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Research article

Periodic measures for a neural field lattice model with state dependent superlinear noise

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Abstract: The primary focus of this paper lies in exploring the limiting dynamics of a neural field lattice model with state dependent superlinear noise. First, we established the well-posedness of solutions to these stochastic systems and subsequently proved the existence of periodic measures for the system in the space of square-summable sequences using Krylov-Bogolyubov's method. The cutoff techniques of uniform estimates on tails of solutions was employed to establish the tightness of a family of probability distributions for the system's solutions.

Keywords: neural field lattice model; superlinear noise; periodic measure

1. Introduction

The objective of this paper is to investigate the existence of periodic measures for a neural field lattice model with state dependent superlinear noise on $\mathbb{Z}^N$ with $N \in \mathbb{N}$:

$$\begin{cases} du_i(t) = \Big(-\alpha u_i(t) + f_i(u_i(t)) + \sum\limits_{j\in\mathbb{Z}^N} k_{i,j}\xi_{i,j}(u_j(t)) + g_i(t)\Big)dt + (\lambda_i(u_i(t)) + h_i(t))dW_i(t), \\ u_i(0) = u_{0,i}, \end{cases} \tag{1.1}$$

where $i = (i_1, i_2, \cdots, i_N) \in \mathbb{Z}^N$, $t > 0$, and $\alpha > 0$. The variable u_i represents the neural activity, specifically the synaptic activity of the ith node. The function $f_i : \mathbb{R} \to \mathbb{R}$ describes the continuous differentiability of neural activity attenuation for the ith node, and $\xi_{i,j} : \mathbb{R} \to \mathbb{R}$ is an activation function that determines a node's output based on its input. The quantity $k_{i,j}$ represents the synaptic strength from the jth to the ith node, which can have positive or negative values indicating excitation or inhibition of the jth neuron on the ith neuron, respectively. The time-dependent functions $g_i : \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}$ describe the external forcing for drift and diffusion at the ith location, respectively. This is represented by a sequence of mutually independent two-sided real-valued Wiener processes

$(W_i(t))_{i\in\mathbb{Z}^N}$, defining on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$, and each Wiener process W_i is associated with a superlinear state-dependent function $\lambda_i : \mathbb{R} \to \mathbb{R}$ within its coefficient.

Differential equations and dynamical systems play a crucial role in the mathematical modeling and analysis of aircraft design, aerospace engineering, materials science, biology, medical engineering, financial engineering, and securities markets, as well as mobile communications, aquatic communications, and other related fields. The field of dynamical systems have witnessed significant achievements by numerous esteemed scholars. Among them, the investigation of traveling wave solutions for such equations have been conducted by [1, 2]. The examination of chaotic properties in the solutions have been carried out by [3, 4]. The existence and uniqueness of solutions and the existence, uniqueness, and upper semi-continuity of attractors have been studied by [5–7]. Additionally, Li et al. conducted an investigation on inverse problems in predator-prey models in [8], while Yin et al. explored a neural network approach for the inversion of turbulence strength in [9]. These studies on inverse problems in mathematical physics have generated a significant impact within academic circles.

The lattice systems are commonly derived from spatial discretizations of partial differential equations. For the asymptotic behavior analysis of lattice systems, we refer readers to [10–13]. Amongst various applications, neural lattice models arising from neural networks have recently gained significant attention. These models can be broadly classified into two types: one is developed as the discretization of continuous neural field models, known as neural field lattice systems; and the other is derived as the limit of finite dimensional discrete neural networks when their sizes become increasingly large. Recent studies on neural lattice models include Faye's investigation on traveling fronts for a class of lattice neural field equations [14], Han and Kloeden's exploration of long-term dynamics for neural field lattice models [15], along with Han et al.'s examination of long-term dynamics for Hopfield-type neural lattice models [16], in addition to Wang et al.'s work [17]. Recently, Wang et al. conducted a study on the existence of weak pullback mean random attractors and invariant measures for a neural lattice model with state-dependent nonlinear noise in their work [18]. In addition, Caraballo et al. investigated the convergence and approximation of invariant measures for neural field lattice models under noise perturbation in their publication [19].

Currently, extensive research has been conducted on the dynamical behavior of differential equations driven by linear noise. To effectively handle stochastic systems with nonlinear noise, Kloeden [20] and Wang [21] introduced the concept of weak pullback mean random attractors. Subsequently, this concept has been widely applied in numerous studies on stochastic systems by various scholars in [22–25]. The periodic measures of stochastic differential equations have been extensively investigated by numerous experts, as documented in [26, 27] and the references therein. Specifically, the existence and limiting behavior of periodic measures for the stochastic reaction-diffusion lattice system were examined in [26], considering both globally Lipschitz continuous nonlinear drift and diffusion terms. However, to the best of our knowledge, the current literature on periodic measures for the neural field lattice model with state-dependent superlinear noise is unfortunately lacking. The existence of periodic measures for the lattice systems (1.1) in ℓ^2 is established through the ingenious Krylov-Bogolyubov's method, which showcases the brilliance of mathematical prowess. By employing the concept of uniform estimates on the tails of solutions, we successfully establish the tightness of a family of distribution laws of the solutions.

The paper is structured as follows: Section 2 introduces the notations and discusses the well-posedness of system (1.1). The subsequent section establishes essential uniform estimates of solutions, which play a pivotal role in demonstrating the main findings in the following section. The primary focus of Section 4 is to investigate the existence of periodic measures for system (1.1) in space ℓ^2.

2. Preliminaries

In this section, we will investigate the well-posedness of the stochastic neural field lattice system (1.1). We begin with the following Banach space ℓ^r, where ℓ^r is defined by

$\ell^r = \left\{u = (u_i)_{i\in\mathbb{Z}^N} | u_i \in \mathbb{R}, \sum_{i\in\mathbb{Z}^N} |u_i|^r < \infty\right\}$ with norm $\|u\|_r^r = \sum_{i\in\mathbb{Z}^N} |u_i|^r$ if $1 \le r < \infty$,

$\ell^\infty = \left\{u = (u_i)_{i\in\mathbb{Z}^N} | u_i \in \mathbb{R}, \sup_{i\in\mathbb{Z}^N} |u_i| < \infty\right\}$ with norm $\|u\|_{\ell^\infty} = \sup_{i\in\mathbb{Z}^N} |u_i|$ if $r = \infty$.

