.2 is consistent. For example, the graph G in Figure 3 satisfies neither assumption (A_1) nor (A_2). By Proposition 4.2 in [2], we know that $\mu_G < \mu_{\mathbb{R}}$.

Finally, it is worth mentioning that the condition $\#V < +\infty$, which means that there exists a finite number of point defects, plays an important role throughout the paper in the proofs of all conclusions in the present paper. In particular, this condition ensures that the ground state energy level $\mathcal{F}_{\alpha,V}(\mu, G)$ is negative for $\mu \in (\mu_G, \mu_{\mathbb{R}}]$ when $|\alpha|$ is small enough.

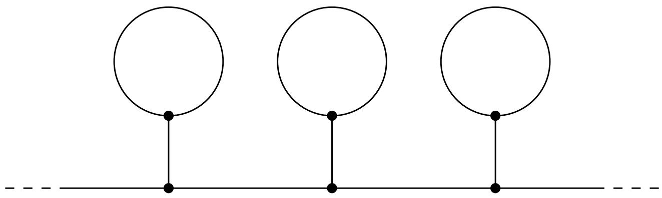

Figure 3. A periodic metric graph satisfying neither assumption (A_1) nor (A_2).

The remainder of the paper is organized as follows: In Section 2, we collect some preliminary results and gives some prior estimates of the ground state energy level $\mathcal{F}_{\alpha,V}(\mu, G)$. Section 3 deals with the proof of Theorem 1.1, which is the existence of ground states when $\alpha > 0$ and $2 < p < 6$. Finally in Section 4, we investigate the role of the topological properties of graph G in the existence of ground states when $\alpha < 0$ and $p = 6$, i.e., the proofs of Theorems 1.2 and 1.3.

2. Preliminaries

We begin here by collecting some useful tools and preliminary estimates that will be helpful in the forthcoming sections.

First of all, let us introduce the rigorous definition of a $Z-$periodic graph borrowed directly from the Section 2 of [25] (see [25] and references there for more details).

Let $\mathcal{K} = (E(\mathcal{K}), V(\mathcal{K}))$ be a connected compact graph, with both the number of edges $\#E(\mathcal{K})$ and the number of vertices $\#V(\mathcal{K})$ finite. Obviously, every edge $e \in E(\mathcal{K})$ has a finite length. Let us denote two non-empty subsets of $V(\mathcal{K})$ as D and R. Define a function $\sigma : D \to R$ such that

(i) $D \cap R = \emptyset$;

(ii) σ is bijective.

Let the compact graph $\mathcal{K}$ reproduce itself along one direction infinitely many times, as shown in Figures 1(a), 2, and 3, but not in Figure 1(b). Consider now the set $\{\mathcal{K}_i\}_{i\in\mathbb{Z}}$, denoted as all the duplicates of $\mathcal{K}$. Corresponding to the two nonempty subsets D and R of $V(\mathcal{K})$, for every $i \in \mathbb{Z}$, let us denote D_i and R_i as the duplicates of D and R in $V(\mathcal{K}_i)$. It is clear that both D_i and R_i are nonempty.

Denote $\mathbb{G} := \bigcup_{i\in\mathbb{Z}} \mathcal{K}_i$ and let σ be a map from D_i to R_{i+1}. We introduce a relation between any two vertices v, w of $\mathbb{G}$ as

$$\mathrm{v} \sim \mathrm{w} \Longleftrightarrow \begin{cases} \mathrm{v} = \mathrm{w}, & \text{if } \ \mathrm{v}, \mathrm{w} \in \mathcal{K}_i, \ \text{for some } i \in \mathbb{Z}, \\ \sigma(\mathrm{v}) = \mathrm{w}, & \text{if } \ \mathrm{v} \in D_i, \mathrm{w} \in R_{i+1}, \ \text{for some } i \in \mathbb{Z}, \\ \sigma(\mathrm{w}) = \mathrm{v}, & \text{if } \ \mathrm{v} \in R_{i+1}, \mathrm{w} \in D_i, \ \text{for some } i \in \mathbb{Z}. \end{cases}$$

It is not difficult to verify that the relation $\mathrm{v} \sim \mathrm{w}$ is well-defined and equivalent on $\mathbb{G}$. Now we can say that the quotient $G := \mathbb{G}/\sim$ is a Z-periodic graph with periodicity cell $\mathcal{K}$ and the pasting rule σ.

Secondly, let us recall several different versions of the Gagliardo−Nirenberg inequalities applicable to this article. For every noncompact (with either at least a half-line or infinitely many bounded edges) metric graph G, we have (see [26]).

$$\|u\|_{L^p(G)}^p \le K_p \|u\|_{L^2(G)}^{\frac{p}{2}+1} \|u'\|_{L^2(G)}^{\frac{p}{2}-1}, \ \forall\, u \in H^1(G) \text{ and } 2 < p < +\infty, \tag{2.1}$$

where $K_p > 0$ is a generic constant that depends only on the exponent p.

When G does not have a terminal edge, an improved version of the Gagliardo−Nirenberg inequality will work (see Lemma 4.4 of [13] and the argument in Section 4 of [2]). For every mass μ in $(0, \mu_{\mathbb{R}}]$ and $u \in H^1_\mu(G)$, there exists a value $\theta_u \in [0, \mu]$ related to function u such that

$$\|u\|_{L^6(G)}^6 \le 3\left(\frac{\mu - \theta_u}{\mu_{\mathbb{R}}}\right)^2 \|u'\|_{L^2(G)}^2 + C_G \theta_u^{\frac{1}{2}}, \tag{2.2}$$

where $C_G > 0$ is a constant depending only on G.

