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Research article

Willmore-type variational problem for foliated hypersurfaces

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Abstract: After Thomas James Willmore, many authors were looking for an immersion of a manifold in Euclidean space or Riemannian manifold, which is the critical point of functionals whose integrands depend on the mean curvature or the norm of the second fundamental form. We study a new Willmore-type variational problem for a foliated hypersurface in Euclidean space. Its general version is the Reilly-type functional, where the integrand depends on elementary symmetric functions of the eigenvalues of the restriction on the leaves of the second fundamental form. We find the 1st and 2nd variations of such functionals and show the conformal invariance of some of them. For a critical hypersurface with a transversally harmonic foliation, we derive the Euler-Lagrange equation and give examples with low-dimensional foliations. We present critical hypersurfaces of revolution and show that they are local minima for special variations of immersion.

Keywords: hypersurface; foliation; second fundamental form; mean curvature; Willmore's type functional; conformal invariant

1. Introduction

Many authors, e.g., [1–5], were looking for an immersion $\phi : M^n \to \bar{M}^{n+1}$ of a smooth manifold M^n ($n \geq 2$) into a Riemannian manifold $(\bar{M}, \bar{g})$ or Euclidean space $\mathbb{R}^{n+1}$, which is a critical point of the following functionals for compactly supported variations of ϕ:

$$W_{n,p} = \int_M H^p \,\mathrm{d}V, \qquad J_{n,p} = \int_M \|h\|^p \,\mathrm{d}V. \tag{1.1}$$

Here, $\mathrm{d}V$ is the volume form of the induced metric $g = \langle\cdot\,,\cdot\rangle$ on M, h is the scalar second fundamental form of $\phi(M)$, $H = \frac{1}{n}\,\mathrm{trace}_g\, h$ is the mean curvature, and $p > 0$. These functionals measure how much $\phi(M)$ differs from a minimal hypersurface ($H = 0$) or a totally geodesic hypersurface ($h = 0$). The actions (1.1) are a particular case of functionals

$$WF_n = \int_M F(H)\,\mathrm{d}V, \quad JF_n = \int_M F(\|h\|)\,\mathrm{d}V,$$

where F is a C^3-regular function of one variable, e.g., [6–9]. For a closed smooth hypersurface M^n in $\mathbb{R}^{n+1}$, we get $W_{n,n} \geq C_n$, where $C_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$ is the area of the unit n-sphere; the equality $W_{n,n} = C_n$ holds if and only if M^n is embedded as a hypersphere; see [1].

Variational problems for (1.1) were first posed by Willmore in [10] for $W_{2,2}$, which belongs to conformal geometry. The Euler-Lagrange equation for $W_{2,2}$ is the well known elliptic PDE

$$\Delta H + 2\,H(H^2 - K) = 0, \tag{1.2}$$

where Δ is the Laplacian and K is the gaussian curvature of $M^2 \subset \mathbb{R}^3$. Solutions of (1.2) are called Willmore surfaces. An important class of Willmore surfaces in $\mathbb{R}^3$ arises from the stereographic projection of minimal surfaces in the 3-sphere. By Lawson's theorem, any compact, orientable surface can be minimally embedded in the 3-sphere. For a closed orientable surface M in $\mathbb{R}^3$, the inequality $W_{2,2} \geq C_2 = 4\pi$ holds with the equality for round spheres. If M^2 is a torus in $\mathbb{R}^3$, then, according to the Willmore conjecture proven by Marques and Neves in [4], we have $W_{2,2} \geq 2\,\pi^2$; the equality holds if and only if the generating curve is a circle and the ratio of radii is $\frac{1}{\sqrt{2}}$. Willmore surfaces have applications in biophysics, computer graphics, materials science, architecture, etc., e.g., [11].

Reilly [12] and some mathematicians studied variations of more general functionals than (1.1):

$$WF_n = \int_M F(\sigma_1, \dots, \sigma_n)\, \mathrm{d}V, \qquad JF_n = \int_M F(\tau_1, \dots, \tau_n)\, \mathrm{d}V, \tag{1.3}$$

where $F \in C^3(\mathbb{R}^n)$. The elementary symmetric functions $\sigma_r = \sum_{i_1<\ldots<i_r} k_{i_1} \ldots k_{i_r}$ $(0 \leq r \leq n)$ of the principal curvatures k_i satisfy the equality $\sum_{r=0}^{n} \sigma_r\, t^r = \det(\mathrm{id}_{TM} + tA)$, where A is the Weingarten operator, i.e., $\langle AX, Y\rangle = h(X, Y)$. The power sums of the principal curvatures, $\tau_i = k_1^i + \ldots + k_n^i = \operatorname{trace} A^i$, can be expressed as polynomials of σ_r using the Newton formulas, e.g., [13]. For example, $\sigma_0 = 1$, $\tau_1 = \sigma_1 = nH$, $\sigma_n = \det A$, $\tau_2 = \|h\|^2$, and $2\,\sigma_2 = \tau_1^2 - \tau_2$. The r-th $(r \leq n)$ order Willmore functional, introduced by Guo in [9],

$$W^{\mathrm{conf}}_{n,r} = \int_M Q_r^{n/r}\, \mathrm{d}V, \quad Q_r = \sum\nolimits_{j=0}^{r} (-1)^{j+1} C_r^j S_1^{r-j} S_j, \tag{1.4}$$

is a special case of (1.3), invariant under the conformal group of $(\bar M, \bar g)$ and vanishing on totally umbilical hypersurfaces. Here, $S_r = \sigma_r / C_n^r$ (where $C_n^r = \frac{n!}{r!(n-r)!}$ is a binomial coefficient) is the r-th mean curvature function of a hypersurface. In particular, $Q_2 = S_1^2 - S_2 = \frac{1}{n^2(n-1)}\,((n-1)\sigma_1^2 - 2\,n\,\sigma_2)$. Examples of hypersurfaces in $\mathbb{R}^{n+1}$ that are critical for (1.4) are given in [7, 8].

An interesting problem is the generalization of the Willmore functional to submanifolds with additional structures, such as foliations or almost products. Let M^n $(n \geq 2)$ equipped with an s-dimensional $(1 \leq s \leq n)$ foliation $\mathcal{F}$ be immersed into a Riemannian manifold $(\bar M, \bar g)$. All leaves of the foliation under consideration have the same dimension. Let $h_{\mathcal{F}}$ be the restriction of the second fundamental form of M on the leaves of $\mathcal{F}$. Denote by $\tau_i^{\mathcal{F}}$ $(1 \leq i \leq s)$ the power sums, $\sigma_r^{\mathcal{F}}$ $(1 \leq r \leq s)$ elementary symmetric functions of the eigenvalues $k_1^{\mathcal{F}} \leq \ldots \leq k_s^{\mathcal{F}}$ of $h_{\mathcal{F}}$, and set $S_r^{\mathcal{F}} = \sigma_r^{\mathcal{F}} / C_s^r$. We have $\tau_1^{\mathcal{F}} = \sigma_1^{\mathcal{F}} = sH_{\mathcal{F}} = \operatorname{trace}_g h_{\mathcal{F}}$, $\tau_2^{\mathcal{F}} = \|h_{\mathcal{F}}\|^2$, $(\tau_1^{\mathcal{F}})^2 - \tau_2^{\mathcal{F}} = 2\,\sigma_2^{\mathcal{F}}$, etc. For foliation theory, we refer to [14]. The extrinsic geometry of foliations was developed in [13]. We study Reilly-type functionals for compactly supported variations of $(M^n, \mathcal{F})$ immersed in $\mathbb{R}^{n+1}$:

$$WF_{n,s} = \int_M F(\sigma_1^{\mathcal{F}}, \dots, \sigma_s^{\mathcal{F}})\, \mathrm{d}V, \qquad JF_{n,s} = \int_M F(\tau_1^{\mathcal{F}}, \dots, \tau_s^{\mathcal{F}})\, \mathrm{d}V, \tag{1.5}$$

which, for $s = n$, reduces to (1.3). For $F = (\sigma^{\mathcal{F}}_1/s)^p$ and $F = (\tau^{\mathcal{F}}_2)^{p/2}$, the actions (1.5) read as

$$W_{n,p,s} = \int_M H^p_{\mathcal{F}}\,\mathrm{d}V, \qquad J_{n,p,s} = \int_M \|h_{\mathcal{F}}\|^p\,\mathrm{d}V, \tag{1.6}$$

which reduces to (1.1) for $s = n$.

Remark 1. A foliated hypersurface in $\mathbb{R}^{n+1}$, whose leaves $\{L\}$ are minimal submanifolds in $\mathbb{R}^{n+1}$ is an example of a minimizer for $W_{n,p,s}$ in (1.6) with even p. A foliated hypersurface in $\mathbb{R}^{n+1}$ with an asymptotic distribution $T\mathcal{F}$ (e.g., a ruled hypersurface) is a minimizer for $J_{n,p,s}$ in (1.6). It is interesting to find critical hypersurfaces for actions (1.6) with $H_{\mathcal{F}} \neq 0$ or $h_{\mathcal{F}} \neq 0$ on an open dense set of M.

The following special case of (1.5) is invariant under the conformal group of $(\bar{M}, \bar{g})$, see Theorem 1:

$$W^{\rm conf}_{n,s,r} = \int_M (Q^{\mathcal{F}}_r)^{n/r}\,\mathrm{d}V, \quad Q^{\mathcal{F}}_r = \sum\nolimits_{j=0}^{r} (-1)^{j+1} C^j_r (S^{\mathcal{F}}_1)^{r-j} S^{\mathcal{F}}_j, \quad r \le s, \tag{1.7}$$

and reduces to (1.4) for $s = n$. Note that $Q^{\mathcal{F}}_2 = (S^{\mathcal{F}}_1)^2 - S^{\mathcal{F}}_2 = \frac{1}{s^2(s-1)}\,((s-1)(\sigma^{\mathcal{F}}_1)^2 - 2s\,\sigma^{\mathcal{F}}_2)$ is true and $Q^{\mathcal{F}}_r = 0$ if $k^{\mathcal{F}}_1 = \ldots = k^{\mathcal{F}}_s$, e.g., for hypersurfaces of revolution in $\mathbb{R}^{n+1}$ foliated by parallels.

We hope that foliated hypersurfaces, which are local minima for (1.5), will be useful for natural sciences and technology related to layered (laminated) or non-isotropic materials.

The paper is organized as follows: Section 2 contains some lemmas that help us calculate variations of Reilly-type functionals on foliated hypersurfaces in $\mathbb{R}^{n+1}$. In Section 3, conformal invariance of (1.7) is shown, the first variations of the functionals (1.5)–(1.7) are found, and the corresponding Euler-Lagrange equations for the case of transversally harmonic (for example, Riemannian) foliation are obtained. Then the second variation on critical hypersurfaces of some Willmore-type functionals is calculated. In Section 4, applications to hypersurfaces with low-dimensional foliations are given, the critical hypersurfaces of revolution for the actions (1.6) are presented, and it is shown that they are local minima for special variations of immersion.

2. Auxiliary results

Let $\mathbf{r} : M^n \to \mathbb{R}^{n+1}$ be an immersion of a manifold M into $\mathbb{R}^{n+1}$ with Euclidean metric $\bar{g}$ and the Levi–Civita connection $\bar\nabla$. We identify M with its image $\mathbf{r}(M)$ and restrict our calculations to a relatively compact neighborhood $U \subset M$ with induced metric $g = \langle\cdot,\cdot\rangle$ and normal coordinates $(x^1, \ldots, x^n)$ centered at a point $x \in M$. Thus, $g_{ij} = \delta_{ij}$ (the Kronecker symbol) and $\Gamma^k_{ij} = 0$ at x. We will denote differentiation of a function f (or a tensor) with respect to the variable x^i by f_i.