Particularly, ℓ^2 is a Hilbert space with the inner product and norm given by

$$(u, v) = \sum_{i\in\mathbb{Z}^N} u_i v_i, \ \|u\|^2 = (u, u), \ u, v \in \ell^2.$$

For the nonlinear drift function $f_i \in C^1(\mathbb{R}, \mathbb{R})$ in system (1.1), we assume that for all $z \in \mathbb{R}$ and $i \in \mathbb{Z}^N$,

$$f_i(z)z \le -\gamma|z|^p + \phi_{1,i}, \phi_1 = (\phi_{1,i})_{i\in\mathbb{Z}^N} \in \ell^1, \tag{2.1}$$

$$|f_i(z)| \le \phi_{2,i}|z|^{p-1} + \phi_{3,i}, \phi_2 = (\phi_{2,i})_{i\in\mathbb{Z}^N} \in \ell^\infty, \phi_3 = (\phi_{3,i})_{i\in\mathbb{Z}^N} \in \ell^2, \tag{2.2}$$

$$|f_i'(z)| \le \phi_{4,i}|z|^{p-2} + \phi_{5,i}, \phi_4 = (\phi_{4,i})_{i\in\mathbb{Z}^N} \in \ell^\infty, \phi_5 = (\phi_{5,i})_{i\in\mathbb{Z}^N} \in \ell^\infty, \tag{2.3}$$

where $p > 2$ and $\gamma > 0$ are constants. For the sequence of continuously differentiable diffusion function λ_i, we assume that for every $z \in \mathbb{R}$ and $i \in \mathbb{Z}^N$,

$$|\lambda_i(z)| \le \varphi_{1,i}|z|^{\frac{q}{2}} + \varphi_{2,i}, \varphi_1 = (\varphi_{1,i})_{i\in\mathbb{Z}^N} \in \ell^{\frac{2p}{p-q}}, \varphi_2 = (\varphi_{2,i})_{i\in\mathbb{Z}^N} \in \ell^2, \tag{2.4}$$

$$|\lambda_i'(z)| \le \varphi_{3,i}|z|^{\frac{q}{2}-1} + \varphi_{4,i}, \varphi_3 = (\varphi_{3,i})_{i\in\mathbb{Z}^N} \in \ell^q, \varphi_4 = (\varphi_{4,i})_{i\in\mathbb{Z}^N} \in \ell^\infty, \tag{2.5}$$

where $q \in [2, p)$ is a constant. Moreover, we assume that there exists a constant $\kappa > 0$ such that

$$\sum_{i\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} k_{i,j}^2 \le \kappa. \tag{2.6}$$

For $i, j \in \mathbb{Z}^N$ and $z \in \mathbb{R}$, we assume that the activation function $\xi_{i,j}$ is globally Lipschitz continuous with Lipchitz constant L_1, and there exist $a_{i,j} \in \mathbb{R}$ and $b_{i,j} > 0$ such that

$$|\xi_{i,j}(z)| \le a_{i,j}|z| + b_{i,j}, \text{ with } \|a\|^2 = \sum_{i\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} |a_{i,j}|^2 < \infty, \|b\|^2 = \sum_{i\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} |b_{i,j}|^2 < \infty. \tag{2.7}$$

In addition, we assume

$$16\kappa\|a\|^2 \le \alpha^2. \tag{2.8}$$

Suppose $G(t), H(t) : \mathbb{R} \to \ell^2$, $G(t) = (g_i(t))_{i\in\mathbb{Z}^N}$, $H(t) = (h_i(t))_{i\in\mathbb{Z}^N}$ are both continuous in $t \in \mathbb{R}$, which shows that for $t \in \mathbb{R}$,

$$\|G(t)\|^2 = \sum_{i\in\mathbb{Z}^N} |g_i(t)|^2 < \infty \text{ and } \|H(t)\|^2 = \sum_{i\in\mathbb{Z}^N} |h_i(t)|^2 < \infty. \tag{2.9}$$

In order to investigate the periodic measures of system (1.1), we assume that all given time-dependent functions are T-periodic in $t \in \mathbb{R}$ for some $T > 0$; that is, for all $t \in \mathbb{R}$,

$$G(t+T) = G(t), \quad H(t+T) = H(t).$$

If $\chi : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function, we denote

$$\bar{\chi} = \max_{0\le t\le T} \chi(t).$$

For all $u = (u_i)_{i\in\mathbb{Z}^N} \in \ell^2$, define the operator F, Λ, and Ξ by

$$\begin{aligned} &F(u) = (f_i(u_i))_{i\in\mathbb{Z}^N}, \;\; \Lambda(u) = (\lambda_i(u_i))_{i\in\mathbb{Z}^N}, \\ &\Xi(u) = (\Xi_i(u_i))_{i\in\mathbb{Z}^N} \text{ with } \Xi_i(u_i) = \sum_{j\in\mathbb{Z}^N} k_{i,j}\xi_{i,j}(u_j). \end{aligned} \tag{2.10}$$