In addition, we can derive a further Gagliardo−Nirenberg−type inequality about the sum of all pointwise nonlinearities at the vertices of the periodic metric graph. That is, for every periodic graph G defined above and $q \in (2, 4)$, there exists $C > 0$, depending on the exponent q and G, such that (see Lemma 2.2 in [25])

$$\sum_{\mathrm{v}\in V(G)} |u(\mathrm{v})|^q \le C\left(\|u\|_{L^q(G)}^q + \|u\|_{L^2(G)}^{\frac{q}{2}} \|u'\|_{L^2(G)}^{\frac{q}{2}}\right), \ \forall\, u \in H^1(G). \tag{2.3}$$

Moreover, the case $\alpha = 0$ has been studied in [2] on periodic metric graphs, with the minimization problem as

$$\mathcal{F}(\mu, G) := \inf_{u\in H^1_\mu(G)} F(u, G),$$

where

$$F(u,G)=\frac{1}{2}\int_G |u'|^2dx-\frac{1}{p}\int_G |u|^p dx.$$

For convenience, we state some results obtained in [2] with the next lemma.

Lemma 2.1 ([2]). *Let G be a periodic metric graph. If $2<p<6$, then we have*

$$\mathcal{F}(\mu,G)\in(-\infty,0)\,,\quad \forall\,\mu>0, \tag{2.4}$$

and ground states exist at every mass μ. If $p=6$ and G satisfies assumptions (A_1) *or* (A_2)*, we have*

$$\mathcal{F}\,(\mu,G)=\begin{cases}0, & \text{for }\ \mu\le\mu_G,\\ -\infty, & \text{for }\ \mu>\mu_G,\end{cases} \tag{2.5}$$

and the infimum is never achieved. If $p=6$, $\mu_G<\mu_{\mathbb{R}}$, and G satisfies neither assumption (A_1) *nor* (A_2)*, then there exist ground states if and only if $\mu\in[\mu_G,\mu_{\mathbb{R}}]$.*

Let us make a simple comparison between these two cases: the case $\alpha\neq 0$ in the present paper and the case $\alpha=0$ in [2]. Although the results in Theorems 1.1 and 1.2 are somewhat similar to the ones in Lemma 2.1, Theorem 1.3 shows significant differences, especially at the mass $\mu=\mu_G$. In Theorem 1.3, ground states do not exist at $\mu=\mu_G$, while exist in Lemma 2.1. Although Theorems 1.1 and 1.2 are similar in conclusion to Lemma 2.1, considering the actual physical background, they are likely to represent different physical phenomena.

In particular, when $p=6$ and $G=\mathbb{R}$, there exists

$$\mathcal{F}\,(\mu,\mathbb{R})=\begin{cases}0, & \text{for }\ \mu\le\mu_{\mathbb{R}},\\ -\infty, & \text{for }\ \mu>\mu_{\mathbb{R}},\end{cases} \tag{2.6}$$

and the infimum $\mathcal{F}\,(\mu,\mathbb{R})$ is attained if and only if $\mu=\mu_{\mathbb{R}}$. When $p=6$ and $G=\mathbb{R}^+$, there exists

$$\mathcal{F}\,(\mu,\mathbb{R}^+)=\begin{cases}0, & \text{for }\ \mu\le\mu_{\mathbb{R}^+},\\ -\infty, & \text{for }\ \mu>\mu_{\mathbb{R}^+},\end{cases} \tag{2.7}$$

and the infimum $\mathcal{F}\,(\mu,\mathbb{R}^+)$ is attained if and only if $\mu=\mu_{\mathbb{R}^+}$.

Finally, as for the case $\alpha<0$ and $p=6$, the next lemma gives an a priori estimate about the minimization energy level $\mathcal{F}_{\alpha,V}(\mu,G)$.

Lemma 2.2. *Let G be a periodic metric graph. $\alpha<0$ and $p=6$. So we have*

$$\mathcal{F}_{\alpha,V}(\mu,G)\le 0,\quad \forall\,\mu>0. \tag{2.8}$$

Proof. Fix $\mu>0$, and for every $n\in\mathbb{N}$, we define a set as

$$S_n:=\{e\in E\,(\mathcal{K}_{n+1})\cup E\,(\mathcal{K}_{-n-1}):\exists\ \mathrm{v}\in D\,(\mathcal{K}_n)\cup R\,(\mathcal{K}_{-n})\ \textit{ such that } e\succ \mathrm{v}\}\,,$$

which contains all the edges, joining either $\mathcal{K}_n$ with $\mathcal{K}_{n+1}$ or $\mathcal{K}_{-n}$ with $\mathcal{K}_{-n-1}$, of G. Obviously, the number of edges in S_n is finite. Then, for every $e\in S_n$, one endpoint of e belongs to $D\,(\mathcal{K}_n)\cup R\,(\mathcal{K}_{-n})$.

