Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 32(6): 4096–4107.
DOI: 10.3934/era.2024184
Received: 16 April 2024
Revised: 14 June 2024
Accepted: 20 June 2024
Published: 26 June 2024

Research article

On a conjecture concerning the exponential Diophantine equation $(an^2 + 1)^x + (bn^2 - 1)^y = (cn)^z$

Shuanglin Fei[1], **Guangyan Zhu**[2,*] **and Rongjun Wu**[3]

[1] School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
[2] School of Mathematics and Statistics, Hubei Minzu University, Enshi 445000, China
[3] School of Mathematics, Southwest Minzu University, Chengdu 610064, China

* **Correspondence:** Email: 2009043@hbmzu.edu.cn.

Abstract: Let a, b, c, and n be positive integers such that $a + b = c^2$, $2 \nmid c$ and $n > 1$. In this paper, we prove that if $\gcd(c, n) = 1$ and $n \geq 117.14c$, then the equation $(an^2 + 1)^x + (bn^2 - 1)^y = (cn)^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2)$ under the assumption $\gcd(an^2 + 1, bn^2 - 1) = 1$. Thus, we affirm that the conjecture proposed by Fujita and Le is true in this case. Moreover, combining the above result with some existing results and a computer search, we show that, for any positive integer n, if $(a, b, c) = (12, 13, 5)$, $(18, 7, 5)$, or $(44, 5, 7)$, then this equation has only the solution $(x, y, z) = (1, 1, 2)$. This result extends the theorem of Terai and Hibino gotten in 2015, that of Alan obtained in 2018, and Hasanalizade's theorem attained recently.

Keywords: linear forms in m-adic logarithms; exponential Diophantine equation; positive integer solution

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ be the set of positive integers, the set of integers, and the set of rational numbers, respectively. Let a, b, c, and n be positive integers such that

$$a + b = c^2, \quad 2 \nmid c, \quad n > 1. \tag{1.1}$$

Under the assumption (1.1), Fujita and Le proposed the following conjecture:

Conjecture 1.1. [1, Conjecture 1.1] *The exponential Diophantine equation*

$$(an^2 + 1)^x + (bn^2 - 1)^y = (cn)^z, \quad x, y, z \in \mathbb{N}, \tag{1.2}$$

has only the solution $(x, y, z) = (1, 1, 2)$.

There is a more general conjecture that readers can refer to [2, Conjecture 1]. In 2012, Terai [3] verified that if $(a, b, c) = (4, 5, 3)$, then Conjecture 1.1 is true except for $n > 20$ and $n \equiv 3 \pmod 6$. The proof of this result is based on elementary methods and Baker's method. In 2014, using some properties of exponential diophantine equations (see [4,5]) and some results on the existence of primitive divisors of Lucas numbers (see [6]), Su and Li [7] proved that if $n > 90$ and $n \equiv 3 \pmod 6$, then Conjecture 1.1 is true for $(a, b, c) = (4, 5, 3)$. Two years later, Bertók [8] showed Conjecture 1.1 is true when $(a, b, c) = (4, 5, 3)$ by completely solving Eq (1.2) for the remaining cases $20 < n < 90$ and $n \equiv 3 \pmod 6$ with the help of exponential congruences. This is a nice application of Bertók and Hajdu [9]. On the other hand, Miyazaki and Terai [10] showed that if $a = 1$ and $c \equiv 3, 5 \pmod 8$, under the condition $n \equiv \pm 1 \pmod c$, then Conjecture 1.1 is true except for the case $(n, b, c) = (1, 8, 3)$. Pan [11] proved that if $a \equiv 4, 5 \pmod 8$, $(\frac{a+1}{c}) = -1$ and $n > 6c^2 \log c$, under the condition $n \equiv \pm 1 \pmod c$, then Conjecture 1.1 is true, where $(\frac{a+1}{c})$ is the Jacobi symbol. Fu and Yang [12] showed that if $a \equiv 0 \pmod 2$ and $n > 36c^3 \log c$, under the condition $c \mid n$, then Conjecture 1.1 is true. Kizildere et al. [13] proved that if $a = b + 1$, $c \equiv 11, 13 \pmod{24}$ and $n > c^2$, under the condition $n \equiv \pm 1 \pmod c$, then Conjecture 1.1 is true. From these works, one can know that studying $c \mid n$ and $n \equiv \pm 1 \pmod c$ for Eq (1.2) plays an important role in solving Conjecture 1.1.