Let $\partial_i = \partial/\partial x^i$ be the coordinate vector fields on U. So, the vectors $\mathbf{r}_i = \bar\nabla_{\partial_i}\mathbf{r}$ form a local coordinate basis for the tangent bundle TM along U, and we get $g = g_{ij}\,dx^i\,dx^j$, where $g_{ij} = \bar{g}(\mathbf{r}_i, \mathbf{r}_j) = \langle\partial_i, \partial_j\rangle$ and the Einstein summation rule is used. Let $\mathbf{N}$ be a unit normal vector field to M on U. The vectors $\mathbf{N}_i = \bar\nabla_{\partial_i}\mathbf{N}$ belong to the tangent space T_xM, i.e., $\langle\mathbf{N}_i, \mathbf{N}\rangle = 0$.

Let h be the scalar second fundamental form of M with respect to unit normal $\mathbf{N}$, $A = -\bar\nabla\,\mathbf{N}$ the Weingarten operator, and $H = \frac{1}{n}\,\mathrm{trace}_g\,h$ the mean curvature. Denote by h^j the symmetric tensor dual to A^j, i.e., $h^j(X,Y) = \langle A^jX, Y\rangle$. For example, $h^2 = g^{kl}h_{li}h_{kj}\,dx^i dx^j = h^k_i h_{kj}\,dx^i dx^j$.

Consider a one-parameter family of hypersurfaces $\mathbf{r}_t = \mathbf{r} + t\,u\,\mathbf{N}$ ($|t| < 1$). We get a variation $\delta\,\mathbf{r} = u\,\mathbf{N}$, where $\delta = (d/dt)|_{t=0}$ is the variational derivative operator, and $u : U \to \mathbb{R}$ is a smooth function supported on a relatively compact neighborhood U. Obviously, $(\delta\,\mathbf{r})_i = u_i\,\mathbf{N} + u\,\mathbf{N}_i$. The Hessian of a function u is a (0,2)-tensor $(\mathrm{Hess}_u)(X,Y) = X(Y(u)) - (\nabla_X Y)u = (u_{ij} - \Gamma^k_{ij}u_k)dx^i dx^j$; see [15], where Γ^k_{ij} are the Christoffel symbols. The Laplacian is $\Delta u = \mathrm{trace}_g\ \mathrm{Hess}_u = g^{ij}u_{ij}$. Note that $\langle g, \mathrm{Hess}_u\rangle = \Delta\,u$. The divergence of a vector field $X = X^i\partial_i$ on M is $\mathrm{div}\,X = \nabla_i X^i$.

Lemma 1 (see [3] for $n = 2$)**.** *The following evolution equations are true:*

$$\delta\,g_{ij} = -2\,u\,h_{ij}, \tag{2.1}$$
$$\delta\,g^{ij} = 2\,u\,h^{ij}, \tag{2.2}$$
$$\delta\,h_{ij} = u_{ij} - uh^l_i\,h_{jl} \ \Leftrightarrow\ \delta\,h = \mathrm{Hess}_u - u\,h^2, \tag{2.3}$$
$$\delta\,\|h\|^2 = 2\,\langle h,\ uh^2 + \mathrm{Hess}_u\rangle, \tag{2.4}$$
$$\delta(nH) = \Delta\,u + u\,\|h\|^2, \tag{2.5}$$
$$\delta\,\mathrm{d}V = -n\,uH\,\mathrm{d}V. \tag{2.6}$$

Proof. Using $\delta\,\mathbf{r}_i = u_i\,\mathbf{N} + u\,\mathbf{N}_i$, we calculate

$$\langle\delta\,\mathbf{r}_i, \mathbf{r}_j\rangle = \langle u_i\,\mathbf{N} + u\,\mathbf{N}_i, \mathbf{r}_j\rangle = u\,\langle\mathbf{N}_i, \mathbf{r}_j\rangle = -u\,\langle\mathbf{N}, \mathbf{r}_{ij}\rangle = -u\,h_{ij}.$$

Thus, since the symmetry $h_{ij} = h_{ji}$ we get the equality (2.1): $\delta\,g_{ij} = \langle\delta\,\mathbf{r}_i, \mathbf{r}_j\rangle + \langle\mathbf{r}_i, \delta\,\mathbf{r}_j\rangle = -2\,u\,h_{ij}$. From $g^{il}g_{lj} = \delta^i_j$, it follows that $(\delta g^{il})g_{lj} = -g^{il}(\delta g_{lj}) = 2\,u\,g^{il}h_{lj}$; hence, (2.2) is true.

We will compute the variation of h. Using $\langle\mathbf{N}, \mathbf{N}_i\rangle = 0$, we find

$$\langle\mathbf{N},\ \delta\,\mathbf{r}_{ij}\rangle = \langle\mathbf{N},\ (u\,\mathbf{N})_{ij}\rangle = u_{ij} - u\,\langle\mathbf{N}_i,\ \mathbf{N}_j\rangle = u_{ij} - u\,\langle h^l_i\,\mathbf{r}_l,\ h^k_j\,\mathbf{r}_k\rangle = u_{ij} - u\,h^l_i\,h_{jl}.$$

Note that $\delta\,\mathbf{N} = c^i\mathbf{r}_i$ for some functions c^i. Using $\langle\mathbf{N}, \mathbf{r}_j\rangle = 0$, we get

$$g_{ij}\,c^i = \langle\delta\,\mathbf{N},\ \mathbf{r}_j\rangle = -\langle\mathbf{N},\ \delta\,\mathbf{r}_j\rangle = -\langle\mathbf{N},\ u_j\,\mathbf{N} + u\,\mathbf{N}_j\rangle = -u_j.$$

It follows that $c^i = -g^{ij}u_j$ and $\delta\,\mathbf{N} = -g^{ij}u_j\mathbf{r}_i = -u^i\mathbf{r}_i$. Using the Gauss equation for a hypersurface in $\mathbb{R}^{n+1}$, we get at x: $\langle\delta\,\mathbf{N},\ \mathbf{r}_{ij}\rangle = \langle -u^i\mathbf{r}_i,\ h_{jl}\mathbf{N} + \Gamma^k_{jl}\,\mathbf{r}_k\rangle = 0$. Thus, (2.3) is true:

$$\delta h_{ij} = \delta\langle\mathbf{r}_{ij},\ \mathbf{N}\rangle = \langle\delta\,\mathbf{N},\ \mathbf{r}_{ij}\rangle + \langle\mathbf{N},\ \delta\,\mathbf{r}_{ij}\rangle = u_{ij} - uh^l_i\,h_{jl}.$$

Calculating the variation of the mean curvature, we get (2.5):

$$\delta\,(nH) = \delta\,(g^{ij}h_{ij}) = 2\,uh^{ij}\,h_{ij} + g^{ij}(u_{ij} - uh^l_i\,h_{jl}) = u\,h_{ij}\,h^{ij} + g^{ij}u_{ij} = \Delta u + u\,\|h\|^2.$$

The formula $\delta(\mathrm{d}V) = \frac{1}{2}\,(\mathrm{trace}_g\,\delta\,g)\,\mathrm{d}V$ for variation of $\mathrm{d}V$ is valid for any variation δg of a metric, for example, [13]. Applying (2.1) to the above gives (2.6). Next, we calculate the variation of $\|h\|^2$:

$$\begin{aligned}\delta\,\|h\|^2 &= \delta\,(g^{ik}g^{jl}h_{kl}h_{ij})\\ &= 2\,u(h^{ik}g^{jl} + h^{jl}g^{ik})h_{kl}h_{ij} + g^{ik}g^{jl}((u_{kl} - uh^q_k h_{lq})h_{ij} + (u_{ij} - uh^q_i h_{jq})h_{kl})\\ &= 2\,(uh_{ij}h^{ik}h^j_k + u_{kl}\,g^{ik}g^{jl}) = 2\,u\,\langle h,\ h^2\rangle + 2\,\langle h, \mathrm{Hess}_u\rangle,\end{aligned}$$

that proves (2.4). □

We carry out further calculations for a foliated hypersurface $(M,\mathcal{F})$ and a foliated neighborhood $U\subset M$ with normal coordinates $(x^1,\ldots,x^n)$ adapted to $\mathcal{F}$, i.e., $(x^1,\ldots,x^s)$ are variables along the leaves; see [14]. Let $\nabla^{\mathcal{F}}: TM\times T\mathcal{F}\to T\mathcal{F}$ be the induced Levi–Civita connection on the leaves of $\mathcal{F}$. The leafwise Laplacian on functions is $\Delta_{\mathcal{F}}=\operatorname{trace}_g\operatorname{Hess}^{\mathcal{F}}=\operatorname{div}_{\mathcal{F}}\circ\nabla^{\mathcal{F}}$, where $\operatorname{Hess}^{\mathcal{F}}$ is the Hessian on the leaves of $\mathcal{F}$. Let $P: TM\to T\mathcal{F}$ be the orthoprojector, thus, $P^2=P$ and P is self-adjoint. For $h_{\mathcal{F}}$ and its dual self-adjoint operator $A_{\mathcal{F}}$, we can write $h_{\mathcal{F}}(X,Y)=h(PX,PY)$ $(X,Y\in\mathfrak{X}_M)$ and $A_{\mathcal{F}}=PAP$. Let $h^j_{\mathcal{F}}$ be the symmetric tensor dual to $A^j_{\mathcal{F}}$, i.e., $h^j_{\mathcal{F}}(X,Y)=\langle A^j_{\mathcal{F}}X,Y\rangle$. The symmetric tensor $h_{\rm mix}$ is given by

$$h_{\rm mix}(X,Y)=\frac{1}{2}\left(h(PX,Y)+h(X,PY)\right)-h(PX,PY),\quad X,Y\in\mathfrak{X}_M.$$

We have $\langle h_{\mathcal{F}},h_{\rm mix}\rangle=0$. The equality $h_{\rm mix}=0$ means that $PA=AP$, i.e., $T\mathcal{F}$ is an invariant subbundle for A. Let $h_{\mathcal{F}^\perp}$ be the restriction of h on the normal distribution to $\mathcal{F}$ in M; then $h^2_{\mathcal{F}^\perp}=g^{\gamma\epsilon}h_{\alpha\gamma}h_{\alpha\epsilon}dx^\alpha dx^\beta$, where $s<\alpha,\beta,\gamma,\epsilon\le n$. Define symmetric tensors $h^2_{\rm mix}=g^{\alpha\beta}h_{\alpha i}h_{\beta j}dx^i dx^j+g^{ij}h_{\alpha i}h_{\beta j}\,dx^\alpha dx^\beta$ and $\operatorname{Hess}^{\rm mix}_u=g^{ij}g^{\alpha\beta}u_{i\alpha}\,dx^j dx^\beta$, where $1\le i,j\le s$ and $s<\alpha,\beta,\gamma,\epsilon\le n$. Let $A_{\rm mix}$ be the (1,1)-tensor dual to $h_{\rm mix}$; then $A^2_{\rm mix}$ is dual to $h^2_{\rm mix}$.

The Newton transformations $T_r(A_{\mathcal{F}})$ of $A_{\mathcal{F}}$ are defined inductively or explicitly by, e.g., [13],

$$\begin{aligned}&T_0(A_{\mathcal{F}})=\operatorname{id}_{T\mathcal{F}},\quad T_r(A_{\mathcal{F}})=\sigma^{\mathcal{F}}_r\operatorname{id}_{T\mathcal{F}}-A_{\mathcal{F}}\,T_{r-1}(A_{\mathcal{F}})\quad(0<r\le s),\\&T_r(A_{\mathcal{F}})=\sum\nolimits_{j=0}^r(-1)^j\sigma^{\mathcal{F}}_{r-j}A^j_{\mathcal{F}}=\sigma^{\mathcal{F}}_r\operatorname{id}_{T\mathcal{F}}-\sigma^{\mathcal{F}}_{r-1}A_{\mathcal{F}}+\ldots+(-1)^rA^r_{\mathcal{F}}.\end{aligned}$$

For example, $T_1(A_{\mathcal{F}})=\sigma^{\mathcal{F}}_1\operatorname{id}_{T\mathcal{F}}-A_{\mathcal{F}}$ and $T_s(A_{\mathcal{F}})=0$, and the following equalities are true:

$$\begin{aligned}&\operatorname{trace}T_r(A_{\mathcal{F}})=(s-r)\,\sigma^{\mathcal{F}}_r,\quad\operatorname{trace}(A_{\mathcal{F}}\,T_r(A_{\mathcal{F}}))=(r+1)\,\sigma^{\mathcal{F}}_{r+1},\\&\operatorname{trace}(A^2_{\mathcal{F}}\,T_r(A_{\mathcal{F}}))=\sigma^{\mathcal{F}}_1\sigma^{\mathcal{F}}_{r+1}-(r+2)\,\sigma^{\mathcal{F}}_{r+2}.\end{aligned}\tag{2.7}$$

The "musical" isomorphism $\sharp: T^*M\to TM$ is used for tensors, e.g., $h^\sharp=A$, and for $(0,2)$-tensors B and C, we have $\langle B,C\rangle=\operatorname{trace}(B^\sharp C^\sharp)=\langle B^\sharp,C^\sharp\rangle$.

