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Theory article

Subdirect Sums of $GSDD_1$ matrices

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Abstract: The class of generalized SDD_1 ($GSDD_1$) matrices is a new subclass of H-matrices. In this paper, we focus on the subdirect sum of $GSDD_1$ matrices, and some sufficient conditions to ensure that the subdirect sum of $GSDD_1$ matrices with strictly diagonally dominant (SDD) matrices is in the class of $GSDD_1$ matrices are given. Moreover, corresponding examples are given to illustrate our results.

Keywords: subdirect sum; H-matrices; $GSDD_1$ matrices; strictly diagonally dominant matrices; sufficient conditions

1. Introduction

In 1999, the concept of k-subdirect sums of square matrices was proposed by Fallat and Johnson [1], which is a generalization of the usual sum of matrices [2]. The subdirect sum of matrices plays an important role in many areas, such as matrix completion problems, global stiffness matrices in finite elements and overlapping subdomains in domain decomposition methods [1–5].

An important question for subdirect sums is whether the k-subdirect sum of two square matrices in one class of matrices lies in the same class. This question has attracted widespread attention in different classes of matrices and produced a variety of results. In 2005, Bru et al. gave sufficient conditions ensuring that the subdirect sum of two nonsingular M-matrices was also a nonsingular M-matrix [3]. Then the following year, they further came to the conclusion of the k-subdirect sum of S-SDD matrices is also an S-SDD matrix [2]. In [6], Chen and Wang succeeded in producing some sufficient conditions that the k-subdirect sum of SDD_1 matrices is an SDD_1 matrix. In [7], Li et al. gave some sufficient conditions such that the k-subdirect sum of doubly strictly diagonally dominant ($DSDD$) matrices is in the class of $DSDD$ matrices. In addition, the k-subdirect sum of other classes of matrices were mentioned, such as Nekrasov matrices [8–10], quasi-Nekrasov (QN) matrices [11], $SDD(p)$ matrices [12], weakly chained diagonally dominant matrices [13], Ostrowski-Brauer Sparse (OBS) matrices [14], $\{i_0\}$-Nekrasov matrices [15], $\{p_1, p_2\}$-Nekrasov matrices [16], Dashnic-Zusmanovich (DZ) matrices [17], and B-matrices [18, 19].

$GSDD_1$ matrices as a new subclass of H-matrices was proposed by Dai et al. in 2023 [20]. In this paper, we focus on the subdirect sum of $GSDD_1$ matrices, and some sufficient conditions such that the k-subdirect sum of $GSDD_1$ matrices with SDD matrices belong to $GSDD_1$ matrices are given. Numerical examples are presented to illustrate the corresponding results.

Now, some definitions are listed as follows.

Definition 1.1. *([2]) Let A and B be two square matrices of order n_1 and n_2, respectively, and k be an integer such that $1 \le k \le \min\{n_1, n_2\}$, and let A and B be partitioned into 2×2 blocks as follows:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{1.1}$$

where A_{22} and B_{11} are square matrices of order k. Following [1], we call the square matrix of order $n = n_1 + n_2 - k$ given by

$$C = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{pmatrix}$$

the k-subdirect sum of A and B, denoted by $C = A \oplus_k B$. We can use the elements in A and B to represent any element in C. Before that, let us define the following set of indices:

$$S_1 = \{1, 2, ..., n_1 - k\}, \quad S_2 = \{n_1 - k + 1, n_1 - k + 2, ..., n_1\}, \quad S_3 = \{n_1 + 1, ..., n\}. \tag{1.2}$$

Obviously, $S_1 \cup S_2 \cup S_3 = N := \{1, 2, ..., n\}$. Denoting $C = A \oplus_k B = \left[c_{ij}\right]$, $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, then

$$c_{ij} = \begin{cases} a_{ij}, & i \in S_1, \ j \in S_1 \cup S_2, \\ 0, & i \in S_1, \ j \in S_3, \\ a_{ij}, & i \in S_2, \ j \in S_1, \\ a_{ij} + b_{i-n_1+k, j-n_1+k}, & i \in S_2, \ j \in S_2, \\ b_{i-n_1+k, j-n_1+k}, & i \in S_2, \ j \in S_3, \\ 0, & i \in S_3, \ j \in S_1, \\ b_{i-n_1+k, j-n_1+k}, & i \in S_3, \ j \in S_2 \cup S_3. \end{cases}$$

Definition 1.2. *([20]) Given a matrix $A = \left[a_{ij}\right] \in C^{n \times n}$, where $C^{n \times n}$ is the set of complex matrices. Let*

$$r_i(A) = \sum_{j \in N, j \neq i} \left|a_{ij}\right|, \quad i \in N.$$

$$N_A = \{i \mid |a_{ii}| \le r_i(A)\},$$

$$\overline{N_A} = \{i \mid |a_{ii}| > r_i(A)\}.$$

It is easy to obtain that $\overline{N_A}$ is the complement of N_A in N, i.e., $\overline{N_A} = N \backslash N_A$.

Definition 1.3. *([6]) A matrix* $A = \left[a_{ij}\right] \in C^{n\times n}$ *is called a strictly diagonally dominant (SDD) matrix if*

$$|a_{ii}| > r_i(A), \quad i \in N.$$

Definition 1.4. *([20]) A matrix* $A = \left[a_{ij}\right] \in C^{n\times n}$ *is called a* $GSDD_1$ *matrix if*

$$\begin{cases} r_i(A) > p_i^{\overline{N_A}}(A), & i \in \overline{N_A}, \\ \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) > p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A), & i \in \overline{N_A},\ \ j \in N_A, \end{cases}$$

where

$$p_i^{N_A}(A) := \sum_{j\in N_A\backslash\{i\}} \left|a_{ij}\right|, \ \ p_i^{\overline{N_A}}(A) := \sum_{j\in \overline{N_A}\backslash\{i\}} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right|, \ i \in N.$$

Remark 1.1. *From Definitions 1.3 and 1.4, it is easy to obtain that if a matrix A is an SDD matrix with* $r_i(A) > 0$*, then it is a* $GSDD_1$ *matrix.*

2. Main results

First of all, a counterexample is given to show that the subdirect sum of two $GSDD_1$ matrices may not necessarily be a $GSDD_1$ matrix.

Example 2.1. *Consider the following* $GSDD_1$ *matrices A and B, where*

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 4 & 3 \\ 0 & 1 & 3.5 \end{pmatrix}, \quad B = \begin{pmatrix} 2.5 & 2 & 0 \\ 1 & 2 & 1 \\ 2.3 & 1.8 & 4 \end{pmatrix}.$$

and the 1-subdirect sum $C = A \oplus_1 B$ *is*

$$C = \begin{pmatrix} 4 & 3 & 2 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 \\ 0 & 1 & 6 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2.3 & 1.8 & 4 \end{pmatrix}.$$

However, C is not a $GSDD_1$ *matrix because*

$$\left(r_3(C) - p_3^{\overline{N_C}}(C)\right)\left(|c_{11}| - p_1^{N_C}(C)\right) = (3-0)(4-3) = 3 = 3\times 1 = p_3^{N_C}(C)\, p_1^{\overline{N_C}}(C).$$

Example 2.1 shows that the subdirect sum of $GSDD_1$ matrices is not a $GSDD_1$ matrix. Then, a meaningful discussion is concerned with: under what conditions will the subdirect sum of $GSDD_1$ matrices is in the class of $GSDD_1$ matrices?

In order to obtain the main results, several lemmas are introduced that will be used in the sequel.

Lemma 2.1. *If matrix* $A = \left[a_{ij}\right] \in C^{n\times n}$ *is a* $GSDD_1$ *matrix, then* $\left|a_{jj}\right| - p_j^{N_A}(A) > 0$ *holds for all* $j \in N_A$.

Proof. According to the definition of $GSDD_1$ matrices, we get

$$\left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) > p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A).$$

Since $r_i(A) - p_i^{\overline{N_A}}(A) > 0$, $p_i^{N_A}(A)$, and $p_j^{\overline{N_A}}(A)$ are all nonnegative, $\left|a_{jj}\right| - p_j^{N_A}(A) > 0$ is obtained.

Lemma 2.2. *([20]) If $A = \left[a_{ij}\right] \in C^{n\times n}$ is a $GSDD_1$ matrix, then there is at least one entry $a_{ij} \neq 0$, $i \neq j$, $i \in \overline{N_A}$, $j \in N$.*

Lemma 2.3. *([20]) If $A = \left[a_{ij}\right] \in C^{n\times n}$ is a $GSDD_1$ matrix with $N_A = \emptyset$, then A is an SDD matrix, and there is at least one entry $a_{ij} \neq 0$, $i \neq j$, $i \in \overline{N_A}$, $j \in \overline{N_A}$.*

Now, we consider the 1-subdirect sum of $GSDD_1$ matrices.

Theorem 2.1. *Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let $k = 1$, $S_1 = \{1, 2, \ldots, n_1 - 1\}$, $S_2 = \{n_1\}$, and $S_3 = \{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 - 1\}$. We assume that A is a $GSDD_1$ matrix, and B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), $n_1 \in \overline{N_A}$ and*

$$\frac{r_{n_1}(A)}{\left|a_{n_1,n_1}\right|} \geq \frac{r_{n_1}(A) + r_1(B)}{\left|a_{n_1,n_1} + b_{11}\right|},$$

then the 1-subdirect sum $C = A \oplus_1 B$ is a $GSDD_1$ matrix.