By (2.3), we get that there exists $\theta_1 \in (0,1)$ such that for $p > 2$ and $u, v \in \ell^2$,

$$\begin{aligned} \sum_{i\in\mathbb{Z}^N} |f_i(u_i) - f_i(v_i)|^2 &= \sum_{i\in\mathbb{Z}^N} |f_i'(\theta_1 u_i + (1-\theta_1)v_i)|^2 |u_i - v_i|^2 \\ &\le \sum_{i\in\mathbb{Z}^N} (|\phi_{4,i}||\theta_1 u_i + (1-\theta_1)v_i|^{p-2} + |\phi_{5,i}|)^2 |u_i - v_i|^2 \\ &\le (2^{2p-4}\|\phi_4\|^2_{\ell^\infty}(\|u\|^{2p-4} + \|v\|^{2p-4}) + 2\|\phi_5\|^2_{\ell^\infty})\|u - v\|^2, \end{aligned} \tag{2.11}$$

which along with $F(0) \in \ell^2$ and according to (2.2) implies $F(u) \in \ell^2$ for all $u \in \ell^2$. Then, $F : \ell^2 \to \ell^2$ is well-defined. In addition, it follows from (2.11) that $F : \ell^2 \to \ell^2$ is a locally Lipschitz continuous function; that is, for every $c \in \mathbb{N}$, there exists a constant $L_2(c) > 0$ such that for all $u, v \in \ell^2$ with $\|u\| \le c$ and $\|v\| \le c$,

$$\|F(u) - F(v)\| \le L_2(c)\|u - v\|. \tag{2.12}$$

For $u \in \ell^2$, by (2.6), (2.7), and (2.10), we have

$$\begin{aligned} \|\Xi(u)\|^2 &\le \sum_{i\in\mathbb{Z}^N} \Big(\sum_{j\in\mathbb{Z}^N} k_{i,j}(a_{i,j}|u_j| + b_{i,j}) \Big)^2 \le \sum_{i\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} k_{i,j}^2 \sum_{j\in\mathbb{Z}^N} (a_{i,j}|u_j| + b_{i,j})^2 \\ &\le 2\kappa\|a\|^2\|u\|^2 + 2\kappa\|b\|^2. \end{aligned} \tag{2.13}$$

In addition, for all $u, v \in \ell^2$, it follows from the globally Lipschitz continuity of $\xi_{i,j}$, Cauchy's inequality, and (2.6) that

$$\begin{aligned} \|\Xi(u) - \Xi(v)\|^2 &\le L_1^2 \sum_{i\in\mathbb{Z}^N} \Big(\sum_{j\in\mathbb{Z}^N} k_{i,j}|u_j - v_j| \Big)^2 \le L_1^2 \sum_{i\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} k_{i,j}^2 \sum_{j\in\mathbb{Z}^N} |u_j - v_j|^2 \\ &\le L_1^2\kappa\|u - v\|^2, \end{aligned} \tag{2.14}$$

which, along with (2.13), implies that $\Xi(u)$ belongs to ℓ^2 and is a globally Lipschitz continuous function.

In order to rewrite the terms $\lambda_i(u_i) + h_i(t)$ as vectors in ℓ^2, define two sequence of operators Λ_i and H_i by

$$\Lambda_i(u) = (\lambda_i(u_i))e^i, \;\; H_i(t) = (h_i(t))e^i, \;\; i \in \mathbb{Z}^N,$$

where e^i represents the infinite sequence with a value of 1 at position i and a value of 0 elsewhere. Then, we can get that $\Lambda(u) = \sum_{i\in\mathbb{Z}^N} \Lambda_i(u)$ and $H(t) = \sum_{i\in\mathbb{Z}^N} H_i(t)$ for every $u \in \ell^2$ and

$$\|\Lambda(u)\|^2 = \sum_{i\in\mathbb{Z}^N} \|\Lambda_i(u)\|^2, \; \|\Lambda(u) - \Lambda(v)\|^2 = \sum_{i\in\mathbb{Z}^N} \|\Lambda_i(u) - \Lambda_i(v)\|^2. \tag{2.15}$$

For $q \in [2, p)$ and $u \in \ell^2$, we can get from (2.4) and Young's inequality that

$$\begin{aligned}\|\Lambda(u)\|^2 &= \sum_{i\in\mathbb{Z}^N} |\lambda_i(u_i)|^2 \le 2\sum_{i\in\mathbb{Z}^N}(|\varphi_{1,i}|^2|u_i|^q + |\varphi_{2,i}|^2)\\ &\le \frac{\gamma}{2}\sum_{i\in\mathbb{Z}^N}|u_i|^p + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}}2^{\frac{p}{p-q}}\sum_{i\in\mathbb{Z}}|\varphi_{1,i}|^{\frac{2p}{p-q}} + 2\sum_{i\in\mathbb{Z}^N}|\varphi_{2,i}|^2\\ &= \frac{\gamma}{2}\|u\|_p^p + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}}2^{\frac{p}{p-q}}\|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + 2\|\varphi_2\|^2,\end{aligned} \tag{2.16}$$

where γ is the same number in (2.1). By (2.16) and $\ell^2 \subseteq \ell^p$, we get that $\Lambda(u) \in \ell^2$ for all $u \in \ell^2$ and $p > 2$. Then, $\Lambda(u) : \ell^2 \to \ell^2$ is also well-defined. By (2.5), we get that there exists $\theta_2 \in (0, 1)$ such that for $q \in [2, p)$ and $u, v \in \ell^2$,

$$\begin{aligned}\|\Lambda(u) - \Lambda(v)\|^2 &= \sum_{i\in\mathbb{Z}^N}|\Lambda_i(u_i) - \Lambda_i(v_i)|^2\\ &= \sum_{i\in\mathbb{Z}^N}|\lambda_i'(\theta_2 u_i + (1-\theta_2)v_i)|^2|u_i - v_i|^2\\ &\le \sum_{i\in\mathbb{Z}^N}(|\varphi_{3,i}||\theta_2 u_i + (1-\theta_2)v_i)|^{\frac{q}{2}-1} + |\varphi_{4,i}|)^2|u_i - v_i|^2\\ &\le \sum_{i\in\mathbb{Z}^N}(2^{q-2}|\varphi_{3,i}|^2(|u_i|^{q-2} + |v_i|^{q-2}) + 2|\varphi_{4,i}|^2)|u_i - v_i|^2\\ &\le \sum_{i\in\mathbb{Z}^N}\Big(2^{q-2}\Big(\frac{4}{q}|\varphi_{3,i}|^q + \frac{q-2}{q}|u_i|^q + \frac{q-2}{q}|v_i|^q\Big) + 2|\varphi_{4,i}|^2\Big)|u_i - v_i|^2\\ &\le (2^{q-1}(\|\varphi_3\|_q^q + \|u\|^q + \|v\|^q) + 2\|\varphi_4\|_{\ell^\infty}^2)\|u - v\|^2,\end{aligned} \tag{2.17}$$