In this paper, using an elementary approach and a deep result on linear forms in two m-adic logarithms due to Bugeaud [14], we investigate Conjecture 1.1 by handling the case $\gcd(c, n) = 1$ (which contains the case $n \equiv \pm 1 \pmod c$) for Eq (1.2), and prove the following result:

Theorem 1.2. *Let a, b, c, and n be positive integers with* $a + b = c^2$, $2 \nmid c$ *and* $\gcd(an^2 + 1, bn^2 - 1) = 1$. *If* $\gcd(c, n) = 1$ *and* $n \geq 117.14c$, *then Conjecture* 1.1 *is true.*

Evidently, Theorem 1.2 improves the result (see [1, Theorem 1.2]) of Fujita and Le when $\gcd(c, n) = 1$. Notice that $6c \log c > 117.14$ for any positive integer $c \geq 9$, and the condition $\gcd(c, n) = 1$ contains $n \equiv \pm 1 \pmod c$. One can easily check that Theorem 1.2 extends the result of Pan [11] when $c \geq 9$. We point out that the condition $\gcd(an^2 + 1, bn^2 - 1) = 1$ in Theorem 1.2 is equivalent to the existence of a positive integer n such that $\gcd(an^2 + 1, bn^2 - 1) = 1$. Therefore, the condition $\gcd(an^2 + 1, bn^2 - 1) = 1$ is weaker than [1, Lemma 2.3].

When $(a, b, c) = (12, 13, 5)$, Terai and Hibino [15] proved that (1.2) has only the solution $(x, y, z) = (1, 1, 2)$ except for $n \equiv 17, 33 \pmod{40}$. When $(a, b, c) = (18, 7, 5)$, Alan [16] showed that (1.2) has only the solution $(x, y, z) = (1, 1, 2)$ except for $n \equiv 23, 47, 63, 87 \pmod{120}$. Recently, Hasanalizade [17] proved that if $(a, b, c) = (44, 5, 7)$ and $n \equiv 2 \pmod 5$ or $n \equiv 0, \pm 1, \pm 3 \pmod 7$, then (1.2) has only the solution $(x, y, z) = (1, 1, 2)$. The proofs of these results are based on elementary methods and Baker's method. On the other hand, Miyazaki and Terai [10], Bertók [8], and Terai and Hibino [18] completely solve Eq (1.2) when $(a, b, c) = (1, 8, 3)$, $(4, 5, 3)$, and $(10, 15, 5)$, respectively. With the help of Theorem 1.2, we can completely solve the Eq (1.2) for $(a, b, c) = (12, 13, 5)$, $(18, 7, 5)$, and $(44, 5, 7)$ without any assumption on n. Namely, we show the following result:

Theorem 1.3. *For any positive integer n, if* $(a, b, c) = (44, 5, 7)$, $(12, 13, 5)$, *or* $(18, 7, 5)$, *then* (1.2) *has only the solution* $(x, y, z) = (1, 1, 2)$.

As an immediate result of Theorem 1.3, we can obtain the following corollary:

Corollary 1.4. *If* $(a, b, c) = (44, 5, 7)$, $(12, 13, 5)$, *or* $(18, 7, 5)$, *then Conjecture* 1.1 *is true.*

This paper is organized as follows: In Section 2, we present several lemmas that will be useful for the proofs of the main results. In Sections 3 and 4, we provide the proofs of Theorems 1.2 and 1.3, respectively.

2. Preliminary results

For a prime number p and a nonzero integer x, we write $v_p(x)$ for the largest power of p dividing x, and, for nonzero rational $\frac{r}{t}$, set $v_p(\frac{r}{t}) = v_p(r) - v_p(t)$. First of all, we need the following known result:

Lemma 2.1. [19, Lemma 2.8] *Let a and b be distinct coprime rational integers, and let q be an odd prime.*

(i). $\gcd(a+b, \frac{a^q+b^q}{a+b}) = 1$ or q.

(ii). *If $q \mid (a+b)$, then* $v_q(\frac{a^q+b^q}{a+b}) = 1$.

Definition 2.2. *Two nonzero complex numbers α and β are called multiplicatively independent if the only solution of the equation $\alpha^X \beta^Y = 1$ in $\mathbb{Z}$ is $X = Y = 0$.*

Next, we quote a result on linear forms in two m-adic logarithms due to Bugeaud [14], which is crucial to the proof of Theorem 1.2. In order to describe this result, we introduce a related notation. If r is a nonzero rational number with $r = \frac{s}{t}$, and s and t being coprime integers, we define the logarithmic height of r as $h(r) := \max\{\log|s|, \log|t|, 1\}$. Note that if $m = p_1^{j_1} \cdots p_k^{j_k}$, where the p_i's are distinct primes and $j_i \in \mathbb{N}$, for any nonzero integer x, we define the arithmetic function v_m by

$$v_m(x) = \min_{1\le i\le k} [\frac{v_{p_i}(x)}{j_i}],$$

where $[\cdot]$ denotes the integer part.