Proof. According to the 1-subdirect sum $C = A \oplus_1 B$, we have

$$r_{n_1}(C) = r_{n_1}(A) + r_1(B).$$

From $n_1 \in \overline{N_A}$, we know $\left|a_{n_1,n_1}\right| > r_{n_1}(A)$. Because all diagonal entries of A_{22} and B_{11} are positive (or negative), we have

$$\left|c_{n_1,n_1}\right| = \left|a_{n_1,n_1} + b_{11}\right| = \left|a_{n_1,n_1}\right| + \left|b_{11}\right| > r_{n_1}(A) + r_1(B) = r_{n_1}(C).$$

Since A is a $GSDD_1$ matrix, B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$, $C = A \oplus_1 B$, and according to Lemmas 2.2 and 2.3, we know that $r_i(C) \neq 0$ for all $i \in \overline{N_C}$. Therefore, for any $i \in \overline{N_C}$,

$$r_i(C) = \sum_{j\in N\setminus\{i\}} \left|c_{ij}\right| > \sum_{j\in\overline{N_C}\setminus\{i\}} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| = p_i^{\overline{N_C}}(C).$$

For any $j \in N_C$, we easily get $j \in N_C \cap S_1 = N_A \cap S_1 \subset S_1$. For the three different selection ranges of i, that is, $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $i \in \overline{N_C} \cap S_2 = \{n_1\}$, and $i \in \overline{N_C} \cap S_3 \subset S_3$, therefore, we divide the proof into three cases.

Case 1. For $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $j \in N_C$, we have

$$r_i(C) = r_i(A),$$

$$\begin{aligned} p_i^{\overline{N_C}}(C) &= \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_1} \frac{r_j(C)}{|c_{jj}|}|c_{ij}| + \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_2} \frac{r_j(C)}{|c_{jj}|}|c_{ij}| + \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_3} \frac{r_j(C)}{|c_{jj}|}|c_{ij}| \\ &= \sum_{j\in\overline{N_A}\backslash\{i\}, j\in S_1} \frac{r_j(A)}{|a_{jj}|}|a_{ij}| + \frac{r_{n_1}(A)+r_1(B)}{|a_{n_1,n_1}+b_{11}|}|a_{i,n_1}| + 0 \\ &\leq \sum_{j\in\overline{N_A}\backslash\{i\}, j\in S_1} \frac{r_j(A)}{|a_{jj}|}|a_{ij}| + \frac{r_{n_1}(A)}{|a_{n_1,n_1}|}|a_{i,n_1}| \\ &= p_i^{\overline{N_A}}(A), \end{aligned}$$

$$|c_{jj}| = |a_{jj}|, \tag{2.1}$$

$$p_j^{N_C}(C) = \sum_{j'\in N_C\backslash\{j\}} |c_{jj'}| = \sum_{j'\in N_A\backslash\{j\}} |a_{jj'}| = p_j^{N_A}(A), \tag{2.2}$$

$$p_i^{N_C}(C) = \sum_{j\in N_C\backslash\{i\}} |c_{ij}| = \sum_{j\in N_A\backslash\{i\}} |a_{ij}| = p_i^{N_A}(A),$$

$$\begin{aligned} p_j^{\overline{N_C}}(C) &= \sum_{j'\in\overline{N_C}\backslash\{j\}, j'\in S_1} \frac{r_{j'}(C)}{|c_{j'j'}|}|c_{jj'}| + \sum_{j'\in\overline{N_C}\backslash\{j\}, j'\in S_2} \frac{r_{j'}(C)}{|c_{j'j'}|}|c_{jj'}| + \sum_{j'\in\overline{N_C}\backslash\{j\}, j'\in S_3} \frac{r_{j'}(C)}{|c_{j'j'}|}|c_{jj'}| \\ &= \sum_{j'\in\overline{N_A}\backslash\{j\}, j'\in S_1} \frac{r_{j'}(A)}{|a_{j'j'}|}|a_{jj'}| + \frac{r_{n_1}(A)+r_1(B)}{|a_{n_1,n_1}+b_{11}|}|a_{j,n_1}| + 0 \\ &\leq \sum_{j'\in\overline{N_A}\backslash\{j\}, j'\in S_1} \frac{r_{j'}(A)}{|a_{j'j'}|}|a_{jj'}| + \frac{r_{n_1}(A)}{|a_{n_1,n_1}|}|a_{j,n_1}| \\ &= p_j^{\overline{N_A}}(A). \end{aligned} \tag{2.3}$$

Therefore, we obtain that

$$\begin{aligned} \left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(|c_{jj}| - p_j^{N_C}(C)\right) &\geq \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(|a_{jj}| - p_j^{N_A}(A)\right) \\ &> p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A) \\ &\geq p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C). \end{aligned}$$

Case 2. For $i \in \overline{N_C} \cap S_2 = \{n_1\}$, $j \in N_C$,

$$r_{n_1}(C) = r_{n_1}(A) + r_1(B),$$

$$p_{n_1}^{N_C}(C) = \sum_{j\in N_C\backslash\{n_1\}} |c_{n_1,j}| = \sum_{j\in N_A\backslash\{n_1\}} |a_{n_1,j}| = p_{n_1}^{N_A}(A).$$

$$\begin{aligned}
p_{n_1}{}^{\overline{N_C}}(C) &= \sum_{j\in\overline{N_C}\setminus\{n_1\}, j\in S_1} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{n_1,j}\right| + \sum_{j\in\overline{N_C}\setminus\{n_1\}, j\in S_3} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{n_1,j}\right| \\
&= \sum_{j\in\overline{N_A}\setminus\{n_1\}} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{n_1,j}\right| + \sum_{j\in\overline{N_B}\setminus\{1\}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{1j}\right| \\
&= p_{n_1}{}^{\overline{N_A}}(A) + p_1{}^{\overline{N_B}}(B).
\end{aligned}$$

We know that the results of the $\left|c_{jj}\right|$, $p_j{}^{N_C}(C)$, and $p_j{}^{\overline{N_C}}(C)$ are the same as (2.1), (2.2), and (2.3). Because B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$, we clearly get

$$r_1(B) - p_1{}^{\overline{N_B}}(B) > 0.$$

Hence,

$$\begin{aligned}
\left(r_{n_1}(C) - p_{n_1}{}^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j{}^{N_C}(C)\right) &= \left(r_{n_1}(A) + r_1(B) - p_{n_1}{}^{\overline{N_A}}(A) - p_1{}^{\overline{N_B}}(B)\right)\left(\left|a_{jj}\right| - p_j{}^{N_A}(A)\right) \\
&> \left(r_{n_1}(A) - p_{n_1}{}^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - p_j{}^{N_A}(A)\right) \\
&> p_{n_1}{}^{N_A}(A)\, p_j{}^{\overline{N_A}}(A) \\
&\geq p_{n_1}{}^{N_C}(C)\, p_j{}^{\overline{N_C}}(C)\,.
\end{aligned}$$

Case 3. For $i \in \overline{N_C} \cap S_3 \subset S_3$, $j \in N_C$, in particular, we obtain that

$$p_i{}^{N_C}(C) = \sum_{j\in N_C\setminus\{i\}} \left|c_{ij}\right| = 0.$$

So we easily come up with

$$\begin{aligned}
\left(r_{n_1}(C) - p_{n_1}{}^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j{}^{N_C}(C)\right) &= \left(r_{n_1}(C) - p_{n_1}{}^{\overline{N_C}}(C)\right)\left(\left|a_{jj}\right| - p_j{}^{N_A}(A)\right) \\
&> 0 \\
&= p_i{}^{N_C}(C)\, p_j{}^{\overline{N_C}}(C)\,.
\end{aligned}$$

From Cases 1–3, we have that for any $i \in \overline{N_C}$ and $j \in N_C$, the C matrix satisfies the definition of the $GSDD_1$ matrix. The conclusion is as follows.

Theorem 2.2. *Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let k, S_1, S_2, and S_3 be as in Theorem 2.1. Likewise, we assume A is a $GSDD_1$ matrix, and B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), $n_1 \in N_A$, $r_{n_1}(A) + r_1(B) \geq \left|a_{n_1,n_1}\right| + |b_{11}|$ and*

$$\min_{2\leq l\leq n_2}(r_l(B) - p_l{}^{\overline{N_B}}(B)) \geq \max_{m\in\overline{N_A}}(r_m(A) - p_m{}^{\overline{N_A}}(A)),$$

$$\min_{m\in\overline{N_A}} p_m{}^{N_A}(A) \geq \max_{2\leq l\leq n_2} |b_{l1}|\,,$$

then $C = A\oplus_1 B$ is a $GSDD_1$ matrix.

Proof. Since A is a $GSDD_1$ matrix, B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$, $n_1 \in N_A$, and $r_{n_1}(A) + r_1(B) \geq \left|a_{n_1,n_1}\right| + |b_{11}|$, we get $n_1 \in N_C$ and then $N_C = N_A$.

For any $i \in \overline{N_C}$, by Lemmas 2.2 and 2.3, we have $r_i(C) \neq 0$ and then

$$r_i(C) = \sum_{j \in N \backslash \{i\}} \left|c_{ij}\right| > \sum_{j \in \overline{N_C} \backslash \{i\}} \frac{r_j(C)}{\left|c_{jj}\right|} \left|c_{ij}\right| = p_i^{\overline{N_C}}(C).$$

Since $n_1 \in N_C$, i.e., $i \in \overline{N_C} \cap S_2 = \emptyset$, we prove it according to the two different selection ranges of i, namely $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$ and $i \in \overline{N_C} \cap S_3 \subset S_3$. For any $j \in N_C$, that is, $j \in N_C \cap S_1 = N_A \cap S_1 \subset S_1$ and $j \in N_C \cap S_2 = N_A \cap S_2 = \{n_1\}$. Therefore, we prove it from the following cases.