At this time, let x_e be the coordinate on $e := [0, l_e]$. We set $x_e(0) = \mathrm{v}$ and then construct a function $u_n \in H^1_\mu(G)$ as

$$u_n(x) = \begin{cases} a_n, & \text{if } x \in \mathcal{K}_i, \text{ for } i \in \{-n, \dots, n\}, \\ \frac{a_n}{l_e}(l_e - x), & \text{if } x \in e, \text{ for } e \in S_n, \\ 0, & \text{otherwise on } G, \end{cases} \quad , \tag{2.9}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is chosen to satisfy

$$\mu = \|u_n\|^2_{L^2(G)} = 2n \int_{\mathcal{K}_n} |u_n|^2 \, dx + \sum_{e\in S_n} \int_e |u_n|^2 \, dx = 2nL_1 a_n^2 + \frac{L_2}{3} a_n^2, \tag{2.10}$$

for every $n \in \mathbb{N}$. Here, L_1 represents the measure of $\mathcal{K}$, and L_2 represents the total length of all the edges in S_n, i.e.,

$$L_1 = \sum_{e\in\mathcal{K}} l_e, \text{ and } L_2 = \sum_{e\in S_n} l_e.$$

Noting that both L_1 and L_2 are finite, (2.10) entails

$$a_n \to 0, \text{ as } n \to \infty,$$

and furthermore

$$\lim_{n\to\infty} n a_n^2 = \frac{\mu}{2L_1}. \tag{2.11}$$

Since $\#V < +\infty$, then by the definition of (2.9), as $n \to \infty$, one can check that

$$u_n(\mathrm{v}) = a_n, \quad \forall\, \mathrm{v} \in V.$$

Hence, we have

$$\begin{aligned} F_{\alpha,V}(u_n, G) &= \frac{1}{2}\int_G |u_n'|^2 dx - \frac{1}{6}\int_G |u_n|^6 dx - \frac{\alpha}{q}\sum_{\mathrm{v}\in\mathrm{V}} |u_n(\mathrm{v})|^q \\ &= \left(\sum_{e\in S_n} \frac{1}{2l_e}\right) a_n^2 - \frac{L_1}{3} n a_n^6 - \frac{L_2}{42} a_n^6 + \frac{|\alpha|}{q} a_n^q \to 0, \text{ as } n \to \infty, \end{aligned} \tag{2.12}$$

where we use the facts that $a_n \to 0$ and the limit in (2.11). By (2.12), it is immediate to see that (2.8) holds.

3. The case $\alpha > 0$ and $2 < p < 6$

In this section, we focus on searching for a function $u \in H^1_\mu(G)$ such that $F_{\alpha,V}(u, G) = \mathcal{F}_{\alpha,V}(\mu, G)$ when $\alpha > 0$, $2 < p < 6$, and $2 < q < 4$. That is the proof of Theorem 1.1.

We begin with the estimate of the minimization energy level $\mathcal{F}_{\alpha,V}(\mu, G)$ in the next lemma.

Lemma 3.1. *Let G be a periodic metric graph.* $2 < p < 6$ *and* $2 < q < 4$. *Then, for every* $\alpha > 0$, *we have*

$$\mathcal{F}_{\alpha,V}(\mu, G) \in (-\infty, 0), \ \forall\, \mu > 0. \tag{3.1}$$

Proof. Fix $\mu > 0$. On the one hand, since $\alpha > 0$, for every $u \in H^1_\mu(G)$, it holds

$$F_{\alpha,V}(u,G) = F(u,G) - \frac{\alpha}{q}\sum_{\mathrm{v}\in\mathrm{V}} |u(\mathrm{v})|^q \le F(u,G),$$

which indicates that

$$\mathcal{F}_{\alpha,V}(\mu,G) \le \mathcal{F}(u,G). \tag{3.2}$$

Combining (2.4) with (3.2), we have

$$\mathcal{F}_{\alpha,V}(\mu,G) < 0. \tag{3.3}$$

On the other hand, by the inequalities (2.1) and (2.3), for every $u \in H^1_\mu(G)$ we get

$$\begin{aligned} F_{\alpha,V}(u,G) &= \frac{1}{2}\|u'\|^2_{L^2(G)} - \frac{1}{p}\|u\|^p_{L^p(G)} - \frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q \\ &\ge \frac{1}{2}\|u'\|^2_{L^2(G)} - \frac{K_p}{p}\|u\|^{\frac{p}{2}+1}_{L^2(G)}\|u'\|^{\frac{p}{2}-1}_{L^2(G)} - \frac{\alpha}{q}\sum_{\mathrm{v}\in V(G)}|u(\mathrm{v})|^q \\ &\ge \frac{1}{2}\left(1 - C_1\|u'\|^{\frac{p}{2}-3}_{L^2(G)} - C_2\|u'\|^{\frac{q}{2}-3}_{L^2(G)} - C_3\|u'\|^{\frac{q}{2}-2}_{L^2(G)}\right)\|u'\|^2_{L^2(G)}, \end{aligned} \tag{3.4}$$

where

$$C_1 = \frac{1}{p}K_p\mu^{\frac{p+2}{4}},\ C_2 = \frac{1}{q}C\alpha K_q\mu^{\frac{q+2}{4}} \text{ and } C_3 = \frac{1}{q}\alpha\mu^{\frac{q}{4}}.$$

Observe that C is the constant obtained in (2.3). Since $p < 6$ and $q < 4$, (3.4) implies that $F_{\alpha,V}(u,G)$ is bounded from below, and we immediately obtain

$$\mathcal{F}_{\alpha,V}(\mu,G) > -\infty. \tag{3.5}$$

The proof is complete.

Proof of Theorem 1.1. For every $\mu > 0$, the estimate of the minimization energy level $\mathcal{F}_{\alpha,V}(\mu,G)$ has been given in Lemma 3.1. We are left to verify the existence of a function $u \in H^1_\mu(G)$ such that $F_{\alpha,V}(u,G) = \mathcal{F}_{\alpha,V}(\mu,G)$, i.e., ground states of mass μ always exist.