Proposition 2.3. [14, Theorem 2] *Let α_1 and α_2 be two nonzero rational numbers with $\alpha_1 \neq \pm 1$, b_1, and b_2 being positive integers, and set*

$$\Lambda := \alpha_1^{b_1} - \alpha_2^{b_2}.$$

For any set of distinct primes $p_1, \cdots, p_k$ and positive integers $j_1, \cdots, j_k$, we set $m = p_1^{j_1} \cdots p_k^{j_k}$ and suppose that there exists a positive integer g such that for each i, we have

$$v_{p_i}(\alpha_1^g - 1) \ge j_i, \quad v_{p_i}(\alpha_2^g - 1) \ge 1, \quad for \ \ any \ \ p_i,$$

and also

$$v_{p_i}(\alpha_1^g - 1) \ge 2, \quad v_{p_i}(\alpha_2^g - 1) \ge 2, \quad for \ \ p_i = 2.$$

Then, if m, b_1, and b_2 are relatively prime, we have

$$v_m(\Lambda) \le \frac{66.8g}{(\log m)^4}(\max\{\log\Gamma + \log(\log m) + 0.64, \ 4\log m\})^2 \log A_1 \log A_2,$$

where

$$\Gamma := \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1},$$

and

$$\log A_i \ge \max\{h(\alpha_i), \log m\}.$$

The assumptions of Proposition 2.3 (which is a consequence of [14, Theorem 2] with $(y_1, y_2) = (1, 1)$ and $\mu = 4$) are very restrictive, but are satisfied (and easy to check) in our context. Proposition 2.3 has many applications in studying the Diophantine equation and related Diophantine problems; readers can refer to [14,20–24]. We stress that α_1 and α_2 need not be multiplicatively independent in Proposition 2.3. Indeed, under the additional assumption that α_1 and α_2 are multiplicatively independent, we have the following result:

Remark 2.4. [14] *The constant* 66.8 *in the upper bound for* $v_m(\Lambda)$ *may be improved to* 53.6.

If $z \le 2$, then it is clear that Conjecture 1.1 is true. Thus, we will assume that $z \ge 3$ in what follows.

Lemma 2.5. *If* (x, y, z) *is a solution of Eq* (1.2)*, then y is odd.*

Proof. Taking (1.2) modulo n^2 implies that

$$1 + (-1)^y \equiv 0 \pmod{n^2}.$$

Hence, y is odd since $n \ge 2$.

Lemma 2.6. *If* $2 \mid a$ *and* $2 \mid n$*, then* (1.2) *has only the solution* $(x, y, z) = (1, 1, 2)$.

Proof. Taking (1.2) modulo n^3 implies that

$$1 + an^2x - 1 + bn^2y \equiv 0 \pmod{n^3}.$$

Therefore, $ax + by \equiv 0 \pmod{n}$, which is impossible since $2 \mid a$, $2 \mid n$, y is odd and c is odd.

Lemma 2.7. *Let a, b, c, and n be positive integers such that* $2 \nmid c$ *and* $a + b = c^2$*. If* $n \ge 3c$*, then each of the following is true.*

(i)

$$\delta_1(n) := \log\left(\frac{\log n}{\log(an^2 + 1)} + \frac{\log n}{\log(bn^2 - 1)}\right) \le 0. \tag{2.1}$$

(ii)

$$\delta_2(n) := \frac{\log(cn) \cdot \max\{\log(an^2 + 1), \log(bn^2 - 1)\}}{(\log n)^4} < 0.933. \tag{2.2}$$

(iii)

$$\delta_3(n) := \frac{\log(an^2 + 1)\log(bn^2 - 1)}{(\log n)^2} < 16. \tag{2.3}$$

Proof. (i) We divide the proof into the following two cases:

Case 1: $b \ge 2$. Since $\log(an^2+1) \ge \log(n^2+1) > 2\log n$ and $\log(bn^2-1) \ge 2\log n$, we get $\delta_1(n) \le 0$.

Case 2: $b = 1$. Since c is odd, we have $a \ge 2$ and $\log(an^2 + 1) \ge \log(2n^2 + 1)$. Then for any integer $n \ge 9$, we can deduce that

$$\delta_1(n) \le \log\left(\frac{\log n}{\log(2n^2 + 1)} + \frac{\log n}{\log(n^2 - 1)}\right) \le 0.$$

Part (i) is proven.

(ii) Since $a + b = c^2$, we have

$$\max\{\log(an^2 + 1), \log(bn^2 - 1)\} < \log(an^2 + bn^2) = 2\log(cn). \quad (2.4)$$

From $c \leq n/3$ and $n \geq 3c \geq 9$, we get that

$$\delta_2(n) < 2 \cdot \frac{(\log(n^2/3))^2}{(\log n)^4} \leq 2 \cdot \frac{(3\log 3)^2}{(2\log 3)^4} = \frac{9}{8(\log 3)^2} < 0.933, \quad (2.5)$$

as desired. Part (ii) is proven.