Lemma 2. *The variations of $\tau^{\mathcal{F}}_i$ and $\sigma^{\mathcal{F}}_r$ are the following:*

$$\frac{1}{i}\,\delta\,\tau^{\mathcal{F}}_i=\langle h^{i-1}_{\mathcal{F}},\operatorname{Hess}^{\mathcal{F}}_u\rangle+u\,(\tau^{\mathcal{F}}_{i+1}+\langle h^{i-1}_{\mathcal{F}},h^2_{\rm mix}\rangle),\tag{2.8a}$$

$$\delta\,\sigma^{\mathcal{F}}_r=\langle T_{r-1}(A_{\mathcal{F}}),\operatorname{Hess}^{\mathcal{F}\sharp}_u\rangle+u(\sigma^{\mathcal{F}}_1\sigma^{\mathcal{F}}_{r-1}-(r+1)\,\sigma^{\mathcal{F}}_{r+1}+\langle T_{r-1}(A_{\mathcal{F}}),\,A^2_{\rm mix}\rangle).\tag{2.8b}$$

Proof. By (2.3), we obtain $\delta A_{\mathcal{F}}=\operatorname{Hess}^{\mathcal{F}\sharp}_u+u(A^2_{\mathcal{F}}+A^2_{\rm mix})$. Using this, (2.7) and the following variations of $\tau^{\mathcal{F}}_i$ and $\sigma^{\mathcal{F}}_r$; see [13]:

$$\delta\,\tau^{\mathcal{F}}_i=i\operatorname{trace}(A^{i-1}_{\mathcal{F}}\delta A_{\mathcal{F}}),\quad\delta\,\sigma^{\mathcal{F}}_r=\operatorname{trace}(T_{r-1}(A_{\mathcal{F}})\,\delta A_{\mathcal{F}}),$$

we get (2.8a,b). □

Lemma 3. *The following evolution equations are true:*

$$\delta\,(s\,H_{\mathcal{F}})=\Delta_{\mathcal{F}}\,u+u\,(\|h_{\mathcal{F}}\|^2-\|h_{\rm mix}\|^2),\tag{2.9}$$

$$\delta\,\|h_{\mathcal{F}}\|^2=2\,\langle h_{\mathcal{F}},\;u\,(h^2_{\mathcal{F}}+h^2_{\rm mix})+\operatorname{Hess}^{\mathcal{F}}_u\rangle,\tag{2.10}$$

$$\delta\,\|h_{\rm mix}\|^2 = u\,\langle h_{\mathcal F} + h_{\mathcal F^\perp}, h^2_{\rm mix}\rangle + 2\,\langle h_{\rm mix}, {\rm Hess}^{\,\rm mix}_u\rangle. \tag{2.11}$$

For any smooth function $f : M\times\mathbb{R}\to\mathbb{R}$, the following evolution equation is true:

$$\delta\,(\Delta_{\mathcal F} f) = \Delta_{\mathcal F}\dot f + 2\,u\langle h_{\mathcal F}, {\rm Hess}^{\mathcal F}_f\rangle + su\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} f\rangle + 2\,h(\nabla^{\mathcal F} u, \nabla^{\mathcal F} f) - sH_{\mathcal F}\langle\nabla^{\mathcal F} u, \nabla^{\mathcal F} f\rangle. \tag{2.12}$$

Proof. The Eqs (2.9) and (2.10) can be deduced from Lemma 2, but we will prove them directly. First, using (2.1) and (2.3), we get for $1\le i,j\le s$, and $1\le q\le n$,

$$\delta\,(s\,H_{\mathcal F}) = \delta\,(g^{ij}h_{ij}) = 2\,uh^{ij}\,h_{ij} + g^{ij}(u_{ij} - uh^q_i\,h_{jq}) = \Delta_{\mathcal F}\,u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2),$$

that proves (2.9). Also for $1\le i,j,k,l\le s$ and $1\le q\le n$, we obtain (2.10):

$$\delta\,\|h_{\mathcal F}\|^2 = \delta(g^{ik}g^{jl}h_{kl}h_{ij}) = 4\,u\,\langle h_{\mathcal F}, h^2_{\mathcal F}\rangle + 2\,\langle h_{\mathcal F}, {\rm Hess}^{\mathcal F}_u\rangle - 2\,u\,\langle h_{\mathcal F}, h^2_{\mathcal F}\rangle - 2\,u\,\langle h_{\mathcal F}, h^2_{\rm mix}\rangle.$$

For $1\le i,j,k,l\le s$, $s<\alpha,\beta,\gamma\le n$, and $1\le q\le n$, we obtain (2.11):

$$\begin{aligned}\delta\,\|h_{\rm mix}\|^2 &= \delta(g^{ij}g^{\alpha\beta}h_{i\alpha}h_{j\beta})\\ &= 2\,u(h^{ij}g^{\alpha\beta} + g^{ij}h^{\alpha\beta})h_{i\alpha}h_{j\beta} + g^{ij}g^{\alpha\beta}(u_{i\alpha}h_{j\beta} + u_{j\beta}h_{i\alpha}) - u\,g^{ij}g^{\alpha\beta}(h^l_i h_{\alpha l}h_{j\beta} + h^l_j h_{\alpha i}h_{l\beta})\\ &= \langle h_{\mathcal F} + h_{\mathcal F^\perp}, u\,h^2_{\rm mix}\rangle + 2\,\langle h_{\rm mix}, {\rm Hess}^{\,\rm mix}_u\rangle.\end{aligned}$$

The proof of (2.12) is similar to the proof of (19) in [3]: instead of M^2, we consider s-dimensional leaves of $\mathcal F$. The variation of the Christoffel symbols is the following tensor, e.g., [3]:

$$\delta\,\Gamma^k_{ij} = -u\,g^{kl}(h_{jl,i} + h_{il,j} - h_{ij,l}) - g^{kl}(u_i h_{jl} + u_j h_{il} - u_l h_{ij}). \tag{2.13}$$

For the Laplacian $\Delta_{\mathcal F}\,f = g^{ij}\,(f_{ij} - \Gamma^k_{ij}\,f_k)$ with $1\le i,j,k\le s$ it follows that

$$\delta\,(\Delta_{\mathcal F}\,f) = \delta\,(g^{ij}f_{ij}) - \delta\,(g^{ij}\Gamma^k_{ij}f_k). \tag{2.14}$$

For the first term, we get (for $1\le i,j\le s$)

$$\delta\,(g^{ij}f_{ij}) = 2\,u\,h^{ij}f_{ij} + g^{ij}\dot f_{ij} = 2\,u\,\langle h_{\mathcal F},\,{\rm Hess}^{\mathcal F}_f\rangle + \Delta_{\mathcal F}\,\dot f. \tag{2.15}$$

For the second term, using (2.13) and $\Gamma^k_{ij} = 0$ at x, we get for $1\le i,j,k\le s$, and $1\le q\le n$,

$$\begin{aligned}\delta\,(g^{ij}\,\Gamma^k_{ij}f_k) &= g^{ij}\delta\,(\Gamma^k_{ij})f_k = -g^{ij}g^{kq}\{u\,(h_{jq,i} + h_{iq,j} - h_{ij,q})f_k - (u_i h_{jq} + u_j h_{iq} - u_q h_{ij})f_k\}\\ &= -2\,u\,g^{ij}\,g^{kq}h_{jq,i}f_k + u\,g^{ij}\,g^{kq}h_{ij,q}f_k - 2\,g^{ij}\,g^{kq}u_i h_{jq}f_k + g^{ij}\,g^{kq}\,u_q h_{ij}f_k.\end{aligned} \tag{2.16}$$

Using the Codazzi–Mainardi equation $\nabla_k\,h_{ij} - \nabla_j\,h_{ik} = 0$, e.g., [15], we get for $1\le i,j,k,l\le s$,

$$g^{ij}\,g^{kl}(\nabla_j\,h_{il})f_k = g^{kl}(g^{ij}\nabla_l\,h_{ij})f_k = s\,(\nabla^k H_{\mathcal F})f_k = s\,\langle\nabla^{\mathcal F} H_{\mathcal F},\,\nabla^{\mathcal F} f\rangle.$$

Thus, using normal coordinates and $-2\,u\,g^{ij}g^{kl}h_{jl,i}\,f_k + u\,g^{ij}g^{kl}h_{ij,l}f_k = -u\,g^{kl}(g^{ij}h_{ij,l})f_k$, we get $s\,(H_{\mathcal F})_l = g^{ij}\,\nabla_l\,h_{ij}$ for $1\le i,j,l\le s$. Therefore, (2.16) becomes

$$\begin{aligned}\delta\,(g^{ij}\,\Gamma^k_{ij}f_k) &= -u\,g^{kl}(g^{ij}h_{ij,l})f_k - 2\,g^{ij}g^{kl}u_i h_{jl}f_k + g^{kl}u_l(g^{ij}h_{ij})f_k\\ &= -s\,u\,\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} f\rangle - 2\,h(\nabla^{\mathcal F} u, \nabla^{\mathcal F} f) + s\,H_{\mathcal F}\langle\nabla^{\mathcal F} u, \nabla^{\mathcal F} f\rangle.\end{aligned} \tag{2.17}$$

Applying (2.15) and (2.17) to (2.14) completes the proof of (2.12). □

Remark 2. To find the second variation of the functionals (1.5), we also need the variation $\delta\langle h, \mathrm{Hess}^{\mathcal F}_u\rangle$, but we omit this calculation and consider the second variation of the functionals (1.6) only.

The following property helps to find the Euler-Lagrange equations using the first variation of the functionals (1.3)–(1.6):

$$(\operatorname{div} P)\circ P = 0. \tag{2.18}$$

Here, $(\operatorname{div} P)X = \sum_{i=1}^n \langle(\nabla_{e_i}P)X, e_i\rangle$, where $e_1,\ldots,e_n$ is a local orthonormal basis on M. Note that Riemannian foliations (and the leaves of twisted products, e.g., [13]) satisfy (2.18).

Lemma 4. *A foliated Riemannian manifold* $(M, g, \mathcal F)$ *satisfies* (2.18) *if and only if* $\mathcal F$ *is transversally harmonic, i.e., the normal distribution has zero mean curvature.*

Proof. Using a local orthonormal frame on M such that $e_i \in T\mathcal F$ $(1\le i\le s)$, we calculate:

$$\begin{aligned}(\operatorname{div} P)(PX) &= \sum\nolimits_{i=1}^n \langle(\nabla_{e_i}P)(PX), e_i\rangle = \sum\nolimits_{i=1}^n \{\langle\nabla_{e_i}(P^2X), e_i\rangle - \langle P\nabla_{e_i}(PX), e_i\rangle\}\\ &= \sum\nolimits_{i=1}^n \{\langle\nabla_{e_i}(PX), e_i\rangle - \langle\nabla_{e_i}(PX), Pe_i\rangle\} = \sum\nolimits_{i>s}\langle\nabla_{e_i}(PX), e_i\rangle = -\langle X, (n-s)H^\perp\rangle,\end{aligned}$$

where $(n-s)H^\perp = P\sum_{i>s}\nabla_{e_i}e_i$ is the mean curvature vector of $(T\mathcal F)^\perp$ and $X\in\mathfrak X_M$. □

For any 2-tensor B on M, define the adjoint of the covariant derivative $\nabla^* B = -\sum_i(\nabla_i B)(e_i,\cdot)$; see [15]. We have the formula $\int_M\langle B,\nabla B'\rangle\,\mathrm dV = \int_M\langle\nabla^*B, B'\rangle\,\mathrm dV$; see [15]; thus,

$$\int_M\langle B, \mathrm{Hess}_u\rangle\,\mathrm dV = \int_M\langle B, \nabla(\nabla u)\rangle\,\mathrm dV = \int_M\langle\nabla^*B, \nabla u\rangle\,\mathrm dV = \int_M u\,(\nabla^*)^2(B)\,\mathrm dV. \tag{2.19}$$

The next lemma generalizes (2.19) and the well-known Green's formula, e.g., [15].