Let $\{u_n\}$ be a minimizing sequence for $\mathcal{F}_{\alpha,V}(\mu,G)$. Then, for n large enough, by (3.1) and (3.4), we immediately obtain

$$0 > F_{\alpha,V}(u_n,G) \ge \frac{1}{2}\left(1 - C_1\|u_n'\|^{\frac{p}{2}-3}_{L^2(G)} - C_2\left\|u_n'\right\|^{\frac{q}{2}-3}_{L^2(G)} - C_3\left\|u_n'\right\|^{\frac{q}{2}-2}_{L^2(G)}\right)\left\|u_n'\right\|^2_{L^2(G)}.$$

It follows from the facts $p < 6$ and $q < 4$ that $\{u_n\}$ is bounded in $H^1(G)$. Thereby, up to subsequences, there exists a weak limit of $\{u_n\}$ in $H^1(G)$, denoted as u so that

$$u_n \rightharpoonup u, \text{ in } H^1(G).$$

Moreover, we have

$$u_n \to u, \text{ in } L^\infty_{loc}(G).$$

It follows from the weak lower semicontinuity that

$$0 \le \gamma =: \|u\|_{L^2(G)}^2 \le \liminf_{n\to\infty} \|u_n\|_{L^2(G)}^2 = \mu.$$

Our aim is to prove $\gamma = \mu$, i.e., $u \in H_\mu^1(G)$.

Let us first show that $\gamma \ne 0$. If that is not the case, we have $\gamma = 0$, i.e., $u \equiv 0$ on G. Noting the fact that $u_n \to u$ in $L_{loc}^\infty(G)$, for every $n \in \mathbb{N}$, the function u_n can achieve its L^∞ norm in any periodicity cell such as $\mathcal{K}_1$. In other words, there exists $x^* \in \mathcal{K}_1$ such that

$$\|u_n\|_{L^\infty(G)} = u_n(x^*) \to 0.$$

Then we have

$$\|u_n\|_{L^p(G)}^p \le \mu \|u_n\|_{L^\infty(G)}^{p-2} \to 0, \text{ as } n \to \infty, \tag{3.6}$$

and

$$\sum_{\mathrm{v}\in V} |u_n(\mathrm{v})|^q \le (\#V) \|u_n\|_{L^\infty(G)}^q \to 0, \text{ as } n \to \infty. \tag{3.7}$$

Coupling (3.6) and (3.7) yields

$$0 > \mathcal{F}_{\alpha,V}(\mu, G) = \lim_{n\to\infty} F_{\alpha,V}(u_n, G) = \lim_{n\to\infty} \frac{1}{2} \left\|u_n'\right\|_{L^2(G)}^2 \ge 0,$$

which leads to a contradiction.

On the other hand, if we assume that $0 < \gamma < \mu$, then according to the Brezis−Lieb lemma [27] and $u_n \to u$ in $L_{loc}^\infty(G)$, one can see that

$$F_{\alpha,V}(u_n, G) = F_{\alpha,V}(u_n - u, G) + F_{\alpha,V}(u, G) + o(1), \text{ as } n \to \infty. \tag{3.8}$$

Observing that $\{u_n\}$ is bounded in $H^1(G)$, we have

$$\begin{aligned} \|u_n - u\|_{L^2(G)}^2 &= \|u_n\|_{L^2(G)}^2 - 2\langle u_n, u\rangle_{L^2(G)} + \|u\|_{L^2(G)}^2 + o(1) \\ &= \mu - \|u\|_{L^2(G)}^2 + o(1) \to \mu - \gamma, \text{ as } n \to \infty. \end{aligned} \tag{3.9}$$

For n large enough, it follows from $0 < \gamma < \mu$ that

$$0 < \|u_n - u\|_{L^2(G)}^2 < \mu,$$

and let us denote

$$\varphi_n := \frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} (u_n - u).$$

It is obvious that $\varphi_n \in H_\mu^1(G)$. Thus, by the definition of $\mathcal{F}_{\alpha,V}(\mu, G)$, it can be obtained that

$$\begin{aligned} \mathcal{F}_{\alpha,V}(\mu, G) \le F_{\alpha,V}(\varphi_n, G) &= F_{\alpha,V}\left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} (u_n - u), G\right) \\ &= \left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}}\right)^2 \frac{1}{2} \left\|u_n' - u'\right\|_{L^2(G)}^2 - \left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}}\right)^p \frac{1}{p} \|u_n - u\|_{L^p(G)}^p \\ &\quad - \left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}}\right)^q \frac{\alpha}{q} \sum_{\mathrm{v}\in V} |(u_n - u)(\mathrm{v})|^q \\ &< \frac{\mu}{\|u_n - u\|_{L^2(G)}^2} F_{\alpha,V}(u_n - u, G), \end{aligned} \tag{3.10}$$

where we use the facts that $\|u_n - u\|_{L^2(G)}^2 < \mu$, $p > 2$, $q > 2$, and $\alpha > 0$. Combining (3.9) with (3.10), we have

$$\liminf_{n\to\infty} F_{\alpha,V}(u_n - u, G) \geq \frac{\mu-\gamma}{\mu}\mathcal{F}_{\alpha,V}(\mu, G). \tag{3.11}$$

Noting that $\sqrt{\frac{\mu}{\gamma}}u \in H_\mu^1(G)$ and $\sqrt{\frac{\mu}{\gamma}} > 1$, then by similar calculations in (3.10), we have

$$\mathcal{F}_{\alpha,V}(\mu, G) \leq F_{\alpha,V}\left(\sqrt{\frac{\mu}{\gamma}}u, G\right) < \frac{\mu}{\gamma}F_{\alpha,V}(u, G),$$

that is

$$F_{\alpha,V}(u, G) > \frac{\gamma}{\mu}\mathcal{F}_{\alpha,V}(\mu, G). \tag{3.12}$$