(iii) By (2.4), we have

$$\delta_3(n) < 4 \cdot (\frac{\log cn}{\log n})^2 \leq 4 \cdot (\frac{\log(n^2/3)}{\log n})^2 = 16 \cdot (\frac{\log(n^2/3)}{\log n^2})^2 < 16,$$

as desired. Part (iii) is proven.

This completes the proof of Lemma 2.7.

Lemma 2.8. *Let a, b, c, and n be positive integers such that* $2 \nmid c$ *and* $a + b = c^2$*, and let* (x, y, z) *be a solution of* (1.2)*. If* $n > c$*, then*

$$z > \frac{\log\beta(n)}{\log(cn)} \cdot N > N,$$

where $N = \max\{x, y\}$ *and* $\beta(n) := \min\{an^2 + 1, bn^2 - 1\}$.

Proof. From (1.2), we have

$$(cn)^z = (an^2 + 1)^x + (bn^2 - 1)^y \geq \beta(n)^x + \beta(n)^y > \beta(n)^N.$$

A direct computation gives us that

$$z > \frac{\log\beta(n)}{\log(cn)} \cdot N. \quad (2.6)$$

Notice that

$$\beta(n) = \min\{an^2 + 1, bn^2 - 1\} \geq \min\{n^2 + 1, n^2 - 1\} > cn + 1. \quad (2.7)$$

From (2.6) and (2.7), one can deduce that

$$z > \frac{\log\beta(n)}{\log(cn)} \cdot N > \frac{\log(cn + 1)}{\log(cn)} \cdot N > N,$$

as required.

This completes the proof of Lemma 2.8.

To establish an upper bound on z, we need to prove the following result:

Lemma 2.9. *Let* $\gcd(c, n) = 1$*, and let* (x, y, z) *be a solution of* (1.2)*. If* $n \geq \lambda c$*, then* $\gcd(x, y, n) = 1$*, where* λ *is any constant with* $\lambda > 2^{\frac{5}{8}}$.

Proof. If $z \le 2$, by (1.2), one can easily check that $x = y = 1$. Hence,

$$\gcd(x, y, n) = 1,$$

as required.

If $z \ge 3$ and x is a power of 2, then by Lemma 2.5, one can derive that

$$\gcd(x, y, n) = 1.$$

Next, we may consider only the case where $z \ge 3$ and x has an odd prime factor. Let us assume that $\gcd(x, y, n) > 1$. Then there exists an odd prime p such that $x = px_1$, $y = py_1$, and $n = pn_1$, since y is odd by Lemma 2.5. By (1.2), we can deduce that

$$\begin{aligned}(cn)^z =&(an^2 + 1)^{px_1} + (bn^2 - 1)^{py_1}\\ =&((an^2 + 1)^{x_1} + (bn^2 - 1)^{y_1}) \cdot \frac{(an^2 + 1)^{px_1} + (bn^2 - 1)^{py_1}}{(an^2 + 1)^{x_1} + (bn^2 - 1)^{y_1}}\\ =&C \cdot D,\end{aligned} \tag{2.8}$$

where $C = (an^2+1)^{x_1}+(bn^2-1)^{y_1}$ and $D = \frac{(an^2+1)^{px_1}+(bn^2-1)^{py_1}}{(an^2+1)^{x_1}+(bn^2-1)^{y_1}}$. Therefore, we have $(an^2+1)^{x_1}-(bn^2-1)^{y_1} \equiv 2 \pmod{n^2}$. Since $n^2 \ge p^2 \ge 9$, we can get that $\left((an^2 + 1)^{x_1} - (bn^2 - 1)^{y_1}\right)^2 \ge 4$ and

$$\begin{aligned}D \ge& \frac{(an^2 + 1)^{3x_1} + (bn^2 - 1)^{3y_1}}{(an^2 + 1)^{x_1} + (bn^2 - 1)^{y_1}}\\ =& (an^2 + 1)^{2x_1} - (an^2 + 1)^{x_1} \cdot (bn^2 - 1)^{y_1} + (bn^2 - 1)^{2y_1}\\ \ge& (an^2 + 1)^{x_1} \cdot (bn^2 - 1)^{y_1} + 4\\ \ge& 2((an^2 + 1)^{x_1} + (bn^2 - 1)^{y_1}) = 2C.\end{aligned}$$

Now, using Lemma 2.1(i), we divide the proof into the following two cases:

Case 1: $\gcd(C, D) = 1$. Since $C \equiv 0 \pmod{n}$ and $\gcd(c, n) = 1$, one can easily get that $C = n^z c_1^z$ and $D = c_2^z$, where $c = c_1c_2$ and $\gcd(c_1, c_2) = 1$. By condition $n \ge \lambda c$ and $c_2^z = D \ge 2C = 2(c_1n)^z$, one has $4 \le 2(\lambda c_1^2)^z \le 1$, which is impossible. Thus, $\gcd(x, y, n) = 1$ in this case.