Case 1. For $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $j \in N_C \cap S_1 = N_A \cap S_1 \subset S_1$, we obtain that

$$r_i(C) = r_i(A), \tag{2.4}$$

$$\begin{aligned} p_i^{\overline{N_C}}(C) &= \sum_{j \in \overline{N_C} \backslash \{i\}, j \in S_1} \frac{r_j(C)}{\left|c_{jj}\right|} \left|c_{ij}\right| + \sum_{j \in \overline{N_C} \backslash \{i\}, j \in S_3} \frac{r_j(C)}{\left|c_{jj}\right|} \left|c_{ij}\right| \\ &= \sum_{j \in \overline{N_A} \backslash \{i\}} \frac{r_j(A)}{\left|a_{jj}\right|} \left|a_{ij}\right| + 0 \\ &= p_i^{\overline{N_A}}(A), \end{aligned} \tag{2.5}$$

$$\left|c_{jj}\right| = \left|a_{jj}\right|, \tag{2.6}$$

$$p_j^{N_C}(C) = \sum_{j' \in N_C \backslash \{j\}} \left|c_{jj'}\right| = \sum_{j' \in N_A \backslash \{j\}} \left|a_{jj'}\right| = p_j^{N_A}(A), \tag{2.7}$$

$$p_i^{N_C}(C) = \sum_{j \in N_C \backslash \{i\}} \left|c_{ij}\right| = \sum_{j \in N_A \backslash \{i\}} \left|a_{ij}\right| = p_i^{N_A}(A), \tag{2.8}$$

$$\begin{aligned} p_j^{\overline{N_C}}(C) &= \sum_{j' \in \overline{N_C} \backslash \{j\}, j' \in S_1} \frac{r_{j'}(C)}{\left|c_{j'j'}\right|} \left|c_{jj'}\right| + \sum_{j' \in \overline{N_C} \backslash \{j\}, j' \in S_3} \frac{r_{j'}(C)}{\left|c_{j'j'}\right|} \left|c_{jj'}\right| \\ &= \sum_{j' \in \overline{N_A} \backslash \{j\}} \frac{r_{j'}(A)}{\left|a_{j'j'}\right|} \left|a_{jj'}\right| + 0 \\ &= p_j^{\overline{N_A}}(A). \end{aligned} \tag{2.9}$$

Therefore,

$$\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) = \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) > p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A) = p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C).$$

Case 2. For $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $j \in N_C \cap S_2 = N_A \cap S_2 = \{n_1\}$, we know that $r_i(C)$, $p_i^{N_C}(C)$, and $p_i^{\overline{N_C}}(C)$ have the same results as (2.4), (2.8), and (2.5). Moreover,

$$\left|c_{n_1,n_1}\right| = \left|a_{n_1,n_1} + b_{11}\right| = \left|a_{n_1,n_1}\right| + |b_{11}|, \tag{2.10}$$

$$p_{n_1}{}^{N_C}(C) = \sum_{j\prime \in N_C \backslash \{n_1\}} \left|c_{n_1,j\prime}\right| = \sum_{j\prime \in N_A \backslash \{n_1\}} \left|a_{n_1,j\prime}\right| = p_{n_1}{}^{N_A}(A), \tag{2.11}$$

$$\begin{aligned} p_{n_1}{}^{\overline{N_C}}(C) &= \sum_{j\prime \in \overline{N_C} \backslash \{n_1\}, j\prime \in S_1} \frac{r_{j\prime}(C)}{\left|c_{j\prime j\prime}\right|}\left|c_{n_1,j\prime}\right| + \sum_{j\prime \in \overline{N_C} \backslash \{n_1\}, j\prime \in S_3} \frac{r_{j\prime}(C)}{\left|c_{j\prime j\prime}\right|}\left|c_{n_1,j\prime}\right| \\ &= \sum_{j\prime \in \overline{N_A} \backslash \{n_1\}} \frac{r_{j\prime}(A)}{\left|a_{j\prime j\prime}\right|}\left|a_{n_1,j\prime}\right| + \sum_{j\prime \in \overline{N_B} \backslash \{1\}} \frac{r_{j\prime}(B)}{\left|b_{j\prime j\prime}\right|}\left|b_{1j\prime}\right| \\ &= p_{n_1}{}^{\overline{N_A}}(A) + p_1{}^{\overline{N_B}}(B). \end{aligned} \tag{2.12}$$

Hence, we obtain that

$$\begin{aligned} \left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{n_1,n_1}\right| - p_{n_1}{}^{N_C}(C)\right) &= \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{n_1,n_1}\right| + |b_{11}| - p_{n_1}{}^{N_A}(A)\right) \\ &= \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{n_1,n_1}\right| - p_{n_1}{}^{N_A}(A)\right) \\ &+ \left(r_i(A) - p_i^{\overline{N_A}}(A)\right) \cdot |b_{11}| \\ &> p_i^{N_A}(A)\, p_{n_1}{}^{\overline{N_A}}(A) + p_i^{N_A}(A)\, p_1{}^{\overline{N_B}}(B) \\ &= p_i^{N_C}(C)\, p_{n_1}{}^{\overline{N_C}}(C). \end{aligned}$$

Case 3. For $i \in \overline{N_C} \cap S_3 \subset S_3$, $j \in N_C \cap S_1 = N_A \cap S_1 \subset S_1$, we have

$$r_i(C) = r_{i-n_1+1}(B) = r_l(B), \tag{2.13}$$

$$\begin{aligned} p_i^{\overline{N_C}}(C) &= \sum_{j \in \overline{N_C} \backslash \{i\}, j \in S_1} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| + \sum_{j \in \overline{N_C} \backslash \{i\}, j \in S_3} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| \\ &= 0 + \sum_{j \in \overline{N_B} \backslash \{l\}, j \in \{2,\ldots,n_2\}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{lj}\right| \\ &\leq p_l^{\overline{N_B}}(B), \end{aligned} \tag{2.14}$$

$$p_i^{N_C}(C) = \sum_{j\in N_C\backslash\{i\}, j\in S_1} |c_{ij}| + \sum_{j\in N_C\backslash\{i\}, j=n_1} |c_{i,n_1}| = 0 + |b_{l1}| = |b_{l1}|, \tag{2.15}$$

where $l = i - n_1 + 1$. We have the same values of $|c_{jj}|$, $p_j^{N_C}(C)$, and $p_j^{\overline{N_C}}(C)$ as (2.6), (2.7), and (2.9). Therefore,

$$\begin{aligned}
\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(|c_{jj}| - p_j^{N_C}(C)\right) &\geq \left(r_l(B) - p_l^{\overline{N_B}}(B)\right)\left(|a_{jj}| - p_j^{N_A}(A)\right) \\
&\geq \left(r_m(A) - p_m^{\overline{N_A}}(A)\right)\left(|a_{jj}| - p_j^{N_A}(A)\right) \\
&> p_m^{N_A}(A)\, p_j^{\overline{N_A}}(A) \\
&\geq |b_{l1}| \cdot p_j^{\overline{N_A}}(A) \\
&= p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C).
\end{aligned}$$

Case 4. For $i \in \overline{N_C} \cap S_3 \subset S_3$, $j \in N_C \cap S_2 = N_A \cap S_2 = \{n_1\}$, we get that the values of $r_i(C)$, $p_i^{N_C}(C)$, and $p_i^{\overline{N_C}}(C)$ are the same as (2.13), (2.15), and (2.14). Moreover, the results of $|c_{jj}|$, $p_j^{N_C}(C)$, and $p_j^{\overline{N_C}}(C)$ are the same as (2.10), (2.11), and (2.12). Hence, we obtain that

$$\begin{aligned}
\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(|c_{n_1,n_1}| - p_{n_1}^{N_C}(C)\right) &\geq \left(r_l(B) - p_l^{\overline{N_B}}(B)\right)\left(|a_{n_1,n_1}| + |b_{11}| - p_{n_1}^{N_A}(A)\right) \\
&= \left(r_l(B) - p_l^{\overline{N_B}}(B)\right) \cdot |b_{11}| \\
&+ \left(r_l(B) - p_l^{\overline{N_B}}(B)\right)\left(|a_{n_1,n_1}| - p_{n_1}^{N_A}(A)\right) \\
&\geq \left(r_m(A) - p_m^{\overline{N_A}}(A)\right) \cdot |b_{11}| \\
&+ \left(r_m(A) - p_m^{\overline{N_A}}(A)\right)\left(|a_{n_1,n_1}| - p_{n_1}^{N_A}(A)\right) \\
&> p_m^{N_A}(A)\, p_1^{\overline{N_B}}(B) + p_m^{N_A}(A)\, p_{n_1}^{\overline{N_A}}(A) \\
&\geq |b_{l1}| \cdot \left(p_1^{\overline{N_B}}(B) + p_{n_1}^{\overline{N_A}}(A)\right) \\
&= p_i^{N_C}(C)\, p_{n_1}^{\overline{N_C}}(C).
\end{aligned}$$

From Cases 1–4, we definitively get that C is a $GSDD_1$ matrix.