Combining (3.8) with (3.11) and (3.12), it then follows that

$$\begin{aligned}\mathcal{F}_{\alpha,V}(\mu, G) &= \lim_{n\to\infty} F_{\alpha,V}(u_n, G)\\ &> \frac{\mu-\gamma}{\mu}\mathcal{F}_{\alpha,V}(\mu, G) + \frac{\gamma}{\mu}\mathcal{F}_{\alpha,V}(\mu, G) = \mathcal{F}_{\alpha,V}(\mu, G),\end{aligned}$$

which leads to a contradiction, and thus $\gamma = \mu$, i.e., $u \in H_\mu^1(G)$ is a ground state for $\mathcal{F}_{\alpha,V}(\mu, G)$. The proof is complete.

4. The case $\alpha < 0$ and $p = 6$

This section is devoted to the proof of Theorems 1.2 and 1.3. First of all, we spit the proof of Theorem 1.2 into the following two lemmas.

For the graphs satisfying assumption (A_1), we have the following nonexistence result:

Lemma 4.1. *Let G be a periodic metric graph. $p = 6$ and $2 < q < 4$. If G satisfies assumption* (A_1)*, then for every $\alpha < 0$, we have*

$$\mathcal{F}_{\alpha,V}(\mu, G) = \begin{cases} 0, & \text{for } \mu \leq \mu_{\mathbb{R}},\\ -\infty, & \text{for } \mu > \mu_{\mathbb{R}},\end{cases} \tag{4.1}$$

and the infimum is never achieved.

Proof. Let G satisfy assumption (A_1). By (1.9), we have

$$\mu_G = \mu_{\mathbb{R}}.$$

When $\mu \in (0, \mu_{\mathbb{R}}]$, by substituting (1.8) into (1.1), then for every $u \in H_\mu^1(G)$ we get

$$\begin{aligned}F_{\alpha,V}(u, G) &= \frac{1}{2}\|u'\|_{L^2(G)}^2 - \frac{1}{6}\|u\|_{L^6(G)}^6 - \frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q\\ &\geq \frac{1}{2}\left(1 - \left(\frac{\mu}{\mu_G}\right)^2\right)\|u'\|_{L^2(G)}^2 - \frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q,\end{aligned} \tag{4.2}$$

which implies that

$$\mathcal{F}_{\alpha,V}(\mu, G) \geq 0. \tag{4.3}$$

Combining (2.8) and (4.3), we have

$$\mathcal{F}_{\alpha,V}(\mu, G) = 0, \;\; \forall\, \mu \leq \mu_{\mathbb{R}}. \tag{4.4}$$

When $\mu > \mu_{\mathbb{R}}$, by (2.6), there exists $\psi \in H^1_\mu(\mathbb{R})$, supported on $[0, 1]$, such that

$$F(\psi, \mathbb{R}) < 0.$$

Denote

$$\psi_\lambda(x) := \sqrt{\lambda}\psi(\lambda x), \;\forall\, \lambda > 0.$$

It is obvious that $\psi_\lambda(x) \in H^1_\mu(\mathbb{R})$ and ψ_λ is supported on $\left[0, \frac{1}{\lambda}\right]$.

Now given any $e \in E(G)$ with its length $l_e := |e|$, i.e., $e = [0, l_e]$. Let $\lambda_0 := \frac{1}{l_e}$ and for every $\lambda \geq \lambda_0$ we have $\psi_\lambda \in H^1_\mu(0, l_e)$. Thus, we construct functions $\{\psi_\lambda\}_{\lambda\geq\lambda_0}$, supported on e, which can be considered as elements in $H^1_\mu(G)$. Furthermore, it holds

$$\begin{aligned} F_{\alpha,V}(\psi_\lambda, G) = F_{\alpha,V}(\psi_\lambda, e) &\rightarrow \lambda^2 F(\psi, \mathbb{R}) - \frac{\alpha}{q}\sum_{e\succ \mathrm{v}} |\psi_\lambda(\mathrm{v})|^q \\ &= \lambda^2 F(\psi, \mathbb{R}) \rightarrow -\infty, \text{ as } \lambda \rightarrow +\infty, \end{aligned}$$

since $\psi_\lambda(0) = \psi_\lambda(l_e) = 0$ and $F(\psi, \mathbb{R}) < 0$. This implies that

$$\mathcal{F}_{\alpha,V}(\mu, G) = -\infty, \;\; \forall\, \mu > \mu_{\mathbb{R}}. \tag{4.5}$$

Finally, let us explain that the infimum is not achieved for any $\mu > 0$. If $\mu > \mu_{\mathbb{R}}$, the result is trivial. If $\mu < \mu_{\mathbb{R}}$, we just need to show that the inequality in (4.2) is strict. Indeed, if, on the contrary, we get $\|u'\|^2_{L^2(G)} = 0$, and u is a constant on G. This is impossible since G is noncompact. If $\mu = \mu_{\mathbb{R}}$, suppose by contradiction that $u \in H^1_\mu(G)$ is a global minimizer of (1.1) such that

$$F_{\alpha,V}(u, G) = F(u, G) - \frac{\alpha}{q}\sum_{\mathrm{v}\in V} |u(\mathrm{v})|^q = 0. \tag{4.6}$$

By a similar analysis in Proposition 3.3 in [5], together with the corresponding boundary conditions in (1.5) and (1.6), we can immediately obtain

$$u > 0, \;\; \text{on } G. \tag{4.7}$$

Combining (4.6) with (4.7), we have

$$F(u, G) = \frac{\alpha}{q}\sum_{\mathrm{v}\in V} |u(\mathrm{v})|^q < 0.$$

It follows that

$$\mathcal{F}(\mu, G) < 0, \tag{4.8}$$

which contradicts the fact that $\mathcal{F}(\mu, G) = 0$ in (2.5), and the proof is complete.