Case 2: $\gcd(C, D) = p$. Since $C \equiv 0 \pmod{n}$ and $n = pn_1$, by Lemma 2.1(ii), one can easily get that $C = p^{-1} \cdot n^z \cdot c_1^z$ and $D = p \cdot c_2^z$, where $c = c_1c_2$ and $\gcd(c_1, c_2) = 1$. Notice that $p \cdot c^z \ge D \ge 2C \ge 2p^{-1}n^z \ge 2p^{-1}(\lambda c)^z$. Therefore $p \ge 2^{\frac{1}{2}}\lambda^{\frac{z}{2}}$. Putting $z > \max\{x, y\}$ of Lemma 2.8 into $p \ge 2^{\frac{1}{2}}\lambda^{\frac{z}{2}}$ and using our assumption yields that

$$z \ge \max\{x, y\} + 1 \ge p + 1 \ge 1 + 2^{\frac{1}{2}}\lambda^{\frac{z}{2}},$$

which is impossible for $z \ge 3$. So one arrives at $\gcd(x, y, n) = 1$ in this case.

This concludes the proof of Lemma 2.9.

Lemma 2.10. *Let* $\gcd(c, n) = 1$*, and let* (x, y, z) *be a solution of* (1.2)*. If* $n \ge c + \frac{1}{n}$ *and* $z \le 3$*, then* (1.2) *has only the solution* $(x, y, z) = (1, 1, 2)$.

Proof. If $x \geq 3$ or $y \geq 3$, then

$$(cn)^3 = (an^2 + 1)^x + (bn^2 - 1)^y \geq ((c + \frac{1}{n})n + 1)^x + ((c + \frac{1}{n})n - 1)^y > (cn)^3,$$

which is impossible. Therefore, Lemma 2.5 tells us that $y = 1$ and $x = 2$ or $x = 1$.

If $x = y = 1$, then (1.2) gives us that $(an^2 + 1) + (bn^2 - 1) = (cn)^2$, which is impossible since $z = 3$.

If $x = 2$ and $y = 1$, then by (1.2), one can deduce that

$$(cn)^3 = a^2n^4 + an^2 + c^2n^2.$$

A simple transformation gives us that

$$c^2(cn - 1) = a(an^2 + 1). \tag{2.9}$$

By the condition $\gcd(an^2 + 1, bn^2 - 1) = 1$ of Theorem 1.2, we have $\gcd(an^2 + 1, cn) = 1$. Further, $\gcd(an^2 + 1, c) = 1$. On the other hand, $c^2 \mid a(an^2 + 1)$, thus we have $c^2 \mid a$, which contradicts the assumption $a < a + b = c^2$.

This concludes the proof of Lemma 2.10.

3. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2.

Proof of Theorem 1.2. First of all, suppose that (x, y, z) is a solution of (1.2). We apply Proposition 2.3 and Remark 2.4 to get an upper bound for z. For this, we set

$$\alpha_1 := an^2 + 1, \quad \alpha_2 := 1 - bn^2, \quad b_1 := x, \quad b_2 := y,$$

and

$$\Lambda := (an^2 + 1)^x - (1 - bn^2)^y.$$

Evidently, one has $\alpha_1 = an^2 + 1 \neq \pm 1$. Let $n = p_1^{j_1} \cdots p_k^{j_k}$, and let $g = 1$. One can easily check that

$$v_{p_i}(\alpha_1 - 1) = 2j_i \geq j_i, \quad v_{p_i}(\alpha_2 - 1) = 2j_i \geq 1, \quad for\ \ all\ p_i,$$
$$v_{p_i}(\alpha_1 - 1) \geq 2j_i \geq 2, \quad v_{p_i}(\alpha_2 - 1) \geq 2j_i \geq 2, \quad for\ \ p_i = 2,$$

and $A_1 = \alpha_1, A_2 = -\alpha_2$ satisfy the assumption of Proposition 2.3. Lemma 2.9 gives us that n, x, and y are relatively prime, and we know from Definition 2.2 that $an^2 + 1$ and $bn^2 - 1$ are multiplicatively independent. Thus, by Proposition 2.3 and Remark 2.4, we have