Lemma 5. *If a foliated Riemannian manifold* $(M, g, \mathcal F)$ *satisfies* (2.18)*, then the following formulas are valid for any compactly supported functions* u, f*, and 2-tensor* B*:*

$$\int_M f(\Delta_{\mathcal F}\,u)\,\mathrm dV = \int_M u(\Delta_{\mathcal F}\,f)\,\mathrm dV, \tag{2.20}$$

$$\int_M\langle B, \mathrm{Hess}^{\mathcal F}_u\rangle\,\mathrm dV = \int_M u\,(\nabla^{\mathcal F *})^2(B)\,\mathrm dV. \tag{2.21}$$

Proof. We have $\Delta_{\mathcal F}\,f_2 = \operatorname{div}_{\mathcal F}(\nabla^{\mathcal F} f_2)$. One can show that $\operatorname{div}_{\mathcal F}(PX) = \operatorname{div}(PX) - (\operatorname{div} P)(PX)$ for all $X\in\mathfrak X_M$. Hence, using $\nabla^{\mathcal F} f = P\nabla f$ and (2.18), we get

$$\begin{aligned}f_1\Delta_{\mathcal F}\,f_2 &= f_1\operatorname{div}_{\mathcal F}(\nabla^{\mathcal F} f_2) = f_1\{\operatorname{div}(P\nabla f_2) - (\operatorname{div} P)(P\nabla f_2)\}\\ &= \operatorname{div}(f_1\,P\nabla f_2) - \langle P\nabla f_1, P\nabla f_2\rangle = \operatorname{div}(f_1\,P\nabla f_2) - \langle\nabla^{\mathcal F} f_1, \nabla^{\mathcal F} f_2\rangle.\end{aligned}$$

Using the divergence theorem, gives

$$\int_M f_1(\Delta_{\mathcal F}\,f_2)\,\mathrm dV = -\int_M\langle\nabla^{\mathcal F} f_1, \nabla^{\mathcal F} f_2\rangle\,\mathrm dV. \tag{2.22}$$

By this, the formula (2.20) is true. Next, using (2.18), we will prove

$$\int_M \langle \varphi_1, \nabla^{\mathcal{F}} \varphi_2 \rangle \, \mathrm{d}V = \int_M \langle \nabla^{\mathcal{F}*} \varphi_1, \varphi_2 \rangle \, \mathrm{d}V \tag{2.23}$$

for any compactly supported (s,t)-tensor φ_1 and $(s,t+1)$-tensor φ_2. Define a compactly supported 1-form ω by $\omega(Y) = \langle \iota_Y \varphi_2, \varphi_1 \rangle$ for $Y \in \mathfrak{X}_M$. Take an orthonormal frame (e_i) such that $\nabla_Y e_i = 0$ for all $Y \in T_x M$ and some $x \in M$. To simplify calculations, assume that $s = t = 1$, and then at $x \in M$,

$$\begin{aligned}
-\nabla^{*\mathcal{F}} \omega &= \sum\nolimits_j (\nabla^{\mathcal{F}}_{e_j} \omega)(e_j) = \sum\nolimits_{i,j} \langle P e_j, e_i \rangle (\nabla_{e_i} \omega)(e_j) \\
&= \sum\nolimits_{i,j,c} \langle P e_j, e_i \rangle (\langle \nabla_{e_i} \varphi_2(e_j, e_c), \varphi_1(e_c) \rangle + \langle \varphi_2(e_j, e_c), \nabla_{e_i} \varphi_1(e_c) \rangle) \\
&= \sum\nolimits_{i,j,c} [\langle \langle P e_j, e_i \rangle \nabla_{e_i} \varphi_2(e_j, e_c), \varphi_1(e_c) \rangle + \langle \varphi_2(e_j, e_c), \langle P e_j, e_i \rangle \nabla_{e_i} \varphi_1(e_c) \rangle] \\
&= \sum\nolimits_{j,c} [\langle \nabla^{\mathcal{F}}_{e_j} \varphi_2(e_j, e_c), \varphi_1(e_c) \rangle + \langle \varphi_2(e_j, e_c), \nabla^{\mathcal{F}}_{e_j} \varphi_1(e_c) \rangle] = \langle \varphi_2, \nabla^{\mathcal{F}} \varphi_1 \rangle - \langle \nabla^{*\mathcal{F}} \varphi_2, \varphi_1 \rangle.
\end{aligned}$$

The $\nabla^{\mathcal{F}*}$ is related to the $\mathcal{F}$-divergence of a vector field $\omega^\sharp$ by $\operatorname{div}_{\mathcal{F}} \omega^\sharp = -\nabla^{\mathcal{F}*} \omega$. By the above and $\int_M (\operatorname{div}_{\mathcal{F}} \omega^\sharp) \, \mathrm{d}V = \int_M \operatorname{div}(P\omega^\sharp) \, \mathrm{d}V = 0$, we obtain (2.23). Applying this twice, we get (2.21). □

3. Main results

In Section 3.1, we find the Euler-Lagrange equations (and first variations) for (1.5)–(1.7), and in Section 3.2, we find the second variations of (1.5) and (1.6). First, we check the conformity of (1.7).

Theorem 1. *The functional $W^{\rm conf}_{n,s,r}$ is a conformal invariant of a foliated hypersurface $(M, \mathcal{F})$ in a Riemannian manifold $(\bar{M}, \bar{g})$.*

Proof. Define a new Riemannian metric on $\bar{M}$ by $\bar{g}^c = \mu^2 \bar{g}$ for some positive function $\mu \in C^3(\bar{M})$. Then $g^c = \mu^2 g$ is the new induced metric on M; thus, the new volume form of M is $\mathrm{d}V^c = \mu^n \, \mathrm{d}V$. If X is a $\bar{g}$-unit vector, then $X^c = X/\mu$ is a g^c-unit vector. By the well known formula for the Levi–Civita connection, e.g., [13], we get $2 \bar{\nabla}^c_X Y = 2 \bar{\nabla}_X Y + \mu^{-2}(X(\mu^2) Y + Y(\mu^2) X - \langle X, Y \rangle \bar{\nabla} \mu^2)$. By this, with $X \in T\mathcal{F}$ and $Y = \mathbf{N}^c$, the operators A and A^c are related by $A^c = \frac{1}{\mu}(A - \frac{1}{\mu} \langle \bar{\nabla} \mu, N \rangle \, \mathrm{id}_{T\mathcal{F}})$, see also [13]. By the above and $A_{\mathcal{F}} = PAP$, $A^c_{\mathcal{F}} = PA^cP$, we get

$$A^c_{\mathcal{F}} = \frac{1}{\mu}(A_{\mathcal{F}} - \frac{1}{\mu} \langle \bar{\nabla} \mu, N \rangle \, \mathrm{id}_{T\mathcal{F}}), \quad H^c_{\mathcal{F}} = \frac{1}{s} \operatorname{trace} A^c_{\mathcal{F}} = \frac{1}{\mu}(H_{\mathcal{F}} - \frac{1}{\mu} \langle \bar{\nabla} \mu, N \rangle).$$

Set $B_{\mathcal{F}} = H_{\mathcal{F}} \, \mathrm{id}_{T\mathcal{F}} - A_{\mathcal{F}}$. Let λ^B_i be the eigenvalues of $B_{\mathcal{F}}$ on $\mathcal{F}$ and σ^B_r be the elementary symmetric functions of $B_{\mathcal{F}}$. Obviously, $B^c_{\mathcal{F}} = \frac{1}{\mu} B_{\mathcal{F}}$ holds; hence, $\lambda^{B,c}_i = \frac{1}{\mu} \lambda^B_i$. One can show that $Q^{\mathcal{F}}_r = -\sigma^B_r / C^s_r$ is true; see [9]. By the above, $\mu^r Q^{\mathcal{F},c}_r = Q^{\mathcal{F}}_r$ holds. Hence, $(Q^{\mathcal{F}}_r)^{n/r} \mathrm{d}V$ is a conformal invariant of $(M, \mathcal{F})$ in $(\bar{M}, \bar{g})$: $(Q^{\mathcal{F},c}_r)^{n/r} \mathrm{d}V^c = (Q^{\mathcal{F}}_r)^{n/r} \mathrm{d}V$. Note that if $A_{\mathcal{F}}$ is a conformal operator on $T\mathcal{F}$ (i.e., proportional to $\mathrm{id}_{T\mathcal{F}}$), then $B_{\mathcal{F}} = 0$, hence, $Q^{\mathcal{F}}_r = 0$. □

3.1. The first variation

We can state our main theorem.

Theorem 2. *If* (2.18) *is valid, then Euler-Lagrange equations for the functionals* (1.5) *are*

$$\sum\nolimits_{r=1}^{s}\{(\nabla^{\mathcal{F}*})^2(F'_r\cdot T_{r-1}(A_{\mathcal{F}}))+F'_r(\sigma^{\mathcal{F}}_1\sigma^{\mathcal{F}}_{r-1}-(r+1)\sigma^{\mathcal{F}}_{r+1}+\langle T_{r-1}(A_{\mathcal{F}}),A^2_{\rm mix}\rangle)\}-nFH=0, \qquad (3.1a)$$

$$\sum\nolimits_{i=1}^{s}\frac{1}{i}\{(\nabla^{\mathcal{F}*})^2(F'_i\cdot A^{i-1}_{\mathcal{F}})+F'_i(\tau^{\mathcal{F}}_{i+1}+\langle h^{i-1}_{\mathcal{F}},h^2_{\rm mix}\rangle)\}-nFH=0. \qquad (3.1b)$$

Proof. Using (2.6), we get the following:

$$\delta\,WF_{n,s}=\int_M\delta\,(F(\sigma^{\mathcal{F}}_1,\ldots,\sigma^{\mathcal{F}}_s)\,{\rm d}V)=\int_M\Big\{\sum\nolimits_{r=1}^{s}F'_r\cdot\delta\,\sigma^{\mathcal{F}}_r-n\,uFH\Big\}{\rm d}V,$$
$$\delta\,JF_{n,s}=\int_M\delta\,(F(\tau^{\mathcal{F}}_1,\ldots,\tau^{\mathcal{F}}_s)\,{\rm d}V)=\int_M\Big\{\sum\nolimits_{i=1}^{s}F'_i\cdot\delta\,\tau^{\mathcal{F}}_i-n\,uFH\Big\}{\rm d}V. \qquad (3.2)$$

From (3.2) and (2.8a,b), we find the first variations of functionals (1.5):

$$\delta\,WF_{n,s}=\int_M\Big\{\sum\nolimits_{r=1}^{s}\langle F'_r\cdot T_{r-1}(A_{\mathcal{F}}),{\rm Hess}^{\mathcal{F}\sharp}_u\rangle \qquad (3.3a)$$
$$+\,u\sum\nolimits_{r=1}^{s}F'_r(\sigma^{\mathcal{F}}_1\sigma^{\mathcal{F}}_{r-1}-(r+1)\sigma^{\mathcal{F}}_{r+1}+\langle T_{r-1}(A_{\mathcal{F}}),A^2_{\rm mix}\rangle)-n\,uFH\Big\}{\rm d}V,$$
$$\delta\,JF_{n,s}=\int_M\Big\{\sum\nolimits_{i=1}^{s}\frac{1}{i}\,F'_i(\langle h^{i-1}_{\mathcal{F}},{\rm Hess}^{\mathcal{F}}_u\rangle+u\,(\tau^{\mathcal{F}}_{i+1}+\langle h^{i-1}_{\mathcal{F}},h^2_{\rm mix}\rangle))-n\,uFH\Big\}{\rm d}V. \qquad (3.3b)$$

From (3.3a,b), using (2.18) and (2.20), we obtain (3.1a,b). □

Equations (3.4) and (3.5) of the next statement follow from Theorem 2, but we will prove them.