For the graphs satisfying assumption (A_2), we have the next result.

Lemma 4.2. *Let G be a periodic metric graph. $p = 6$ and $2 < q < 4$. If G satisfies assumption* (A_2)*, then for any $\alpha < 0$, we have*

$$\mathcal{F}_{\alpha,V}(\mu, G) = \begin{cases} 0, & \text{for } \mu \leq \mu_{\mathbb{R}^+}, \\ -\infty, & \text{for } \mu > \mu_{\mathbb{R}^+}, \end{cases} \tag{4.9}$$

and the infimum is never achieved.

Proof. Let G satisfy assumption (A_2). There exists

$$\mu_G = \mu_{\mathbb{R}^+}.$$

Then, by a completely analogous analysis in Lemma 4.1, with $\mu_{\mathbb{R}}$ being replaced by $\mu_{\mathbb{R}^+}$, we conclude that the result of Lemma 4.2 is valid.

Proof of Theorem 1.2. According to the results of Lemmas 4.1 and 4.2, one can check that the conclusion in Theorem 1.2 is clearly valid.

Next, for the graphs satisfying neither assumption (A_1) nor assumption (A_2), we give the following lemma concerning the mass $\mu \leq \mu_G$ and $\mu > \mu_{\mathbb{R}}$, at which ground states do not exist.

Lemma 4.3. *Let G be a periodic metric graph. $p = 6$ and $2 < q < 4$. If G satisfies neither assumption,* (A_1) *nor* (A_2)*, then for every $\alpha < 0$, we have*

$$\mathcal{F}_{\alpha,V}(\mu, G) = \begin{cases} 0, & \text{for } \mu \leq \mu_G, \\ -\infty, & \text{for } \mu > \mu_{\mathbb{R}}, \end{cases} \tag{4.10}$$

and the infimum is never achieved.

Proof. When $\mu \leq \mu_G$ and $\mu > \mu_{\mathbb{R}}$, through a similar proof in Lemma 4.1, we obtain

$$\mathcal{F}_{\alpha,V}(\mu, G) = 0, \ \forall \, \mu \leq \mu_G,$$

and

$$\mathcal{F}_{\alpha,V}(\mu, G) = -\infty, \ \forall \, \mu > \mu_{\mathbb{R}},$$

thus (4.10) holds. Meanwhile, the infimum is not achieved.

In order to show that ground states exist at the mass $\mu \in (\mu_G, \mu_{\mathbb{R}}]$, the following lemma gives a preliminary estimate about the minimization energy level $\mathcal{F}_{\alpha,V}(\mu, G)$.

Lemma 4.4. *Let G be a periodic metric graph. $p = 6$ and $2 < q < 4$. If $\mu_G < \mu_{\mathbb{R}}$, then for every $\mu \in (\mu_G, \mu_{\mathbb{R}}]$, there exists $\tilde{\alpha} < 0$ (possibly equal to $-\infty$) depending on μ, q, V and G so that*

$$\mathcal{F}_{\alpha,V}(\mu, G) \in (-\infty, 0)\,, \text{ for } \alpha \in (\tilde{\alpha}, 0)\,. \tag{4.11}$$

Proof. Given $\mu \in (\mu_G, \mu_{\mathbb{R}}]$. On the one hand, for every $u \in H^1_\mu(G)$, it follows from (2.2) that there exists $\theta_u \in [0, \mu]$ such that

$$\begin{aligned} F_{\alpha,V}(u,G) &= \frac{1}{2}\|u'\|^2_{L^2(G)} - \frac{1}{6}\|u\|^6_{L^6(G)} - \frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q \\ &\geq \frac{1}{2}\left(1-\left(\frac{\mu-\theta_u}{\mu_{\mathbb{R}}}\right)^2\right)\|u'\|^2_{L^2(G)} - \frac{C_G}{6}\theta_u^{\frac{1}{2}} - \frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q \\ &\geq \frac{1}{2}\left(1-\left(\frac{\mu}{\mu_{\mathbb{R}}}\right)^2\right)\|u'\|^2_{L^2(G)} - \frac{C_G}{6}\mu^{\frac{1}{2}} - \frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q, \end{aligned}$$

which indicates that

$$\mathcal{F}_{\alpha,V}(\mu,G) > -\infty, \ \forall\, \alpha < 0. \tag{4.12}$$

On the other hand, by Lemma 2.2, we have

$$\mathcal{F}_{\alpha,V}(\mu,G) \leq 0, \ \ \forall\, \alpha < 0.$$

By the monotonicity of the ground state energy level $\mathcal{F}_{\alpha,V}(\mu,G)$ with respect to α, we know that $\alpha \mapsto \mathcal{F}_{\alpha,V}(\mu,G)$ is monotone non-increasing. Denote

$$\tilde{\alpha} = \sup\{\alpha < 0 : \mathcal{F}_{\alpha,V}(\mu,G) = 0\}, \tag{4.13}$$

which depends on μ, q, V and G. To proceed with the proof, let us consider the sharp constant K_G of the inequality in (1.8), i.e.,

$$K_G := \sup_{u\in H^1(G)\setminus\{0\}} \frac{\|u\|^6_{L^6(G)}}{\|u\|^4_{L^2(G)}\|u'\|^2_{L^2(G)}}. \tag{4.14}$$