$$z = v_n(\Lambda) \leq \frac{53.6}{(\log n)^4}(\max\{\log \Gamma + \log\log n + 0.64, 4\log n\})^2 \log(an^2 + 1)\log(bn^2 - 1), \tag{3.1}$$

where

$$\Gamma := \frac{x}{\log(bn^2 - 1)} + \frac{y}{\log(an^2 + 1)}.$$

Let $N = \max\{x, y\}$. Assume that

$$\log N + 0.64 > 4\log n. \tag{3.2}$$

By (3.1), Lemma 2.8, and the definition of $\delta_2(n)$, we have

$$\frac{\log\beta(n)}{\log(cn)}N < z \leq \frac{53.6}{(\log n)^4}(\max\{\log\Gamma + \log\log n + 0.64, 4\log n\})^2 \log(an^2+1)\log(bn^2-1),$$

implies that

$$N \leq 53.6 \cdot \delta_2(n) \cdot (\log N + \delta_1(n) + 0.64)^2. \tag{3.3}$$

From the proof of Lemma 2.7(ii) and (i), one can get $\delta_2(n) \leq \frac{9}{8(\log 3)^2}$ and

$$\log\Gamma + \log\log n \leq \log N + \delta_1(n) \leq \log N. \tag{3.4}$$

Applying Lemma 2.7(i), (3.3), and (3.4) gives us that

$$N < 53.6 \cdot \frac{9}{8(\log 3)^2} \cdot (\log N + 0.64)^2, \tag{3.5}$$

which implies $N < 3985$. Therefore

$$n < e^{0.16}N^{0.25} = 9.32382\cdots < 4c,$$

a contradiction. The claim is proven. Hence, one must have

$$\log N + 0.64 \leq 4\log n. \tag{3.6}$$

Putting (3.6) into (3.1), we can deduce that

$$z < 53.6 \cdot 16 \cdot \frac{\log(an^2+1)\log(bn^2-1)}{(\log n)^2} = 53.6 \cdot 16 \cdot \delta_3(n).$$

Now, applying (iii) of Lemma 2.7, one has

$$z < 53.6 \cdot 16^2. \tag{3.7}$$

Suppose that $z \geq 4$. Taking Eq (1.2) modulo n^4, one can arrive at

$$an^2x + 1 + bn^2y - 1 \equiv 0 \pmod{n^4},$$

so

$$ax + by \equiv 0 \pmod{n^2}.$$

This implies that

$$n^2 \leq ax + by. \tag{3.8}$$

Further, applying Lemma 2.8 and (3.8), we have

$$n^2 \leq ax + by < (a+b)\cdot N < (a+b)\cdot z = c^2 \cdot z. \tag{3.9}$$

Therefore, using (3.7) and (3.9), one can deduce that

$$n^2 < 53.6 \cdot 16^2 \cdot c^2,$$

which contradicts the assumption $n \geq 117.14c$.

Finally, we conclude that $z \leq 3$. By Lemma 2.10, one can easily know that (1.2) has only the solution $(x, y, z) = (1, 1, 2)$.

This concludes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

In this section, we present the proof of Theorem 1.3.
Proof of Theorem 1.3 (i) $(a, b, c) = (44, 5, 7)$. Suppose that (x, y, z) is a solution of the equation

$$(44n^2 + 1)^x + (5n^2 - 1)^y = (7n)^z.$$

First of all, we show that $\gcd(44n^2 + 1, 5n^2 - 1) = 1$. In fact, from $5 \cdot (44n^2 + 1) - 44 \cdot (5n^2 - 1) = 49$, it follows that $\gcd(44n^2 + 1, 5n^2 - 1) \mid 49$. Since $44n^2 + 1 \not\equiv 0 \pmod 7$ for any positive integer n, we get $\gcd(44n^2 + 1, 5n^2 - 1) = 1$. By Theorem 1.2, [17, Example 1 and Lemma 12], and Lemma 2.6, we only solve the remaining cases of $19 \le n \le 820$ and $n \not\equiv 0 \pmod 2$ and $n \not\equiv 0 \pmod 7$. Since $n \ge 18 \ge \lambda c$, Lemma 2.9 tells us that n, x, and y are relatively prime. Using (3.1) of Theorem 1.2, one can immediately deduce that

$$z \le \frac{53.6}{(\log n)^4}(\max\{\log \Gamma_1 + \log\log n + 0.64, 4\log n\})^2 \log(44n^2 + 1)\log(5n^2 - 1), \tag{4.1}$$

where

$$\Gamma_1 = \frac{x}{\log(5n^2 - 1)} + \frac{y}{\log(44n^2 + 1)}.$$

For brevity, we let $N = \max\{x, y\}$ and

$$\Omega(n) = \frac{1}{\log(5n^2 - 1)} + \frac{1}{\log(44n^2 + 1)}.$$

Then we have

$$\Gamma_1 \le \frac{N}{\log(5n^2 - 1)} + \frac{N}{\log(44n^2 + 1)} = N \cdot \Omega(n). \tag{4.2}$$