The following Example 2.2 shows that Theorem 2.1 may not necessarily hold when $k \geq 2$.

Example 2.2. *Consider the following matrices:*

$$A = \begin{pmatrix} 3 & 1 & 1.7 & 1 \\ 1 & 4 & 1 & 1 \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix},$$

where A is a $GSDD_1$ matrix and B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$. It is easy to verify that A and B satisfy the conditions of Theorem 2.1 and $A \oplus_1 B$ is a $GSDD_1$ matrix. However, $C = A \oplus_2 B$ is not a $GSDD_1$ matrix. In fact,

$$C = \begin{pmatrix} 3 & 1 & 1.7 & 1 & 0 \\ 1 & 4 & 1 & 1 & 0 \\ 2 & 2 & 7 & 2 & 1 \\ 0 & 1 & 1 & 5 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{pmatrix}.$$

By computation,

$$\overline{N_C} = \{2, 4, 5\}, \quad N_C = \{1, 3\},$$

$$\left(r_5(C) - p_5^{\overline{N_C}}(C)\right)\left(|c_{11}| - p_1^{N_C}(C)\right) = (1-0)(3-1.7) = 1.3 < 1.35 = 1 \times 1.35 = p_5^{N_C}(C)\, p_1^{\overline{N_C}}(C).$$

Therefore, $C = A \oplus_2 B$ is not a $GSDD_1$ matrix.

The following Example 2.3 shows that Theorem 2.2 may not necessarily hold when $k \geq 2$.

Example 2.3. *Consider the following matrices:*

$$A = \begin{pmatrix} 5 & 2 & 2 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 15 & 4.3 \\ 0.9 & -5.1 & 17 \end{pmatrix},$$

where A is a $GSDD_1$ matrix and B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$. It is easy to verify that A and B satisfy the conditions of Theorem 2.2 and $A \oplus_1 B$ is a $GSDD_1$ matrix. However, $C = A \oplus_2 B$ is not a $GSDD_1$. In fact,

$$C = \begin{pmatrix} 5 & 2 & 2 & 1 & 0 \\ 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 1 & 1 & 2 & 17 & 4.3 \\ 0 & 0 & 0.9 & -5.1 & 17 \end{pmatrix}.$$

By computation, $r_3(C) - p_3^{\overline{N_C}}(C) = 0$, therefore, $C = A \oplus_2 B$ is not a $GSDD_1$ matrix.

Those are sufficient conditions to ensure that the 1-subdirect sum of $GSDD_1$ matrices with SDD matrices is a $GSDD_1$ matrix. In fact, as the value of k increases, the situation becomes more complicated, so that the adequate conditions we give will also be more complicated.

Next, some sufficient conditions ensuring that the k-subdirect ($k \geq 2$) sum of $GSDD_1$ matrices with SDD matrices is a $GSDD_1$ matrix are given.

Theorem 2.3. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let $2 \leq k \leq \min\{n_1, n_2\}$, S_1, S_2, and S_3 be as in (1.2). We assume A is a $GSDD_1$ matrix and B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), $i \in \overline{N_A}$ for any $i \in S_2$ and*

$$\sum_{j \in \overline{N_A} \setminus \{i\}, j \in S_2} \frac{\lambda_j}{\left|a_{jj} + b_{j-n_1+k, j-n_1+k}\right|} \left|a_{ij}\right| \leq \sum_{j \in \overline{N_A} \setminus \{i\}, j \in S_2} \frac{r_j(A)}{\left|a_{jj}\right|} \left|a_{ij}\right|, \quad (i \in S_1 \cup S_2)$$

$$\sum_{\substack{j \in \overline{N_B} \setminus \{i-n_1+k\} \\ j \in \{1,\ldots,k\}}} \frac{\lambda_{j+n_1-k}}{\left|a_{j+n_1-k, j+n_1-k} + b_{jj}\right|} \left|b_{i-n_1+k, j}\right| \leq \sum_{\substack{j \in \overline{N_B} \setminus \{i-n_1+k\} \\ j \in \{1,\ldots,k\}}} \frac{r_j(B)}{\left|b_{jj}\right|} \left|b_{i-n_1+k, j}\right|, \quad (i \in S_2)$$

$$\lambda_i \geq r_i(A) + p_{i-n_1+k}^{\overline{N_B}}(B), \quad (i \in S_2)$$

where $\lambda_i = r_i(A) + r_{i-n_1+k}(B) + \sum_{\substack{j=n_1-k+1 \\ j\neq i}}^{n_1} \left|a_{ij} + b_{i-n_1+k,j-n_1+k}\right| - \sum_{\substack{j=n_1-k+1 \\ j\neq i}}^{n_1} \left(\left|a_{ij}\right| + \left|b_{i-n_1+k,j-n_1+k}\right|\right)$, *then the k-subdirect sum* $C = A\oplus_k B$ *is a* $GSDD_1$ *matrix.*

Proof. Since A is a $GSDD_1$ matrix with $i \in \overline{N_A}$ for any $i \in S_2$, we get $|a_{ii}| > r_i(A)$. According to the k-subdirect sum $C = A\oplus_k B$, we have $r_i(C) = \lambda_i \leq r_i(A) + r_{i-n_1+k}(B)$. Because all diagonal entries of A_{22} and B_{11} are positive (or negative), we get $|c_{ii}| = |a_{ii}| + \left|b_{i-n_1+k,i-n_1+k}\right|$. Therefore, we obtain that $|c_{ii}| > r_i(C)$, that is, for any $i \in S_2$, $i \in \overline{N_C}$. Since A is a $GSDD_1$ matrix, B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$, and $C = A\oplus_k B$, by Lemmas 2.2 and 2.3 we know that $r_i(C) \neq 0$ for $i \in \overline{N_C} \cap S_1 \cup S_3 = \overline{N_A} \cap S_1 \cup S_3$. For $i \in S_2$, by sufficient conditions, we have $\lambda_i \geq r_i(A) + p_{i-n_1+k}^{\overline{N_B}}(B)$, which means that $\lambda_i > 0$. Therefore, for any $i \in \overline{N_C}$, we obtain that

$$r_i(C) = \sum_{j\in N\backslash\{i\}} \left|c_{ij}\right| > \sum_{j\in\overline{N_C}\backslash\{i\}} \frac{r_j(C)}{\left|c_{jj}\right|} \left|c_{ij}\right| = p_i^{\overline{N_C}}(C).$$

Moreover, for any $j \in N_C$, we get $j \in N_C \cap S_1 = N_A \cap S_1 \subset S_1$. For any $i \in \overline{N_C}$, similarly, we prove it from the following three cases, which are $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $i \in \overline{N_C} \cap S_2 = \overline{N_A} \cap S_2 \subset S_2$, and $i \in \overline{N_C} \cap S_3 \subset S_3$.

Case 1. For $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $j \in N_C$, we have

$$r_i(C) = r_i(A),$$

$$\begin{aligned}
p_i^{\overline{N_C}}(C) &= \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_1} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| + \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_2} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| + \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_3} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| \\
&= \sum_{j\in\overline{N_A}\backslash\{i\}, j\in S_1} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right| + \sum_{j\in\overline{N_A}\backslash\{i\}, j\in S_2} \frac{\lambda_j}{\left|a_{jj} + b_{j-n_1+k,j-n_1+k}\right|}\left|a_{ij}\right| + 0 \\
&\leq \sum_{j\in\overline{N_A}\backslash\{i\}, j\in S_1} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right| + \sum_{j\in\overline{N_A}\backslash\{i\}, j\in S_2} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right| \\
&= p_i^{\overline{N_A}}(A),
\end{aligned}$$

$$\left|c_{jj}\right| = \left|a_{jj}\right|, \tag{2.16}$$

$$p_j^{N_C}(C) = \sum_{j'\in N_C\backslash\{j\}} \left|c_{jj'}\right| = \sum_{j'\in N_A\backslash\{j\}} \left|a_{jj'}\right| = p_j^{N_A}(A), \tag{2.17}$$

$$p_i^{N_C}(C) = \sum_{j\in N_C\backslash\{i\}} \left|c_{ij}\right| = \sum_{j\in N_A\backslash\{i\}} \left|a_{ij}\right| = p_i^{N_A}(A),$$

$$\begin{aligned}
p_j^{\overline{N_C}}(C) &= \sum_{j'\in\overline{N_C}\backslash\{j\},j'\in S_1} \frac{r_{j'}(C)}{\left|c_{j'j'}\right|}\left|c_{jj'}\right| + \sum_{j'\in\overline{N_C}\backslash\{j\},j'\in S_2} \frac{r_{j'}(C)}{\left|c_{j'j'}\right|}\left|c_{jj'}\right| + \sum_{j'\in\overline{N_C}\backslash\{j\},j'\in S_3} \frac{r_{j'}(C)}{\left|c_{j'j'}\right|}\left|c_{jj'}\right| \quad (2.18)\\
&= \sum_{j'\in\overline{N_A}\backslash\{j\},j'\in S_1} \frac{r_{j'}(A)}{\left|a_{j'j'}\right|}\left|a_{jj'}\right| + \sum_{j'\in\overline{N_A}\backslash\{j\},j'\in S_2} \frac{\lambda_{j'}}{\left|a_{j'j'}+b_{j'-n_1+k,j'-n_1+k}\right|}\left|a_{jj'}\right| + 0\\
&\le \sum_{j'\in\overline{N_A}\backslash\{j\},j'\in S_1} \frac{r_{j'}(A)}{\left|a_{j'j'}\right|}\left|a_{jj'}\right| + \sum_{j'\in\overline{N_A}\backslash\{j\},j'\in S_2} \frac{r_{j'}(A)}{\left|a_{j'j'}\right|}\left|a_{jj'}\right|\\
&= p_j^{\overline{N_A}}(A).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) &\ge \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right)\\
&> p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A)\\
&\ge p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C).
\end{aligned}$$