Corollary 1. *If* (2.18) *is valid, then the Euler-Lagrange equations for the functionals* $W_{n,p,s}$, $J_{n,p,s}$, *see* (1.6)*, and* $W^{\rm conf}_{n,s,2}$*, see* (1.7)*, are, respectively, the following:*

$$\Delta_{\mathcal{F}}\,(H^{p-1}_{\mathcal{F}})+H^{p-1}_{\mathcal{F}}(\|h_{\mathcal{F}}\|^2-\|h_{\rm mix}\|^2-\frac{n\,s}{p}\,HH_{\mathcal{F}})=0, \qquad (3.4)$$

$$(\nabla^{\mathcal{F}*})^2(\|h_{\mathcal{F}}\|^{p-2}h_{\mathcal{F}})+\|h_{\mathcal{F}}\|^{p-2}(\langle h_{\mathcal{F}},\ h^2_{\mathcal{F}}+h^2_{\rm mix}\rangle-\frac{n}{p}\,\|h_{\mathcal{F}}\|^2\,H)=0, \qquad (3.5)$$

$$\Delta_{\mathcal{F}}\,((Q^{\mathcal{F}}_2)^{n/2-1}\sigma^{\mathcal{F}}_1)-\frac{s}{s-1}\,(\nabla^{\mathcal{F}*})^2((Q^{\mathcal{F}}_2)^{n/2-1}T_1(A_{\mathcal{F}}))+\{\sigma^{\mathcal{F}}_1(\sigma^{\mathcal{F}}_1-2\,\sigma^{\mathcal{F}}_2+\|A_{\rm mix}\|^2)$$
$$-\,\frac{s}{s-1}\,(\sigma^{\mathcal{F}}_1\sigma^{\mathcal{F}}_2-3\,\sigma^{\mathcal{F}}_3+\langle T_1(A_{\mathcal{F}}),\ A^2_{\rm mix}\rangle)-s^2Q^{\mathcal{F}}_2\,H\}(Q^{\mathcal{F}}_2)^{n/2-1}=0. \qquad (3.6)$$

Proof. Using (2.6), (2.9), and (2.10), we calculate the variation

$$\delta\,(H^p_{\mathcal{F}}\,{\rm d}V)=H^{p-1}_{\mathcal{F}}\Big\{\frac{p}{s}\,(\Delta_{\mathcal{F}}\,u+u\,(\|h_{\mathcal{F}}\|^2-\|h_{\rm mix}\|^2))-nuHH_{\mathcal{F}}\Big\}{\rm d}V,$$
$$\delta\,(\|h_{\mathcal{F}}\|^p\,{\rm d}V)=\|h_{\mathcal{F}}\|^{p-2}\{p\,\langle h_{\mathcal{F}},\ u\,(h^2_{\mathcal{F}}+h^2_{\rm mix})+{\rm Hess}^{\mathcal{F}}_u\rangle-n\,u\,\|h_{\mathcal{F}}\|^2\,H\}{\rm d}V.$$

Hence, using (2.18), (2.20), (2.21), and (2.6), we find the first variation of the actions (1.6):

$$\delta\,W_{n,p,s}=\int_M\delta\,(H^p_{\mathcal{F}}\,{\rm d}V)=\int_M H^{p-1}_{\mathcal{F}}\Big\{\frac{p}{s}\,(\Delta_{\mathcal{F}}\,u+u\,(\|h_{\mathcal{F}}\|^2-\|h_{\rm mix}\|^2))-n\,u\,H_{\mathcal{F}}H\Big\}{\rm d}V$$
$$=\frac{p}{s}\int_M u\Big\{\Delta_{\mathcal{F}}(H^{p-1}_{\mathcal{F}})+H^{p-1}_{\mathcal{F}}\,(\|h_{\mathcal{F}}\|^2-\|h_{\rm mix}\|^2-\frac{n\,s}{p}\,HH_{\mathcal{F}})\Big\}{\rm d}V, \qquad (3.7)$$

$$\delta J_{n,p,s} = \int_M \|h_{\mathcal F}\|^{p-2}\big\{p\,\langle h_{\mathcal F},\, u\,(h^2_{\mathcal F} + h^2_{\rm mix}) + {\rm Hess}^{\mathcal F}_u\rangle - n\,u\,\|h_{\mathcal F}\|^2\,H\big\}{\rm d}V$$
$$= \int_M u\big\{p\,(\nabla^{\mathcal F *})^2(\|h_{\mathcal F}\|^{p-2}h_{\mathcal F}) + p\,\|h_{\mathcal F}\|^{p-2}(\langle h_{\mathcal F}, h^2_{\mathcal F} + h^2_{\rm mix}\rangle - n\,\|h_{\mathcal F}\|^2\,H)\big\}{\rm d}V. \tag{3.8}$$

From (3.7) and (3.8), the Euler-Lagrange equations (3.4) and (3.5) follow. By (2.8b) we get

$$\delta\,\sigma^{\mathcal F}_1 = \Delta_{\mathcal F}\,u + u(\sigma^{\mathcal F}_1 - 2\,\sigma^{\mathcal F}_2 + \|A_{\,\rm mix}\|^2,$$
$$\delta\,\sigma^{\mathcal F}_2 = \langle T_1(A_{\mathcal F}), {\rm Hess}^{\mathcal F\sharp}_u\rangle + u(\sigma^{\mathcal F}_1\sigma^{\mathcal F}_2 - 3\,\sigma^{\mathcal F}_3 + \langle T_1(A_{\mathcal F}),\, A^2_{\rm mix}\rangle).$$

Using $Q^{\mathcal F}_2 = \frac{1}{s^2(s-1)}\,((s-1)(\sigma^{\mathcal F}_1)^2 - 2\,s\,\sigma^{\mathcal F}_2)$ and (2.8b) for $r = 1, 2$, we get

$$s^2(s-1)\delta\,Q^{\mathcal F}_2 = 2(s-1)\sigma^{\mathcal F}_1\,\delta\,\sigma^{\mathcal F}_1 - 2\,s\,\delta\,\sigma^{\mathcal F}_2 = 2(s-1)\sigma^{\mathcal F}_1\,(\Delta_{\mathcal F}\,u + u(\sigma^{\mathcal F}_1 - 2\,\sigma^{\mathcal F}_2 + \|A_{\,\rm mix}\|^2))$$
$$- 2\,s(\langle T_1(A_{\mathcal F}), {\rm Hess}^{\mathcal F\sharp}_u\rangle + u\,(\sigma^{\mathcal F}_1\sigma^{\mathcal F}_2 - 3\,\sigma^{\mathcal F}_3 + \langle T_1(A_{\mathcal F}),\, A^2_{\rm mix}\rangle)).$$

Hence

$$\delta\,W^{\rm conf}_{n,s,2} = \frac{n}{2}\int_M (Q^{\mathcal F}_2)^{n/2-1}\Big\{\frac{1}{s^2(s-1)}[2(s-1)\sigma^{\mathcal F}_1\,(\Delta_{\mathcal F}\,u + u(\sigma^{\mathcal F}_1 - 2\,\sigma^{\mathcal F}_2 + \|A_{\,\rm mix}\|^2))$$
$$- 2\,s(\langle T_1(A_{\mathcal F}), {\rm Hess}^{\mathcal F\sharp}_u\rangle + u\,(\sigma^{\mathcal F}_1\sigma^{\mathcal F}_2 - 3\,\sigma^{\mathcal F}_3 + \langle T_1(A_{\mathcal F}), A^2_{\rm mix}\rangle))] - 2\,Q^{\mathcal F}_2\,uH\Big\}\,{\rm d}V. \tag{3.9}$$

Using (2.20) with $f = (Q^{\mathcal F}_2)^{\frac{n}{2}-1}\sigma^{\mathcal F}_1$ and (2.21) with $B = (Q^{\mathcal F}_2)^{\frac{n}{2}-1}T_1(A_{\mathcal F})$ in (3.9), we get (3.6). □

Remark 3. (i) For a hypersurface $M \subset \mathbb{R}^{n+1}$ equipped with a line field (i.e., $s = 1$) of the normal curvature κ, the functionals (1.5) and (1.6) coincide with $WF_{n,1} = \int_M F(\kappa)\,{\rm d}V$ and $W_{n,p,1} = \int_M \kappa^p\,{\rm d}V$. For $W_{n,2,1}$ and $J_{n,2,1}$, from (3.4) and (3.5) with $p = 2$ and $s = 1$, using $(\nabla^{\mathcal F *})^2 h_{\mathcal F} = \Delta_{\mathcal F}\,\kappa$, we get the following leaf-wise elliptic PDE: $\Delta_{\mathcal F}\,\kappa + (\kappa^2 - \|h_{\rm mix}\|^2 - \frac{n}{2}\,H\kappa)\,\kappa = 0$.

(ii) The first variation of the functional $W^{\rm conf}_{n.s,r}$ and the Euler-Lagrange equation can be obtained from (3.1a) and (3.3a), similarly to the corresponding equations in [9] for $W^{\rm conf}_{n.r}$.

(iii) By (3.7) and (3.8) with $s = n$, the first variations of functionals (1.1) are given by

$$\delta\,W_{n,p} = \int_M H^{p-1}\Big\{\frac{p}{n}\,(\Delta\,u + u\,\|h\|^2) - nuH^2\Big\}{\rm d}V,$$
$$\delta\,J_{n,p} = \int_M \|h\|^{p-2}\big\{p\,\langle h,\ u\,h^2 + {\rm Hess}_u\rangle - n\,u\,\|h\|^2 H\big\}{\rm d}V.$$

The corresponding Euler-Lagrange equations are well known:

$$\Delta\,H^{p-1} + H^{p-1}\,(\|h\|^2 - \frac{n^2}{p}\,H^2) = 0, \tag{3.10}$$
$$(\nabla^*)^2(\|h\,\|^{p-2}h) + \|h\,\|^{p-2}(\langle h, h^2\rangle - \frac{n}{p}\,\|h\,\|^2 H) = 0, \tag{3.11}$$

for example, [3], where $n = 2$ and $M^2 \subset \mathbb{R}^3$. For $p = n = 2$, we can use the identity $\|h\|^2 - 2\,H^2 = \frac{1}{2}\,(k_1 - k_2)^2 = 2\,(H^2 - K)$, where k_1 and k_2 are the principal curvatures, $H = (k_1 + k_2)/2$, and $K = k_1 k_2$ is the gaussian curvature of a surface $M^2 \subset \mathbb{R}^3$. In this case, the Euler-Lagrange equation (3.10) reduces to (1.2). Using the identity $\langle h, h^2\rangle = 8\,H^3 - 6\,HK$, the Euler-Lagrange equation (3.11) for $p = n = 2$ reads as $(\nabla^*)^2 h + 4\,H(H^2 - K) = 0$.

3.2. The second variation

The following statement generalizes Corollary 1 in [3] when $M^2 \subset \mathbb{R}^3$.