Subsequently, suppose that

$$\log \Gamma_1 + \log\log n + 0.64 > 4\log n. \tag{4.3}$$

Because $n \ge 19$, from (4.2), we derive

$$N \ge \frac{\Gamma_1}{\Omega(n)} > \frac{n^4}{\Omega(n) \cdot e^{0.64} \cdot \log n} > 98575. \tag{4.4}$$

On the other hand, because $n \ge 19$, according to Lemma 2.8, one must have

$$1.533 \cdot N < N \cdot \frac{\log(5n^2 - 1)}{\log(7n)} < z. \tag{4.5}$$

Putting (4.5) into (4.1) gives us that

$$N \le 53.6 \cdot (\log N + \vartheta_1(n) + 0.64)^2 \cdot \vartheta_2(n), \tag{4.6}$$

where $\vartheta_1(n)$ and $\vartheta_2(n)$ are given by

$$\vartheta_1(n) = \log\left(\frac{\log n}{\log(5n^2 - 1)} + \frac{\log n}{\log(44n^2 + 1)}\right),$$

and

$$\vartheta_2(n) = \frac{\log(7n)\log(44n^2+1)}{(\log n)^4}.$$

Notice that $\vartheta_1(n)$ and $\vartheta_2(n)$ decrease as n increases in the interval $[19, \infty)$. Then

$$\vartheta_1(n) \leq \vartheta_1(19) < 0 \quad \text{and} \quad \vartheta_2(n) \leq \vartheta_2(19) < 0.6294.$$

By (4.6), one can immediately deduce that

$$N < 53.6 \cdot (\log N + 0.64)^2 \cdot 0.6294, \tag{4.7}$$

which implies that $N < 2392$. This contradicts $N > 98575$.

Therefore, one must have

$$\log \Gamma_1 + \log\log n + 0.64 \leq 4 \log n. \tag{4.8}$$

Because $n \geq 19$, one arrives at

$$\vartheta(n) = \frac{\log(44n^2+1)\log(5n^2-1)}{(\log n)^2} < 8.3656. \tag{4.9}$$

Thus, from (4.1), (4.8), and (4.9), one can immediately deduce that

$$z < 53.6 \cdot 16 \cdot \vartheta(n) < 7175. \tag{4.10}$$

Further, by (4.5) and (4.10), we see that

$$N \leq 4680. \tag{4.11}$$

Hence, all of x, y, and z are bounded. Using program search, we now show that $z \leq 3$ by following two steps:

Step 1: Under the hypotheses $19 \leq n \leq 820$, $n \not\equiv 0 \pmod 2$, $n \not\equiv 0 \pmod 7$, $N \leq 4680$, and $2 \nmid y$, one can check that $z \leq 5$.

Step 2: Under the hypotheses $19 \leq n \leq 820$, $n \not\equiv 0 \pmod 2$, $n \not\equiv 0 \pmod 7$, $\max\{x, y\} < z$, and $4 \leq z \leq 5$, one can deduce Eq (1.2) has no positive integer solution (x, y, z).

Finally, we conclude that $z \leq 3$. By Lemma 2.10, one can easily know that (1.2) has only the solution $(x, y, z) = (1, 1, 2)$.

When $(a, b, c) = (12, 13, 5)$ and $(18, 7, 5)$, using a similar way as in the proof of Theorem 1.3(i), the desired result follows immediately.