which shows that $\Lambda : \ell^2 \to \ell^2$ is a locally Lipschitz continuous function; that is, for every $c \in \mathbb{N}$, we can find a constant $L_3(c) > 0$ such that for all $u, v \in \ell^2$ with $\|u\| \le c$ and $\|v\| \le c$,

$$\|\Lambda(u) - \Lambda(v)\|^2 \le L_3^2(c)\|u - v\|^2. \tag{2.18}$$

By the above notation, system (1.1) can be rewritten as follows: For all $t > 0$,

$$\begin{cases} du(t) = \Big(-\alpha u(t) + F(u(t)) + \Xi(u(t)) + G(t)\Big)dt + \sum_{i\in\mathbb{Z}^N}(\Lambda_i(u(t)) + H_i(t))dW_i(t),\\ u(0) = u_0.\end{cases} \tag{2.19}$$

Let $u_0 \in L^2(\Omega, \ell^2)$ be $\mathcal{F}_0$-measurable. A continuous ℓ^2-valued $\mathcal{F}_t$-adapted stochastic process $u(t)$ is called a solution of system (2.19) if $u(t) \in L^2(\Omega, C([0,T], \ell^2)) \bigcap L^p(\Omega, L^p(0,T,\ell^p))$ for all $T > 0$, $t \geq 0$ and for almost all $\omega \in \Omega$,

$$u(t) = u_0 + \int_0^t \Big(-\alpha u(s) + F(u(s)) + \Xi(u(s)) + G(s)\Big) ds + \sum_{i \in \mathbb{Z}^N} \int_0^t (\Lambda_i(u(s)) + H_i(s)) dW_i(s).$$

By (2.1)–(2.9) and the theory of functional differential equations, we can get that for any $u_0 \in L^2(\Omega, \ell^2)$, system (2.19) has local solutions $u(t) \in L^2(\Omega, C([0,T], \ell^2)) \bigcap L^p(\Omega, L^p(0,T,\ell^p))$ for every $T > 0$. Moreover, similar to [24], we can get that the local solutions to system (2.19) are also global.

3. Uniform estimates

In this section, we establish uniform estimates for the solutions to system (2.19), which play a crucial role in proving the existence of periodic measures. Specifically, we will demonstrate the compactness of a family of probability distributions related to $u(t)$ in ℓ^2.

Lemma 3.1. *Suppose (2.1)–(2.9) hold. Let $u_0 \in L^2(\Omega, \ell^2)$ be the initial data of system (2.19), then the solution $u(t, 0, u_0)$ of the system (2.19) satisfies*

$$\begin{aligned} &\mathbb{E}[\|u(t,0,u_0)\|^2] + \int_0^t e^{\alpha(r-t)} \mathbb{E}[\|u(r,0,u_0)\|_p^p] dr \\ &\leq M_1 (\mathbb{E}[\|u_0\|^2] + \|\phi_1\|_1 + \|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + \|\varphi_2\|^2 + \|\bar{H}\| + \|\bar{G}\|), \end{aligned} \tag{3.1}$$

where M_1 is a positive constant independent of u_0.

Proof. By (2.19) and Itô's formula, we get that for all $t \geq 0$,

$$\begin{aligned} d\|u\|^2 &= -2\alpha \|u\|^2 dt + 2(F(u) + \Xi(u) + G(t), u) dt + \|\Lambda(u) + H(t)\|^2 dt \\ &\quad + 2 \sum_{i \in \mathbb{Z}^N} (u, \Lambda_i(u) + H_i(t)) dW_i(t). \end{aligned} \tag{3.2}$$

Taking the expectation, we obtain that for $t \geq 0$,

$$\frac{d}{dt}\mathbb{E}[\|u\|^2] = -2\alpha \mathbb{E}[\|u\|^2] + 2\mathbb{E}[(F(u), u)] + 2\mathbb{E}[(\Xi(u), u)] + 2\mathbb{E}[(G(t), u)] + \mathbb{E}[\|\Lambda(u) + H(t)\|^2]. \tag{3.3}$$

By (2.1), we have

$$2\mathbb{E}[(F(u), u)] \leq -2\gamma \mathbb{E}[\|u\|_p^p] + 2\|\phi_1\|_1. \tag{3.4}$$

By (2.7) and Young's inequality, we get

$$\begin{aligned} 2\mathbb{E}[(\Xi(u), u)] &\leq 2\mathbb{E}\Big[\sum_{i \in \mathbb{Z}^N} u_i \sum_{j \in \mathbb{Z}^N} k_{i,j} (a_{i,j}|u_j| + b_{i,j}) \Big] \\ &\leq \frac{\alpha}{4} \mathbb{E}[\|u\|^2] + \frac{4}{\alpha} \mathbb{E}\Big[\sum_{i \in \mathbb{Z}^N} \Big(\sum_{j \in \mathbb{Z}^N} k_{i,j}(a_{i,j}|u_j| + b_{i,j}) \Big)^2 \Big] \\ &\leq \frac{\alpha}{4} \mathbb{E}[\|u\|^2] + \frac{8\kappa}{\alpha} \Big(\|a\|^2 \mathbb{E}[\|u\|^2] + \|b\|^2 \Big) \\ &= \frac{\alpha}{4} \mathbb{E}[\|u\|^2] + \frac{8\kappa \|a\|^2}{\alpha} \mathbb{E}[\|u\|^2] + \frac{8\kappa \|b\|^2}{\alpha}. \end{aligned} \tag{3.5}$$