Case 2. For $i \in \overline{N_C} \cap S_2 = \overline{N_A} \cap S_2 \subset S_2$, $j \in N_C$, we obtain that

$$r_i(C) = \lambda_i,$$

$$\begin{aligned}
p_i^{\overline{N_C}}(C) &= \sum_{j\in\overline{N_C}\backslash\{i\},j\in S_1} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| + \sum_{j\in\overline{N_C}\backslash\{i\},j\in S_2} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| + \sum_{j\in\overline{N_C}\backslash\{i\},j\in S_3} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right|\\
&= \sum_{j\in\overline{N_A}\backslash\{i\},j\in S_1} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right| + \sum_{j\in\overline{N_A}\backslash\{i\},j\in S_2} \frac{\lambda_j}{\left|a_{jj}+b_{j-n_1+k,j-n_1+k}\right|}\left|a_{ij}+b_{i-n_1+k,j-n_1+k}\right|\\
&+ \sum_{\substack{j\in\overline{N_B}\backslash\{i-n_1+k\}\\ j\in\{k+1,\dots,n_2\}}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{i-n_1+k,j}\right|\\
&\le \sum_{j\in\overline{N_A}\backslash\{i\},j\in S_1} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right| + \sum_{j\in\overline{N_A}\backslash\{i\},j\in S_2} \frac{\lambda_j}{\left|a_{jj}+b_{j-n_1+k,j-n_1+k}\right|}\left|a_{ij}\right|\\
&+ \sum_{j\in\overline{N_A}\backslash\{i\},j\in S_2} \frac{\lambda_j}{\left|a_{jj}+b_{j-n_1+k,j-n_1+k}\right|}\left|b_{i-n_1+k,j-n_1+k}\right| + \sum_{\substack{j\in\overline{N_B}\backslash\{i-n_1+k\}\\ j\in\{k+1,\dots,n_2\}}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{i-n_1+k,j}\right|\\
&\le \sum_{j\in\overline{N_A}\backslash\{i\},j\in S_1} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right| + \sum_{j\in\overline{N_A}\backslash\{i\},j\in S_2} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right|\\
&+ \sum_{\substack{j\in\overline{N_B}\backslash\{i-n_1+k\}\\ j\in\{1,\dots,k\}}} \frac{\lambda_{j+n_1-k}}{\left|a_{j+n_1-k,j+n_1-k}+b_{jj}\right|}\left|b_{i-n_1+k,j}\right| + \sum_{\substack{j\in\overline{N_B}\backslash\{i-n_1+k\}\\ j\in\{k+1,\dots,n_2\}}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{i-n_1+k,j}\right|\\
&\le p_i^{\overline{N_A}}(A) + \sum_{\substack{j\in\overline{N_B}\backslash\{i-n_1+k\}\\ j\in\{1,\dots,k\}}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{i-n_1+k,j}\right| + \sum_{\substack{j\in\overline{N_B}\backslash\{i-n_1+k\}\\ j\in\{k+1,\dots,n_2\}}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{i-n_1+k,j}\right|
\end{aligned}$$

$$= \quad p_i^{\overline{N_A}}(A) + p_{i-n_1+k}^{\overline{N_B}}(B),$$

$$p_i^{N_C}(C) = \sum_{j\in N_C\setminus\{i\}} \left|c_{ij}\right| = \sum_{j\in N_A\setminus\{i\}} \left|a_{ij}\right| = p_i^{N_A}(A).$$

We know that $\left|c_{jj}\right|$, $p_j^{N_C}(C)$, and $p_j^{\overline{N_C}}(C)$ are the same as (2.16), (2.17), and (2.18). Therefore,

$$\begin{aligned}
\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) &\geq \left(\lambda_i - p_i^{\overline{N_A}}(A) - p_{i-n_1+k}^{\overline{N_B}}(B)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) \\
&\geq r_i(A) + p_{i-n_1+k}^{\overline{N_B}}(B) - p_i^{\overline{N_A}}(A) - p_{i-n_1+k}^{\overline{N_B}}(B) \\
&\times \left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) \\
&= \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) \\
&> p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A) \\
&\geq p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C).
\end{aligned}$$

Case 3. For $i \in \overline{N_C} \cap S_3 \subset S_3$, $j \in N_C$, specifically, we obtain that

$$p_i^{N_C}(C) = \sum_{j\in N_C/\{i\}} \left|c_{ij}\right| = 0.$$

Hence,

$$\begin{aligned}
\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) &= \left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) \\
&> 0 \\
&= p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C).
\end{aligned}$$

Therefore, we get that $r_i(C) - p_i^{\overline{N_C}}(C) > 0$ and $\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) > p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C)$ for any $i \in \overline{N_C}$, $j \in N_C$.

Corollary 2.1. *Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let $2 \leq k \leq \min\{n_1, n_2\}$, S_1, S_2, and S_3 be as in (1.2). We assume A is a $GSDD_1$ matrix and B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), $i \in \overline{N_A}$ for any $i \in S_2$ and*

$$\frac{\lambda_j}{\left|a_{jj} + b_{j-n_1+k,j-n_1+k}\right|} \leq \min\left\{\frac{r_j(A)}{\left|a_{jj}\right|}, \frac{r_{j-n_1+k}(B)}{\left|b_{j-n_1+k,j-n_1+k}\right|}\right\}, \quad (j \in S_2)$$

$$\lambda_i \geq r_i(A) + p_{i-n_1+k}^{\overline{N_B}}(B),$$

where λ_i is the same as λ_i of Theorem 2.3 and $i \in S_2$, then the k-subdirect sum $C = A \oplus_k B$ is a $GSDD_1$ matrix.

Proof. For the inequality

$$\frac{\lambda_i}{\left|a_{ii}+b_{i-n_1+k,i-n_1+k}\right|}\le\frac{r_i\left(A\right)}{|a_{ii}|},$$

multiplying both sides of this inequality by $\left|a_{ij}\right|$ $(i\in S_1\cup S_2,\ \ j\ne i)$ and summing for every $j\in\overline{N_A}\setminus\{i\}$ $(j\in S_2)$, we have

$$\sum_{j\in\overline{N_A}\setminus\{i\},j\in S_2}\frac{\lambda_j}{\left|a_{jj}+b_{j-n_1+k,j-n_1+k}\right|}\left|a_{ij}\right|\le\sum_{j\in\overline{N_A}\setminus\{i\},j\in S_2}\frac{r_j\left(A\right)}{\left|a_{jj}\right|}\left|a_{ij}\right|.$$

Similarly, for $i\in S_2$, we obtain that

$$\sum_{\substack{j\in\overline{N_B}\setminus\{i-n_1+k\}\\ j\in\{1,\ldots,k\}}}\frac{\lambda_{j+n_1-k}}{\left|a_{j+n_1-k,j+n_1-k}+b_{jj}\right|}\left|b_{i-n_1+k,j}\right|\le\sum_{\substack{j\in\overline{N_B}\setminus\{i-n_1+k\}\\ j\in\{1,\ldots,k\}}}\frac{r_j\left(B\right)}{\left|b_{jj}\right|}\left|b_{i-n_1+k,j}\right|.$$

By Theorem 2.3, we obtain that the k-subdirect sum $C=A\oplus_k B$ is a $GSDD_1$ matrix.

Theorem 2.4. *Let $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let $2\le k\le\min\{n_1,n_2\}$, S_1, S_2, and S_3 be as in (1.2). We assume A is a $GSDD_1$ matrix and B is an SDD matrix with $r_i\left(B\right)>0$ for all $i\in\overline{N_B}$. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), $i\in N_A$ for any $i\in S_2$, $|a_{ii}|+\left|b_{i-n_1+k,i-n_1+k}\right|\le\lambda_i$ and*

$$\left|b_{j-n_1+k,j-n_1+k}\right|-\sum_{\substack{j\prime\in\overline{N_B}\setminus\{j-n_1+k\}\\ j\prime\in\{1,\ldots,k\}}}\left|a_{j,j\prime+n_1-k}+b_{j-n_1+k,j\prime}\right|\ge {p_{j-n_1+k}}^{\overline{N_B}}\left(B\right),\ \ (j\in S_2)$$

$$\min_{k+1\le l\le n_2}(r_l(B)-{p_l}^{\overline{N_B}}(B))\ge\max_{m\in\overline{N_A}}(r_m(A)-{p_m}^{\overline{N_A}}(A)),$$

$$\min_{m\in\overline{N_A}}{p_m}^{N_A}(A)\ge\max_{k+1\le l\le n_2}\sum_{j\in\{1,\ldots,k\}}\left|b_{lj}\right|,$$

where λ_i is the same as λ_i of Theorem 2.3, then the k-subdirect sum $C=A\oplus_k B$ is a $GSDD_1$ matrix.