The proof of Theorem 1.3 is complete.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported in part by the Southwest Minzu University Research Startup Funds (Grant No. RQD2021100), and also supported in part by Natural Science Foundation of Sichuan Province (No. 2022NSFSC1830).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. Y. Fujita, M. H. Le, A parametric family of ternary purely exponential Diophantine equation $A^x + B^y = C^z$, *Turk. J. Math.*, **46** (2022), 1224–1232. https://doi.org/10.55730/1300-0098.3153
2. N. Terai, Y. Shinsh, On the exponential Diophantine equation $(3m^2 + 1)^x + (qm^2 - 1)^y = (rm)^z$, *SUT J. Math.*, **56** (2020), 147–158. https://doi.org/10.55937/sut/1611009430
3. N. Terai, On the exponential Diophantine equation $(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z$, *Int. J. Algebra*, **6** (2012), 1135–1146.
4. Y. Bugeaud, T. N. Shorey, On the number of solutions of the generalized Ramanujan-Nagell equation, *J. Reine Angew. Math.*, **539** (2001), 55–74. https://doi.org/10.1515/crll.2001.079
5. M. H. Le, Some exponential Diophantine equations I: The equation $D_1x^2 - D_2y^2 = \lambda k^z$, *J. Number Theory*, **55** (1995), 209–221. https://doi.org/10.1006/jnth.1995.1138
6. Y. Bilu, G. Hanrot, P. M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers, with an appendix by M. Mignotte, *J. Reine Angew. Math.*, **539** (2001), 75–122. https://doi.org/10.1515/crll.2001.080
7. J. L. Su, X. X. Li, The Diophantine equation $(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z$, *Abstr. Appl. Anal.*, **2014** (2014), 670175. https://doi.org/10.1155/2014/670175
8. C. Bertók, The complete solution of Diophantine equation $(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z$, *Period. Math. Hung.*, **72** (2016), 37–42. https://doi.org/10.1007/s10998-016-0111-x
9. C. Bertók, L. Hajdu, A Hasse-type principle for exponential Diophantine equations and its applications, *Math. Comp.*, **85** (2016), 849–860. http://doi.org/10.1090/mcom/3002
10. T. Miyazaki, N. Terai, On the exponential Diophantine equation $(m^2 + 1)^x + (cm^2 - 1)^y = (am)^z$, *Bull. Aust. Math. Soc.*, **90** (2014), 9–19. https://doi.org/10.1017/S0004972713000956
11. X. W. Pan, A note on the exponential Diophantine equation $(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z$, *Colloq. Math.*, **149** (2017), 265–273. https://doi.org/10.4064/cm6878-10-2016
12. R. Q. Fu, H. Yang, On the exponential Diophantine equation $(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z$ with $c \mid m$, *Period. Math. Hung.*, **75** (2017), 143–149. https://doi.org/10.1007/s10998-016-0170-z
13. E. Kizildere, T. Miyazaki, G. Soydan, On the Diophantine equation $((c + 1)m^2 + 1)^x + (cm^2 - 1)^y = (am)^z$, *Turk. J. Math.*, **42** (2018), 2690–2698. https://doi.org/10.3906/mat-1803-14
14. Y. Bugeaud, Linear forms in two m-adic logarithms and applications to Diophantine problems, *Compos. Math.*, **132** (2002), 137–158. https://doi.org/10.1023/A:1015825809661
15. N. Terai, T. Hibino, On the exponential Diophantine equation $(12m^2 + 1)^x + (13m^2 - 1)^y = (5m)^z$, *Int. J. Algebra*, **9** (2015), 261–272. http://doi.org/10.12988/ija.2015.5529
16. M. Alan, On the exponential Diophantine equation $(18m^2 + 1)^x + (7m^2 - 1)^y = (5m)^z$, *Turk. J. Math.*, **42** (2018), 1990–1999. https://doi.org/10.3906/mat-1801-76

17. E. Hasanalizade, A note on the exponential Diophantine equation $(44m^2 + 1)^x + (5m^2 - 1)^y = (7m)^z$, *Integers*, **23** (2023), 1–12. https://doi.org/10.5281/zenodo.8399672
18. N. Terai, T. Hibino, On the exponential Diophantine equation $(3pm^2-1)^x+(p(p-3)m^2+1)^y = (pm)^z$, *Period. Math. Hung.*, **74** (2017), 227–234. https://doi.org/10.1007/s10998-016-0162-z
19. Y. F. Bilu, Y. Bugeaud, M. Mignotte, *The Problem of Catalan*, Springer-Verlag, New York, 2014. https://doi.org/10.1007/978-3-319-10094-4
20. M. Alan, On the exponential Diophantine equation $(m^2 + m + 1)^x + m^y = (m + 1)^z$, *Mediterr. J. Math.*, **17** (2020). https://doi.org/10.1007/s00009-020-01613-4
21. M. A. Bennett, Y. Bugeaud, Perfect powers with three digits, *Mathematika*, **60** (2014), 66–84. https://doi.org/10.1112/S0025579313000107
22. M. A. Bennett, Y. Bugeaud, M. Mignotte, Perfect powers with few binary digits and related Diophantine problems, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **XII** (2013), 941–953. https://doi.org/10.2422/2036-2145.201110_006
23. Y. Bugeaud, T. N. Shorey, On the Diophantine equation $\frac{x^m-1}{x-1} = \frac{y^m-1}{y-1}$, *Pacific J. Math.*, **207** (2002), 61–75. http://doi.org/10.2140/pjm.2002.207.61
24. T. Miyazaki, M. Sudo, N. Terai, A purely exponential Diophantine equation in three unknowns, *Period. Math. Hung.*, **84** (2022), 287–298. https://doi.org/10.1007/s10998-021-00405-x