Note that

$$2\mathbb{E}[(G(t),u)] \le \frac{\alpha}{4}\mathbb{E}[\|u\|^2] + \frac{4}{\alpha}\mathbb{E}[\|G(t)\|^2]. \tag{3.6}$$

By (2.4), we obtain

$$\begin{aligned}\mathbb{E}[\|\Lambda(u)+H(t)\|^2] &\le 2\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N}\lambda_i^2(u_i)\Big] + 2\mathbb{E}[\|H(t)\|^2]\\ &\le 2\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N}(\varphi_{1,i}|u_i|^{\frac{q}{2}}+\varphi_{2,i})^2\Big] + 2\mathbb{E}[\|H(t)\|^2]\\ &\le 4\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N}(\varphi_{1,i}^2|u_i|^q+\varphi_{2,i}^2)\Big] + 2\mathbb{E}[\|H(t)\|^2]\\ &\le \frac{\gamma}{2}\mathbb{E}[\|u\|_p^p] + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}}4^{\frac{p}{p-q}}\|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + 4\|\varphi_2\|^2 + 2\mathbb{E}[\|H(t)\|^2].\end{aligned} \tag{3.7}$$

It follows from (3.3)–(3.7) and (2.8) that

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}[\|u(t)\|^2] + \alpha\mathbb{E}[\|u(t)\|^2] + \frac{3\gamma}{2}\mathbb{E}[\|u(t)\|_p^p]\\ &\le 2\|\phi_1\|_1 + \frac{p-q}{p}(\frac{p\gamma}{2q})^{-\frac{q}{p-q}}4^{\frac{p}{p-q}}\|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + 4\|\varphi_2\|^2 + \frac{8\kappa\|b\|^2}{\alpha} + 2\mathbb{E}[\|H(t)\|^2] + \frac{4}{\alpha}\mathbb{E}[\|G(t)\|^2],\end{aligned} \tag{3.8}$$

which implies that for $t \ge 0$,

$$\mathbb{E}[\|u(t,0,u_0)\|^2] + \frac{3\gamma}{2}\int_0^t e^{\alpha(r-t)}\mathbb{E}[\|u(r,0,u_0)\|_p^p]dr \le e^{-\alpha t}\mathbb{E}[\|u_0\|^2] + C_1\int_0^t e^{\alpha(r-t)}dr, \tag{3.9}$$

where $C_1 = 2\|\phi_1\|_1 + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}}4^{\frac{p}{p-q}}\|\varphi_1\|_{\frac{2p}{p-q}}^{\frac{2p}{p-q}} + 4\|\varphi_2\|^2 + \frac{8\kappa\|b\|^2}{\alpha} + 2\|\bar{H}\|^2 + \frac{4}{\alpha}\|\bar{G}\|^2$. This completes the proof. □

The subsequent step involves obtaining uniform estimates on the tails of solutions to the stochastic lattice system (2.19).

Lemma 3.2. *Suppose (2.1)–(2.9) hold. For compact subset $\mathcal{K} \in \ell^2$, there is a number $N_0 = N_0(\mathcal{K}) \in \mathbb{N}$ such that the solution $u(t,0,u_0)$ of the system (2.19) satisfies, for all $n \ge N_0$ and $t \ge 0$,*

$$\mathbb{E}\Big[\sum_{\|i\|\ge n}|u_i(t,0,u_0)|^2\Big] + \int_0^t e^{\alpha(r-t)}\mathbb{E}\Big[\sum_{\|i\|\ge n}|u_i(r,0,u_0)|^p\Big]dr \le \varepsilon, \tag{3.10}$$

where $u_0 \in \mathcal{K}$ and $\|i\| := \max_{1\le j\le N}|i_j|$.

Proof. Let ϑ be a smooth function which is defined on $\mathbb{R}$ such that $0 \le \vartheta(z) \le 1$ for all $z \in \mathbb{R}$, and

$$\vartheta(z) = \begin{cases} 0, & 0 \le |z| \le 1;\\ 1, & |z| \ge 2.\end{cases}$$

For $n \in N$, set $\vartheta_n = \Big(\vartheta\Big(\frac{\|i\|}{n}\Big)\Big)_{i\in\mathbb{Z}^N}$ and $\vartheta_n u = \Big(\vartheta\Big(\frac{\|i\|}{n}\Big)u_i\Big)_{i\in\mathbb{Z}^N}$. By (2.19), we have

$$d(\vartheta_n u) = (-\alpha\vartheta_n u + \vartheta_n F(u) + \vartheta_n\Xi(u) + \vartheta_n G(t))dt + \sum_{i\in\mathbb{Z}^N}(\vartheta_n\Lambda_i(u) + \vartheta_n H_i(t))dW_i(t),$$

which along with Itô's formula implies that

$$
\begin{aligned}
d\|\vartheta_n u\|^2 =& -2\alpha\|\vartheta_n u\|^2 dt + 2(\vartheta_n F(u), \vartheta_n u)dt + 2(\vartheta_n \Xi(u), \vartheta_n u)dt + 2(\vartheta_n G(t), \vartheta_n u)dt \\
&+ \sum_{i\in\mathbb{Z}^N} \|\vartheta_n \Lambda_i(u) + \vartheta_n H_i(t)\|^2 dt + 2\sum_{i\in\mathbb{Z}^N} (\vartheta_n \Lambda_i(u) + \vartheta_n H_i(t), \vartheta_n u) dW_i(t).
\end{aligned}
\tag{3.11}
$$

Then, we get that for all $t \geq 0$,

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}[\|\vartheta_n u\|^2] =& -2\alpha\mathbb{E}[\|\vartheta_n u\|^2] + 2\mathbb{E}[(\vartheta_n F(u), \vartheta_n u)] + 2\mathbb{E}[(\vartheta_n \Xi(u), \vartheta_n u)] \\
&+ 2\mathbb{E}[(\vartheta_n G(t), \vartheta_n u)] + \mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \|\vartheta_n \Lambda_i(u) + \vartheta_n H_i(t)\|^2\Big].
\end{aligned}
\tag{3.12}
$$