Proof. Since A is a $GSDD_1$ matrix with $i\in N_A$ for any $i\in S_2$ and $|a_{ii}|+\left|b_{i-n_1+k,i-n_1+k}\right|\le\lambda_i$, we have $|a_{ii}|\le r_i\left(A\right)$ and $|c_{ii}|=|a_{ii}|+\left|b_{i-n_1+k,i-n_1+k}\right|\le\lambda_i=r_i\left(C\right)$, that is, for any $i\in S_2$, we have $i\in N_C$. Moreover, we know that $i\in N_C\cap S_1=N_A\cap S_1\subset S_1$, which means that $N_C=N_A$. Combining Lemmas 2.2 and 2.3, we get that $r_i\left(C\right)\ne 0$ for $i\in\overline{N_C}\cap S_1\cup S_3=\overline{N_A}\cap S_1\cup S_3$. Therefore, for any $i\in\overline{N_C}$, we obtain that

$$r_i\left(C\right)=\sum_{j\in N\setminus\{i\}}\left|c_{ij}\right|>\sum_{j\in\overline{N_C}\setminus\{i\}}\frac{r_j\left(C\right)}{\left|c_{jj}\right|}\left|c_{ij}\right|={p_i}^{\overline{N_C}}\left(C\right).$$

Since $i\in\overline{N_C}\cap S_2=\emptyset$, we prove it from the following two aspects, which are $i\in\overline{N_C}\cap S_1=\overline{N_A}\cap S_1\subset S_1$ and $i\in\overline{N_C}\cap S_3\subset S_3$. For any $j\in N_C$, that is, $j\in N_C\cap S_1=N_A\cap S_1\subset S_1$ and $j\in N_C\cap S_2=N_A\cap S_2\subset S_2$. Therefore, we prove it from the following cases.

Case 1. For $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $j \in N_C \cap S_1 = N_A \cap S_1 \subset S_1$, we get

$$r_i(C) = r_i(A), \tag{2.19}$$

$$\begin{aligned} p_i^{\overline{N_C}}(C) &= \sum_{j \in \overline{N_C} \setminus \{i\}, j \in S_1} \frac{r_j(C)}{|c_{jj}|} |c_{ij}| + \sum_{j \in \overline{N_C} \setminus \{i\}, j \in S_3} \frac{r_j(C)}{|c_{jj}|} |c_{ij}| \\ &= \sum_{j \in \overline{N_A} \setminus \{i\}} \frac{r_j(A)}{|a_{jj}|} |a_{ij}| + 0 \\ &= p_i^{\overline{N_A}}(A), \end{aligned} \tag{2.20}$$

$$|c_{jj}| = |a_{jj}|, \tag{2.21}$$

$$p_j^{N_C}(C) = \sum_{j\prime \in N_C \setminus \{j\}} |c_{jj\prime}| = \sum_{j\prime \in N_A \setminus \{j\}} |a_{jj\prime}| = p_j^{N_A}(A), \tag{2.22}$$

$$p_i^{N_C}(C) = \sum_{j \in N_C \setminus \{i\}} |c_{ij}| = \sum_{j \in N_A \setminus \{i\}} |a_{ij}| = p_i^{N_A}(A), \tag{2.23}$$

$$\begin{aligned} p_j^{\overline{N_C}}(C) &= \sum_{j\prime \in \overline{N_C} \setminus \{j\}, j\prime \in S_1} \frac{r_{j\prime}(C)}{|c_{j\prime j\prime}|} |c_{jj\prime}| + \sum_{j\prime \in \overline{N_C} \setminus \{j\}, j\prime \in S_3} \frac{r_{j\prime}(C)}{|c_{j\prime j\prime}|} |c_{jj\prime}| \\ &= \sum_{j\prime \in \overline{N_A} \setminus \{j\}} \frac{r_{j\prime}(A)}{|a_{j\prime j\prime}|} |a_{jj\prime}| + 0 \\ &= p_j^{\overline{N_A}}(A). \end{aligned} \tag{2.24}$$

Therefore, we obtain that

$$\begin{aligned} \left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(|c_{jj}| - p_j^{N_C}(C)\right) &= \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(|a_{jj}| - p_j^{N_A}(A)\right) \\ &> p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A) \\ &= p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C). \end{aligned}$$

Case 2. For $i \in \overline{N_C} \cap S_1 = \overline{N_A} \cap S_1 \subset S_1$, $j \in N_C \cap S_2 = N_A \cap S_2 \subset S_2$, we know that $r_i(C)$ and (2.19) are equal, $p_i^{\overline{N_C}}(C)$ and (2.20) are equal, and $p_i^{N_C}(C)$ and (2.23) are equal. Moreover,

$$|c_{jj}| = |a_{jj} + b_{j-n_1+k, j-n_1+k}| = |a_{jj}| + |b_{j-n_1+k, j-n_1+k}|, \tag{2.25}$$

$$p_j^{N_C}(C) = \sum_{j\prime \in N_C \setminus \{j\}, j\prime \in S_1} |c_{jj\prime}| + \sum_{j\prime \in N_C \setminus \{j\}, j\prime \in S_2} |c_{jj\prime}| \tag{2.26}$$

$$= \sum_{j\prime\in N_A\backslash\{j\}, j\prime\in S_1} \left|a_{jj\prime}\right| + \sum_{\substack{j\prime\in\overline{N_B}\backslash\{j-n_1+k\}\\ j\prime\in\{1,\ldots,k\}}} \left|a_{j,j\prime+n_1-k} + b_{j-n_1+k,j\prime}\right|,$$

$$\begin{aligned} p_j^{\overline{N_C}}(C) &= \sum_{j\prime\in\overline{N_C}\backslash\{j\}, j\prime\in S_1} \frac{r_{j\prime}(C)}{\left|c_{j\prime j\prime}\right|}\left|c_{jj\prime}\right| + \sum_{j\prime\in\overline{N_C}\backslash\{j\}, j\prime\in S_3} \frac{r_{j\prime}(C)}{\left|c_{j\prime j\prime}\right|}\left|c_{jj\prime}\right| \\ &= \sum_{j\prime\in\overline{N_A}\backslash\{j\}} \frac{r_{j\prime}(A)}{\left|a_{j\prime j\prime}\right|}\left|a_{jj\prime}\right| + \sum_{j\prime\in\{k+1,\ldots,n_2\}} \frac{r_{j\prime}(B)}{\left|b_{j\prime j\prime}\right|}\left|b_{j-n_1+k,j\prime}\right| \\ &\leq p_j^{\overline{N_A}}(A) + p_{j-n_1+k}^{\overline{N_B}}(B). \end{aligned} \tag{2.27}$$

Hence,

$$\begin{aligned} &\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) \\ &= \left(r_i(A) - p_i^{\overline{N_A}}(A)\right) \\ &\times\left(\left|a_{jj}\right| + \left|b_{j-n_1+k,j-n_1+k}\right| - \sum_{j\prime\in N_A\backslash\{j\}, j\prime\in S_1}\left|a_{jj\prime}\right| - \sum_{\substack{j\prime\in\overline{N_B}\backslash\{j-n_1+k\}\\ j\prime\in\{1,\ldots,k\}}}\left|a_{j,j\prime+n_1-k} + b_{j-n_1+k,j\prime}\right|\right) \\ &= \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - \sum_{j\prime\in N_A\backslash\{j\}, j\prime\in S_1}\left|a_{jj\prime}\right|\right) \\ &+ \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|b_{j-n_1+k,j-n_1+k}\right| - \sum_{\substack{j\prime\in\overline{N_B}\backslash\{j-n_1+k\}\\ j\prime\in\{1,\ldots,k\}}}\left|a_{j,j\prime+n_1-k} + b_{j-n_1+k,j\prime}\right|\right) \\ &\geq \left(r_i(A) - p_i^{\overline{N_A}}(A)\right)\left(\left|a_{jj}\right| - p_j^{N_A}(A)\right) + p_i^{N_A}(A)\, p_{j-n_1+k}^{\overline{N_B}}(B) \\ &> p_i^{N_A}(A)\, p_j^{\overline{N_A}}(A) + p_i^{N_A}(A)\, p_{j-n_1+k}^{\overline{N_B}}(B) \\ &\geq p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C). \end{aligned}$$

Case 3. For $i \in \overline{N_C} \cap S_3 \subset S_3$, $j \in N_C \cap S_1 = N_A \cap S_1 \subset S_1$, we obtain that

$$r_i(C) = r_{i-n_1+k}(B) = r_l(B), \tag{2.28}$$

$$\begin{aligned} p_i^{\overline{N_C}}(C) &= \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_1} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| + \sum_{j\in\overline{N_C}\backslash\{i\}, j\in S_3} \frac{r_j(C)}{\left|c_{jj}\right|}\left|c_{ij}\right| \\ &= 0 + \sum_{\substack{j\in\overline{N_B}\backslash\{l\}\\ j\in\{k+1,\ldots,n_2\}}} \frac{r_j(B)}{\left|b_{jj}\right|}\left|b_{lj}\right| \\ &\leq p_l^{\overline{N_B}}(B), \end{aligned} \tag{2.29}$$

$$p_i^{N_C}(C) = \sum_{j\in N_C\setminus\{i\}, j\in S_1} |c_{ij}| + \sum_{j\in N_C\setminus\{i\}, j\in S_2} |c_{ij}| = 0 + \sum_{j\in\{1,\ldots,k\}} |b_{i-n_1+k,j}| = \sum_{j\in\{1,\ldots,k\}} |b_{lj}|, \tag{2.30}$$

where $l = i - n_1 + k$. We know that $|c_{jj}|$, $p_j^{N_C}(C)$, and $p_j^{\overline{N_C}}(C)$ are the same as (2.21), (2.22), and (2.24). Therefore,

$$\begin{aligned}
\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(|c_{jj}| - p_j^{N_C}(C)\right) &\geq \left(r_l(B) - p_l^{\overline{N_B}}(B)\right)\left(|a_{jj}| - p_j^{N_A}(A)\right) \\
&\geq \left(r_m(A) - p_m^{\overline{N_A}}(A)\right)\left(|a_{jj}| - p_j^{N_A}(A)\right) \\
&> p_m^{N_A}(A)\, p_j^{\overline{N_A}}(A) \\
&\geq \sum_{j\in\{1,\ldots,k\}} |b_{lj}| \cdot p_j^{\overline{N_A}}(A) \\
&= p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C).
\end{aligned}$$