For every $\epsilon > 0$, by the above definition in (4.14), one can see that there exists $u \in H^1_\mu(G)$ satisfying

$$\|u\|^6_{L^6(G)} > (K_G - \epsilon)\|u\|^4_{L^2(G)}\|u'\|^2_{L^2(G)} = (K_G-\epsilon)\mu^2\|u'\|^2_{L^2(G)}.$$

Then we have

$$\begin{aligned} F_{\alpha,V}(u,G) &= \frac{1}{2}\|u'\|^2_{L^2(G)} - \frac{1}{6}\|u\|^6_{L^6(G)} - \frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q \\ &< \frac{1}{2}\left(1-\frac{(K_G-\epsilon)}{3}\mu^2\right)\|u'\|^2_{L^2(G)} + \frac{|\alpha|}{q}\sum_{\mathrm{v}\in V}|u(\mathrm{v})|^q. \end{aligned} \tag{4.15}$$

Since $\mu > \mu_G$, then as long as we pick ϵ small enough, it holds

$$\frac{1}{2}\|u'\|^2_{L^2(G)}\left(1-\frac{(K_G-\epsilon)}{3}\mu^2\right) < 0. \tag{4.16}$$

Note that $\#V < +\infty$, combining (4.15) with (4.16), we have

$$F_{\alpha,V}(u,G) < 0, \ \text{as} \ |\alpha| \ \text{is small enough,}$$

which implies that

$$\mathcal{F}_{\alpha,V}(\mu,G)<0,\ \text{as } |\alpha| \text{ is small enough.} \tag{4.17}$$

It is readily seen that $\tilde{\alpha} < 0$ by the definition in (4.13) and the monotonicity of $\mathcal{F}_{\alpha,V}(\mu,G)$ with respect to α. Moreover, we immediately have

$$\mathcal{F}_{\alpha,V}(\mu,G)<0,\ \text{for } \alpha\in(\tilde{\alpha},0). \tag{4.18}$$

Combining (4.12) with (4.18), we have that (4.11) holds.

Proof of Theorem 1.3. When $\mu \le \mu_G$ and $\mu > \mu_{\mathbb{R}}$, the result is given in Lemma 4.3. When $\mu \in (\mu_G, \mu_{\mathbb{R}}]$ provided $\mu_G < \mu_{\mathbb{R}}$, the estimate has been given in Lemma 4.4. We are left to prove the existence of ground states when $\mu \in (\mu_G, \mu_{\mathbb{R}}]$ and $\alpha \in (\tilde{\alpha}, 0)$.

Let $\{u_n\}$ be a minimizing sequence for $\mathcal{F}_{\alpha,V}(\mu,G)$. Then, for n large enough, it follows from the result in (4.11) and the inequality (2.2) that there exists $\theta_{u_n}\in[0,\mu]$ satisfying

$$\begin{aligned}0 > F_{\alpha,V}(u_n,G) &\ge \frac{1}{2}\left(1-\left(\frac{\mu-\theta_{u_n}}{\mu_{\mathbb{R}}}\right)^2\right)\left\|u_n'\right\|_{L^2(G)}^2-\frac{C_G}{6}\theta_{u_n}^{\frac{1}{2}}-\frac{\alpha}{q}\sum_{\mathrm{v}\in V}|u_n(\mathrm{v})|^q\\ &\ge \frac{1}{2}\left(1-\left(\frac{\mu_{\mathbb{R}}-\theta_{u_n}}{\mu_{\mathbb{R}}}\right)^2\right)\left\|u_n'\right\|_{L^2(G)}^2-\frac{C_G}{6}\theta_{u_n}^{\frac{1}{2}}\\ &=\frac{\theta_{u_n}}{2\mu_{\mathbb{R}}}\left(2-\frac{\theta_{u_n}}{\mu_{\mathbb{R}}}\right)\left\|u_n'\right\|_{L^2(G)}^2-\frac{C_G}{6}\theta_{u_n}^{\frac{1}{2}}.\end{aligned} \tag{4.19}$$

Noting the fact that $\theta_{u_n}\to 0$ contradicts (4.19), as a result, there exists a constant $c>0$, depending on α,μ,G, such that

$$\theta_{u_n}\ge c.$$

By (4.19), there exists

$$\frac{c}{2\mu_{\mathbb{R}}}\left\|u_n'\right\|_{L^2(G)}^2-\frac{C_G}{6}\mu^{\frac{1}{2}}\le\frac{\theta_{u_n}}{2\mu_{\mathbb{R}}}\left(2-\frac{\theta_{u_n}}{\mu_{\mathbb{R}}}\right)\left\|u_n'\right\|_{L^2(G)}^2-\frac{C_G}{6}\theta_{u_n}^{\frac{1}{2}}<0,$$

which directly indicates that $\{u_n\}$ is bounded in $H^1(G)$. Thereby, up to subsequences, there exists a weak limit of $\{u_n\}$ in $H^1(G)$, denoted as u, such that

$$u_n\rightharpoonup u,\ \text{ in } H^1(G),$$

and

$$u_n\to u,\ \text{ in } L^\infty_{loc}(G).$$

Based on weak lower semicontinuity, it holds

$$0\le\gamma:=\|u\|_{L^2(G)}^2\le\liminf_{n\to\infty}\|u_n\|_{L^2(G)}^2=\mu.$$

If $\gamma=0$, i.e., $u\equiv 0$ on G, since $u_n\to u$ in $L^\infty_{loc}(G)$, then for every $n\in\mathbb{N}$, the function u_n can achieve its L^∞ norm in any periodicity cell such as $\mathcal{K}_1$. In other words, there exists $x^{**}\in\mathcal{K}_1$ such that