Theorem 3. *If* (2.18) *is valid, then the Euler-Lagrange equation for* $WF_{n,s}$ *of* (1.5) *with* $F = F(H_{\mathcal F})$ *is*

$$\Delta_{\mathcal F} F' + F'(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - s\, n\, F H = 0. \tag{3.12}$$

At a critical hypersurface satisfying (2.18)*, the second variation of* $WF_{n,s}$ *with* $F = F(H_{\mathcal F})$ *is*

$$\begin{aligned}
\delta^2 WF_{n,s} &= -\int_M \frac{n}{s}\big\{F'\Delta_{\mathcal F}\, u - u\,\Delta_{\mathcal F} F'\big\} uH\,{\rm d}V + \int_M \big\{\frac{F'}{s}\,(2\,u\langle h_{\mathcal F}, {\rm Hess}^{\mathcal F}_u\rangle + s\,u\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} u\rangle\\
&+ 2\,h(\nabla^{\mathcal F} u, \nabla^{\mathcal F} u) - s\,H_{\mathcal F}\|\nabla^{\mathcal F} u\|^2) + \frac{F''}{s^2}\,\Delta_{\mathcal F}\, u(\Delta_{\mathcal F}\, u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2))\big\}{\rm d}V\\
&+ \int_M u\big\{(\frac{F''}{s^2}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - \frac{n}{s}\,HF')(\Delta_{\mathcal F}\, u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2)) - F(\Delta\, u + u\|h\|^2)\\
&+ \frac{F'}{s}\,(2\,\langle h_{\mathcal F},\ u\,(h^2_{\mathcal F} + h^2_{\rm mix}) + {\rm Hess}^{\mathcal F}_u\rangle - u\,\langle h_{\mathcal F} + h_{\mathcal F^\perp}, h^2_{\rm mix}\rangle - 2\,\langle h_{\rm mix}, {\rm Hess}^{\rm mix}_u\rangle)\big\}{\rm d}V.
\end{aligned} \tag{3.13}$$

Proof. By (3.3a) with $F = F(\sigma^{\mathcal F}_1/s)$, using $\langle {\rm id}_{TM}, h^2_{\rm mix}\rangle = \|h_{\rm mix}\|^2$ and $\langle {\rm id}_{T\mathcal F}, {\rm Hess}^{\mathcal F}_u\rangle = \Delta_{\mathcal F}\, u$, we find the first variation of the functional $WF_{n,s}$ with $F = F(H_{\mathcal F})$, see (1.5):

$$\delta\, WF_{n,s} = \int_M \big\{\frac{F'}{s}\,\Delta_{\mathcal F}\, u + (\frac{F'}{s}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - n\,FH)\,u\big\}{\rm d}V = 0. \tag{3.14}$$

If (2.18) is valid, then using (3.14) and (2.20), we obtain (3.12). Our next aim is to calculate

$$\begin{aligned}
\delta^2 WF_{n,s} &= \delta\int_M \big\{\frac{F'}{s}\,\Delta_{\mathcal F}\, u + (\frac{F'}{s}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - n\,FH)\,u\big\}{\rm d}V\\
&= -\int_M \big\{\frac{F'}{s}\,\Delta_{\mathcal F}\, u + (\frac{F'}{s}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - n\,FH)\,u\big\} n\,uH\,{\rm d}V\\
&+ \int_M \delta\big(\frac{F'}{s}\,\Delta_{\mathcal F}\, u\big){\rm d}V + \int_M \delta\big\{\big(\frac{F'}{s}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - n\,FH\big)\,u\big\}{\rm d}V.
\end{aligned} \tag{3.15}$$

For the first integral in the last line of (3.15), using (2.9), (2.12), and $\delta u = 0$, we get

$$\begin{aligned}
\int_M \delta\big(\frac{F'}{s}\,\Delta_{\mathcal F}\, u\big){\rm d}V &= \int_M \big\{\frac{F'}{s}\,(2\,u\langle h_{\mathcal F}, {\rm Hess}^{\mathcal F}_u\rangle + s\,u\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} u\rangle\\
&+ 2\,h(\nabla^{\mathcal F} u, \nabla^{\mathcal F} u) - sH_{\mathcal F}\langle\nabla^{\mathcal F} u, \nabla^{\mathcal F} u\rangle) + \frac{F''}{s^2}\,\Delta_{\mathcal F}\, u(\Delta_{\mathcal F}\, u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2))\big\}{\rm d}V.
\end{aligned} \tag{3.16}$$

For the second integral in the last line of (3.15), using (2.11), we get

$$\begin{aligned}
&\int_M \delta\big\{\big(\frac{F'}{s}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - n\,FH\big)\,u\big\}{\rm d}V\\
&= \int_M u\big\{(\frac{F''}{s^2}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - \frac{nF'}{s}\,H)(\Delta_{\mathcal F}\, u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2)) - F(\Delta\, u + u\|h\|^2)\\
&+ \frac{F'}{s}\,(2\,\langle h_{\mathcal F},\ u\,(h^2_{\mathcal F} + h^2_{\rm mix}) + {\rm Hess}^{\mathcal F}_u\rangle - u\,\langle h_{\mathcal F} + h_{\mathcal F^\perp}, h^2_{\rm mix}\rangle - 2\,\langle h_{\rm mix}, {\rm Hess}^{\rm mix}_u\rangle)\big\}{\rm d}V.
\end{aligned} \tag{3.17}$$

By (3.15), (3.16), and (3.17), noting that the first variation vanishes at a critical immersion, we get

$$
\begin{aligned}
\delta^2 WF_{n,s} = &-\int_M \frac{n}{s}\big\{F'\Delta_{\mathcal F}\, u + (F'(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - s\, n\, FH)\, u\big\} uH\,{\rm d}V\\
&+\int_M \big\{\frac{F'}{s}\,(2\, u\langle h_{\mathcal F}, {\rm Hess}^{\mathcal F}_u\rangle + s\, u\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} u\rangle + 2\, h(\nabla^{\mathcal F} u, \nabla^{\mathcal F} u) - s\, H_{\mathcal F}\|\nabla^{\mathcal F} u\|^2)\\
&+\frac{1}{s^2}\, F''\Delta_{\mathcal F}\, u(\Delta_{\mathcal F}\, u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2))\big\}\,{\rm d}V\\
&+\int_M u\big\{(\frac{F''}{s^2}\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - \frac{nF'}{s}\, H)(\Delta_{\mathcal F}\, u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2)) - F(\Delta\, u + u\|h\|^2)\\
&+\frac{F'}{s}\,(2\,\langle h_{\mathcal F},\ u\,(h^2_{\mathcal F} + h^2_{\rm mix}) + {\rm Hess}^{\mathcal F}_u\rangle - u\,\langle h_{\mathcal F} + h_{\mathcal F^\perp}, h^2_{\rm mix}\rangle - 2\,\langle h_{\rm mix}, {\rm Hess}^{\,\rm mix}_u\rangle)\big\}{\rm d}V. \qquad (3.18)
\end{aligned}
$$

From (3.18) and (3.12), at a critical hypersurface, we get (3.13). □

Similarly, one can obtain the Euler-Lagrange equation for the functional $JF_{n,s}$ with $F = F(\|h_{\mathcal F}\|)$, see (1.5), but we do not present it here. From Theorem 3, with $F = H^p_{\mathcal F}$, we obtain the following.

Corollary 2. *At a critical hypersurface satisfying* (2.18)*, the second variation of the action* $W_{n,p,s}$ *in* (1.6) *is*

$$
\begin{aligned}
\delta^2 W_{n,p,s} = &-\int_M \frac{np}{s}\big\{H^{p-1}_{\mathcal F}\Delta_{\mathcal F}\, u - u\,\Delta_{\mathcal F}\,(H^{p-1}_{\mathcal F})\big\} uH\,{\rm d}V + \int_M \frac{p}{s}\, H^{p-2}_{\mathcal F}\big\{H_{\mathcal F}(2\, u\langle h_{\mathcal F}, {\rm Hess}^{\mathcal F}_u\rangle\\
&+ s\, u\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} u\rangle + 2h(\nabla^{\mathcal F} u, \nabla^{\mathcal F} u) - sH_{\mathcal F}\|\nabla^{\mathcal F} u\|^2) + \frac{p-1}{s}\Delta_{\mathcal F} u(\Delta_{\mathcal F} u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2))\big\}{\rm d}V\\
&+\int_M H^{p-2}_{\mathcal F} u\big\{\frac{p}{s}\big(\frac{p-1}{s}(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2) - n\, HH_{\mathcal F}\big)(\Delta_{\mathcal F}\, u + u\,(\|h_{\mathcal F}\|^2 - \|h_{\rm mix}\|^2)) - H^2_{\mathcal F}(\Delta\, u + u\|h\|^2)\\
&+\frac{p}{s}\, H_{\mathcal F}(2\,\langle h_{\mathcal F}, u\,(h^2_{\mathcal F} + h^2_{\rm mix}) + {\rm Hess}^{\mathcal F}_u\rangle - u\,\langle h_{\mathcal F} + h_{\mathcal F^\perp}, h^2_{\rm mix}\rangle - 2\,\langle h_{\rm mix}, {\rm Hess}^{\,\rm mix}_u\rangle)\big\}{\rm d}V. \qquad (3.19)
\end{aligned}
$$

Proof. Substituting $F' = p\, H^{p-1}_{\mathcal F}$ and $F'' = p(p-1)\, H^{p-2}_{\mathcal F}$ in (3.13), we obtain (3.19). □

Remark 4. By (3.18) with $s = n$, the second variation of the action $WF_n = \int_M F(H)\,{\rm d}V$ is

$$
\begin{aligned}
\delta^2 WF_n = &-\int_M \big\{F'\,\Delta\, u + (F'\|h\|^2 - n^2\, FH)\, u\big\} u\, H\,{\rm d}V + \int_M \big\{\frac{F'}{n}(2\, u\langle h, {\rm Hess}_u\rangle + n\, u\langle\nabla\, H, \nabla u\rangle\\
&+ 2\, h(\nabla\, u, \nabla u) - nH\,\|\nabla u\|^2) + \frac{F''}{n^2}\Delta\, u(\Delta\, u + u\,\|h\|^2)\big\}{\rm d}V\\
&+\int_M u\big\{(\frac{F''}{n^2}\,\|h\|^2 - HF' - F)(\Delta\, u + u\,\|h\|^2) + \frac{2\,F'}{n}\,\langle h,\ uh^2 + {\rm Hess}_u\rangle\big\}{\rm d}V. \qquad (3.20)
\end{aligned}
$$

This is compatible with a special case of Eq (7) in [3] for $n = 2$. As a special case of (3.20), the second variation of the functional $W_{n,p}$ in (1.1) with $n = 2$ has the following form compatible with [3]:

$$
\begin{aligned}
\delta^2 W_{2,p} = &\int_M H^{p-2}\big\{\frac{p(p-1)}{4}\,(\Delta\, u)^2 + p\, H(h(\nabla\, u, \nabla\, u) + 2\, u\langle h, {\rm Hess}_u\rangle + u\langle\nabla\, H, \nabla u\rangle - H\|\nabla u\|^2)\\
&+ ((2p^2 - 4p - 1)H^2 - p(p-1)K)u\Delta\, u + (4p(p-1)H^4 - 2(p-1)(2p+1)KH^2 + p(p-1)K^2)u^2\big\}{\rm d}V.
\end{aligned}
$$

4. Applications

We consider critical hypersurfaces equipped with two-dimensional foliations (i.e., $s = 2$) in Section 4.1, and discuss critical hypersurfaces of revolution and their stability in Section 4.2.