Case 4. For $i \in \overline{N_C} \cap S_3 \subset S_3$, $j \in N_C \cap S_2 = N_A \cap S_2 \subset S_2$, we obtain that the values of $r_i(C)$, $p_i^{N_C}(C)$, and $p_i^{\overline{N_C}}(C)$ are equal to (2.28), (2.30), and (2.29). Moreover, the results of $|c_{jj}|$, $p_j^{N_C}(C)$, and $p_j^{\overline{N_C}}(C)$ are the same as (2.25), (2.26), and (2.27). Hence, we arrive at

$$\begin{aligned}
&\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(|c_{jj}| - p_j^{N_C}(C)\right) \\
&\geq \left(r_l(B) - p_l^{\overline{N_B}}(B)\right) \\
&\times\left(|a_{jj}| + |b_{j-n_1+k,j-n_1+k}| - \left(\sum_{j'\in N_A\setminus\{j\}, j'\in S_1} |a_{jj'}| + \sum_{\substack{j'\in\overline{N_B}\setminus\{j-n_1+k\}\\ j'\in\{1,\ldots,k\}}} |a_{j,j'+n_1-k} + b_{j-n_1+k,j'}|\right)\right) \\
&= \left(r_l(B) - p_l^{\overline{N_B}}(B)\right)\left(|a_{jj}| - \sum_{j'\in N_A\setminus\{j\}, j'\in S_1} |a_{jj'}|\right) \\
&+ \left(r_l(B) - p_l^{\overline{N_B}}(B)\right)\left(|b_{j-n_1+k,j-n_1+k}| - \sum_{\substack{j'\in\overline{N_B}\setminus\{j-n_1+k\}\\ j'\in\{1,\ldots,k\}}} |a_{j,j'+n_1-k} + b_{j-n_1+k,j'}|\right) \\
&\geq \left(r_m(A) - p_m^{\overline{N_A}}(A)\right)\left(|a_{jj}| - p_j^{N_A}(A)\right) + \left(r_m(A) - p_m^{\overline{N_A}}(A)\right) p_{j-n_1+k}^{\overline{N_B}}(B) \\
&> p_m^{N_A}(A)\, p_j^{\overline{N_A}}(A) + p_m^{N_A}(A)\, p_{j-n_1+k}^{\overline{N_B}}(B) \\
&= p_m^{N_A}(A)\left(p_j^{\overline{N_A}}(A) + p_{j-n_1+k}^{\overline{N_B}}(B)\right) \\
&\geq \sum_{j\in\{1,\ldots,k\}} |b_{lj}| \cdot \left(p_j^{\overline{N_A}}(A) + p_{j-n_1+k}^{\overline{N_B}}(B)\right) \\
&\geq p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C).
\end{aligned}$$

In conclusion, for any $i \in \overline{N_C}$, $j \in N_C$, we successfully derive that $r_i(C) - p_i^{\overline{N_C}}(C) > 0$ and $\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(|c_{jj}| - p_j^{N_C}(C)\right) > p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C)$. Therefore, $C = A \oplus_k B$ is a $GSDD_1$ matrix.

Example 2.4. *Consider the following matrices:*

$$A=\begin{pmatrix} 7.5 & 1 & 2 & 2 & 1 & 2.5\\ 1 & 7 & 0.3 & 1 & 2 & 0.2\\ 1.1 & 1.3 & 5 & 1 & 0.8 & 1\\ 0.4 & 1 & 0.2 & 6.5 & 1.2 & 0.9\\ 0.3 & 1 & 0.2 & -0.9 & 6.6 & 1.4\\ 0.7 & 0.9 & 0.1 & 1.2 & -1 & 8 \end{pmatrix},\quad B=\begin{pmatrix} 65 & -1.5 & -2 & 1 & 1.5\\ 1.2 & 66 & -2.3 & 1.6 & 0.9\\ -1.4 & 2 & 67 & 1.3 & 1.2\\ 3 & 3.4 & 2 & 66 & 0.6\\ 0.4 & 2.1 & 1 & 1.8 & 77 \end{pmatrix},$$

where A is a $GSDD_1$ matrix with $i \in \overline{N_A}$ for all $i \in S_2$, and B is an SDD matrix with $r_i(B) > 0$ for all $i \in \overline{N_B}$. By computation, we derive $N_A = \{1, 3\}$, $\overline{N_A} = \{2, 4, 5, 6\}$. Moreover,

$$\frac{\lambda_4}{|a_{44}+b_{11}|}=\frac{5.5}{71.5}\approx 0.077<0.569\approx\frac{3.7}{6.5}=\frac{r_4(A)}{|a_{44}|},\quad \frac{\lambda_5}{|a_{55}+b_{22}|}=\frac{5.2}{72.6}\approx 0.072<0.576\approx\frac{3.8}{6.6}=\frac{r_5(A)}{|a_{55}|},$$

$$\frac{\lambda_6}{|a_{66}+b_{33}|}=\frac{5.4}{75}=0.072<0.488\approx\frac{3.9}{8}=\frac{r_6(A)}{|a_{66}|},$$

we get that $\sum\limits_{j\in\overline{N_A}\backslash\{i\}, j\in S_2}\frac{\lambda_j}{|a_{jj}+b_{j-n_1+k,j-n_1+k}|}\left|a_{ij}\right| \le \sum\limits_{j\in\overline{N_A}\backslash\{i\}, j\in S_2}\frac{r_j(A)}{|a_{jj}|}\left|a_{ij}\right|$ *is true for $i \in S_1 \cup S_2$.*

$$\frac{\lambda_4}{|a_{44}+b_{11}|}\approx 0.077<0.092\approx\frac{6}{65}=\frac{r_1(B)}{|b_{11}|},\quad \frac{\lambda_5}{|a_{55}+b_{22}|}\approx 0.072<0.091\approx\frac{6}{66}=\frac{r_2(B)}{|b_{22}|},$$

$$\frac{\lambda_6}{|a_{66}+b_{33}|}=0.072<0.088\approx\frac{5.9}{67}=\frac{r_3(B)}{|b_{33}|},$$

we have that the second sufficient condition in Theorem 2.3 is true.

$$\lambda_4 = 5.5 > 4.252 = 3.7 + 0.552 \approx r_4(A) + p_1^{\overline{N_B}}(B),\quad \lambda_5 = 5.2 > 4.393 = 3.8 + 0.593 \approx r_5(A) + p_2^{\overline{N_B}}(B),$$

$$\lambda_6 = 5.4 > 4.471 = 3.9 + 0.571 \approx r_6(A) + p_3^{\overline{N_B}}(B),$$

we get that the third sufficient condition in Theorem 2.3 is met. Therefore, by Theorem 2.3, $C = A\oplus_3 B$ is a $GSDD_1$ matrix. In fact,

$$C=\begin{pmatrix} 7.5 & 1 & 2 & 2 & 1 & 2.5 & 0 & 0\\ 1 & 7 & 0.3 & 1 & 2 & 0.2 & 0 & 0\\ 1.1 & 1.3 & 5 & 1 & 0.8 & 1 & 0 & 0\\ 0.4 & 1 & 0.2 & 71.5 & -0.3 & -1.1 & 1 & 1.5\\ 0.3 & 1 & 0.2 & 0.3 & 72.6 & -0.9 & 1.6 & 0.9\\ 0.7 & 0.9 & 0.1 & -0.2 & 1 & 75 & 1.3 & 1.2\\ 0 & 0 & 0 & 3 & 3.4 & 2 & 66 & 0.6\\ 0 & 0 & 0 & 0.4 & 2.1 & 1 & 1.8 & 77 \end{pmatrix},$$

where $N_C = \{1, 3\}$, $\overline{N_C} = \{2, 4, 5, 6, 7, 8\}$. *By computation,*

$$r_2(C) = 4.5,\ \ p_2^{\overline{N_C}}(C) \approx 0.235,\ \ p_2^{N_C}(C) = 1.3;\ \ r_4(C) = 5.5,\ \ p_4^{\overline{N_C}}(C) \approx 0.983,\ \ p_4^{N_C}(C) = 0.6;$$

$$r_5(C) = 5.2,\ \ p_5^{\overline{N_C}}(C) \approx 1.011,\ \ p_5^{N_C}(C) = 0.5;\ \ r_6(C) = 5.4,\ \ p_6^{\overline{N_C}}(C) \approx 0.925,\ \ p_6^{N_C}(C) = 0.8;$$

$$r_7(C) = 9,\ \ p_7^{\overline{N_C}}(C) \approx 0.66,\ \ p_7^{N_C}(C) = 0;\ \ r_8(C) = 5.3,\ \ p_8^{\overline{N_C}}(C) \approx 0.499,\ \ p_8^{N_C}(C) = 0;$$

$$|c_{11}| = 7.5,\ \ p_1^{N_C}(C) = 2,\ \ p_1^{\overline{N_C}}(C) \approx 1.048;\ \ |c_{33}| = 5,\ \ p_3^{N_C}(C) = 1.1,\ \ p_3^{\overline{N_C}}(C) \approx 1.042.$$