By (2.1), we find

$$
\begin{aligned}
2\mathbb{E}[(\vartheta_n F(u), \vartheta_n u)] &\leq 2\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \vartheta^2\Big(\frac{\|i\|}{n}\Big)(-\gamma|u_i|^p + \phi_{1,i})\Big] \\
&\leq -2\gamma\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \vartheta^2\Big(\frac{\|i\|}{n}\Big)|u_i|^p\Big] + 2\sum_{\|i\|\geq n} \phi_{1,i}.
\end{aligned}
\tag{3.13}
$$

By (2.7) and Young's inequality, we have

$$
\begin{aligned}
2\mathbb{E}[(\vartheta_n \Xi(u), \vartheta_n u)] &\leq \frac{\alpha}{4}\mathbb{E}[\|\vartheta_n u\|^2] + \frac{4}{\alpha}\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \vartheta^2\Big(\frac{\|i\|}{n}\Big)\Big(\sum_{j\in\mathbb{Z}^N} k_{i,j}\xi_{i,j}(u_j)\Big)^2\Big] \\
&\leq \frac{\alpha}{4}\mathbb{E}[\|\vartheta_n u\|^2] + \frac{8\kappa}{\alpha}\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \vartheta^2\Big(\frac{\|i\|}{n}\Big)\Big(\sum_{j\in\mathbb{Z}^N} |a_{i,j}u_j|^2 + |b_{i,j}|^2\Big)\Big] \\
&= \frac{\alpha}{4}\mathbb{E}[\|\vartheta_n u\|^2] + \frac{8\kappa\|a\|^2}{\alpha}\mathbb{E}[\|\vartheta_n u\|^2] + \frac{8\kappa}{\alpha}\sum_{\|i\|\geq n}\sum_{j\in\mathbb{Z}^N} |b_{i,j}|^2.
\end{aligned}
\tag{3.14}
$$

Note that

$$
2\mathbb{E}[(\vartheta_n G(t), \vartheta_n u)] \leq \frac{\alpha}{4}\mathbb{E}[\|\vartheta_n u\|^2] + \frac{4}{\alpha}\mathbb{E}[\|\vartheta_n G(t)\|^2].
\tag{3.15}
$$

For the last term of (3.12), by (2.4), we get

$$
\begin{aligned}
\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \|\vartheta_n \Lambda(u) + \vartheta_n H(t)\|^2\Big] &\leq 2\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \vartheta^2\Big(\frac{\|i\|}{n}\Big)(\varphi_{1,i}|u_i|^{\frac{q}{2}} + \varphi_{2,i})^2\Big] + 2\mathbb{E}\Big[\|\vartheta_n H(t)\|^2\Big] \\
&\leq 4\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \vartheta^2\Big(\frac{\|i\|}{n}\Big)(\varphi_{1,i}^2|u_i|^q + \varphi_{2,i}^2)\Big] + 2\mathbb{E}\Big[\|\vartheta_n H(t)\|^2\Big] \\
&\leq \frac{\gamma}{2}\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N} \vartheta^2\Big(\frac{\|i\|}{n}\Big)|u_i|^p\Big] + 2\mathbb{E}\Big[\|\vartheta_n H(t)\|^2\Big] \\
&\quad + 4\sum_{\|i\|\geq n} |\varphi_{2,i}|^2 + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}} 4^{\frac{p}{p-q}} \sum_{\|i\|\geq n} |\varphi_{1,i}|^{\frac{2p}{p-q}}.
\end{aligned}
\tag{3.16}
$$

It follows from (3.12)–(3.16) and (2.8) that

$$\frac{d}{dt}\mathbb{E}[\|\vartheta_n u\|^2] + \alpha\mathbb{E}[\|\vartheta_n u\|^2] + \frac{3\gamma}{2}\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N}\vartheta^2\Big(\frac{\|i\|}{n}\Big)|u_i|^p\Big]$$
$$\le 2\sum_{\|i\|\ge n}\phi_{1,i} + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}}4^{\frac{p}{p-q}}\sum_{\|i\|\ge n}|\varphi_{1,i}|^{\frac{2p}{p-q}} + 4\sum_{\|i\|\ge n}|\varphi_{2,i}|^2$$
$$+ 2\sum_{\|i\|\ge n}|\bar{H}_i|^2 + \frac{4}{\alpha}\sum_{\|i\|\ge n}|\bar{G}_i|^2 + \frac{8\kappa}{\alpha}\sum_{\|i\|\ge n}\sum_{j\in\mathbb{Z}^N}|b_{i,j}|^2,$$

which implies that

$$\begin{aligned}
&\mathbb{E}[\|\vartheta_n u(t,0,u_0)\|^2] + \frac{3\gamma}{2}\int_0^t e^{\alpha(r-t)}\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N}\vartheta^2\Big(\frac{\|i\|}{n}\Big)|u_i(r,0,u_0)|^p\Big]dr\\
&\le e^{-\alpha t}\mathbb{E}[\|\vartheta_n u_0\|^2] + \frac{1}{\alpha}\Big(2\sum_{\|i\|\ge n}\phi_{1,i} + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}}4^{\frac{p}{p-q}}\sum_{\|i\|\ge n}|\varphi_{1,i}|^{\frac{2p}{p-q}}\\
&\quad + 4\sum_{\|i\|\ge n}|\varphi_{2,i}|^2 + 2\sum_{\|i\|\ge n}|\bar{H}_i|^2 + \frac{4}{\alpha}\sum_{\|i\|\ge n}|\bar{G}_i|^2 + \frac{8\kappa}{\alpha}\sum_{\|i\|\ge n}\sum_{j\in\mathbb{Z}^N}|b_{i,j}|^2\Big).
\end{aligned}\tag{3.17}$$