$$\|u_n\|_{L^\infty(G)}=u_n(x^{**})\to 0.$$

Thus, we have

$$\|u_n\|_{L^6(G)}^6 \le \|u_n\|_{L^\infty(G)}^4 \mu \to 0, \text{ as } n \to \infty. \tag{4.20}$$

and

$$\sum_{v\in V} |u_n(v)|^q \le (\#V)\, \|u_n\|_{L^\infty(G)}^q \to 0, \text{ as } n \to \infty. \tag{4.21}$$

It follows from (4.20) and (4.21) that

$$0 > \mathcal{F}_{\alpha,V}(\mu, G) = \lim_{n\to\infty} F_{\alpha,V}(u_n, G) = \lim_{n\to\infty} \frac{1}{2} \left\| u_n' \right\|_{L^2(G)}^2 \ge 0,$$

which leads to a contradiction.

If $0 < \gamma < \mu$, by the Brezis−Lieb lemma, one can see that

$$F_{\alpha,V}(u_n, G) = F_{\alpha,V}(u_n - u, G) + F_{\alpha,V}(u, G) + o(1), \text{ as } n \to \infty. \tag{4.22}$$

Since $\{u_n\}$ is bounded in $H^1(G)$, we have

$$\begin{aligned} \|u_n - u\|_{L^2(G)}^2 &= \|u_n\|_{L^2(G)}^2 - 2\langle u_n, u\rangle_{L^2(G)} + \|u\|_{L^2(G)}^2 + o(1) \\ &= \mu - \|u\|_{L^2(G)}^2 + o(1) \to \mu - \gamma, \text{ as } n \to \infty. \end{aligned} \tag{4.23}$$

For n large enough, since $0 < \gamma < \mu$, we have

$$0 < \|u_n - u\|_{L^2(G)}^2 < \mu.$$

We still denote

$$\varphi_n := \frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} (u_n - u) \in H_\mu^1(G).$$

Thus, by the definition of $\mathcal{F}_{\alpha,V}(\mu, G)$ there exists

$$\begin{aligned} \mathcal{F}_{\alpha,V}(\mu, G) \le F_{\alpha,V}(\varphi_n, G) &= F_{\alpha,V}\left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} (u_n - u), G \right) \\ &= \left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} \right)^2 \frac{1}{2} \left\| u_n' - u' \right\|_{L^2(G)}^2 - \left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} \right)^6 \frac{1}{6} \|u_n - u\|_{L^6(G)}^6 \\ &\quad - \left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} \right)^q \frac{\alpha}{q} \sum_{v\in V} |(u_n - u)(v)|^q \\ &< \left(\frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}} \right)^q F_{\alpha,V}(u_n - u, G), \end{aligned} \tag{4.24}$$

where we use the facts that $\|u_n - u\|_{L^2(G)}^2 < \mu$, $2 < p < 6$, $2 < q < 4$, and $\alpha < 0$. Combining (4.23) with (4.24), we have

$$\liminf_{n\to\infty} F_{\alpha,V}(u_n - u, G) \ge \left(\sqrt{\frac{\mu - \gamma}{\mu}} \right)^q \mathcal{F}_{\alpha,V}(\mu, G). \tag{4.25}$$

Noting that $\sqrt{\frac{\mu}{\gamma}} u \in H_\mu^1(G)$ and $\sqrt{\frac{\mu}{\gamma}} > 1$, through similar calculations in (4.24), we have

$$\mathcal{F}_{\alpha,V}(\mu, G) \le F_{\alpha,V}\left(\sqrt{\frac{\mu}{\gamma}} u, G \right) < \left(\sqrt{\frac{\mu}{\gamma}} \right)^q F_{\alpha,V}(u, G),$$

that is

$$F_{\alpha,V}(u,G) > \left(\sqrt{\frac{\gamma}{\mu}}\right)^q \mathcal{F}_{\alpha,V}(\mu,G). \tag{4.26}$$

Coupling (4.22) with (4.25) and (4.26), it then follows that

$$\begin{aligned}\mathcal{F}_{\alpha,V}(\mu,G) &= \lim_{n\to\infty} F_{\alpha,V}(u_n,G)\\ &> \left(\sqrt{\frac{\mu-\gamma}{\mu}}\right)^q \mathcal{F}_{\alpha,V}(\mu,G) + \left(\sqrt{\frac{\gamma}{\mu}}\right)^q \mathcal{F}_{\alpha,V}(\mu,G)\\ &> \left(\sqrt{\frac{\mu-\gamma}{\mu}}\right)^2 \mathcal{F}_{\alpha,V}(\mu,G) + \left(\sqrt{\frac{\gamma}{\mu}}\right)^2 \mathcal{F}_{\alpha,V}(\mu,G) = \mathcal{F}_{\alpha,V}(\mu,G),\end{aligned}$$

where we use the facts that $0 < \gamma < \mu$, $q > 2$ and $\mathcal{F}_{\alpha,V}(\mu,G) < 0$. This leads to a contradiction.

To sum up, we conclude that $\gamma = \mu$, i.e., $u \in H^1_\mu(G)$ is a ground state for $\mathcal{F}_{\alpha,V}(\mu,G)$. The proof is complete.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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