4.1. Hypersurfaces with two-dimensional foliations

For $s = 2$, it is natural to present the functionals (1.5) in the following form:

$$WF_{n,2} = \int_M F(H_\mathcal{F}, K_\mathcal{F})\, \mathrm{d}V. \tag{4.1}$$

where $K_\mathcal{F} = \det A_\mathcal{F} = \sigma_2^\mathcal{F}$ is the Gaussian curvature of the leaves. For $n = s = 2$, (4.1) reduces to the functional $WF_2 = \int_M F(H, K)\, \mathrm{d}V$ seen in [3]. The following equalities are true:

$$\|h_\mathcal{F}\|^2 = k_{\mathcal{F},1}^2 + k_{\mathcal{F},2}^2 = 4\, H_\mathcal{F}^2 - 2\, K_\mathcal{F}, \quad \langle h_\mathcal{F},\ h_\mathcal{F}^2\rangle = k_{\mathcal{F},1}^3 + k_{\mathcal{F},2}^3 = 8\, H_\mathcal{F}^3 - 6\, H_\mathcal{F} K_\mathcal{F}.$$

From (2.9) and (2.10) with $s = 2$, we obtain the following evolution equations:

$$\delta\, (2\, H_\mathcal{F}) = \Delta_\mathcal{F}\, u + u\, (4\, H_\mathcal{F}^2 - 2\, K_\mathcal{F} - \|h_{\rm mix}\|^2), \tag{4.2}$$

$$\delta\, \|h_\mathcal{F}\|^2 = 4\, u\, (4\, H_\mathcal{F}^3 - 3\, H_\mathcal{F} K_\mathcal{F}) + 2\, \langle h_\mathcal{F},\ u\, h_{\rm mix}^2 + {\rm Hess}_u^\mathcal{F}\rangle. \tag{4.3}$$

Using (4.2) and (4.3) in $\delta\, K_\mathcal{F} = 2\, H_\mathcal{F}\, \delta\, (2\, H_\mathcal{F}) - (1/2)\delta\, (\|h_\mathcal{F}\|^2)$, we get the evolution equation

$$\delta\, K_\mathcal{F} = 2\, H_\mathcal{F}\, \Delta_\mathcal{F}\, u - \langle h_\mathcal{F},\ u\, h_{\rm mix}^2 + {\rm Hess}_u^\mathcal{F}\rangle + 2\, u\, H_\mathcal{F}\, (K_\mathcal{F} - \|h_{\rm mix}\|^2). \tag{4.4}$$

For $n = 2$, (4.2)–(4.4) reduce to the equations in [3].

The next statement for $(M^n, \mathcal{F}^2)$ immersed in $\mathbb{R}^{n+1}$ generalizes Theorem 1 in [3] with $n = 2$.

Theorem 4. *If* (2.18) *is valid, then the Euler–Lagrange equation for the action* (4.1) *with* $s = 2$ *is*

$$\begin{aligned}&\Delta_\mathcal{F}\Big(\frac{1}{2}\, F'_H + 2\, H_\mathcal{F} F'_K\Big) - (\nabla^{\mathcal{F}*})^2 (F'_K h_\mathcal{F}) + F'_H (2\, H_\mathcal{F}^2 - K_\mathcal{F} - \frac{1}{2}\, \|h_{\rm mix}\|^2)\\ &\quad + F'_K (2\, H_\mathcal{F}\, (K_\mathcal{F} - \|h_{\rm mix}\|^2) - \langle h_\mathcal{F},\ h_{\rm mix}^2\rangle) - n\, F H = 0,\end{aligned} \tag{4.5}$$

where F'_H, F'_K *denote partial derivatives of* $F(H_\mathcal{F}, K_\mathcal{F})$ *with respect to* $H_\mathcal{F}$ *and* $K_\mathcal{F}$*. At a critical hypersurface foliated by surfaces* ($s = 2$) *and satisfying* (2.18)*, the second variation of the functional* (4.1) *with* $F = F(H_\mathcal{F})$ *is given by*

$$\begin{aligned}\delta^2 WF_{n,2} = &-\int_M \frac{n}{2} \big\{F' \Delta_\mathcal{F}\, u - u\, \Delta_\mathcal{F}\, F'\big\} u H\, \mathrm{d}V + \int_M \Big\{\frac{F'}{2} (2\, u \langle h_\mathcal{F}, {\rm Hess}_u^\mathcal{F}\rangle + 2\, u \langle \nabla^\mathcal{F} H_\mathcal{F}, \nabla^\mathcal{F} u\rangle\\ &+ 2\, h(\nabla^\mathcal{F} u, \nabla^\mathcal{F} u) - 2\, H_\mathcal{F} \|\nabla^\mathcal{F} u\|^2) + \frac{F''}{4}\, \Delta_\mathcal{F}\, u \big(\Delta_\mathcal{F}\, u + u\, (4\, H_\mathcal{F}^2 - 2\, K_\mathcal{F} - \|h_{\rm mix}\|^2)\big)\Big\} \mathrm{d}V\\ &+ \int_M u \Big\{\Big(\frac{F''}{4}\, (4\, H_\mathcal{F}^2 - 2\, K_\mathcal{F} - \|h_{\rm mix}\|^2) - \frac{nF'}{2}\, H\Big) (\Delta_\mathcal{F}\, u + u\, (4\, H_\mathcal{F}^2 - 2\, K_\mathcal{F} - \|h_{\rm mix}\|^2))\\ &+ \frac{F'}{2} (4\, u\, H_\mathcal{F} (4 H_\mathcal{F}^2 - 3\, K_\mathcal{F}) + 2\, u\, \langle h_\mathcal{F}, h_{\rm mix}^2\rangle + 2\, \langle h_\mathcal{F}, {\rm Hess}_u^\mathcal{F}\rangle - F(\Delta\, u + u \|h\|^2)\\ &- u \langle h_\mathcal{F} + h_{\mathcal{F}^\perp}, h_{\rm mix}^2\rangle - 2 \langle h_{\rm mix}, {\rm Hess}_u^{\rm mix}\rangle)\Big\} \mathrm{d}V.\end{aligned} \tag{4.6}$$

Proof. Using (4.2) and (4.4) in $\delta F(H_{\mathcal F}, K_{\mathcal F}) = \frac12 F'_H \cdot \delta(2H_{\mathcal F}) + F'_K\,\delta K_{\mathcal F}$, and applying (2.6), we calculate the first variation of the functional (4.1) with $s = 2$:

$$\begin{aligned}\delta WF_{n,2} &= \int_M \delta(F(H_{\mathcal F}, K_{\mathcal F})\,\mathrm{d}V) = \int_M \Big\{\frac12 F'_H(\Delta_{\mathcal F} u + u(4H^2_{\mathcal F} - 2K_{\mathcal F} - \|h_{\rm mix}\|^2))\\ &+ F'_K(2H_{\mathcal F}\Delta_{\mathcal F} u - \langle h_{\mathcal F},\ u\,h^2_{\rm mix} + \mathrm{Hess}^{\mathcal F}_u\rangle + 2u\,H_{\mathcal F}(K_{\mathcal F} - \|h_{\rm mix}\|^2)) - n\,u\,HF\Big\}\mathrm{d}V\\ &= \int_M \Big\{(\frac12 F'_H + 2H_{\mathcal F}F'_K)\,\Delta_{\mathcal F} u - F'_K\langle h_{\mathcal F},\ \mathrm{Hess}^{\mathcal F}_u\rangle + (F'_H(2H^2_{\mathcal F} - K_{\mathcal F} - \frac12\|h_{\rm mix}\|^2)\\ &+ F'_K(2H_{\mathcal F}(K_{\mathcal F} - \|h_{\rm mix}\|^2) - \langle h_{\mathcal F},\ h^2_{\rm mix}\rangle) - n\,FH)\,u\Big\}\mathrm{d}V. \end{aligned} \tag{4.7}$$

From (4.7), using (2.21), we get (4.5). From Theorem 3 with $s = 2$ we get (4.6). □

Remark 5. Let $F = F(H_{\mathcal F})$, then from (4.5) we obtain

$$\Delta_{\mathcal F}(F'_H) + F'_H(4H^2_{\mathcal F} - 2K_{\mathcal F} - \|h_{\rm mix}\|^2) - 2nFH = 0.$$

From this, with $F = H^2_{\mathcal F}$, or from (3.4), we get the Euler-Lagrange equation for $W_{n,2,2}$, see (1.6):

$$\Delta_{\mathcal F} H_{\mathcal F} + (4H^2_{\mathcal F} - 2K_{\mathcal F} - \|h_{\rm mix}\|^2 - n\,HH_{\mathcal F})H_{\mathcal F} = 0.$$

The following particular case of (4.6), or Corollary 2 with $s = 2$, is true.

Corollary 3. *At a critical hypersurface satisfying* (2.18)*, the second variation of $W_{n,p,2}$ is*

$$\begin{aligned}\delta^2 W_{n,p,2} &= -\int_M \frac{np}{2}\Big\{H^{p-1}_{\mathcal F}\Delta_{\mathcal F} u - u\,\Delta_{\mathcal F}(H^{p-1}_{\mathcal F})\Big\}uH\,\mathrm{d}V + \int_M \frac{p}{2}H^{p-2}_{\mathcal F}\Big\{2H_{\mathcal F}(u\langle h_{\mathcal F}, \mathrm{Hess}^{\mathcal F}_u\rangle\\ &+ u\langle\nabla^{\mathcal F} H_{\mathcal F}, \nabla^{\mathcal F} u\rangle + h(\nabla^{\mathcal F} u, \nabla^{\mathcal F} u) - H_{\mathcal F}\|\nabla^{\mathcal F} u\|^2) + \frac{p-1}{2}\Delta_{\mathcal F} u(\Delta_{\mathcal F} u + u(4H^2_{\mathcal F} - 2K_{\mathcal F} - \|h_{\rm mix}\|^2))\Big\}\mathrm{d}V\\ &+ \int_M H^{p-2}_{\mathcal F} u\Big\{\frac{p}{2}\Big(\frac{p-1}{2}(4H^2_{\mathcal F} - 2K_{\mathcal F} - \|h_{\rm mix}\|^2) - n\,HH_{\mathcal F}\Big)(\Delta_{\mathcal F} u + u(4H^2_{\mathcal F} - 2K_{\mathcal F} - \|h_{\rm mix}\|^2))\\ &- H^2_{\mathcal F}(\Delta u + u\|h\|^2) + \frac{p}{2}H_{\mathcal F}\Big(4u(4H^3_{\mathcal F} - 3H_{\mathcal F}K^2_{\mathcal F}) + 2\langle h_{\mathcal F},\ u\,h^2_{\rm mix} + \mathrm{Hess}^{\mathcal F}_u\rangle\\ &- u\langle h_{\mathcal F} + h_{\mathcal F^\perp}, h^2_{\rm mix}\rangle - 2\langle h_{\rm mix}, \mathrm{Hess}^{\rm mix}_u\rangle\Big)\Big\}\mathrm{d}V.\end{aligned}$$

The following consequence of Theorem 4 was proven for $n = 2$ in [3].