Since $\mathcal{K}$ is a compact subset of ℓ^2, we get that

$$\lim_{n\to\infty}\sup_{u_0\in\mathcal{K}}\sup_{t\ge 0}e^{-\alpha t}\mathbb{E}[\|\vartheta_n u_0\|^2] \le \lim_{n\to\infty}\sup_{u_0\in\mathcal{K}}\mathbb{E}\Big[\sum_{\|i\|\ge n}|u_{0,i}|^2\Big] = 0. \tag{3.18}$$

By $\phi_1 \in \ell^1$, $\varphi_1 \in \ell^{\frac{2p}{p-q}}$, $\varphi_2 \in \ell^2$, (2.7), and (2.9), we infer that

$$\begin{aligned}
&2\sum_{\|i\|\ge n}\phi_{1,i} + \frac{p-q}{p}\Big(\frac{p\gamma}{2q}\Big)^{-\frac{q}{p-q}}4^{\frac{p}{p-q}}\sum_{\|i\|\ge n}|\varphi_{1,i}|^{\frac{2p}{p-q}} + 4\sum_{\|i\|\ge n}|\varphi_{2,i}|^2\\
&\quad + 2\sum_{\|i\|\ge n}|\bar{H}_i|^2 + \frac{4}{\alpha}\sum_{\|i\|\ge n}|\bar{G}_i|^2 + \frac{8\kappa}{\alpha}\sum_{\|i\|\ge n}\sum_{j\in\mathbb{Z}^N}|b_{i,j}|^2 \to 0 \text{ as } n\to\infty.
\end{aligned}\tag{3.19}$$

It follows from (3.17)–(3.19) that as $n \to \infty$,

$$\sup_{u_0\in\mathcal{K}}\sup_{t\ge 0}\Big(\mathbb{E}[\|\vartheta_n u(t,0,u_0)\|^2] + \int_0^t e^{\alpha(r-t)}\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N}\vartheta^2\Big(\frac{\|i\|}{n}\Big)|u_i(r,0,u_0)|^p\Big]dr\Big) \to 0. \tag{3.20}$$

Then, for every $\varepsilon > 0$, we can find that there exists a number $N_0 = N_0(\mathcal{K}) \in \mathbb{N}$ such that for all $n \ge N_0$ and $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{\|i\|\ge 2n}|u_i(t,0,u_0)|^2\Big] + \int_0^t e^{\alpha(r-t)}\mathbb{E}\Big[\sum_{\|i\|\ge 2n}|u_i(r,0,u_0)|^p\Big]dr\\
&\le \mathbb{E}[\|\vartheta_n u(t,0,u_0)\|^2] + \int_0^t e^{\alpha(r-t)}\mathbb{E}\Big[\sum_{i\in\mathbb{Z}^N}\vartheta^2\Big(\frac{\|i\|}{n}\Big)|u_i(r,0,u_0)|^p\Big]dr \le \varepsilon
\end{aligned}\tag{3.21}$$

uniformly for $u_0 \in \mathcal{K}$ and $t \ge 0$. This concludes the proof. □

Remark 1. In order to establish the existence of periodic measures for stochastic lattice system (2.19), the main challenge lies in deriving the tightness of a family of probability distributions of solutions. Our approach involves approximating solutions in ℓ^2 using finite-dimensional methods. In order to achieve this, it is necessary to establish uniformly small estimates for the "tail ends" of these solutions for $t \geq 0$ as stated in Lemma 3.2. For further elaboration on cutoff techniques related to estimating the "tail ends", please refer to [13, 28].

4. Existence of periodic measures

The primary focus of this section is to establish the existence of periodic measures for the lattice system (2.19) in ℓ^2. First, we introduce the transition operators associated with the lattice system and subsequently provide evidence for the convergence and compactness properties exhibited by a family of probability distributions representing solutions to this particular lattice system.

Suppose $\psi : \ell^2 \to \mathbb{R}$ is a bounded Borel function. For $0 \leq r \leq t$, we set

$$(p_{r,t}\psi)(u_0) = \mathbb{E}[\psi(u(t, r, u_0))], \quad \forall u_0 \in \ell^2.$$

In addition, for $G \in \mathcal{B}(\ell^2)$, $0 \leq r \leq t$, and $u_0 \in \ell^2$, we set

$$p(r, u_0; t, G) = (p_{r,t}1_G)(u_0),$$

where 1_G is the characteristic function of G. Then, we can get that $p(r, u_0; t, \cdot)$ is the probability distribution of $u(t)$ in ℓ^2. Furthermore, the transition operator $p_{0,t}$ is denoted as p_t for the sake of convenience.

Definition 4.1. *A probability measure μ on ℓ^2 is called a periodic measure of lattice system (2.19) if*

$$\int_{\ell^2} (p_{0,t+T}\psi)(u_0)d\mu(u_0) = \int_{\ell^2} (p_{0,t}\psi)(u_0)d\mu(u_0), \quad \forall t \geq 0, T > 0.$$

Now, we show the properties of transition operators $\{p_{r,t}\}_{0\leq r\leq t}$ as follows.

Lemma 4.1. *Suppose (2.1)–(2.9) hold. Then, we have*

(i) The family $\{p_{r,t}\}_{0\leq r\leq t}$ is Feller; that is, if $\psi : \ell^2 \to \mathbb{R}$ is bounded and continuous, then $p_{r,t}\psi : \ell^2 \to \mathbb{R}$ is bounded and continuous.

(ii) The family $\{p_{r,t}\}_{0\leq r\leq t}$ is T-periodic; that is,

$$p(r, u_0; t, \cdot) = p(r + T, u_0; t + T, \cdot), \forall r \in [0, t], u_0 \in \ell^2.$$

(iii) $\{u(t, 0, u_0)\}_{t\geq 0}$ is a ℓ^2-value Markov process.