It is not difficult to find that $r_i(C) - p_i^{\overline{N_C}}(C) > p_i^{N_C}(C)$ *and* $\left|c_{jj}\right| - p_j^{N_C}(C) > p_j^{\overline{N_C}}(C)$ *when* $i \in \overline{N_C}$, $j \in N_C$. *So we deduce that* $r_i(C) > p_i^{\overline{N_C}}(C)$ *and* $\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) > p_i^{N_C}(C)\, p_j^{\overline{N_C}}(C)$ *are true when* $i \in \overline{N_C}$, $j \in N_C$. *Thus,* $C = A \oplus_3 B$ *is a* $GSDD_1$ *matrix.*

Example 2.5. *Consider the following matrices:*

$$A = \begin{pmatrix} 6 & 2 & 0.5 & 1 & 1 & 0.8 & 1.2 \\ 0.1 & 8 & 0.7 & 0.3 & 1 & 1.3 & 0.8 \\ 0.5 & 0.8 & 7.7 & 1.1 & 1.2 & 0.3 & 0.1 \\ 2.1 & 1.5 & 0.9 & 8 & 1.8 & 0.6 & 1.7 \\ 0.3 & 0.7 & 1.4 & 1 & 8.4 & 2.5 & 2.8 \\ 1.6 & 2.5 & 2 & 1 & 1.7 & 9.2 & 1.1 \\ 0.8 & 1.2 & 1.6 & 2.4 & 1.8 & 1.5 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 40 & 2 & 1 & 0.7 & 15 & 20.8 \\ 1.2 & 45 & 3 & 2.5 & 19 & 19.2 \\ 2.5 & 2.1 & 54 & 1.1 & 21 & 26.9 \\ 1.8 & 2.4 & 0.9 & 61 & 25 & 30.8 \\ 0.5 & 1 & 1.3 & 0.2 & 65 & 10 \\ 1.4 & 0.4 & 0.7 & 0.5 & 9.9 & 68 \end{pmatrix},$$

where A *is a* $GSDD_1$ *matrix and* B *is an* SDD *matrix with* $r_i(B) > 0$ *for all* $i \in \overline{N_B}$. *By computation,* $N_A = \{1, 4, 5, 6, 7\}, S_2 = \{4, 5, 6, 7\}$,

$$|a_{44}| + |b_{11}| = 48 < 48.1 = \lambda_4,\ \ |a_{55}| + |b_{22}| = 53.4 < 53.6 = \lambda_5,$$

$$|a_{66}| + |b_{33}| = 63.2 < 63.5 = \lambda_6,\ \ |a_{77}| + |b_{44}| = 70 < 70.2 = \lambda_7.$$

Moreover,

$$|b_{11}| - \sum_{j' \in \overline{N_B} \setminus \{1\}, j' \in \{1,\ldots,4\}} \left|a_{4,j'+3} + b_{1j'}\right| = 32.2 > 10.633 \approx p_1^{\overline{N_B}}(B),$$

$$|b_{22}| - \sum_{j' \in \overline{N_B} \setminus \{2\}, j' \in \{1,\ldots,4\}} \left|a_{5,j'+3} + b_{2j'}\right| = 32 > 14.101 \approx p_2^{\overline{N_B}}(B),$$

$$|b_{33}| - \sum_{j' \in \overline{N_B} \setminus \{3\}, j' \in \{1,\ldots,4\}} \left|a_{6,j'+3} + b_{3j'}\right| = 44.5 > 14.965 \approx p_3^{\overline{N_B}}(B),$$

$$|b_{44}| - \sum_{j\prime\in\overline{N_B}\backslash\{4\},j\prime\in\{1,\ldots,4\}} \left|a_{7,j\prime+3} + b_{4j\prime}\right| = 50.2 > 15.908 \approx p_4^{\overline{N_B}}(B).$$

$$\begin{aligned}\min_{k+1\le l\le n_2}(r_l(B) - p_l^{\overline{N_B}}(B)) &= r_6(B) - p_6^{\overline{N_B}}(B) \approx 7.944\\ &> 3.836 \approx r_2(A) - p_2^{\overline{N_A}}(A) = \max_{m\in\overline{N_A}}(r_m(A) - p_m^{\overline{N_A}}(A)),\end{aligned}$$

$$\min_{m\in\overline{N_A}} p_m^{N_A}(A) = p_3^{N_A}(A) = 3.2 > 3 = \sum_{j\in\{1,\ldots,4\}}\left|b_{5j}\right| = \sum_{j\in\{1,\ldots,4\}}\left|b_{6j}\right| = \max_{k+1\le l\le n_2}\sum_{j\in\{1,\ldots,k\}}\left|b_{lj}\right|.$$

Hence, the conditions in Theorem 2.4 are met. By Theorem 2.4, $C = A\oplus_4 B$ is a $GSDD_1$ matrix. In fact,

$$C = \begin{pmatrix} 6 & 2 & 0.5 & 1 & 1 & 0.8 & 1.2 & 0 & 0\\ 0.1 & 8 & 0.7 & 0.3 & 1 & 1.3 & 0.8 & 0 & 0\\ 0.5 & 0.8 & 7.7 & 1.1 & 1.2 & 0.3 & 0.1 & 0 & 0\\ 2.1 & 1.5 & 0.9 & 48 & 3.8 & 1.6 & 2.4 & 15 & 20.8\\ 0.3 & 0.7 & 1.4 & 2.2 & 53.4 & 5.5 & 5.3 & 19 & 19.2\\ 1.6 & 2.5 & 2 & 3.5 & 3.8 & 63.2 & 2.2 & 21 & 26.9\\ 0.8 & 1.2 & 1.6 & 4.2 & 4.2 & 2.4 & 70 & 25 & 30.8\\ 0 & 0 & 0 & 0.5 & 1 & 1.3 & 0.2 & 65 & 10\\ 0 & 0 & 0 & 1.4 & 0.4 & 0.7 & 0.5 & 9.9 & 68 \end{pmatrix}.$$

By computation, $N_C = \{1,4,5,6,7\}$, $\overline{N_C} = \{2,3,8,9\}$. Moreover,

$$r_2(C) = 4.2,\ \ p_2^{\overline{N_C}}(C) \approx 0.364,\ \ p_2^{N_C}(C) = 3.5;\ \ r_3(C) = 4,\ \ p_3^{\overline{N_C}}(C) = 0.42,\ \ p_3^{N_C}(C) = 3.2;$$

$$r_8(C) = 13,\ \ p_8^{\overline{N_C}}(C) \approx 1.897,\ \ p_8^{N_C}(C) = 3;\ \ r_9(C) = 12.9,\ \ p_9^{\overline{N_C}}(C) = 1.98,\ \ p_9^{N_C}(C) = 3.$$

$$|c_{11}| = 6,\ \ p_1^{N_C}(C) = 4,\ \ p_1^{\overline{N_C}}(C) \approx 1.31;\ \ |c_{44}| = 48,\ \ p_4^{N_C}(C) = 9.9,\ \ p_4^{\overline{N_C}}(C) \approx 8.201;$$

$$|c_{55}| = 53.4,\ \ p_5^{N_C}(C) = 13.3,\ \ p_5^{\overline{N_C}}(C) \approx 8.537;\ \ |c_{66}| = 63.2,\ \ p_6^{N_C}(C) = 11.1,\ \ p_6^{\overline{N_C}}(C) \approx 11.655;$$

$$|c_{77}| = 70,\ \ p_7^{N_C}(C) = 11.6,\ \ p_7^{\overline{N_C}}(C) \approx 12.304.$$

We see that $r_i(C) - p_i^{\overline{N_C}}(C) > p_i^{N_C}(C)$ and $\left|c_{jj}\right| - p_j^{N_C}(C) > p_j^{\overline{N_C}}(C)$ when $i \in \overline{N_C}$, $j \in N_C$. Therefore, we obtain that $r_i(C) > p_i^{\overline{N_C}}(C)$ and $\left(r_i(C) - p_i^{\overline{N_C}}(C)\right)\left(\left|c_{jj}\right| - p_j^{N_C}(C)\right) > p_i^{N_C}(C)\,p_j^{\overline{N_C}}(C)$ are true when $i \in \overline{N_C}$, $j \in N_C$. Therefore, $C = A\oplus_4 B$ is a $GSDD_1$ matrix.

Remark 2.1. *Since the subdirect sum of matrices does not satisfy the commutative law, if we change "A is a $GSDD_1$ matrix, and B is an SDD matrix" to "A is an SDD matrix, and B is a $GSDD_1$ matrix", then we will obtain new sufficient conditions by using similar proofs in this paper.*

3. Conclusions

In this paper, some sufficient conditions are given to show that the subdirect sum of $GSDD_1$ matrices with SDD matrices is in the class of $GSDD_1$ matrices, and these conditions are only dependent on the elements of the given matrices. Furthermore, some numerical examples are also presented to illustrate the corresponding theoretical results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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