Corollary 4. *The Euler-Lagrange equation for the functional WF_2 with $F = F(H, K)$ is given by*

$$\Delta\Big(\frac12 F'_H + 2HF'_K\Big) - (\nabla^*)^2(F'_K h) + (2H^2 - K)F'_H + 2HKF'_K - n\,HF = 0.$$

4.2. Hypersurfaces of revolution

Hypersurfaces of revolution in Euclidean space $\mathbb{R}^{n+1}$ are naturally foliated into $(n-1)$-spheres (parallels) and equipped with rotationally symmetric metrics $g = d\rho^2 + \rho^2 ds^2_{n-1}$ – a special case of a warped product metric; see [15]. Such a hypersurface can be represented as a graph $x_{n+1} = f(\rho)$, where the function $f \in C^2$ is monotonic, $\rho = \sqrt{x_1^2 + \ldots + x_n^2}$ and (x_i) are Cartesian coordinates in $\mathbb{R}^{n+1}$. We obtain the parametrization $\mathbf{r} = \mathbf{r}(\phi_1, \ldots, \phi_{n-1}; f(\rho))$, of M, where $(\phi_1, \ldots, \phi_{n-1}; \rho)$ are cylindrical coordinates in $\mathbb{R}^{n+1}$. The principal curvatures of M (functions of ρ) are $k_1 = \ldots = k_{n-1} = \frac{f'}{\rho\,(1+(f')^2)^{1/2}} \le \frac{1}{\rho}$ for parallels and $k_n = \frac{f''}{(1+(f')^2)^{3/2}}$ for profile curves (geodesics on M). If the profile curve is a straight line ($f'' \equiv 0$), then $k_n \equiv 0$ and M is a cone, a cylinder, or a hyperplane. To exclude these cases, we will assume $f'' \neq 0$. We get

$$\begin{aligned} &n\,H = (n-1)\,k_1 + k_n, \quad H_{\mathcal{F}} = k_1, \quad \|h_{\mathcal{F}}\|^2 = (n-1)k_1^2, \quad \|h\|^2 = (n-1)k_1^2 + k_n^2, \\ &\langle h, h^2\rangle = (n{-}1)\,k_1^3 + k_n^3, \quad \langle h_{\mathcal{F}}, h_{\mathcal{F}}^2\rangle = (n{-}1)k_1^3, \quad h_{\rm mix} = h_{\rm mix}^2 = 0. \end{aligned} \tag{4.8}$$

Recall that $\lambda_j = j(j + n - 2)$ corresponds to the solutions with multiplicities $N_j = C^n_{n+j} = \frac{(n+j)!}{n!\,j!}$ of the eigenvalue problem $\Delta\,u + \lambda\,u = 0$ on a unit $(n-1)$-sphere. Any constant function on the round sphere spans the space of λ_0-eigenfunctions of the Laplacian. Let $\perp$ denote the orthogonality of functions with respect to the L^2 inner product. The sphere $S^2(\rho)$ of radius ρ in $\mathbb{R}^3$ is not a local minimum of $W_{2,p}$ under volume-preserving deformations for $p > 2$. For $p \ge 1$, $S^2(\rho)$ is a local minimum of $W_{2,p}$ under volume-preserving, nonconstant deformations u provided $u \in \{v : \Delta\,v = (2/\rho^2)v\}^\perp$, see Propositions 2 and 3 in [3]. According to (3.12), a hypersurface of revolution in $\mathbb{R}^{n+1}$ foliated by $(n-1)$-spheres-parallels $\{L_\rho\}$ is a critical point of the action $WF_{n,s}$ with $F = F(H_{\mathcal{F}})$, see (1.5), if and only if

$$(F'/F)\,((n-1)k_1^2 + k_n^2) - s\,n\,((n-1)k_1 + k_n) = 0.$$

In this case, k_n and k_1 are functionally related; hence, M is a Weingarten hypersurface.

The following theorem studies the stability of hypersurfaces of revolution critical for (1.6).

Theorem 5. *A hypersurface of revolution $M : x_{n+1} = f(\rho)$ $(f'' \neq 0)$ in $\mathbb{R}^{n+1}$ foliated by $(n-1)$-spheres-parallels $\{L_\rho\}$ is a critical point of the action $W_{n,p,n-1}$ or $J_{n,p,n-1}$, see* (1.6)*, if and only if*

$$f(\rho) = \int \frac{\sqrt{C_1\rho^{2p-2n+2} - \rho^{4p-4n+4}}}{C_1 - \rho^{2p-2n+2}}\,{\rm d}\rho + C_2, \quad C_1, C_2 \in \mathbb{R}. \tag{4.9}$$

A critical hypersurface is not a local minimum of $W_{n,p,n-1}$ for $p > n \ge 2$ with respect to general variations, but it is a local minimum for variations $u = u(\phi_1, \ldots, \phi_{n-1})$ satisfying $u|_{L_\rho} \perp \ker\Delta_{\mathcal{F}}$.

Proof. 1. Let M be critical for the action $W_{n,p,n-1}$ under general deformations. Since all principal curvatures k_i are constant on parallels, from (3.4) and (3.5), we get, respectively,

$$p\,\|h_{\mathcal{F}}\|^2 - n(n-1)\,HH_{\mathcal{F}} = 0, \quad p\,\langle h_{\mathcal{F}},\, h_{\mathcal{F}}^2\rangle - n\,\|h_{\mathcal{F}}\|^2\,H = 0. \tag{4.10}$$

Using (4.8) in (4.10), yields $k_n = (p - n + 1)\,k_1 \neq 0$, which is the differential equation for $f = f(\rho)$,

$$\rho\,f'' = (p - n + 1)\,f'(1 + (f')^2). \tag{4.11}$$

The solution of (4.11) is given by (4.9).

2. Let $u = u(\phi_1, \ldots, \phi_n)$ be the eigenfunction of Δ on $S^{n-1}(1)$ with the eigenvalue λ_j, then $\Delta_{\mathcal{F}}\,u + \lambda_j\,\rho^{-2}\,u = 0$. Since our hypersurface of revolution (M, g) is a warped product, its volume form is decomposed as ${\rm d}V = {\rm d}V_\rho \cdot d\,\rho$, see [16]. For any function $a(\rho)$ we have, see (2.22),

$$\int_M a\|\nabla^{\mathcal{F}} u\|^2 {\rm d}V = \int_{\rho_1}^{\rho_2} a(\int_{L_\rho} \|\nabla^{\mathcal{F}} u\|^2 {\rm d}V_\rho)d\,\rho = -\int_{\rho_1}^{\rho_2} a(\int_{L_\rho} u\,\Delta_{\mathcal{F}}\,u\;{\rm d}V_\rho)d\,\rho = -\int_M a\,u\,\Delta_{\mathcal{F}}\,u\,{\rm d}V.$$

Using (3.19) with $s = n-1$, Example 1, the equalities $\nabla u = \nabla^{\mathcal F} u$ and

$$\langle \nabla k_1, \nabla u\rangle = 0,\ h(\nabla^{\mathcal F} u, \nabla u) = k_1\|\nabla^{\mathcal F} u\|^2,\ \langle h, \mathrm{Hess}_u\rangle = \langle h_{\mathcal F}, \mathrm{Hess}^{\mathcal F}_u\rangle = (n-1)\, H_{\mathcal F}\Delta_{\mathcal F}\, u,$$

we find the second variation of $W_{n,p,n-1}$:

$$\delta^2 W_{n,p,n-1} = \frac{1}{(n-1)^2}\int_M k_1^{p-2}\{p(p-1)(\Delta_{\mathcal F}\, u)^2 \\ + (n-1)(5\,np - n - 9\,p + 1)k_1^2\, u\,\Delta_{\mathcal F}\, u - (n-1)^2(p-n)(p-n+1)k_1^4\, u^2\}\mathrm{d}V. \tag{4.12}$$

If the variation $u = u(\phi_1, \ldots, \phi_{n-1})$ satisfies $\Delta_{\mathcal F}\, u = 0$, then by (4.12) we get

$$\delta^2 W_{n,p,n-1} = -\int_M (p-n)(p-n+1)k_1^{p+2} u^2\, \mathrm{d}V,$$

which is negative for $p > n$; hence, our critical hypersurface is not a local minimum of $W_{n,p,n-1}$.

Let u satisfy $u|_{L_\rho} \perp \ker\Delta_{\mathcal F}$. Using the inequalities $\int_{S^{n-1}(1)} u\,\Delta\, u\,\mathrm{d}V \ge \int_{S^{n-1}(1)} \lambda_1 u^2\,\mathrm{d}V$ and $\int_{S^{n-1}(1)}(\Delta\, u)^2\,\mathrm{d}V \ge \int_{S^{n-1}(1)} \lambda_1 u^2\,\mathrm{d}V$, see, for example, [3], we get

$$\int_{L_\rho} u\,\Delta_{\mathcal F}\, u\,\mathrm{d}V_{L_\rho} \ge \int_{L_\rho} \lambda_1\rho^{-4} u^2\,\mathrm{d}V_{L_\rho}, \qquad \int_{L_\rho} (\Delta_{\mathcal F}\, u)^2\,\mathrm{d}V_{L_\rho} \ge \int_{L_\rho} \lambda_1\rho^{-4} u^2\,\mathrm{d}V_{L_\rho}.$$

By these estimates, (4.12) and the inequality $\rho^{-4} \ge k_1^2$, we find

$$\delta^2 W_{n,p,n-1} \ge \frac{1}{(n-1)^2}\int_M \Big\{p(p-1)(n-1)^2 + (n-1)^2(5\,np - n - 9\,p + 1) \\ - (n-1)^2(p-n)(p-n+1)\Big\}k_1^{p+2}u^2\mathrm{d}V = \int_M \{n(p-n) + p(6\,n - 11) + 1\}k_1^{p+2}u^2\mathrm{d}V.$$

Hence, $\delta^2 W_{n,p,n-1} > 0$ for all $p \ge n \ge 2$. □

Example 1. (i) Let $M^3 : x_4 = f(\rho)$ $(f'' \ne 0)$, $\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}$ be a hypersurface of revolution in $\mathbb{R}^4$ foliated by parallels (2-spheres). We get $H = \frac{1}{3}(2\,k_1 + k_3)$ and $H_{\mathcal F} = k_1$. Let M^3 be critical for the functional $W_{3,p,2}$ or $J_{3,p,2}$ with $p \ge 3$, see (1.6), under general deformations. From (4.10) we get $k_3 = (p-2)k_1 \ne 0$. The solution to (4.11) is $f(\rho) = \int \frac{\sqrt{C_1\rho^{2p+4} - \rho^{4p}}}{\rho^{2p} - C_1\,\rho^4}\,\mathrm{d}\rho + C_2$, where $C_1, C_2 \in \mathbb{R}$.

(ii) Let $M^2 : x_3 = f(\rho)$ $(f'' \ne 0)$, $\rho = \sqrt{x_1^2 + x_2^2}$ be a surface of revolution in $\mathbb{R}^3$ foliated by parallels (circles). The principal curvatures are $k_1 = \frac{f'}{\rho(1+(f')^2)^{1/2}}$ for parallels and $k_2 = \frac{f''}{(1+(f')^2)^{3/2}}$ for profile curves. Let M^2 be critical for the action $W_{2,p,1}$ or $J_{2,p,1}$ with $p \ge 2$ under general deformations. Then $k_2 = (p-1)k_1 \ne 0$; hence, the equality $H^2/K = \frac{p^2}{4\,(p-1)}$ is true. The solution to (4.11) is $f(\rho) = \int \frac{\sqrt{C_1\rho^{2p+2} - \rho^{4p}}}{\rho^{2p} - C_1\,\rho^2}\,\mathrm{d}\rho + C_2$, where $C_1, C_2 \in \mathbb{R}$, it is illustrated on Figure 1 for $p = 2, 3, \ldots, 8$.

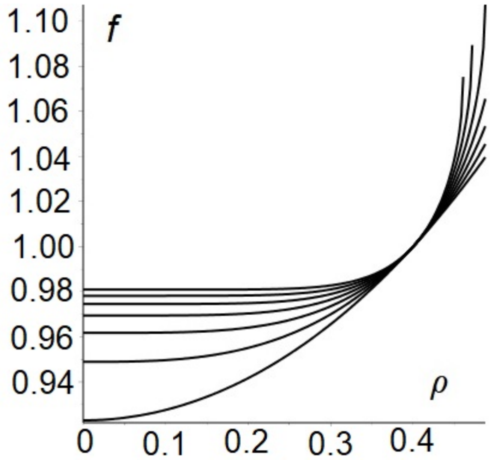

Figure 1. Graphs of $f(\rho)$ for $f(\frac{2}{5}) = 1$, $f'(\frac{2}{5}) = \frac{2}{5}$, $n = 2$ and $p = 2, 3, \ldots, 8$.

5. Conclusions

This paper explores a generalized (for foliated hypersurfaces in a Riemannian manifold) form of the classical Willmore functionals, which is the Reilly-type functional. The 1st and 2nd variations of such functionals in the Euclidean space are computed, and the conformal properties of some of them are shown. Examples of critical hypersurfaces with low-dimensional transversally harmonic foliations and critical hypersurfaces of revolution, which are local minima for a specialized family of variations, are given. The results obtained are important for researchers working in the field of geometric variational problems and for scientists involved in the design of layered (laminated) or non-isotropic materials.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Vladimir Rovenski is a guest editor for Electronic Research Archive and was not involved in the editorial review or the decision to publish this article. The author declares there is no conflict of interest.

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