Lemma 4.2. *Suppose (2.1)–(2.9) hold. Then, the family $\{\mathcal{L}(u(t, 0, u_0)) : t \geq 0\}$ of the distribution laws of the solutions to system (2.19) is tight on ℓ^2.*

Proof. For all $t \geq 0$, by Lemma 3.1 and Chebyshev's inequality, we get that there exists a constant $c_1 > 0$ such that

$$P\{\|u(t)\|^2 \geq R\} \leq \frac{1}{R^2}\mathbb{E}[\|u(t)\|^2] \leq \frac{c_1}{R^2}.$$

Then, for each $\varepsilon > 0$, there exists a constant $R_1 = R_1(\varepsilon) > 0$ such that

$$P\{\|u(t)\|^2 \geq R_1\} \leq \frac{\varepsilon}{2}, \quad \forall t \geq 0. \tag{4.1}$$

By Lemma 3.2, we obtain that for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an integer $n_m = n_m(\varepsilon, m) \geq 1$ such that

$$\mathbb{E}\Big[\sum_{\|i\|\geq n_m} |u_i(t)|^2\Big] \leq \frac{\varepsilon}{2^{2m+2}}, \ \forall t \geq 0. \tag{4.2}$$

Then, for all $t \geq 0$ and $m \in \mathbb{N}$, we get

$$P\Big(\bigcup_{m=1}^{\infty}\Big\{\sum_{\|i\|\geq n_m} |u_i(t)|^2 \geq \frac{1}{2^m}\Big\}\Big) \leq \sum_{m=1}^{\infty} 2^m \mathbb{E}\Big[\sum_{\|i\|\geq n_m} |u_i(t)|^2\Big] \leq \frac{\varepsilon}{4},$$

which shows that for all $t \geq 0$,

$$P\Big(\Big\{\sum_{\|i\|\geq n_m} |u_i(t)|^2 \leq \frac{1}{2^m}, \forall m \in \mathbb{N}\Big\}\Big) > 1 - \frac{1}{2}\varepsilon. \tag{4.3}$$

For $\varepsilon > 0$, set $\mathcal{Z}_\varepsilon = \mathcal{Z}_{1,\varepsilon} \bigcap \mathcal{Z}_{2,\varepsilon}$, where

$$\mathcal{Z}_{1,\varepsilon} = \{v \in \ell^2 : \|v\| \leq R_1(\varepsilon)\}, \tag{4.4}$$

$$\mathcal{Z}_{2,\varepsilon} = \Big\{v \in \ell^2 : \sum_{\|i\|\geq n_m} |v_i(t)|^2 \leq \frac{1}{2^m}, \forall m \in \mathbb{N}\Big\}. \tag{4.5}$$

It follows from (4.1) and (4.3) that for all $t \geq 0$,

$$P(\{u(t) \in \mathcal{Z}_\varepsilon\}) > 1 - \varepsilon. \tag{4.6}$$

Given $\epsilon > 0$, choose an interger $m_0 = m_0(\epsilon) \in \mathbb{N}$ such that $2^{m_0} > \frac{8}{\epsilon^2}$. Then, by (4.5), we get that for all $v \in \mathcal{Z}_\varepsilon$,

$$\sum_{\|i\|\geq n_{m_0}} |v_i(t)|^2 \leq \frac{1}{2^{m_0}} < \frac{\epsilon^2}{8}. \tag{4.7}$$

The set $\{(v_i)_{\|i\|\leq m_0} : v \in \mathcal{Z}_\varepsilon\}$ is bounded in the finite-dimensional space R^{2m_0+1} as shown by (4.4), and therefore is pre-compact. As a result, $\{v : v \in \mathcal{Z}_\varepsilon\}$ has a finite open cover of balls with radius $\frac{\epsilon}{2}$, which combined with (4.7) implies that the set $\{v : v \in \mathcal{Z}_\varepsilon\}$ has a finite open cover of balls with radius ϵ in ℓ^2. Since $\epsilon > 0$ can be chosen arbitrarily, the set $\{v : v \in \mathcal{Z}_\varepsilon\}$ is pre-compact in ℓ^2. This completes the proof. □

Now, the main outcome of this paper has been proved by Krylov-Bogolyubov's method.

Theorem 4.1. *Suppose (2.1)–(2.9) hold. Then, system (2.19) has a periodic measure on* ℓ^2.

Proof. For each $n \in \mathbb{N}$, the probability measure μ_n is defined by

$$\mu_n = \frac{1}{n}\sum_{l=1}^{n} p(0,0;lT,\cdot). \tag{4.8}$$

It follows from Lemma 4.2 that the sequence $(\mu_n)_{n=1}^{\infty}$ is tight in ℓ^2. Consequently, there exist a probability measure μ on ℓ^2 and a subsequence (still denoted by $(\mu_n)_{n=1}^{\infty}$) such that

$$\mu_n \to \mu, \ as\ n \to \infty. \tag{4.9}$$

By (4.8)–(4.9) and Lemma 4.1, we can get that for every $t \geq 0$ and every bounded and continuous function $\psi : \ell^2 \to \mathbb{R}$,

$$\begin{aligned}
\int_{\ell^2} (p_{0,t}\psi)(u_0)d\mu(u_0) &= \int_{\ell^2}\int_{\ell^2} \psi(y)p(0,u_0;t,dy)d\mu(u_0)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2}\int_{\ell^2} \psi(y)p(0,u_0;t,dy)p(0,0;lT,du_0)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2}\int_{\ell^2} \psi(y)p(lT,u_0;t+lT,dy)p(0,0;lT,du_0)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2} \psi(y)p(0,0;t+lT,dy)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2} \psi(y)p(0,0;t+lT+T,dy)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\int_{\ell^2}\int_{\ell^2} \psi(y)p(0,u_0;t+T,dy)p(0,0;lT,du_0)\\
&= \int_{\ell^2}\int_{\ell^2} \psi(y)p(0,u_0;t+T,dy)d\mu(u_0)\\
&= \int_{\ell^2} (p_{0,t+T}\psi)(u_0)d\mu(u_0),
\end{aligned}$$

which implies that μ is a periodic measure of system (2.19). This completes the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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